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Wave Operators for the Coupled Klein-Gordon-Schrödinger Equations in Two Space Dimensions

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Abstract. In this paper, the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions is studied. The scattering problem for this equation belongs to the borderline between the short range case and the long range one. The existence of the wave operators to this equation for small scattered states is proved without any restrictions on the support of the Fourier transform of them.

Key Words and Phrases. Klein-Gordon-Schrödinger equations, Asymptotic behavior of solutions, Scattering theory, Wave operators.

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1. Introduction

We study the scattering theory for the coupled Klein-Gordon-Schrödinger equation with the Yukawa type interaction in two space dimensions:

$$\text{(KGS)} \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = uv, \\ \partial_t^2 v - \Delta v + v = -|u|^2. \end{cases}$$

Here u and v are complex and real valued unknown functions of $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^2$, respectively. In the present paper, we prove the existence of wave operators to the equation (KGS) for small scattered states without any restrictions on the support of the Fourier transform of them.

The equation (KGS) describes a classical model of the Yukawa interaction of complex nucleon field with neutral real meson field. Here u is a complex scalar nucleon field and v is a real scalar meson field.

First we recall some results of the scattering theory for the (decoupled) nonlinear Schrödinger equation

$$\text{(NLS)} \qquad i\partial_t u + \frac{1}{2}\Delta u = |u|^{p-1}u,$$

$(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n$, and the (decoupled) nonlinear Klein-Gordon equation

$$\text{(NLKG)} \qquad \partial_t^2 v - \Delta v + v = -|v|^{p-1}v,$$

$(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^n$. There are amount of papers concerning the asymptotic behavior of solutions for the nonlinear Schrödinger equation (see [3, 6, 9, 10, 11, 12, 14, 17, 18, 22, 24, 25, 26, 27, 35, 36, 37, 38, 39, 40, 41]), and for the nonlinear Klein-Gordon equation (see [4, 5, 13, 20, 19, 21, 23, 24, 28, 30, 34, 35, 36, 37]). We consider the existence of wave operators $W_\pm$. For the equation (NLS), the wave operator W_+ is defined as follows. Let Σ be L^2 or a dense subset of it. Let $\phi \in \Sigma$, and let

$$u_0(t,x) \equiv (U(t)\phi)(x),$$

where

$$U(t) \equiv e^{(it/2)\Delta}.$$

Note that u_0 is a solution to the Cauchy problems of the free Schrödinger equation

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = 0, \\ u(0,x) = \phi(x). \end{cases}$$

If there exists a unique global solution u of the equation (NLS) such that

$$\|u(t) - u_0(t)\|_{L^2} \to 0,$$

as $t \to +\infty$, then a mapping

$$W_+ : \phi \mapsto u(0)$$

is well-defined on Σ. We call the mapping W_+ a wave operator. ϕ is called a scattered state or a scattered data. In the same way as above, we define the wave operator W_- for the case of $t \to -\infty$. We put

$$v_0(t,x) \equiv ((\cos \omega t)\psi_0)(x) + ((\omega^{-1} \sin \omega t)\psi_1)(x),$$

where

$$\omega \equiv (1 - \Delta)^{1/2}.$$

The function v_0 is a unique solution of the free Klein-Gordon equation

$$\begin{cases} \partial_t^2 v - \Delta v + v = 0, \\ v(0,x) = \psi_0(x), \qquad \partial_t v(0,x) = \psi_1(x). \end{cases}$$

For the equation (NLKG) and the coupled equation (KGS), we can define the wave operators $W_\pm$ in the same way.

It is known that when $p > 1 + 2/n$ and $1 \leq n \leq 3$, there exist wave operators for the equations (NLS) and (NLKG), and that when $1 \leq p \leq 1 + 2/n$, we cannot define wave operators of the equations (NLS) and (NLKG) for any nontrivial scattered states in all space dimensions (see, e.g., [3, 10, 13, 18, 37, 38, 39, 40]). Namely, quadratic nonlinearity is the critical power in two space dimensions. Intuitive mean of these facts is as follows. Recalling the well-known time decay estimates $\|u_0(t)\|_{L^2(\boldsymbol{R}^n)}, \|v_0(t)\|_{L^2(\boldsymbol{R}^n)} = O(1)$, and $\|u_0(t)\|_{L^\infty(\boldsymbol{R}^n)}, \|v_0(t)\|_{L^\infty(\boldsymbol{R}^n)} = O(t^{-n/2})$, we see that $\| |u_0(t)|^p\|_{L^2(\boldsymbol{R}^n)}$, $\| |v_0(t)|^p\|_{L^2(\boldsymbol{R}^n)} = O(t^{-n(p-1)/2})$. Roughly speaking, according the linear scattering theory (the Cook-Kuroda method), wave operators exist if and only if $\| |u_0(t)|^p\|_{L^2(\boldsymbol{R}^n)}$ and $\| |v_0(t)|^p\|_{L^2(\boldsymbol{R}^n)}$ are integrable with respect to t at infinity, that is, $p > 1 + 2/n$. In the case of $p = 1 + 2/n$ and $1 \leq n \leq 3$, modified wave operators of the equation (NLS) were constructed for small scattered states (see Ozawa [26] for $n = 1$, and Ginibre and Ozawa [9] for $n = 2$ or 3).

Recently, for the nonlinear Schrödinger equations with the nonlinearity u^2 or $\bar{u}^2$ in two space dimensions, Moriyama, Tonegawa and Tsutsumi [22] have proved the existence of the wave operators for small data. The main part of their proof is as follows. Using the oscillation of the asymptotic form for u_0^2 and $\overline{u_0}^2$, they could construct suitable approximate functions (asymptotic profiles) such that $(i\partial_t + (1/2)\Delta)u_2 - u_0^2$ [resp. $(i\partial_t + (1/2)\Delta)u_2 - \overline{u_0}^2$] decays faster than u_0^2 [resp. $\overline{u_0}^2$] as $t \to \infty$.

We turn to the coupled Klein-Gordon-Schrödinger equation. The time global well-posedness for the equation (KGS) is well-known (see [1, 2, 7, 15]). Fukuda and M. Tsutsumi [8] and Strauss [36] studied the asymptotic behavior of the solutions to the coupled Klein-Gordon-Schrödinger equations with interactions higher than the quadratic order. Their results are similar to those of the decoupled nonlinear Schrödinger and Klein-Gordon equations.

According to the results of the scattering theory for the decoupled nonlinear Schrödinger and Klein-Gordon equations mentioned above, the coupled equation (KGS), which has a quadratic interaction, in two space dimensions has a critical interaction. Thus it is expected that the scattering problem for the equation (KGS) in two space dimensions is more difficult than that of the coupled Klein-Gordon-Schrödinger equations with interactions higher than the quadratic order. This problem was firstly studied by Ozawa and Y. Tsutsumi [29]. They proved the existence of the wave operators for the equation (KGS) in two space dimensions not only under the smallness conditions on scattered states (ϕ, ψ_0, ψ_1) but also under the restriction on support of the Fourier transform of the Schrödinger scattered state ϕ. Their proof is based on the special property of the Yukawa interaction and on the improved decay estimates of the interaction term which take account of the difference between the propagation property of the Schrödinger wave and the Klein-Gordon wave.

Roughly speaking, the reason why they assumed the condition on the support of the Fourier transform of ϕ is as follows. The time decay estimates $\|u_0(t)\|_{L^2(\boldsymbol{R}^2)}, \|v_0(t)\|_{L^2(\boldsymbol{R}^2)} = O(1)$ and $\|u_0(t)\|_{L^\infty(\boldsymbol{R}^2)}, \|v_0(t)\|_{L^\infty(\boldsymbol{R}^2)} = O(t^{-1})$ imply a decay estimate of the interaction term $\|u_0(t)v_0(t)\|_{L^2(\boldsymbol{R}^2)} = O(t^{-1})$, which is not integrable at infinity. Thus the existence of wave operators cannot be proved from this estimate directly. To overcome this difficulty, we note that the property of finite propagation speed for the Klein-Gordon equation implies a time decay estimate $\|v_0(t)\|_{L^\infty(|x|\geq(1+\varepsilon)t)} = O_{\varepsilon,N}(t^{-N})$, for any $\varepsilon, N > 0$. If $\operatorname{supp}\hat{\phi} \subset \{\xi \in \boldsymbol{R}^2 : |\xi| \geq 1+\varepsilon\}$ for some $\varepsilon > 0$, then an improved time decay estimate of the L^2-norm of the interaction term u_0v_0 follows from the estimate above: $\|u_0(t)v_0(t)\|_{L^2(\boldsymbol{R}^2)} \sim t^{-1}\|\hat{\phi}(\cdot/t)v_0(t)\|_{L^2(\boldsymbol{R}^2)} = t^{-1}\|\hat{\phi}(\cdot/t)v_0(t)\|_{L^2(|x|\geq(1+\varepsilon)t)} = O_{\varepsilon,N}(t^{-N})$ as $t \to \infty$ for any $N > 0$.

In this paper, we prove the existence of wave operators of the equation (KGS) under the certain smallness conditions on the scattered states. We do not assume any restrictions on the support of the Fourier transform of the scattered states, which are assumed in [29]. The proof is mainly based on the construction of suitable second approximations (u_2, v_2) of the solution to the equation (KGS) such that $(i\partial_t + (1/2)\Delta)u_2 - u_0v_0$ and $(\partial_t^2 - \Delta + 1)v_2 + |u_0|^2$ decay faster than u_0v_0 and $|u_0|^2$ as $t \to \infty$, respectively, and that the Cook-Kuroda method is applicable (Section 3). This method is also used in [31, 32, 33].

Before stating our main result, we introduce some notations and functions.

Notations. We use the following symbols:

$$\partial_t = \frac{\partial}{\partial t}, \qquad \partial_j = \frac{\partial}{\partial x_j} \qquad \text{for } j = 1, 2,$$

$$\partial^\alpha = \partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2} \qquad \text{for a multi-index } \alpha = (\alpha_1, \alpha_2),$$

$$\nabla = (\partial_1, \partial_2), \qquad \Delta = \partial_1^2 + \partial_2^2,$$

for $t \in \boldsymbol{R}$ and $x = (x_1, x_2) \in \boldsymbol{R}^2$.

Let $L^q \equiv L^q(\boldsymbol{R}^2) = \{\psi : \|\psi\|_{L^q} = (\int_{\boldsymbol{R}^2} |\psi(x)|^q dx)^{1/q} < \infty\}$ for $1 \leq q < \infty$, and let $L^\infty \equiv L^\infty(\boldsymbol{R}^2) = \{\psi : \|\psi\|_{L^\infty} = \operatorname{ess.sup}_{x\in\boldsymbol{R}^2}|\psi(x)| < \infty\}$.

For $w \in L^1(\boldsymbol{R}^n)$, $\hat{w}$ denotes the Fourier transform of w:

$$\hat{w}(\xi) = (2\pi)^{-n/2}\int_{\boldsymbol{R}^n} w(x)e^{-ix\cdot\xi}\,dx.$$

For $w \in \mathscr{S}'$, we denote the Fourier transform of w by $\hat{w}$, wherer $\mathscr{S}'$ be the set of tempered distributions on $\boldsymbol{R}^2$.

For $s, m \in \boldsymbol{R}$, we introduce the weighted Sobolev spaces $H^{s,m}$ corresponding to the Lebesgue space L^2 as follows:

$$H^{s,m} \equiv \{\psi \in \mathscr{S}' : \|\psi\|_{H^{s,m}} \equiv \|(1+|x|^2)^{m/2}(1-\Delta)^{s/2}\psi\|_{L^2} < \infty\}.$$

We also denote $H^{s,0}$ by H^s. For $1 \le p \le \infty$ and a positive integer k, we define the Sobolev space W_p^k corresponding to the Lebesgue space L^p by

$$W_p^k \equiv \left\{\psi \in L^p : \|\psi\|_{W_p^k} \equiv \sum_{|\alpha| \le k} \|\partial^\alpha \psi\|_{L^p} < \infty\right\}.$$

Note that for a positive integer k, $H^k = W_2^k$ and the norms $\|\cdot\|_{H^k}$ and $\|\cdot\|_{W_2^k}$ are equivalent.

We denote various constants by C and so forth. They may differ from line to line, when it does not cause any confusion.

For $|x| < 1$, we introduce the following function:

$$g(x) \equiv \frac{2}{2(1-|x|^2)^{3/2} - |x|^2}. \tag{1.1}$$

The function g appears in construction of the second approximation u_2 (Section 3).

For the scattered states (ϕ, ψ_0, ψ_1), we define

$$\begin{aligned}\delta \equiv\ & \|\phi\|_{H^{2,4}} + \|g^5\omega^4\hat{\phi}\|_{L^2(D)} + \|\psi_0\|_{H^{12,7}} + \|\psi_0\|_{W_1^4} \\ & + \|\psi_1\|_{H^{11,7}} + \|\psi_1\|_{W_1^3} \\ \le\ & 1,\end{aligned} \tag{1.2}$$

where $D \equiv \{x \in \boldsymbol{R}^2 : |x| < 1\}$. Hereafter we assume that δ is sufficiently small.

Our main result is as follows.

Theorem. *Assume that $\phi \in H^{2,4}$ and that $g^5\omega^4\hat{\phi} \in L^2(D)$. Let $\psi_0 \in H^{12,7} \cap W_1^4$ and $\psi_1 \in H^{11,7} \cap W_1^3$. Assume that $\delta > 0$ is sufficiently small. Then there exists a unique solution (u, v) of the equation* (KGS) *such that*

$$u \in C([0,\infty); H^2),$$

$$v \in C([0,\infty); H^2) \cap C^1([0,\infty); H^1),$$

$$\sup_{t\ge 0}\left[(1+t)\left\{\|u(t)-u_0(t)\|_{H^2} + \left(\int_t^\infty \|u(s)-u_0(s)\|_{W_4^2}^4 ds\right)^{1/4}\right\}\right] < \infty,$$

$$\sup_{t \ge 0}[(1+t)(\|v(t)-v_0(t)\|_{H^2} + \|\partial_t v(t) - \partial_t v_0(t)\|_{H^1})] < \infty.$$

A similar result holds for negative time.

Let $\mathscr{V}$ be the set of all functions (ϕ, ψ_0, ψ_1) such that $\phi \in H^{2,4}$, $g^5\omega^4\hat{\phi} \in L^2(D)$, $\psi_0 \in H^{12,7} \cap W_1^4$, $\psi_1 \in H^{11,7} \cap W_1^3$ and that $\delta > 0$ defined in (1.2) is sufficiently small as in Theorem. Namely $\mathscr{V}$ is the set of all scattered states (ϕ, ψ_0, ψ_1) satisfying all the assumptions of Theorem.

The following corollary is an immediate consequence of our theorem.

Corollary. *For the equation* (KGS), *the wave operator* $W_+ : (\phi, \psi_0, \psi_1) \mapsto (u(0), v(0), \partial_t v(0))$ *is well-defined on* $\mathscr{V}$, *where* (u, v) *is the solution to the equation* (KGS) *obtained in Theorem. Similarly the wave operator* W_- *for negative time is also well-defined on* $\mathscr{V}$.

Remark 1.1. We note that the function g has singularities on the circle $C_l = \{x \in D : |x| = l\}$ for some $l < 1$. The assumption $g^5\omega^4\hat{\phi} \in L^2(D)$ in Theorem implies that the function $\omega^4\hat{\phi}$ is almost vanished on the circle C_l.

We explain our idea of the proof of Theorem. Let $(\tilde{u}, \tilde{v})$ be an asymptotic profile generated by the scattered state (ϕ, ψ_0, ψ_1) satisfying $\tilde{u}(t) - u_0(t)$, $\tilde{v}(t) - v_0(t) \to 0$ in H^2, as $t \to \infty$, and let $F = u - \tilde{u}$, $G = v - \tilde{v}$. We note that solving the equation (KGS) with $u(t) - \tilde{u}(t), v(t) - \tilde{v}(t) \to 0$ in H^2, and $\partial_t v(t) - \partial_t \tilde{v}(t) \to 0$ in H^1 as $t \to \infty$ is equivalent to solving the following coupled integral equation

$$\begin{cases} F(t) = i\displaystyle\int_t^\infty U(t-s)[(F(s) + \tilde{u}(s))(G(s) + \tilde{v}(s)) - \mathscr{L}\tilde{u}(s)]ds, \\ G(t) = \displaystyle\int_t^\infty (\omega^{-1} \sin \omega(t-s))[|F(s) + \tilde{u}(t)|^2 + \mathscr{N}\tilde{v}(s)]ds, \end{cases}$$

with $F(t), G(t) \to 0$ in H^2, and $\partial_t G(t) \to 0$ in H^1 as $t \to \infty$. Here $\mathscr{L} = i\partial_t + (1/2)\Delta$, $\mathscr{N} = \partial_t^2 - \Delta + 1$. Roughly speaking, if $\|\mathscr{L}\tilde{u}(t) - \tilde{u}(t)\tilde{v}(t)\|_{H^2}$ and $\|\mathscr{N}\tilde{v}(t) + |\tilde{u}(t)|^2\|_{H^2}$ are integrable $t \to \infty$, there exists wave operator W_+ (the Cook-Kuroda method). It is natural to choose $\tilde{u} = u_0$, $\tilde{v} = v_0$. But in this case, $\|\mathscr{L}\tilde{u}(t) - \tilde{u}(t)\tilde{v}(t)\|_{H^2} = \|u_0(t)v_0(t)\|_{H^2} = O(t^{-1})$, $\|\mathscr{N}\tilde{v}(t) + |\tilde{u}(t)|^2\|_{H^2} = \|\,|u_0(t)|^2\|_{H^2} = O(t^{-1})$, hence they are not integrable $t \to \infty$. To overcome this difficulty, we use an asymptotic form (u_1, v_1) of (u_0, v_0), construct a second approximation (u_2, v_2) of the solution to the equation (KGS) satisfying $\|u_2(t)\|_{H^2}$, $\|v_2(t)\|_{H^2} = O(t^{-1})$, and put $\tilde{u} = u_0 + u_2$, $\tilde{v} = v_0 + v_2$ so that

$$\begin{aligned} \|\mathscr{L}\tilde{u}(t) - \tilde{u}(t)\tilde{v}(t)\|_{H^2} &\le \|\mathscr{L}u_2(t) - u_1(t)v_1(t)\|_{H^2} + \|u_1(t)v_1(t) - u_0(t)v_0(t)\|_{H^2} \\ &\quad + \|u_0(t)v_2(t) + u_2(t)v_0(t) + u_2(t)v_2(t)\|_{H^2} \\ &= O(t^{-2}), \end{aligned}$$

$$\begin{aligned}\|\mathcal{N}\tilde{v}(t)+|\tilde{u}(t)|^2\|_{H^2} &\le \|\mathcal{N}v_2(t)+|u_1(t)|^2\|_{H^2}+\||u_1(t)|^2-|u_0(t)|^2\|_{H^2}\\ &\quad+\|2\,\mathrm{Re}(u_0(t)\overline{u_2(t)})+|u_2(t)|^2\|_{H^2}\\ &=O(t^{-2}),\end{aligned}$$

and hence that they are integrable at infinity. For choices of the functions u_2 and v_2, see Section 3.

Outline of this paper is as follows. We prove the statement for positive time in Theorem. The statement for negative time is proved in the same way. In Section 2, we study time decay estimates of the solutions for the free Schrödinger and Klein-Gordon equations and their asymptotic forms. In Section 3, we construct a second approximation (u_2, v_2) of the solution to the equation (KGS) mentioned above. In Section 4, we prove Theorem by using the Cook-Kuroda method.

2. The first approximation

In this section, we study time decay estimates of the solutions for the free Schrödinger and Klein-Gordon equations and their asymptotic forms.

The time decay estimates of the free solutions u_0 and v_0 are well-known (see, e.g., Section 1 in Strauss [37] or Section 2 in Ozawa and Tsutsumi [29]):

Lemma 2.1. *There exists a constant $C>0$ such that for $t\ge 1$,*

$$\begin{aligned}\|u_0(t)\|_{H^2}&=\|\phi\|_{H^2}\le\delta,\\ \|u_0(t)\|_{W^2_\infty}&\le C\|\phi\|_{W^2_1}t^{-1}\le C\delta t^{-1},\\ \|v_0(t)\|_{W^2_\infty}&\le C(\|\psi_0\|_{W^4_1}+\|\psi_1\|_{W^3_1})t^{-1}\le C\delta t^{-1},\end{aligned}$$

where $\delta>0$ is defined in (1.2).

Instead of the free solution u_0 of the Schrödinger equation, we also consider the following asymptotic form of u_0:

$$\begin{aligned}u_1(t,x)&\equiv(U(t)e^{-(i|\cdot|^2/2t)}\phi)(x)\\ &=(it)^{-1}e^{i|x|^2/2t}\hat{\phi}\left(\frac{x}{t}\right).\end{aligned}$$

The following lemma is well-known (see, e.g., Lemmas 1, 2 and their proof in Moriyama, Tonegawa and Tsutsumi [22]):

Lemma 2.2. *There exists a constant $C>0$ such that for $t\ge 1$,*

$$\|u_0(t)-u_1(t)\|_{H^2}\le C\|\phi\|_{H^{2,2}}t^{-1}\le C\delta t^{-1},$$

$$\|u_1(t)\|_{H^2} \le C(\|\phi\|_{H^2} + \|\phi\|_{H^{0,2}}) \le C\delta,$$

$$\begin{aligned}\|u_1(t)\|_{W^2_\infty} &\le Ct^{-1} \sum_{|\alpha|+|\beta|\le 2} \|x^\beta \partial^\alpha \hat{\phi}\|_{L^\infty}\\ &\le C\|\phi\|_{H^{2,4}} t^{-1}\\ &\le C\delta t^{-1},\end{aligned}$$

where $\delta > 0$ *is defined in* (1.2).

Similarly, instead of the free solution v_0 of the Klein-Gordon equation, we introduce the asymptotic form of v_0:

$$v_1(t,x) \equiv \begin{cases} -\dfrac{1}{it}\dfrac{t^2}{t^2-|x|^2}\hat{\psi}_+\left(-\dfrac{x}{\sqrt{t^2-|x|^2}}\right)e^{i\sqrt{t^2-|x|^2}} \\ \qquad +\dfrac{1}{it}\dfrac{t^2}{t^2-|x|^2}\hat{\psi}_-\left(\dfrac{x}{\sqrt{t^2-|x|^2}}\right)e^{-i\sqrt{t^2-|x|^2}}, \quad \text{if } |x| < t, \\ 0, \quad \text{if } |x| \ge t, \end{cases}$$

where

$$\psi_\pm \equiv \frac{1}{2}(\psi_0 \mp \omega^{-1}\psi_1).$$

The following lemma is also well-known (see, e.g., Section 7.2 in Hörmander [16]).

Lemma 2.3. *There exists a constant* $C > 0$ *such that for* $t \ge 1$,

$$\begin{aligned}\|v_0(t) - v_1(t)\|_{W^2_\infty} &\le C \sum_{|\alpha|\le 5} (\|(1+|x|^2)^2 \partial^\alpha \hat{\psi}_+\|_{L^\infty} + \|(1+|x|^2)^2 \partial^\alpha \hat{\psi}_-\|_{L^\infty}) t^{-2}\\ &\le C(\|\psi_0\|_{H^{4,7}} + \|\psi_1\|_{H^{3,7}}) t^{-2}\\ &\le C\delta t^{-2},\end{aligned}$$

where $\delta > 0$ *is defined in* (1.2).

3. The second approximation

In this section, we construct a suitable second approximation (u_2, v_2) of the solution to the equation (KGS) so that the asymptotic profile $(u_0 + u_2, v_0 + v_2)$

tends to solutions of the equation (KGS) faster than the free solution (u_0, v_0) as $t \to \infty$.

First we construct a second approximation u_2 of the Schrödinger part u. By the definitions of u_1 and v_1, we see

$$(3.1)\quad u_1(t,x)v_1(t,x) = \begin{cases} \dfrac{1}{t^2}\dfrac{1}{1-(|x|/t)^2}\hat{\phi}\left(\dfrac{x}{t}\right)\hat{\psi}_+\left(-\dfrac{x/t}{\sqrt{1-(|x|/t)^2}}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} \\ \quad -\dfrac{1}{t^2}\dfrac{1}{1-(|x|/t)^2}\hat{\phi}\left(\dfrac{x}{t}\right)\hat{\psi}_-\left(\dfrac{x/t}{\sqrt{1-(|x|/t)^2}}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}, \\ \quad \text{if } |x| < t, \\ 0, \quad \text{if } |x| \geq t. \end{cases}$$

This implies the time decay estimate $\|u_1(t)v_1(t)\|_{H^2} = O(t^{-1})$, which is not integrable at infinity. Therefore we cannot prove Theorem directly by using this estimate. To overcome this difficulty, we find a second approximation of the form

$$u_2(t,x) = \begin{cases} t^{-a}P\left(\dfrac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} + t^{-a}Q\left(\dfrac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}, \\ \quad \text{if } |x| < t, \\ 0, \quad \text{if } |x| \geq t, \end{cases}$$

such that $\|\mathscr{L}u_2(t) - u_1(t)v_1(t)\|_{H^2}$ is integrable at infinity. We determine a constant $a > 0$ and functions P and Q on D. Let

$$\mathscr{L} \equiv i\partial_t + \frac{1}{2}\Delta.$$

By a direct calculation, we see that if $|x| < t$,

$$\mathscr{L}u_2(t,x) = -t^{-a}P\left(\frac{x}{t}\right)\left(\sqrt{1-\frac{|x|^2}{t^2}} + \frac{1}{2}\frac{(|x|/t)^2}{1-(|x|/t)^2}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}$$
$$+ t^{-a}Q\left(\frac{x}{t}\right)\left(\sqrt{1-\frac{|x|^2}{t^2}} - \frac{1}{2}\frac{(|x|/t)^2}{1-(|x|/t)^2}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}$$

$$
\begin{aligned}
&- it^{-a-1}\nabla P\left(\frac{x}{t}\right)\cdot\frac{x/t}{\sqrt{1-(|x|/t)^2}}e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}\\
&+ it^{-a-1}\nabla Q\left(\frac{x}{t}\right)\cdot\frac{x/t}{\sqrt{1-(|x|/t)^2}}e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}\\
&- i(a-1)t^{-a-1}P\left(\frac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}\\
&- i(a-1)t^{-a-1}Q\left(\frac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}\\
&- it^{-a-1}P\left(\frac{x}{t}\right)\frac{1-|x|^2/2t^2}{(1-(|x|/t)^2)^{3/2}}e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}\\
&+ it^{-a-1}Q\left(\frac{x}{t}\right)\frac{1-|x|^2/2t^2}{(1-(|x|/t)^2)^{3/2}}e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}\\
&+\frac{1}{2}t^{-a-2}\Delta P\left(\frac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}\\
&+\frac{1}{2}t^{-a-2}\Delta Q\left(\frac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}.
\end{aligned}
$$

We note that the first and second terms decay most slowly of all terms in the right hand side of the equality above. If we choose a, P and Q such that

$$
\begin{aligned}
(3.2)\qquad &-t^{-a}P\left(\frac{x}{t}\right)\left(\sqrt{1-\frac{|x|^2}{t^2}}+\frac{1}{2}\frac{(|x|/t)^2}{1-(|x|/t)^2}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})}\\
&\qquad =\frac{1}{t^2}\frac{1}{1-(|x|/t)^2}\hat{\phi}\left(\frac{x}{t}\right)\hat{\psi}_+\left(-\frac{x/t}{\sqrt{1-(|x|/t)^2}}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})},
\end{aligned}
$$

$$
\begin{aligned}
(3.3)\qquad &t^{-a}Q\left(\frac{x}{t}\right)\left(\sqrt{1-\frac{|x|^2}{t^2}}-\frac{1}{2}\frac{(|x|/t)^2}{1-(|x|/t)^2}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}\\
&\qquad =-\frac{1}{t^2}\frac{1}{1-(|x|/t)^2}\hat{\phi}\left(\frac{x}{t}\right)\hat{\psi}_-\left(\frac{x/t}{\sqrt{1-(|x|/t)^2}}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}
\end{aligned}
$$

holds, then $\mathscr{L}u_2 - u_1v_1$ decays faster than u_1v_1 (recall the equality (3.1)). Actually, if we set

$$a = 2, \tag{3.4}$$

$$P(x) \equiv -f(x)\hat{\phi}(x)\hat{\psi}_+\left(-\frac{x}{\sqrt{1-|x|^2}}\right), \tag{3.5}$$

$$Q(x) \equiv -g(x)\hat{\phi}(x)\hat{\psi}_-\left(\frac{x}{\sqrt{1-|x|^2}}\right), \tag{3.6}$$

for $|x| < 1$, where

$$f(x) \equiv \frac{2}{2(1-|x|^2)^{3/2} + |x|^2},$$

and g is defined in (1.1), then the equalities (3.2) and (3.3) are satisfied. Therefore, as a second approximation for u, we choose the following function,

$$u_2(t,x) \equiv \begin{cases} t^{-2}P\left(\frac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} + t^{-2}Q\left(\frac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}, \\ \qquad \text{if } |x| < t, \\ 0, \quad \text{if } |x| \geq t, \end{cases}$$

where P and Q are defined in (3.5) and (3.6), respectively. Then we see that if $|x| < t$,

$$\begin{aligned} &\mathscr{L}u_2(t,x) - u_1(t,x)v_1(t,x) \\ &\quad = -it^{-3}\nabla P\left(\frac{x}{t}\right)\cdot\frac{x/t}{\sqrt{1-(|x|/t)^2}}e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} \\ &\quad + it^{-3}\nabla Q\left(\frac{x}{t}\right)\cdot\frac{x/t}{\sqrt{1-(|x|/t)^2}}e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})} \\ &\quad - it^{-3}P\left(\frac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} - it^{-3}Q\left(\frac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})} \\ &\quad - it^{-3}P\left(\frac{x}{t}\right)\frac{1-|x|^2/2t^2}{(1-(|x|/t)^2)^{3/2}}e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} \\ &\quad + it^{-3}Q\left(\frac{x}{t}\right)\frac{1-|x|^2/2t^2}{(1-(|x|/t)^2)^{3/2}}e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})} \\ &\quad + \frac{1}{2}t^{-4}\Delta P\left(\frac{x}{t}\right)e^{i(|x|^2/2t+\sqrt{t^2-|x|^2})} + \frac{1}{2}t^{-4}\Delta Q\left(\frac{x}{t}\right)e^{i(|x|^2/2t-\sqrt{t^2-|x|^2})}. \end{aligned} \tag{3.7}$$

The following lemma is time decay estimates for $\mathscr{L}u_2 - u_1 v_1$ and u_2.

Lemma 3.1. *There exists a constant $C > 0$ such that for $t \geq 1$,*

$$\begin{aligned}
\|\mathscr{L}u_2(t) - u_1(t)v_1(t)\|_{H^2} &\leq C\|g^5\omega^4\hat{\phi}\|_{L^2(D)}(\|\psi_0\|_{H^{12,6}} + \|\psi_1\|_{H^{11,6}})t^{-2} \\
&\leq C\delta^2 t^{-2},
\end{aligned}$$

$$\begin{aligned}
\|u_2(t)\|_{H^2} &\leq C\|g^3\omega^2\hat{\phi}\|_{L^2(D)}(\|\psi_0\|_{H^{6,4}} + \|\psi_1\|_{H^{5,4}})t^{-1} \\
&\leq C\delta^2 t^{-1},
\end{aligned}$$

$$\begin{aligned}
\|u_2(t)\|_{W_4^2} &\leq C\|g^4\omega^3\hat{\phi}\|_{L^2(D)}(\|\psi_0\|_{H^{6,4}} + \|\psi_1\|_{H^{5,4}})t^{-3/2} \\
&\leq C\delta^2 t^{-3/2},
\end{aligned}$$

where $\delta > 0$ is defined in (1.2).

Proof. By a direct calculation, we see that there exists a constant $C > 0$ such that for $|x| < 1$,

$$\sum_{|\alpha| \leq 1} |\partial^\alpha f(x)| \leq C,$$

$$\sum_{|\alpha| = k} |\partial^\alpha f(x)| \leq C(1 - |x|^2)^{-(k-3/2)}, \qquad k = 2, 3, 4,$$

$$\sum_{|\alpha| = 1} |\partial^\alpha g(x)| \leq Cg(x)^2,$$

$$\sum_{|\alpha| = k} |\partial^\alpha g(x)| \leq C(1 - |x|^2)^{-(k-3/2)}|g(x)|^{k+1}, \qquad k = 2, 3, 4,$$

$$\sum_{|\alpha| = k} |\partial_x^\alpha e^{it(|x|^2/2 \pm \sqrt{1-|x|^2})}| \leq Ct^k(1 - |x|^2)^{-(k-1/2)}, \qquad k = 1, 2.$$

We note that for $|x| < 1$,

$$\begin{aligned}
(1 - |x|^2)^{-k/2}\hat{\psi}_+(-x/\sqrt{1 - |x|^2}) &= (1 + |y|^2)^{k/2}\hat{\psi}_+(y), \\
(1 - |x|^2)^{-k/2}\hat{\psi}_-(x/\sqrt{1 - |x|^2}) &= (1 + |z|^2)^{k/2}\hat{\psi}_-(z),
\end{aligned}$$

where $y = -x/\sqrt{1 - |x|^2}$ and $z = x/\sqrt{1 - |x|^2}$. Therefore it follows from the equality (3.7), the definition of u_2, the Leibniz formula, Hölder's inequality, and estimates above that

$$\|\mathscr{L}u_2(t) - u_1(t)v_1(t)\|_{H^2}$$
$$\leq C\|g^2\omega\hat{\phi}\|_{L^2(D)} \sum_{|\alpha|\leq 1}(\|(1+|x|^2)^{7/2}\partial^\alpha\hat{\psi}_+\|_{L^\infty} + \|(1+|x|^2)^{7/2}\partial^\alpha\hat{\psi}_-\|_{L^\infty})t^{-2}$$
$$+ C\|g^5\omega^4\hat{\phi}\|_{L^2(D)} \sum_{|\alpha|\leq 4}(\|(1+|x|^2)^{6}\partial^\alpha\hat{\psi}_+\|_{L^\infty} + \|(1+|x|^2)^{6}\partial^\alpha\hat{\psi}_-\|_{L^\infty})t^{-3},$$

$$\|u_2(t)\|_{H^2} \leq C\|g^3\omega^2\hat{\phi}\|_{L^2(D)}$$
$$\times \sum_{|\alpha|\leq 2}(\|(1+|x|^2)^{3}\partial^\alpha\hat{\psi}_+\|_{L^\infty} + \|(1+|x|^2)^{3}\partial^\alpha\hat{\psi}_-\|_{L^\infty})t^{-1},$$

$$\|u_2(t)\|_{W_4^2} \leq C\|g^3\omega^2\hat{\phi}\|_{L^4(D)}$$
$$\times \sum_{|\alpha|\leq 2}(\|(1+|x|^2)^{3}\partial^\alpha\hat{\psi}_+\|_{L^\infty} + \|(1+|x|^2)^{3}\partial^\alpha\hat{\psi}_-\|_{L^\infty})t^{-3/2}.$$

These estimates imply this lemma. □

We next construct a second approximation v_2 for the Kein-Gordon part v. Let

$$\mathscr{N} \equiv \partial_t^2 - \Delta + 1.$$

We consider the following inhomogeneous equation:

$$\mathscr{N}w = h, \tag{3.8}$$

where h is a given function of $(t,x) \in \boldsymbol{R} \times \boldsymbol{R}^2$.

In the exactly same way as Lemma 2.4 in Ozawa and Tsutsumi [29], we obtain the following lemma.

Lemma 3.2. *Assume that* $h \in C^1([1,\infty); H^1)$ *and*

$$\sup_{t\geq 1}\{t\|h(t)\|_{L^2} + t^2(\|\partial_t h(t)\|_{L^2} + \|\nabla h(t)\|_{L^2} + \|\partial_t \nabla h(t)\|_{L^2})\} \leq A.$$

Then there exists a solution $w \in C([1,\infty); H^2) \cap C^1([1,\infty); H^1)$ *of the equation* (3.8) *such that*

$$\sup_{t\geq 1}\{t(\|w(t)\|_{H^2} + \|\partial_t w(t)\|_{H^1})\} \leq CA,$$

for some $C > 0$ *independent of* A.

Using the Sobolev embedding theorem, we see that $h = -|u_1|^2$ satisfies the assumptions of Lemma 3.2 with

$$A = C\|\phi\|_{H^{2,2}}^2 \leq C\delta,$$

by a straightforward calculation. By Lemma 3.2, there exists a function $v_2 \in C([1,\infty);H^2)\cap C^1([1,\infty);H^1)$ satisfying

$$\mathcal{N} v_2 = -|u_1|^2, \tag{3.9}$$

$$\sup_{t\geq 1}\{t(\|v_2(t)\|_{H^2}+\|\partial_t v_2(t)\|_{H^1})\} \leq C\|\phi\|_{H^{2,2}}^2 \leq C\delta. \tag{3.10}$$

We choose this solution v_2 as a second approximation for v.

4. Proof of Theorem

In this section, we prove Theorem for positive time. The statement for negative time in Theorem is proved in the same way. The proof is based on the Cook-Kuroda method. Throughout this section, we always assume that the assumptions of Theorem are satisfied.

We recall the Strichartz estimates for the free Schrödinger evolution group. We define the following linear operator:

$$Sw(t) = \int_t^\infty U(t-s)w(s)ds.$$

The following lemma is proved by, e.g., Ginibre and Velo [11] or Yajima [42].

Lemma 4.1. *Let (q,r) satisfy $1/2-1/q=1/r$ and $2\leq q<\infty$. Then S is a bounded operator from $L^1((T_0,\infty);L^2(\boldsymbol{R}^2))$ into $L^r((T_0,\infty);L^q(\boldsymbol{R}^2))$ with norm uniformly bounded with respect to T_0. Furthermore, if $w\in L^1((T_0,\infty);L^2(\boldsymbol{R}^2))$, then $Sw\in C([T_0,\infty);L^2(\boldsymbol{R}^2))$.*

For $T\geq 1$ and $R\geq 0$, we introduce the following function spaces:

$$X_T \equiv \{(F,G)\in C([T,\infty);H^2)\oplus[C([T,\infty);H^2)\cap C^1([T,\infty);H^1)] : \|(F,G)\|_{X_T}<\infty\},$$

$$B_T(R)\equiv\{(F,G)\in X_T : \|(F,G)\|_{X_T}\leq R\},$$

where

$$\|(F,G)\|_{X_T} \equiv \sup_{t\geq T}\left[t\left\{\|F(t)\|_{H^2}+\left(\int_t^\infty \|F(\tau)\|_{W_4^2}^4 d\tau\right)^{1/4} + \|G(t)\|_{H^2}+\|\partial_t G(t)\|_{H^1}\right\}\right].$$

We note that X_T is a Banach space, and that $B_T(R)$ is a complete metric space in the X_T-norm.

We consider the following system:

$$\begin{cases} i\partial_t F + \frac{1}{2}\Delta F = (F + u_0 + u_2)(G + v_0 + v_2) - \mathscr{L}u_2, \\ \partial_t^2 G - \Delta G + G = -|F + u_0 + u_2|^2 - \mathscr{N}v_2. \end{cases} \tag{4.1}$$

Remark 4.1. We note that if we put $F = u - u_0 - u_2$ and $G = v - v_0 - v_2$, then the system (KGS) is equivalent to the system (4.1).

We define the nonlinear operator K as follows:

$$K(F, G) \equiv (K_1(F, G), K_2(F, G)),$$

where

$$K_1(F, G)(t) \equiv i\int_t^\infty U(t - s)[(F(s) + u_0(s) + u_2(s)) \times (G(s) + v_0(s) + v_2(s)) - \mathscr{L}u_2(s)]ds,$$

$$K_2(F, G)(t) \equiv -\int_t^\infty \omega^{-1}(\sin(t - s)\omega)[-|F(s) + u_0(s) + u_2(s)|^2 - \mathscr{N}v_2(s)]ds.$$

Noting the definition of v_2, we have

$$K_1(F, G)(t) = i\int_t^\infty U(t - s)[F(s)G(s) + F(s)(v_0(s) + v_2(s)) + (u_0(s) + u_2(s))G(s) + q(s)]ds,$$

$$K_2(F, G)(t) = \int_t^\infty \omega^{-1}(\sin(t - s)\omega)[|F(s)|^2 + 2\,\mathrm{Re}(F(s)(\overline{u_0(s) + u_2(s)})) + r(s)]ds,$$

where

$$q = u_2(v_0 + v_2) + u_0 v_2 + (u_0 - u_1)v_0 + u_1(v_0 - v_1) - (\mathscr{L}u_2 - u_1 v_1),$$

$$r = 2\,\mathrm{Re}(u_0\overline{u_2}) + |u_0|^2 - |u_1|^2 + |u_2|^2.$$

Remark 4.2. We note that it follows from the Sobolev embedding theorem that

$$\|w_1 w_2\|_{H^2} \le C\|w_1\|_{H^2}\|w_2\|_{H^2},$$

$$\|w_1 w_2\|_{H^2} \le C\|w_1\|_{W^2_\infty}\|w_2\|_{H^2}.$$

Lemma 4.2. *There exists a constant* $C > 0$ *such that for* $t \geq 1$,

$$\|q(t)\|_{H^2} \leq C\delta^2 t^{-2},$$

$$\|r(t)\|_{H^2} \leq C\delta^2 t^{-2},$$

where $\delta > 0$ *is defined in* (1.2).

Proof. By Lemmas 2.1, 2.2, 2.3, 3.1, the estimate (3.10) and Remark 4.2, we have for $t \geq 1$,

$$\begin{aligned}\|q(t)\|_{H^2} &\leq \|u_2(t)\|_{H^2}(\|v_0(t)\|_{W^2_\infty} + \|v_2(t)\|_{H^2}) \\ &\quad + \|u_0(t)\|_{W^2_\infty}\|v_2(t)\|_{H^2} + \|u_0(t) - u_1(t)\|_{H^2}\|v_0(t)\|_{W^2_\infty} \\ &\quad + \|u_1(t)\|_{H^2}\|v_0(t) - v_1(t)\|_{W^2_\infty} + \|\mathscr{L}u_2(t) - u_1(t)v_1(t)\|_{H^2} \\ &\leq C\delta^2 t^{-2},\end{aligned}$$

and

$$\begin{aligned}\|r(t)\|_{H^2} &\leq \|u_0(s)\|_{W^2_\infty}\|u_2(t)\|_{H^2} + (\|u_0(s)\|_{W^2_\infty} + \|u_1(s)\|_{W^2_\infty}) \\ &\quad \times \|u_0(s) - u_1(s)\|_{H^2} + \|u_2(t)\|^2_{H^2} \\ &\leq C\delta^2 t^{-2}.\end{aligned}$$

□

Proof of Theorem. To prove the existence argument, we show that if $\delta > 0$ is sufficiently small, then K is a contraction mapping on $B_1(R)$ for some $R > 0$.

Let $(F, G) \in B_1(R)$. By Lemmas 2.1, 3.1, 4.2, the estimate (3.10) and Remark 4.2, we have for $t \geq 1$,

(4.2)
$$\begin{aligned}&\|K_1(F, G)(t)\|_{H^2} \\ &\leq C\int_t^\infty \|F(s)G(s) + F(s)(v_0(s) + v_2(s)) \\ &\qquad + (u_0(s) + u_2(s))G(s) + q(s)\|_{H^2}ds \\ &\leq C\int_t^\infty [\|F(s)\|_{H^2}\|G(s)\|_{H^2} + \|F(s)\|_{H^2}(\|v_0(s)\|_{W^2_\infty} + \|v_2(s)\|_{H^2}) \\ &\qquad + (\|u_0(s)\|_{W^2_\infty} + \|u_2(s)\|_{H^2})\|G(s)\|_{H^2} + \|q(s)\|_{H^2}]ds \\ &\leq C\int_t^\infty (R^2 s^{-2} + R\delta s^{-2} + \delta^2 s^{-2})ds \\ &\leq C(R^2 + R\delta + \delta^2)t^{-1}.\end{aligned}$$

Similarly, we see that for $t \geq 1$,

$$\left(\int_t^\infty \|K_1(F,G)(s)\|_{W_4^2}^4 ds\right)^{1/4} \leq C\int_t^\infty \|F(s)G(s) + F(s)(v_0(s)+v_2(s)) + (u_0(s)+u_2(s))G(s) + q(s)\|_{H^2} ds \leq C(R^2 + R\delta + \delta^2)t^{-1}. \tag{4.3}$$

We note that the first inequality in (4.3) follows from Lemma 4.1.

By Lemmas 2.1, 3.1, 4.2 and Remark 4.2, we obtain for $t \geq 1$,

$$\begin{aligned} \|K_2(F,G)(t)\|_{H^2} &\leq C\int_t^\infty [\|F(s)\|_{H^2}^2 + (\|u_0(s)\|_{W_\infty^1} + \|u_2(s)\|_{H^1})\|F(s)\|_{H^2} + \|r(s)\|_{H^1}]ds \\ &\leq C\int_t^\infty (R^2 s^{-2} + R\delta s^{-2} + \delta^2 s^{-2})ds \\ &\leq C(R^2 + R\delta + \delta^2)t^{-1}. \end{aligned} \tag{4.4}$$

Note that

$$\frac{d}{dt}K_2(F,G)(t) = -\int_t^\infty (\cos(t-s)\omega)[|F(s)|^2 + 2\,\mathrm{Re}(F(s)(\overline{u_0(s)+u_2(s)})) + r(s)]ds.$$

Thus, in the same way as above, we see that

$$\left\|\frac{d}{dt}K_2(F,G)(t)\right\|_{H^1} \leq C(R^2 + R\delta + \delta^2)t^{-1}. \tag{4.5}$$

The estimates (4.2), (4.3), (4.4) and (4.5) imply

$$\|K(F,G)\|_{X_1} \leq C(R^2 + R\delta + \delta^2).$$

Let $(F_1, G_1), (F_2, G_2) \in B_1(R)$. Similarly, we can show that

$$\|K(F_1,G_1) - K(F_2,G_2)\|_{X_1} \leq C(R+\delta)\|(F_1,G_1) - (F_2,G_2)\|_{X_1}.$$

We assume that $\delta > 0$ is sufficiently small. Then there exists sufficiently small $R_\delta > 0$ satisfying

$$C(R_\delta^2 + R_\delta\delta + \delta^2) \leq R_\delta,$$

$$C(R_\delta + \delta) \leq \frac{1}{2}.$$

Thus K is a contraction mapping on $B_1(R_\delta)$ and for some sufficiently small $R_\delta > 0$. Therefore there exists a unique fixed point of K in $B_1(R_\delta)$ for sufficiently small $R_\delta > 0$. This means that there exists a solution $(F, G) \in X_1$ of the integral equation

$$(F, G) = K(F, G),$$

and hence of the system (4.1). Here we set $F = u - u_0 - u_2$ and $G = v - v_0 - v_2$. Recalling Lemma 3.1, the estimate (3.10) and Remark 4.1, we see that there exists a solution (u, v) of the equation (KGS) such that $(u - u_0, v - v_0) \in X_1$.

In the exactly same way as in Ozawa and Tsutsumi [29], if $\delta > 0$ is sufficiently small, we can prove the uniqueness of the solution (u, v) to the equation (KGS) satisfying $(u - u_0, v - v_0) \in X_{T_\delta}$ for sufficiently large $T_\delta \geq 1$. Thus we obtain the unique existence of the solution (u, v) for the equation (KGS) such that $(u - u_0, v - v_0) \in X_{T_\delta}$ for sufficiently large $T_\delta \geq 1$.

It is well-known that the equation (KGS) is globally well-posed in $C(\boldsymbol{R}; H^2) \oplus [C(\boldsymbol{R}; H^2) \cap C^1(\boldsymbol{R}; H^1)]$ (see, e.g., Baillon and Chadam [2], Fukuda and Tsutsumi [7] and Hayashi and von Wahl [15]). This implies that the unique local solution (u, v) on $[T_\delta, \infty)$, which is obtained above, can be extended to all times. Therefore the proof of Theorem is complete. □

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