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Periodic Solutions of a Certain Generalized Liénard Equation

By

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Abstract. We are concerned with the existence of at least one periodic solution of a generalized nonlinear Liénard equation with a periodic forcing term. The main tool is a continuation theorem by Capietto, Mawhin and Zanolin. A priori bounds for the periodic solutions are obtained either by studying the behavior of the trajectories of a new equivalent system or by determining the nature of singular points at infinity of suitable autonomous systems in the usual phase plane.

Key Words and Phrases. Generalized Liénard equation, A priori bounds, Periodic solutions.

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1. Introduction

In this paper we consider the problem of existence of periodic solutions for the forced generalized Liénard equation:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = e(t), \tag{1}$$

where the forcing term $e(t)$ is a T-periodic continuous function. We remark that our results are still valid if the forcing term is a bounded function $e(t, x, \dot{x})$ depending also on x and $\dot{x}$.

The study of this equation comes from the following motivation. In a paper of 90's Freedman and Kuang [4] considered the equation:

$$\ddot{x} + f(x, \dot{x})\dot{x} + g(x) = 0, \tag{2}$$

where:

$$f(x, \dot{x}) = \sum_{i=0}^{n} f_i(x)\dot{x}^i.$$

Such an equation comes from a Gause-type predator-prey problem, and the special case $f(x, \dot{x}) = f_1(x) + f_2(x)\dot{x}$ has been widely investigated by several authors [4, 6, 8, 9, 10]. The phase plane approach used in those paper was based on a transformation which sends the equation:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0 \tag{3}$$

in a classical Liénard equation:

$$\ddot{x} + \tilde{f}(x)\dot{x} + \tilde{g}(x) = 0.$$

Hence all the classical results can be invoked (for the transformation see Remark 4.7 or [10]). However for the forced problem this method cannot be used because the forcing term is drastically changed by the transformation. For this reason we investigate directly the problem (1). There are two different approaches to attack the problem: the first one is based on the classical Brower fixed point theorem. It is necessary to construct an invariant region for the Poicaré map and this is obtained by considering some auxiliary autonomous system (see [16, 2, 19]): however in this case the assumptions are in a certain sense severe because the invariant region is constructed for *all* the solutions and not only the periodic ones.

The second approach relies on some continuation result. Classical theorems by Mawhin, Lazer, Cesari, Zanolin (see [12, 13, 23] and the references therein contained) investigated from the 70's, although very powerful for the classical Liénard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = e(t),$$

seems not suitable in our case because the desired a priori bounds are not easily obtained (see [20] for a more precise discussion). In a recent paper by Mawhin, Capietto and Zanolin [3] a new continuation method was proposed and this method has application also in this case. We state here for immediate reference the version of that theorem we are going to apply.

Theorem 1.1 (Capietto, Mawhin, Zanolin [3]). *Assume that*:

1. $g(x)x > 0$ *for all* $|x| > \bar{x}$;
2. *there exists a constant* $C > 0$ *such that every* T*-periodic solution* x *of*:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = \lambda e(t), \qquad \lambda \in]0,1[,$$

satisfies the a priori bound:

$$\|x\|_\infty + \|\dot{x}\|_\infty \le C.$$

Then equation (1) *has at least a* T*-periodic solution.*

The main difference between this new result and the previous continuation theorems is based on the fact that the parameter λ appears only in the forcing term. Hence in most situations the geometrical properties of the solutions are preserved and may be used in order to get the desired a priori bounds. Recent results in this direction may be found in [18, 19, 20, 21, 14].

In this paper we combine this approach and the use of a new plane for the equivalent planar system of the scalar equation (3).

In the general case it is necessary to produce a suitable bound for both $\|x\|_\infty$ and $\|\dot{x}\|_\infty$, while, due to the particular structure of this case, it will be enough to get the bound just for $\|x\|_\infty$ or, respectively, for $\|\dot{x}\|_\infty$.

The plane of the paper is the following:

1. in section 2 we consider the autonomous case, introduce the new system and study its portrait
2. the new plane will be actually used in section 3 to get a bound for $\|\dot{x}\|_\infty$
3. the inspection of the nature of the critical points at infinity in the phase plane will be performed in section 4 to get the bound for $\|x\|_\infty$ in the line of the results of [20, 22].

2. The autonomous case

We consider at the moment the autonomous equation:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = 0,$$

which is equivalent to the system:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f_1(x)y - f_2(x)y^2 - g(x). \end{cases} \tag{4}$$

We impose the standard regularity assumptions which guarantee the uniqueness of the initial value problems associated to the system. Moreover we assume that $g(x)x > 0$ for all x different from 0. Hence the origin is the only singular point and the trajectories turn clockwise around the origin.

System (4) is equivalent to the system:

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -f_2(x)[y - F(x)]^2 - g(x), \end{cases} \tag{5}$$

where $F(x) = \int_0^x f_1(s)ds$.

The nonlinear transformation:

$$(x, y) \mapsto (x, y + F(x))$$

takes the trajectories of the phase plane into the trajectories of the new system. The x-coordinate does not change, while the y-axis is transformed into the curve $y = F(x)$ (such a transformation plays the same role of the similar one, which for the classical Liénard equation maps the phase plane in the Liénard plane). Hence the geometrical properties of the solutions, like for instance limit cycles, can be read in both planes.

We are interested in the 0-isocline of the new plane, that is the points in which $\dot{y} = 0$. Straightforward calculations show that the 0-isocline is given by the curve:

$$y = F(x) \pm \sqrt{-\frac{g(x)}{f_2(x)}}.$$

If we assume that f_2 is positive, such a curve is contained in the half plane $x \leq 0$ and passes through the origin. It is trivial to prove the following lemma which will be useful in the next section.

Lemma 2.1. *Assume that f_1, f_2 and g are locally Lipschitz continuous functions (or any other kind of assumption ensuring the uniqueness for the initial value problems associated to* (5)*). Assume also that*:

1. $g(x)x > 0$ *for all* $x \neq 0$;
2. $F(x)$ *is bounded*;
3. $f_2(x) > 0$ *for all* x;
4. $\liminf_{x \to -\infty} \frac{g(x)}{f_2(x)} > -\infty$.

Then the 0*-isocline is bounded in the* y*-coordinate.*

We just note that a similar result holds with minor changes if we consider the function $\tilde{g}(x) = g(x) + c$, being c a suitable constant, provided that $g(x) > -c$ for all $x > \bar{x}$.

3. The forced case: a bound for y

Following the result of [3], we must find a priori bounds for T-periodic solutions of the family of equations:

$$\ddot{x} + f_1(x)\dot{x} + f_2(x)\dot{x}^2 + g(x) = \lambda e(t), \qquad \lambda \in [0, 1]. \tag{6}$$

This goal will be reached studying the periodic orbits of the equivalent system:

$$\begin{cases} \dot{x} = y - F(x) + \lambda E_0(t), \\ \dot{y} = -f_2(x)[y - F(x) + \lambda E_0(t)]^2 - g(x) + \lambda \bar{e}, \end{cases} \qquad \lambda \in [0, 1], \tag{7}$$

where:

$$\bar{e} = \frac{1}{T}\int_0^T e(s)ds$$

is the mean value of e and:

$$E_0(t) = \int_0^t [e(s) - \bar{e}]ds.$$

Such a system is introduced having in mind the previous one for the autonomous case and we are going to get a result similar to Lemma 2.1.

Lemma 3.1. *Assume that f_1, f_2, g and e are continuous functions and let $e(t)$ be T-periodic with mean value $\bar{e}$. Assume also that:*

1. $g(x) > \|e\|_\infty$ *for all* $x > R$ *and* $g(x) < -\|e\|_\infty$ *for all* $x < -R$;
2. $F(x)$ *is bounded*;
3. $f_2(x) > 0$ *for all* x;
4. $\liminf_{x \to -\infty} \dfrac{g(x) - \bar{e}}{f_2(x)} > -\infty$.

Then there exists a constant $C > 0$ such that every T-periodic solution x of (6) *satisfies*:

$$\|\dot{x}\|_\infty \leq C.$$

Proof. Let (x, y) be a T-periodic solution of the equivalent system (7) for some $\lambda \in [0, 1]$ and let t^* be such that $|y(t^*)| = \|y\|_\infty$. Hence $\dot{y}(t^*) = 0$ and:

$$-f_2(x(t^*))[y(t^*) - F(x(t^*)) + \lambda E_0(t^*)]^2 - g(x(t^*)) + \lambda\bar{e} = 0.$$

As stated in section 1, we have:

$$y(t^*) = F(x(t^*)) - \lambda E_0(t^*) \pm \sqrt{-\frac{g(x(t^*)) - \lambda\bar{e}}{f_2(x(t^*))}}.$$

Now, since E_0 is bounded being T-periodic, we are in the case of Lemma 2.1 and $y(t^*)$ is uniformly bounded.

To finish the proof it remains to notice that:

$$\dot{x}(t) = y(t) - F(x(t)) + \lambda E_0(t)$$

and this is clearly bounded since F and E_0 are bounded. ■

Proposition 3.2. *Under the assumptions of Lemma* 3.1, *equation* (1) *has at least a T-periodic solution.*

Proof. In order to apply the continuation theorem, it remains to be proved that $\|x\|_\infty$ is uniformly bounded. This comes from Lagrange's theorem. Indeed $x(t_1) - x(t_2) = \dot{x}(\bar{t})(t_1 - t_2)$ so that $|x(t_1) - x(t_2)| \leq CT$ for all $t_1, t_2 \in [0, T]$ and this gives a bound for the oscillation of x. If s_1 and s_2 are respectively the times in which x attains its minimum and maximum values, then $\ddot{x}(s_1) \geq 0$, $\ddot{x}(s_2) \leq 0$ and $\lambda e(s_i) - g(x(s_i)) = \ddot{x}(s_i)$; therefore, the first condition on g in Lemma 3.1 implies that the intervals $[\min x, \max x]$ and $[R, R]$ have nonempty intersection and, thus, there is a $\hat{t}$ such that $|x(\hat{t})| \leq R$. Hence $|x(t)| \leq R + CT$ for all t. ■

In order to get our main result we are going to relax the sign condition on f_2.

Theorem 3.3. *Assume that* f_1, f_2, g *and* e *are continuous functions and let* $e(t)$ *be* T*-periodic with mean value* $\bar{e}$. *Assume also that*:

1. $g(x) > \|e\|_\infty$ *for all* $x > R$ *and* $g(x) < -\|e\|_\infty$ *for all* $x < -R$;
2. $F(x)$ *is bounded*;
3. $f_2(x) > 0$ *for all* $|x| \geq R$;
4. $\liminf_{x \to -\infty} \dfrac{g(x) - \bar{e}}{f_2(x)} > -\infty$.

Then equation (1) *has at least a* T*-periodic solution.*

Proof. Let (x, y) be a T-periodic solution of (7) for some $\lambda \in]0, 1[$.

At first we study the behavior of the solution when $x(t) \leq -R$. From the second equation in (7), if:

$$y(t) < F(x(t)) - \lambda E_0(t) - \sqrt{-\frac{g(x(t)) - \lambda\bar{e}}{f_2(x(t))}}$$

or if:

$$y(t) > F(x(t)) - \lambda E_0(t) + \sqrt{-\frac{g(x(t)) - \lambda\bar{e}}{f_2(x(t))}},$$

then $\dot{y}(t) < 0$. From the assumptions of the theorem the quantities:

$$F(x(t)), \qquad \lambda E_0(t), \qquad \sqrt{-\frac{g(x(t)) - \lambda\bar{e}}{f_2(x(t))}},$$

are bounded by a constant K; hence the periodic solutions, if any, must lie in the strip $\{|y| \leq 3K\}$, as long as $x(t) \leq -R$.

On the other hand, in the half plane $\{x \geq R\}$ we have that $\dot{y} < 0$ so that the absolute maximum and minimum of y cannot lie there.

It remains to investigate the behavior of y in the strip $\{|x| \leq R\}$. In this region we estimate the slope of the solution given by:

$$\frac{dy}{dx} = -f_2(x)[y - F(x) + \lambda E_0(t)] + \frac{\lambda\bar{e} - g(x)}{y - F(x) + \lambda E_0(t)}. \tag{8}$$

Let:

$$A = \sup_{|x| \leq R} |f_2(x)|, \qquad B = 2AK + \frac{1}{K}\left(\bar{e} + \sup_{|x| \leq R} |g(x)|\right),$$

$$C = \left(3K + \frac{B}{A}\right)e^{2AR}.$$

We claim that $\|y\|_\infty \le C$ and we argue by contradiction. Assume that $|y(t^*)| = \|y\|_\infty > C$ for some t^*. By the previous discussion we have also that $|x(t^*)| < R$, since $C > 3K$ and y cannot have critical points where $x \ge R$. Let us suppose that actually $y(t^*) > C$ (the other case $y(t^*) < -C$ can be treated in an analogous way) and let:

$$t_* = \inf\{t_0 : t_0 < t^* \text{ and } y(t) > 3K \text{ for all } t \in [t_0, t^*]\};$$

note that t_* is finite, $y(t_*) = 3K$ and $x(t_*) \ge -R$, since there must be a time $t_1 < t^*$ in which $\dot{x}(t_1) = 0$ and, from the first equation in (7), $y(t_1) = F(x(t_1)) - \lambda E_0(t_1) \le 2K$. Again from the same equation we deduce that $\dot{x}(t) \ge K$ and equation (8) has meaning when t varies in $[t_*, t^*]$, giving rise to the following bound:

$$\frac{1}{Ay + B}\frac{dy}{dx} \le 1 \qquad \text{as } t \in [t_*, t^*].$$

Integrating this inequality for x varying in $[x(t_*), x(t^*)]$, one obtains:

$$y(t^*) \le \left(3K + \frac{B}{A}\right)e^{2AR} - \frac{B}{A} < C,$$

that is a contradiction.

The rest of the proof is the same as in the previous Proposition 3.2. ■

Remark 3.4. We just note that a dual result can be easily obtained replacing assumptions 3 and 4 with:

3*. $f_2(x) < 0$ for all $|x| \ge R$;

4*. $\liminf_{x \to +\infty} \frac{g(x) - \bar{e}}{f_2(x)} > -\infty$.

4. The forced case: a bound for x

In this section we attack the problem with a complete different approach. Therefore no assumptions on the sign of f_2 or on the boundedness of F will be required. Again we start investigating the autonomous case, but now we will work in the standard phase plane and consider system (4). The basic idea is that, if the system possesses a separatrix which tends to the singular point at infinity on the x-axis, then such a separatrix is run in infinite time. A continuity argument shows that the trajectories near the separatrix are run in an arbitrarily large time. This argument was presented for the first time in [21]. In particular T-periodic solutions of (6) must be confined in a region bounded in the x-direction.

In fact we have the following result, which we take entirely from [20] and it is valid for the more general equation:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -F(x, y) + \lambda e(t). \end{cases} \tag{9}$$

Lemma 4.1 (Villari, Zanolin [20, 22]). *Assume that system*:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -F(x, y) + E, \end{cases} \tag{10}$$

with $E = \max e(t)$, *has a separatrix* Γ *which lies in the third quadrant and crosses the negative* y*-axis. If the time along the separatrix from* $x = 0$ *to the point at infinity is larger than* T, *then there is* $R > 0$ *such that*:

$$\min x(t) > -R$$

for every T*-periodic solution of* (9).

Proof. The complete proof of this lemma will appear in [22], while a sketched proof can be found in [20]. ■

The same conclusion holds if the separatrix lies in the second quadrant. A dual result can be proved for a separatrix in the first or in the fourth quadrant obtaining an upper bound for $x(t)$. For the complete proof we refer to the forthcoming paper [22].

The lemma provides only a one-sided bound. In this light we need to have a phase portrait with two separatrices on opposite sides with respect to the y-axis. If this happens one has the desired bound on x, say $|x(t)| \leq R$. In the general case this is not enough for having a bound on $\dot{x}$ and some kind of Nagumo-type conditions is necessary (see [7, Ch. XII, Lemma 5.1, p. 428]); indeed, if we set:

$$A = \sup_{|x| \leq R} |f_2(x)|, \qquad B = \sup_{|x| \leq R} |f_1(x)|,$$

$$C = \sup_{|x| \leq R} |g(x)| + \sup_{0 \leq t \leq T} |e(t)|, \qquad \varphi(s) = As^2 + Bs + C,$$

then $\int^{\infty} s/\varphi(s)ds = +\infty$ and $|\ddot{x}| \leq \varphi(|\dot{x}|)$ from (6). Therefore the above mentioned Lemma in [7] directly applies to our T-periodic solutions and the missing bound on $\dot{x}$ can be deduced. However, in this particular case, the conclusion directly comes from the formula which gives the slope of the solutions in the phase plane:

$$\frac{dy}{dx} = -f_2(x)y - f_1(x) - \frac{g(x) - \lambda e(t)}{y}.$$

Indeed, from this expression we get:

$$\left|\frac{dy}{dx}\right| \le A|y| + B + \frac{C}{|y|},$$

where the right hand side is sublinear in $|y|$, provided that y is bounded away from zero, so that an argument like that in the proof of Theorem 3.3 could be exploited.

For this reason we concentrate on the problem of existence of separatrices. We investigate the behavior at infinity by studying the nature of the singular points at infinity of the autonomous system (10). This will be performed introducing homogeneous coordinates so that we restrict ourselves to the polynomial case. In the following subsections, where f_1, f_2 and g are polynomials, the nature of the singular points at infinity will be determined by imposing conditions only on the coefficients of the highest degree terms and, therefore, it will not be affected at all by the constant term $+E$ in (10).

The problem of investigating the nature of singular points at infinity is a classical one and we refer to the papers by Bendixson [1] and Gomory [5] which are milestones in this field.

A preliminary result was obtained by Tempesti in her thesis for the generalized Liénard equation:

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = \lambda e(t). \tag{11}$$

Theorem 4.2 (Tempesti [15]). *If the point at infinity of the x-axis is a saddle point for*:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -f(x,\dot{x})\dot{x} - g(x), \end{cases}$$

then there are two separatrices from opposite sides with respect to the y-axis and, hence, $\|x\|_\infty \le C$ *for every T-periodic solution of* (11).

Being our equation a particular case of (11), one has the following corollary.

Corollary 4.3 (Tempesti [15]). *If the point at infinity of the x-axis of* (4) *is a saddle point, the equation* (1) *has at least a T-periodic solution.*

Since this result is abstract, in the remaining part of the paper we will give some condition ensuring that the point at infinity of the x-axis of (4) is actually a saddle point. Moreover we will investigate other situations in which the singular point considered is not a saddle, but a bound can be actually reached.

We assume that the functions f_1, f_2 and g in the system (4) are polynomials:

$$f_1(x) = \sum_{i=0}^{l} a_i x^i, \qquad f_2(x) = \sum_{i=0}^{m} b_i x^i, \qquad g(x) = \sum_{i=0}^{n} c_i x^i,$$

with $a_l \neq 0$, $b_m \neq 0$ and $c_n \neq 0$. In view of the first assumption on g in Theorem 1.1, we assume also that n is an odd integer and $c_n > 0$. Let $k = \max\{l, m+1\}$ and suppose that $k \geq n$ (this condition implies that the point at infinity of the x-axis is actually a critical point), so that the degree of $f_1(x)y + f_2(x)y^2 + g(x)$ is exactly $k+1$. In order to extend system (4) to the projective plane, we follow [5] (see also the book [11, XI, §9, pp. 307–311]) and introduce homogeneous coordinates (x, y, z) such that the point $(x, y, 1)$ represents the point (x, y) in the affine plane; with this identification, (4) and the following system:

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x}{z}\right) = \dfrac{y}{z}, \\ \dfrac{d}{dt}\left(\dfrac{y}{z}\right) = -f_2\left(\dfrac{x}{z}\right)\left(\dfrac{y}{z}\right)^2 - f_1\left(\dfrac{x}{z}\right)\dfrac{y}{z} - g\left(\dfrac{x}{z}\right), \end{cases} \tag{12}$$

have the same solutions, but the last one can be extended to cover also the line at infinity $z = 0$; indeed, multiplying both the equations of (12) by z^{k+2} and considering the homogeneous extension of the polynomials f_1, f_2 and g, namely:

$$f_1^H(x,z) = z^l f_1\left(\frac{x}{z}\right), \qquad f_2^H(x,z) = z^m f_2\left(\frac{x}{z}\right), \qquad g^H(x,z) = z^n g\left(\frac{x}{z}\right),$$

we obtain:

$$\begin{cases} z^k\left(z\dfrac{dx}{dt} - x\dfrac{dz}{dt}\right) = yz^{k+1}, \\ z^k\left(z\dfrac{dy}{dt} - y\dfrac{dz}{dt}\right) = -f_2^H(x,z)z^{k-m}y^2 - f_1^H(x,z)z^{k-l+1}y - g^H(x,z)z^{k-n+2}. \end{cases} \tag{13}$$

Using coordinates $(x, 1, z)$ and a new time-variable τ such that $dt = z^k\, d\tau$, the trajectories of (13) in the open region $y \neq 0$ of the projective plane are described by the system:

$$\begin{cases} \dfrac{dz}{d\tau} = f_1^H(x,z)z^{k-l+1} + f_2^H(x,z)z^{k-m} + g^H(x,z)z^{k-n+2}, \\ \dfrac{dx}{d\tau} = f_1^H(x,z)xz^{k-l} + f_2^H(x,z)xz^{k-m-1} + g^H(x,z)xz^{k-n+1} + z^k. \end{cases} \tag{14}$$

On the other hand, using coordinates $(1, y, z)$ and a new time-variable σ such that $dt = z^k\, d\sigma$, the trajectories of (13) in the open region $x \neq 0$ are described by the system:

$$(15)\quad \begin{cases} \dfrac{dz}{d\sigma} = -yz^{k+1}, \\ \dfrac{dy}{d\sigma} = -[f_2^H(1,z)z^{k-m-1} + z^k]y^2 - f_1^H(1,z)z^{k-l}y - g^H(1,z)z^{k-n+1}. \end{cases}$$

In particular, the point at infinity of the x-axis is the critical point $(0,0)$ of the last system (15). Therefore we look for conditions ensuring that the origin is a saddle for (15).

4.1. The case $k = l \geq m+1$

If we suppose that $k = l \geq m+1$, then system (15) can be written in the following form:

$$(16)\quad \begin{cases} \dfrac{dz}{d\sigma} = -yz^{l+1}, \\ \dfrac{dy}{d\sigma} = -a_l y - \gamma z + Y_2(z, y), \end{cases}$$

where $Y_2(z, y)$ contains terms with degree grater than one and:

$$\gamma = \begin{cases} c_n & \text{if } l = n, \\ 0 & \text{if } l > n. \end{cases}$$

This kind of system was already studied by Bendixson in [1, §35–40], where it is proved that the behavior of the trajectories of (16) around $(0,0)$ is almost completely determined by the Poincaré index i of the field in $(0,0)$. It turns out that the origin is a saddle for (16) if and only if $i = 1$, therefore now we are going to calculate this number. We use its original definition: if we set $Z(z, y) = -yz^{l+1}$ and $Y(z, y) = -[f_2^H(1,z)z^{k-m-1} + z^k]y^2 - f_1^H(1,z)z^{k-l}y - g^H(1,z)z^{k-n+1}$ and if we consider a small curve which turns around $(0,0)$, then the index of the field $(Z(z, y), Y(z, y))$ around $(0,0)$ is given by one half of the difference between the number of times that Y/Z passes from $-\infty$ to $+\infty$ and the number of times it passes from $+\infty$ to $-\infty$, as the curve is run counterclockwise:

$$i = \frac{\#\{Y/Z \text{ passes from } -\infty \text{ to } +\infty\} - \#\{Y/Z \text{ passes from } +\infty \text{ to } -\infty\}}{2}.$$

Since $Z(z, y)$ vanishes only on the coordinate axes $z = 0$ and $y = 0$, we can choose the curve to be a small circle and check the sign of:

$$Y(z, 0) = -c_n z^{l-n+1} + \text{h.o.t.'s} \qquad \text{and} \qquad Y(0, y) = -a_m y - \beta y^2,$$

(with $\beta \neq 0$ only if $l = n$) in a neighborhood of the origin. Thus the value of i depends only on the sign of a_l and c_n as well as on the parity of l and n and it is straightforward to check that actually we have:

$$i = \begin{cases} 1 & \text{if } n \text{ is odd and } c_n > 0, \\ -1 & \text{if } n \text{ is odd and } c_n < 0, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Hence we are exactly in the case in which the point at infinity of the x-axis is a saddle and Corollary 4.3 implies the following result.

Theorem 4.4. *Let l, m and n be positive integers, a_l, b_m and c_n be non-zero real numbers and $e(t)$ be a continuous T-periodic function. If $l \geq m+1$, $l \geq n$, n is odd and $c_n > 0$, then the equation*:

$$\ddot{x} + \left(\sum_{i=0}^{l} a_i x^i\right)\dot{x} + \left(\sum_{i=0}^{m} b_i x^i\right)\dot{x}^2 + \sum_{i=0}^{n} c_i x^i = e(t) \tag{17}$$

has a T-periodic solution.

4.2. The case $k = m + 1 > l$

Assume now that $m \geq l$ and let:

$$\mu = m - l + 1 \qquad \text{and} \qquad \nu = m - n + 2,$$

so that (15) can be rewritten as:

$$\begin{cases} \dfrac{dz}{d\sigma} = -yz^{m+2}, \\ \dfrac{dy}{d\sigma} = -[f_2^H(1, z) + z^{m+1}]y^2 - f_1^H(1, z)z^{\mu}y - g^H(1, z)z^{\nu}. \end{cases} \tag{18}$$

Let us put in evidence the terms with degree 1 and 2 in the second equation:

$$\frac{dy}{d\sigma} = Y(z, y) := -b_m y^2 - \alpha zy - \beta z - \gamma z^2 + Y_3(z, y), \tag{19}$$

where:

$$\alpha = \begin{cases} a_l & \text{if } m = l, \\ 0 & \text{if } m \geq l+1, \end{cases} \qquad \beta = \begin{cases} c_n & \text{if } m = n-1, \\ 0 & \text{if } m \geq n, \end{cases}$$

$$\gamma = \begin{cases} c_{n-1} & \text{if } m = n-1, \\ c_n & \text{if } m = n, \\ 0 & \text{if } m \geq n+1. \end{cases}$$

The possible directions (z, y) of approach to $(0,0)$ satisfy the homogeneous equation $z[Y(z,y)]_{\text{lowest degree terms}} = 0$, therefore we will have to consider only the direction $z = 0$, if $\beta \neq 0$ (that is if $v = 1$), and maybe also other directions satisfying $b_m y^2 + \alpha zy + \gamma z^2 = 0$, if, otherwise, $\beta = 0$, (that is if $v \geq 2$). We note that in both cases the linear part of the field does not help in the study of the nature of the singular point $(0,0)$ since it is identically zero or its eigenvalues are both zero, thus we are led to apply some change of variables suitably developed in [1].

In order to examine the trajectories approaching $(0,0)$ with the limiting direction $z = 0$, we consider the following positions:

$$y = \xi \qquad \text{and} \qquad z = \xi\eta,$$

which transform (18) into:

$$\text{(20)} \quad \begin{cases} \dfrac{d\xi}{d\sigma} = -[f_2^H(1,\xi\eta) + (\xi\eta)^{m+1}]\xi^2 - f_1^H(1,\xi\eta)\xi^{\mu+1}\eta^{\mu} - g^H(1,\xi\eta)\xi^{v}\eta^{v}, \\ \dfrac{d\eta}{d\sigma} = f_2^H(1,\xi\eta)\xi\eta + f_1^H(1,\xi\eta)\xi^{\mu}\eta^{\mu+1} + g^H(1,\xi\eta)\xi^{v-1}\eta^{v+1}. \end{cases}$$

The critical point $(0,0)$ of (18) corresponds to the trajectories on $\xi = 0$ of (20) and all the other characteristics approaching $(0,0)$ in (20) are in one-to-one correspondence with the characteristics of (18) approaching $(0,0)$ with limiting direction $z = 0$.

Now, if $v \geq 2$, we can introduce a new parameter σ_1 such that $d\sigma_1 = \xi\, d\sigma$ and the system becomes:

$$\begin{cases} \dfrac{d\xi}{d\sigma_1} = -b_m\xi + \text{h.o.t.'s}, \\ \dfrac{d\eta}{d\sigma_1} = b_m\eta + \text{h.o.t.'s}, \end{cases}$$

for which the origin is clearly a saddle. Therefore, if $m \geq n$, there are no other orbits of (18) which goes towards $(0,0)$ with limiting direction $z = 0$, besides those on $z = 0$ itself.

If $v = 1$, then $m = n - 1$ and system (20) has the form:

$$\begin{cases} \dfrac{d\xi}{d\sigma} = -\xi(b_m\xi + c_n\eta) + \text{h.o.t.'s}, \\ \dfrac{d\eta}{d\sigma} = \eta(b_m\xi + c_n\eta) + \text{h.o.t.'s}, \end{cases}$$

hence the possible directions to approach $(0,0)$ solve $2\xi\eta(b_m\xi + c_n\eta) = 0$ and are the following three: $\xi = 0$, $\eta = 0$ and $\eta/\xi = -b_m/c_n$. The substitution:

$$\xi = \xi_1, \qquad \eta = \xi_1\eta_1, \qquad d\sigma_1 = \xi_1\, d\sigma,$$

transforms (20) into:

$$\text{(21)} \qquad \begin{cases} \dfrac{d\xi_1}{d\sigma_1} = -b_m\xi_1 - c_n\xi_1\eta_1 + \xi_1^2\eta_1\phi(\xi_1,\eta_1), \\ \dfrac{d\eta_1}{d\sigma_1} = 2\eta_1(c_n\eta_1 + b_m) + \xi_1\eta_1^2\psi(\xi_1,\eta_1), \end{cases}$$

where ϕ and ψ are polynomials. This system has on the line $\xi_1 = 0$ two critical points $(0,0)$ and $(0,-b_m/c_m)$, which respectively correspond to the limiting directions $\eta = 0$ and $\eta/\xi = -b_m/c_n$ for (20). Now, $(0,0)$ is clearly a saddle and, therefore, the characteristics on $\eta = 0$ are the only two approaching the origin in (20) with limiting direction $\eta = 0$. On the other hand the Jacobian of the field of (21) in $(0,-b_m/c_n)$ is:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2b_m \end{pmatrix}$$

therefore again the Poincaré index helps in the determination of the nature of the critical point $(0,-b_m/c_m)$. Using the fact that n is odd, $c_n > 0$ and m even, it can be shown that the index is 1 and the point is a saddle. Finally, the substitution:

$$\eta = \xi_2, \qquad \xi = \xi_2\eta_2, \qquad d\sigma_2 = \xi_2\, d\sigma,$$

shows that for system (20) only the two characteristics on $\xi = 0$ approach $(0,0)$ with direction $\xi = 0$.

Hence, when $m = n - 1$, there are only 4 orbits of (18) tending to $(0,0)$ with limit tangency $z = 0$: two of them are the trivial ones on $z = 0$, while the other two correspond to the trajectories of (21) approaching $(0,-b_m/c_n)$ and, therefore, lie in $z > 0$, if $b_m < 0$, and in $z < 0$, if $b_m > 0$. In both cases, however, the origin cannot be a saddle for (18). We will consider this and other similar situations in Subsection 4.3.

Let us study now the case in which $m \geq n$, so that $\beta = 0$ in (19), and consider the quantity $\Delta := \alpha^2 - 4b_m\gamma$. We already showed that in this case the trajectories on $z = 0$ are the only two to approach the origin with limiting direction $z = 0$. If $\Delta < 0$, there are no other possible directions of approach to $(0,0)$ and, then, no other characteristics going to $(0,0)$ at all. The situation in which $\Delta = 0$ is a delicate one and we will consider it later. Therefore assume that $\Delta > 0$. Just note that in this case at least one of α and γ must be different from zero; thus either $m = l$ (and $\mu = 1$) or $m = n$ (and $\nu = 2$).

Let us make the following change of variables:

$$z = \xi, \qquad y = \xi\eta, \qquad d\sigma_1 = \xi\, d\sigma, \tag{22}$$

which transforms system (18) into the following one:

$$\begin{cases} \dfrac{d\xi}{d\sigma_1} = -\xi^{m+2}\eta, \\ \dfrac{d\eta}{d\sigma_1} = -f_2^H(1,\xi)\eta^2 - f_1^H(1,\xi)\xi^{\mu-1}\eta - g^H(1,\xi)\xi^{\nu-2}, \end{cases} \tag{23}$$

and, specifying the lower order terms in the second equation, we have:

$$\frac{d\eta}{d\sigma_1} = -b_m\eta^2 - \alpha\eta - \gamma + \xi\psi(\xi,\eta) = -b_m(\eta - k_1)(\eta - k_2) + \xi\psi(\xi,\eta),$$

where ψ is a polynomial and $|k_1| \le |k_2|$. Therefore system (23) has on the η-axis two critical points, $(0,k_1)$ and $(0,k_2)$, which, as explained above, can be studied in order to determine the characteristics of (18) approaching the origin with slope $y/z = k_1$ and $y/z = k_2$, respectively.

In both these critical points the linear approximation of system (23) has one null eigenvalue and the other eigenvalue is $\pm\sqrt{\Delta} \neq 0$; hence the behavior of the characteristics approaching them is completely determined by the Poincaré index of the field in $(0,k_1)$ and $(0,k_2)$. It can be easily checked that the index of $(0,k_1)$ depends on the parity of n and on the sign of c_n and that, in our case (that is n odd and $c_n > 0$) it is always 1. Therefore it is a saddle and, in terms of the original system (18), it means that there are exactly two characteristics approaching $(0,0)$ with limiting direction $y/z = k_1$, one in the half plane $z > 0$ and one in $z < 0$.

In the same way it can be shown that the index of $(0,k_2)$ depends on the parity of m and on the sign of b_m, but, actually, we do not care about it, since $(0,k_2)$ can be a saddle (index 1), a node (index -1) or a saddle–node (index 0), but in any case there are at least two characteristics which go to $(0,k_2)$, one in $\xi > 0$ and one in $\xi < 0$. Hence the original system (18) has *at least* two orbit (one in $z > 0$ and one in $z < 0$) which approach $(0,0)$ with limiting direction $y/z = k_2$ and, in conclusion, it cannot be a classical saddle and Corollary 4.3 cannot be applied.

However, in this case we can use directly Lemma 4.1. Indeed the desired a priori bounds on the T-periodic solutions of (9) can be deduced also if we show the existence of *at least* two unbounded trajectories (one in either the first or the fourth quadrant and the other either in the second or in the third quadrant) of (10) which are run in infinite time and we already accomplished this task when we said that the index of $(0, k_1)$ is 1 under our assumptions.

Let us consider now the case in which $\Delta = 0$. This can happen in two ways. The first one is when $m = n = l$ (so $\mu = 1$ and $\nu = 2$) and $a_l^2 - 4b_m c_n = 0$, so that the second equation in (18) becomes:

$$\frac{dy}{d\sigma} = -b_m(y - kz)^2 + Y_3(x, y) \qquad \text{with } k = -\frac{a_l}{2b_m}$$

and the unique limiting direction of approach to $(0,0)$ left to be studied is $y/z = k$. The change of variables:

$$z = \xi, \qquad y = \xi(\eta + k), \qquad d\sigma_1 = \xi\, d\sigma,$$

leads to the following system:

$$\begin{cases} \dfrac{d\xi}{d\sigma_1} = -\xi^{m+2}(\eta + k), \\ \dfrac{d\eta}{d\sigma_1} = -f_2^H(1, \xi)(\eta + k)^2 - f_1^H(1, \xi)(\eta + k) - g^H(1, \xi), \end{cases}$$

and, using the condition $\Delta = 0$, the second equation becomes:

$$\frac{d\eta}{d\sigma_1} = -f_2^H(1, \xi)\eta^2 - \sum_{i=1}^{m}(2kb_{m-i} + a_{m-i})\xi^i\eta - \sum_{i=1}^{m}(k^2 b_{m-i} + ka_{m-i} + c_{m-i})\xi^i.$$

It is clear now that one should impose some further condition on the coefficient of the terms with lower degree in the polynomials f_1, f_2 and g in order to determine the structure of the origin for the new system, but we are not going to do this here.

We shall discuss the other case in which $\Delta = 0$, that is when $m \geq l + 1$ and $m \geq n + 1$ and, therefore, $\mu \geq 2$, $\nu \geq 3$ and $\alpha = \gamma = 0$. We have to study therefore the characteristics of (18) approaching the origin with limit tangency to the z-axis. We note that the change of variables (22) transforms (18) into the new system (23) in which the first equation is unchanged, while in the second one the degree of some terms decreases; hence Bendixson's idea is to iterate that change of variable until we obtain an already studied case. In particular, if we iterate the transformation i times, that is if we set:

$$z = \xi_i, \qquad y = \xi_i^i \eta_i, \qquad d\sigma_i = z^i\, d\sigma,$$

we get the system:

$$\begin{cases} \dfrac{d\xi_i}{d\sigma_i} = -\xi_i^{m+2}\eta_i, \\ \dfrac{d\eta_i}{d\sigma_i} = -[f_2^H(1,\xi_i) - (i-1)\xi_i^{m+1}]\eta_i^2 - f_1^H(1,\xi_i)\xi_i^{\mu-i}\eta_i - g^H(1,\xi_i)\xi_i^{\nu-2i}, \end{cases}$$

which we studied above if either $\mu - i = 1$ or $\nu - 2i \in \{1,2\}$, since in these cases the second equation has the form of (19) with some coefficient different from zero among α, β and γ. Therefore we distinguish the following cases:

1. *if* $\nu \geq 2\mu + 1$, that is if $2l \geq m + n + 1$, we choose $i = \mu - 1$ and we obtain the second equation of the form (19) with $\beta = \gamma = 0$ and $\alpha = a_l$; thus Δ is positive and the existence of the desired characteristics is ensured;
2. *if* $\nu = 2\mu$, that is if $2l = m + n$, the choice $i = \mu - 1$ leads to a case in which $\beta = 0$, $\alpha = a_l$ and $\gamma = c_n$ and we have to ask that $a_l^2 - 4b_m c_n > 0$ in order to obtain the existence of the two characteristics;
3. *if* $\nu \leq 2\mu - 1$ *and* ν *is even*, then actually $\nu \leq 2\mu - 2$, $2l \leq m + n - 2$, m is odd and we choose $i = \nu/2 - 1$ obtaining the case $\alpha = \beta = 0$ and $\gamma = c_n$; therefore we have to impose that $b_m c_n < 0$, that is, $b_m < 0$, to satisfy the requirement $\Delta > 0$;
4. *if* $\nu \leq 2\mu - 1$ *and* ν *is odd*, that is if $2l \leq m + n - 1$ and m is even, then the choice $i = (\nu - 1)/2$ leads to the case $\beta = c_n$ in which the characteristics towards the origin exists only in one of the two half planes $z \leq 0$ and $z \geq 0$, depending on $b_m > 0$ or $b_m < 0$, respectively, thus Lemma 4.1 gives the bound for x only on one side; we treat this case in the next subsection.

The discussion above can be summarized by the following result.

Theorem 4.5. *Let l, m and n be positive integers, a_l, b_m and c_n be non-zero real numbers and $e(t)$ be a continuous T-periodic function and assume also that $m \geq l$, $m \geq n$, n is odd and $c_n > 0$. If one of the following three conditions is satisfied*:

1. $2l \geq m + n + 1$,
2. $2l = m + n$ *and* $a_l^2 - 4b_m c_n > 0$,
3. $2l \leq m + n - 1$, m *is odd and* $b_m < 0$,

then the equation (17) *has a T-periodic solution.*

4.3. The case $m \geq l$, $2l \leq m + n - 1$ and m even

If $2l \leq m + n - 1$ and m is even then we saw that we are able to obtain the characteristics towards the point at infinity of the x-axis for (4) only in one of the two half planes generated by the y-axis. Therefore Lemma 4.1 provides an a priori bound only for either the minimum or the maximum of the x-component of the periodic solutions. However, in this case the other bound can be deduced through the study of the other critical point at infinity of (4), that is the point at infinity of the y-axis or, with the notation of the preceding section, the point $(0,0)$ in system (14), which now, since $k = m+1$, becomes:

$$\begin{cases} \dfrac{dz}{d\tau} = f_1^H(x,z)z^{m-l+2} + f_2^H(x,z)z + g^H(x,z)z^{m-n+3}, \\ \dfrac{dx}{d\tau} = f_1^H(x,z)xz^{m-l+1} + f_2^H(x,z)x + g^H(x,z)xz^{m-n+2} + z^{m+1}. \end{cases} \tag{24}$$

The unique direction of approach to the origin is easily seen to be $z = 0$, hence we perform the usual change of variables:

$$x = \xi, \qquad z = \xi\eta, \qquad d\tau_1 = \xi^m\, d\tau,$$

to get the system:

$$\begin{cases} \dfrac{d\xi}{d\tau_1} = [f_2^H(1,\eta) + \eta^{m+1}]\xi + f_1^H(1,\eta)\eta^{m-l+1}\xi^2 + g^H(1,\eta)\eta^{m-n+2}\xi^3, \\ \dfrac{d\eta}{d\tau_1} = -\eta^{m+2}, \end{cases}$$

whose Jacobian in $(0,0)$ is:

$$\begin{pmatrix} b_m & 0 \\ 0 & 0 \end{pmatrix}.$$

Since in this case the Poincaré index can be easily calculated to be 0, we are in presence of a saddle–node. A picture of the directions in which the orbits on $\xi = 0$ and on $\eta = 0$ of (24) are run, shows immediately that the zone $\eta > 0$ is the saddle one if $b_m > 0$, while it is the nodal one if $b_m < 0$. Hence it turns out that in the system (24) the origin is a repulsive 1-node, if $b_m > 0$, and an attractive one, if $b_m < 0$.

Just to fix the idea, let us consider the case $b_m > 0$ and take the point $(1,1,0)$ in the projective plane. By the discussion above, there is an $\varepsilon > 0$ such that the characteristic passing by $(1,1,z_0)$ tends to $(0,1,0)$ in the past if $0 < z_0 \le \varepsilon$. On the other hand, since in this case system (18) has no characteristics approaching $(0,0)$ from the half plane $z > 0$, if ε is taken sufficiently small then the characteristic passing by $(1,1,\varepsilon)$ crosses the x-axis in a point $(1,0,\delta)$, with $\delta > 0$, and the space between this characteristic and the one on

$z=0$ is filled up by characteristics joining points $(1,1,z_0)$ and $(1,0,z_1)$ with $0<z_0\le\varepsilon$ and $0<z_1\le\delta$. Therefore all the characteristics passing through $(1,0,z_1)$ tends to $(0,1,0)$ in the past if $0<z_1\le\delta$. If we read this fact in the affine plane, we deduce that all the orbits of (4) crossing the x-axis to the right of $x=1/\delta$ have the y-component which tends to $+\infty$ for negative times (in the same way it can be proved that it also goes to $-\infty$ as time increases): we use these orbits as a "barrier" for the periodic solutions of the forced version of (4).

Indeed, if $|e(t)|\le M$ then all the solution curves of the system:

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-f_1(x)y-f_2(x)y^2-g(x)+\lambda e(t), \end{cases} \quad \lambda\in[0,1], \tag{25}$$

crosses the characteristics of:

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-f_1(x)y-f_2(x)y^2-g(x)+M, \end{cases}$$

passing from above to below as time increases, as a comparison of the slopes immediately shows. Since the constant M does not affect at all the nature of the critical points at infinity, we deduce that the periodic solutions of (25) can cross the x-axis only before $1/\delta$, which therefore constitutes the missing a priori bound.

The case $b_m<0$ can be treated in a similar way and we can state the following theorem.

Theorem 4.6. *Let l,m and n be positive integers such that m is even, n is odd, $m\ge l$, $m\ge n-1$ and $2l\le m+n-1$; let a_l,b_m and c_n be non-zero real numbers such that $c_n>0$; let $e(t)$ be a continuous T-periodic function. Then equation* (17) *has a T-periodic solution.*

Remark 4.7. The remaining a priori bound could be deduced in this case also without the study of the point at infinity of the y-axis. Indeed we could first transform our equation in a classical Liénard equation:

$$\ddot{x}+f(x)\dot{x}+\tilde{g}(x)=0$$

by the positions:

$$f(x)=f_1(x)e^{\int_0^x f_2(s)ds}, \qquad \tilde{g}(x)=g(x)e^{2\int_0^x f_2(s)ds}.$$

Then we could apply results like those in [17] or [19] which give necessary and sufficient conditions ensuring that all the trajectories of a Liénard equation cross the x-axis in the phase plane (or the graph of $F(x)=\int_0^x f(s)ds$ in the Liénard plane).

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