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Global Attractivity for Non-Autonomous Linear Delay Systems*

By

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Abstract. In this paper, we study the asymptotic behavior of solutions of a non-autonomous delay differential system. It is shown that every solutions tends to zero provided a certain matrix derived from the coefficients and the delays of the system is a M-matrix.

Key Words and Phrases. Asymptotic behavior, Non-autonoumous delay linear system, M-matrix.

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1. Introduction

Consider a system of delayed linear differential equations with variable coefficients of the form

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_{ij}(t)), \qquad i = 1, 2, \dots, n, \tag{1.1}$$

where $a_{ij}(t), \tau_{ij}(t)$, $i, j = 1, 2, \dots, n$ are continuous on $[t_0, \infty)$, and the delays $\tau_{ij}(t)$ $(i, j = 1, \dots, n)$ satisfy

$$\tau_{ij}(t) \geq 0 \quad \text{and} \quad t - \tau_{ij}(t) \uparrow \infty \qquad \text{as } t \to \infty. \tag{1.2}$$

When $a_{ij}(t)$ and $\tau_{ij}(t)$ are constants, that is, $a_{ij}(t) \equiv a_{ij}$, $\tau_{ij}(t) \equiv \tau_{ij}$ $(i, j = 1, 2, \dots, n)$, (1.1) reduces to an autonomous system

$$\dot{x}_i(t) = -\sum_{j=1}^{n} a_{ij}x_j(t-\tau_{ij}), \qquad i = 1, 2, \dots, n. \tag{1.3}$$

For system (1.3), it is well known the trivial solution $x(t) \equiv 0$ is asymptotically stable if and only if all the roots λ of its characteristic equation, namely,

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\text{(1.4)} \qquad \det\begin{pmatrix} a_{11}e^{-\lambda\tau_{11}} - \lambda & a_{12}e^{-\lambda\tau_{12}} & \cdots & a_{1n}e^{-\lambda\tau_{1n}} \\ a_{21}e^{-\lambda\tau_{21}} & a_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & a_{2n}e^{-\lambda\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}e^{-\lambda\tau_{n1}} & a_{n2}e^{-\lambda\tau_{n2}} & \cdots & a_{nn}e^{-\lambda\tau_{nn}} - \lambda \end{pmatrix} = 0
$$

have negative real parts (c.f. [5]). However, in general, it is quite difficult and often an analytically almost impossible task to decide if all the roots of (1.4) have negative real parts. When $\tau_{ij} = 0$, for all $i, j = 1, 2, \dots, n$, (1.3) becomes an ordinary differential system. In that case, $x(t) \equiv 0$ is asymptotically stable if and only if the coefficients matrix $A = (a_{ij})$ is a positively stable matrix (i.e. all its eigenvalues have positive real parts) and the Routh-Hurwitz criterion is applicable.

When $\tau_{ii} = 0$ for $i = 1, 2, \dots, n$, by studying the roots of (1.4) using Rouché's theorem and the implicit function theorem, Hofbauer and So [6] established the following result.

Theorem 1.1. *Assume that $\tau_{ii} = 0$ for $i = 1, 2, \dots, n$. Then $x(t) \equiv 0$ is asymptotically stable for (1.3) for all choice of $\tau_{ij} \geq 0$ $(i \neq j)$ if and only if $a_{ii} > 0$ for $i = 1, 2, \dots, n$,* det $A \neq 0$ *and A is weakly diagonally dominant. (A is said to be weakly diagonally dominant if all the principal minors of $\hat{A} = (\hat{a}_{ij})$ are non-negative, where $\hat{a}_{ii} = a_{ii}$, $\hat{a}_{ij} = -|a_{ij}|$, $i \neq j$).*

In the case of a quasi-monotone matrix A (i.e. $a_{ij} \geq 0$ for $i \neq j$), Györi [4] obtained a similar result.

When $\tau_{ii} \neq 0$, $i = 1, \dots, n$, i.e. when instantaneous feedback is absent, (1.3) becomes a system of "pure-delay-type". As was pointed out by Gopalsamy and He [3], He [6] and Kuang [8], the stability problem for such a system becomes much harder, even for the autonomous case. Recently, So, Tang and Zou [9] obtained a criterion for the stability of (1.3) by using M-matrix and some inequality techniques. The criterion is related to a form of 3/2 estimate for the diagonal delays τ_{ii} $(i = 1, \dots, n)$. In order to state this result, first we introduce the matrix $\tilde{A} = (\tilde{a}_{ij})$ defined by

$$\tilde{a}_{ii} = a_{ii}, \qquad \text{for } i = 1, 2, \dots, n$$

and

$$\tilde{a}_{ij} = -\frac{1 + \frac{1}{9}a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})}{1 - \frac{1}{9}a_{ii}\tau_{ii}(3 + 2a_{ii}\tau_{ii})}|a_{ij}|, \qquad \text{for } 1 \leq i \neq j \leq n.$$

Also, for the sake of convenience, we recall the concept of a non-singular M-matrix (c.f. Fiedler [1, p. 114]).

Definition 1.1. A square matrix $B = (b_{ij})$ of order n is called a non-singular M-matrix if (i) $b_{ij} \leq 0$, for all $i \neq j$, and (ii) all the principal minors of B are positive.

There are many equivalent formulation of this concept (c.f. [1, Theorem 5.1]). In particular, if B is a non-singular M-matrix, then B^{-1} is a positive matrix. The following theorem is proved in So, Tang and Zou [9].

Theorem 1.2. *Assume that*

$$a_{ii}\tau_{ii} < \frac{3}{2}, \qquad \text{for all } i = 1, 2, \ldots, n.$$

If $\tilde{A}$ *is a non-singular M-matrix, then every solution* $(x_1(t), x_2(t), \ldots, x_n(t))$ *of* (1.3) *tends to* 0, *as* $t \to \infty$.

Now, back to the non-autonomous system (1.1). Recall that in the scalar case, there are also $3/2$ type criteria in the form of an integral (c.f. [10], [11]). One naturally wonders if a result similar to Theorem 1.2 can be established for (1.1). This paper will answer this question in the affirmative (see Section 2). As will be seen in what follows, due to the non-autonomous nature of (1.1), one also needs some additional tricks on integration, besides M-matrix and inequalities.

2. Main results

Theorem 2.1. *Assume that*

$$a_{ii}(t) \geq 0, \quad i = 1, 2, \ldots, n, \qquad |a_{ij}(t)| \leq b_{ij} a_{ii}(t), \quad 1 \leq i \neq j \leq n \tag{2.1}$$

and

$$d_i := \limsup_{t \to \infty} \int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)\,ds < \frac{3}{2}, \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

Let $\tilde{B} = (\tilde{b}_{ij})$ *be the* $n \times n$ *matrix with entries*

$$\tilde{b}_{ii} = 1, \qquad i = 1, 2, \ldots, n \tag{2.3}$$

and

$$\tilde{b}_{ij} = \begin{cases} -\frac{2+d_i^2}{2-d_i^2} b_{ij}, & \text{if } d_i < 1, \\ -\frac{1+2d_i}{3-2d_i} b_{ij}, & \text{if } d_i \geq 1, \end{cases} \qquad i \neq j. \tag{2.4}$$

If $\tilde{B}$ *is a non-singular M-matrix, then every (forward) solution* $x(t)$ *of* (1.1) *is bounded.*

Proof. Assume that $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ is a solution of (1.1) on $[t_0, \infty)$. Let $\hat{t}_0 = \min_{1 \le i,j \le n}\{t_0 - \tau_{ij}(t_0)\}$. We may assume that $x(t)$ is defined and continuous on $[\hat{t}_0, \infty)$. Let $\tilde{B}(\varepsilon) = (\tilde{b}_{ij}(\varepsilon))$ be the matrix with entries $\tilde{b}_{ii}(\varepsilon) = 1$ for $i = 1, 2, \dots, n$ and

$$\tilde{b}_{ij}(\varepsilon) = \begin{cases} -\frac{2+(d_i+\varepsilon)^2}{2-(d_i+\varepsilon)^2} b_{ij}, & \text{if } d_i < 1, \\ -\frac{1+2(d_i+\varepsilon)}{3-2(d_i+\varepsilon)} b_{ij}, & \text{if } d_i \ge 1, \end{cases} \qquad i \ne j.$$

Since $\tilde{B}(0) = \tilde{B}$ is a non-singular M-matrix, it follows that there exists an $\varepsilon_0 > 0$ such that $\tilde{B}(\varepsilon_0)$ is also a non-singular M-matrix, $d_i + \varepsilon_0 < 3/2$ for all i and $d_i + \varepsilon_0 < 1$ whenever $d_i < 1$. Note that for $i \ne j$, we have, $0 \le b_{ij} \le -\tilde{b}_{ij}(\varepsilon_0)$, where we set $b_{ij} = 1$ if $a_{ii}(t) \equiv 0$. For the given ε_0, there exists $T_0 > t_0$ such that

$$\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds \le d_i + \varepsilon_0, \qquad \text{for all } t \ge T_0,\ i = 1, 2, \dots, n. \tag{2.5}$$

We shall prove that $\max\{|x_i(t)| : i = 1, 2, \dots, n\}$ is bounded. Otherwise, we may assume, without loss of generality, that

$$\limsup_{t\to\infty} |x_i(t)| = \infty \qquad \text{for } i = 1, 2, \dots, k (\le n) \tag{2.6}$$

and

$$|x_i(t)| \le M \qquad \text{for } t \ge \hat{t}_0,\ i = k+1, \dots, n. \tag{2.7}$$

First, choose a sequence $\{t_{1m}\}_{m=1}^{\infty}$ such that $t_{1m} \ge T_0$, $t_{1m} \uparrow \infty$, $|x_1(t_{1m})| \uparrow \infty$ as $m \to \infty$ and $|x_1(t_{1m})| = \max\{|x_1(s)| : \hat{t}_0 \le s \le t_{1m}\}$. For each $i = 2, \dots, k$, let t_{im} be the leftmost maximum point of the function $|x_i(t)|$ on the interval $[\hat{t}_0, t_{1m}]$. Hence, we have obtained k sequences $\{t_{im}\}_{m=1}^{\infty}$, $i = 1, 2, \dots, k$ such that

$$\begin{cases} t_{im} - \tau_{ij}(t_{im}) \ge t_0, \quad t_{im} \ge T_0 \\ t_{im} \uparrow \infty,\ |x_i(t_{im})| \uparrow \infty \quad \text{as } m \to \infty, \quad i = 1, 2, \dots, k,\ j = 1, 2, \dots, n \\ |x_i(t)| \le |x_i(t_{im})| \quad \text{for } \hat{t}_0 \le t \le t_{1m}, \end{cases}$$

We may assume $|x_i(t_{im})| = x_i(t_{im})$ (if necessary, we can apply the following argument to $-x_i(t)$ instead of $x_i(t)$ and $-a_{ij}(t)$ instead of $a_{ij}(t)$ for $j \ne i$). Then

$$|x_i(t)| \le x_i(t_{im}), \quad t_0 \le t \le t_{1m}, \quad \dot{x}_i(t_{im}) \ge 0, \qquad i = 1, 2, \dots, k. \tag{2.8}$$

It follows from (1.1) that

(2.9)

$$\dot{x}_i(t) \le a_{ii}(t)\left[-x_i(t - \tau_{ii}(t)) + \sum_{j \ne i}^{k} b_{ij}|x_j(t_{jm})| + M \sum_{j=k+1}^{n} b_{ij}\right], \qquad t_0 \le t \le t_{1m}.$$

Set

$$\alpha_i = \sum_{j\neq i}^{k} b_{ij}|x_j(t_{jm})| + M\sum_{j=k+1}^{n} b_{ij}, \qquad i = 1,2,\ldots,k. \tag{2.10}$$

We claim that

$$x_i(t_{im} - \tau_{ii}(t_{im})) \leq \alpha_i, \qquad i = 1,2,\ldots,k. \tag{2.11}$$

If (2.11) is not true for some i $(i = 1,\ldots,k)$, then there exists a $h > 0$ such

$$-x_i(t - \tau_{ii}(t)) + \alpha_i < 0 \qquad \text{for } t_{im} - h \leq t \leq t_{im},$$

which, together with (2.9) and (2.10), implies that

$$\dot{x}_i(t) \leq 0 \qquad \text{for } t_{im} - h \leq t \leq t_{im}.$$

This contradicts to the choice of t_{im} as the leftmost maximum point. Hence (2.11) holds.

Next, we show that

$$x_i(t_{im}) + \sum_{j\neq i}^{n} \tilde{b}_{ij}(\varepsilon_0)|x_j(t_{jm})| \leq M\sum_{j=k+1}^{n} |\tilde{b}_{ij}(\varepsilon_0)|, \qquad i = 1,2,\ldots,k. \tag{2.12}$$

If $x_i(t_{im}) \leq \alpha_i$, then (2.12) obviously holds. On the other hand, if $x_i(t_{im}) > \alpha_i$, then it follows from (2.11) that there exists $\xi_{im} \in [t_{im} - \tau_{ii}(t_{im}), t_{im}]$ such that $x_i(\xi_{im}) = \alpha_i$. From (2.9) we have

$$\dot{x}_i(t) \leq a_{ii}(t)[-x_i(t - \tau_{ii}(t)) + \alpha_i] \leq a_{ii}(t)(|x_i(t_{im})| + \alpha_i), \qquad t_0 \leq t \leq t_{1m}. \tag{2.13}$$

For $t \in [\xi_{im}, t_{im})$, we have, $t - \tau_{ii}(t) \leq \xi_{im}$. Integrating (2.13) from $t - \tau_{ii}(t)$ to ξ_{im}, we have

$$\alpha_i - x_i(t - \tau_{ii}(t)) \leq (|x_i(t_{im})| + \alpha_i)\int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds, \qquad \xi_{im} \leq t \leq t_{im}.$$

Substituting this into the first inequality in (2.13), we obtain

$$\dot{x}_i(t) \leq (|x_i(t_{im})| + \alpha_i)a_{ii}(t)\int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds, \qquad \xi_{im} \leq t \leq t_{im}.$$

Combining this and (2.13), we have

$$\dot{x}_i(t) \leq (|x_i(t_{im})| + \alpha_i)a_{ii}(t)\min\left\{1, \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds\right\}, \qquad \xi_{im} \leq t \leq t_{im}. \tag{2.14}$$

We consider the following three cases:

Case 1. $d_i + \varepsilon_0 \le 1$. In this case, by (2.5) and (2.14) we have

$$\begin{aligned} &x_i(t_{im}) - x_i(\xi_{im}) \\ &\quad \le (|x_i(t_{im})| + \alpha_i) \int_{\xi_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \\ &\quad = (|x_i(t_{im})| + \alpha_i) \int_{\xi_{im}}^{t_{im}} a_{ii}(t) \left(\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds - \int_{\xi_{im}}^{t} a_{ii}(s)ds \right) dt \\ &\quad \le (|x_i(t_{im})| + \alpha_i) \left[(d_i + \varepsilon_0) \int_{\xi_{im}}^{t_{im}} a_{ii}(t)dt - \int_{\xi_{im}}^{t_{im}} a_{ii}(t) \int_{\xi_{im}}^{t} a_{ii}(s)dsdt \right] \\ &\quad = (|x_i(t_{im})| + \alpha_i) \left[(d_i + \varepsilon_0) \int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds - \frac{1}{2} \left(\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds \right)^2 \right] \\ &\quad \le \frac{1}{2}(d_i + \varepsilon_0)^2 (|x_i(t_{im})| + \alpha_i), \end{aligned}$$

since $z \mapsto \delta z - z^2/2$ is increasing for $z \in [0, \delta]$ (where $\delta = d_i + \varepsilon_0$ and $z = \int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds$). We have used the fact that

$$\int_{\xi_{im}}^{t_{im}} \int_{\xi_{im}}^{t} a_{ii}(t)a_{ii}(s)dsdt = \int_{\xi_{im}}^{t_{im}} \int_{\xi_{im}}^{s} a_{ii}(t)a_{ii}(s)dtds = \frac{1}{2} \int_{\xi_{im}}^{t_{im}} \int_{\xi_{im}}^{t_{im}} a_{ii}(t)a_{ii}(s)dsdt.$$

Case 2. $d_i + \varepsilon_0 > 1$ and $\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds \le 1$. In this case, by (2.5) and (2.14) we have

$$\begin{aligned} x_i(t_{im}) - x_i(\xi_{im}) &\le (|x_i(t_{im})| + \alpha_i) \int_{\xi_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \\ &\le (|x_i(t_{im})| + \alpha_i) \left[(d_i + \varepsilon_0) \int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds - \frac{1}{2} \left(\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds \right)^2 \right] \\ &\le \frac{1}{2}[2(d_i + \varepsilon_0) - 1](|x_i(t_{im})| + \alpha_i), \end{aligned}$$

since $z \mapsto \delta z - z^2/2$ is increasing for $z \in [0, 1]$, $\delta > 1$ (where $\delta = d_i + \varepsilon_0$ and $z = \int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds$).

Case 3. $d_i + \varepsilon_0 > 1$ and $\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds > 1$. In this case, let $\eta_{im} \in [\xi_{im}, t_{im}]$ such that $\int_{\eta_{im}}^{t_{im}} a_{ii}(s)ds = 1$. Then by (2.5) and (2.14) we have

$$\begin{aligned} &x_i(t_{im}) - x_i(\xi_{im}) \\ &\quad \le (|x_i(t_{im})| + \alpha_i) \left[\int_{\xi_{im}}^{\eta_{im}} a_{ii}(s)ds + \int_{\eta_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \right] \end{aligned}$$

$$= (|x_i(t_{im})| + \alpha_i)\left[\left(1 - \int_{\eta_{im}}^{t_{im}} a_{ii}(s)ds\right)\int_{\xi_{im}}^{\eta_{im}} a_{ii}(s)ds + \int_{\eta_{im}}^{t_{im}} a_{ii}(t)\int_{t-\tau_{ii}(t)}^{\eta_{im}} a_{ii}(s)dsdt\right]$$

$$= (|x_i(t_{im})| + \alpha_i)\left[\int_{\eta_{im}}^{t_{im}} a_{ii}(t)\left(\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds - \int_{\eta_{im}}^{t} a_{ii}(s)ds\right)dt\right]$$

$$\leq (|x_i(t_{im})| + \alpha_i)\left[(d_i + \varepsilon_0)\int_{\eta_{im}}^{t_{im}} a_{ii}(t)dt - \frac{1}{2}\left(\int_{\eta_{im}}^{t_{im}} a_{ii}(s)ds\right)^2\right]$$

$$= \frac{1}{2}[2(d_i + \varepsilon_0) - 1](|x_i(t_{im})| + \alpha_i).$$

Combining Cases 1, 2 and 3, we have for $i = 1, 2, \ldots, k$

$$x_i(t_{im}) \leq \frac{2 + (d_i + \varepsilon_0)^2}{2 - (d_i + \varepsilon_0)^2}\left[\sum_{j \neq i}^{k} b_{ij}|x_j(t_{jm})| + M\sum_{j=k+1}^{n} b_{ij}\right], \qquad \text{if } d_i + \varepsilon_0 < 1,$$

or

$$x_i(t_{im}) \leq \frac{1 + 2(d_i + \varepsilon_0)}{3 - 2(d_i + \varepsilon_0)}\left[\sum_{j \neq i}^{k} b_{ij}|x_j(t_{jm})| + M\sum_{j=k+1}^{n} b_{ij}\right], \qquad \text{if } d_i + \varepsilon_0 \geq 1.$$

This implies (2.12) is true.

Let $\tilde{B}_k(\varepsilon_0) = (\tilde{b}_{ij}(\varepsilon_0))_{k\times k}$ denote the kth leading principal submatrix of $\tilde{B}(\varepsilon_0)$. Then $\tilde{B}_k(\varepsilon_0)$ is a non-singular M-matrix of order k, and so $\tilde{B}_k^{-1}(\varepsilon_0) \geq 0$. Hence, by writing (2.12) as a matrix inequality and multiplying it on both sides by $\tilde{B}_k^{-1}(\varepsilon_0)$, we have

$$(x_1(t_{1m}), x_2(t_{2m}), \ldots, x_k(t_{km}))^T$$

$$\leq M\tilde{B}_k^{-1}(\varepsilon_0)\left(\sum_{j=k+1}^{n}|\tilde{b}_{1j}(\varepsilon_0)|, \sum_{j=k+1}^{n}|\tilde{b}_{2j}(\varepsilon_0)|, \ldots, \sum_{j=k+1}^{n}|\tilde{b}_{kj}(\varepsilon_0)|\right)^T, \qquad m = 1, 2, \ldots.$$

Note that if $k = n$, then the right hand side (above) is just the zero vector. From this, we conclude that

$$\limsup_{m\to\infty}|x_i(t_{im})| < \infty, \qquad i = 1, 2, \ldots, k,$$

which contradicts the fact that $\lim_{m\to\infty}|x_i(t_{im})| = \infty$ for $i = 1, 2, \ldots, k$, and the proof is complete.

Remark. The hypotheses in Theorem 2.1 are different from that of Theorem 1.2 even for the special case of (1.3).

Theorem 2.2. *If, in addition to the hypotheses of Theorem* 2.1, *we further assume that*

$$\int_{t_0}^{\infty} a_{ii}(s)ds = \infty, \qquad \text{for all } i = 1, 2, \ldots, n, \tag{2.15}$$

then every solution $x(t)$ *of* (1.1) *tends to* 0 *as* $t \to \infty$.

Proof. Assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))$ satisfies (1.1) on $[t_0, \infty)$. We will prove that

$$\lim_{t\to\infty} x_i(t) = 0, \qquad \text{for all } i = 1, 2, \ldots, n. \tag{2.16}$$

Set $d_* = \max\{d_i : d_i < 1, i = 1, 2, \ldots, n\}$ and $d^* = \max\{d_i : d_i \geq 1, i = 1, 2, \ldots, n\}$. For any $\varepsilon \in (0, \min\{1 - d_*, 3/2 - d^*\})$, there exists $T_0 > t_0$ such that

$$\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds \leq d_i + \varepsilon, \qquad \text{for all } t \geq T_0,\ i = 1, 2, \ldots, n. \tag{2.17}$$

Note that ε is chosen small enough so that for each i with $d_i < 1$, we have, $d_i + \varepsilon < 1$ and for each i with $d_i \geq 1$, we have, $d_i + \varepsilon \leq 3/2$.

Case 1. The functions $\sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t))$ $(i = 1, 2, \ldots, n)$ are all non-oscillatory, meaning that they are non-zero for t sufficiently large. Then $\dot{x}_i(t)$ $(i = 1, 2, \ldots, n)$ are eventually sign-definite, and so the limits $\lim_{t\to\infty} x_i(t) = c_i$ exist, for all $i = 1, 2, \ldots, n$. Without loss of generality, we may assume that $\dot{x}_i(t) > 0$ eventually for $i = 1, 2, \ldots, n$ (if necessary, we use $-x_i(t)$ instead of $x_i(t)$ and $-a_{ij}(t)$ instead of $a_{ij}(t)$ for $j \neq i$). Then, for $i = 1, 2, \ldots, n$,

$$\begin{aligned}
c_i - x_i(t) &= \int_t^{\infty} \left[-\sum_{j=1}^{n} a_{ij}(s)x_j(s - \tau_{ij}(s)) \right] ds = \int_t^{\infty} \left| \sum_{j=1}^{n} a_{ij}(s)x_j(s - \tau_{ij}(s)) \right| ds \\
&= \int_t^{\infty} \left| a_{ii}(s)x_i(s - \tau_{ii}(s)) + \sum_{j\neq i} a_{ij}(s)x_j(s - \tau_{ij}(s)) \right| ds \\
&\geq \int_t^{\infty} |a_{ii}(s)x_i(s - \tau_{ii}(s))| - \left| \sum_{j\neq i} a_{ij}(s)x_j(s - \tau_{ij}(s)) \right| ds \\
&\geq \int_t^{\infty} a_{ii}(s) \left[|x_i(s - \tau_{ii}(s))| - \sum_{j\neq i}^{n} b_{ij}|x_j(s - \tau_{ij}(s))| \right] ds.
\end{aligned}$$

Since

$$\lim_{t\to\infty} \left[|x_i(t - \tau_{ii}(t))| - \sum_{j\neq i}^{n} b_{ij}|x_j(t - \tau_{ij}(t))| \right] = |c_i| - \sum_{j\neq i}^{n} b_{ij}|c_j|,$$

it follows from (2.15) that

$$|c_i| - \sum_{j \neq i}^{n} b_{ij}|c_j| \leq 0, \qquad i = 1, 2, \ldots, n,$$

and hence,

$$\sum_{j=1}^{n} \tilde{b}_{ij}|c_j| \leq 0, \qquad i = 1, 2, \ldots, n, \tag{2.18}$$

since $\tilde{b}_{ij} \leq -b_{ij} \leq 0$, for $i \neq j$. Now, by the positivity of $\tilde{B}^{-1}$, we conclude that $|c_1| = |c_2| = \cdots = |c_n| = 0$.

Case 2. At least one of the functions $\sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t))$ $(i = 1, 2, \ldots, n)$ is oscillatory. Set

$$U_i = \limsup_{t \to \infty} |x_i(t)|, \qquad i = 1, 2, \ldots, n.$$

By Theorem 2.1, $0 \leq U_i < \infty$, $i = 1, 2, \ldots, n$. It suffices to prove that $U_1 = U_2 = \cdots = U_n = 0$. Without loss of generality, assume that the functions $\sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t))$ $(i = 1, 2, \ldots, k)$ are oscillatory and the functions $\sum_{j=1}^{n} a_{ij}(t)x_j(t - \tau_{ij}(t))$ $(i = k+1, k+2, \ldots, n)$ are non-oscillatory. It follows from (1.1) that $\dot{x}_i(t)$ is oscillatory for $i = 1, 2, \ldots, k$ and

$$\dot{x}_i(t) \text{ is non-oscillatory and } \lim_{t \to \infty} |x_i(t)| = U_i, \qquad i = k+1, \ldots, n. \tag{2.19}$$

Let $\varepsilon > 0$ be sufficiently small. Then, there exist k sequences $\{t_{im}\}_{m=1}^{\infty}$, $i = 1, 2, \ldots, k$ with $t_{im} - \tau_{ij}(t_{im}) > T_0$ such that t_{im} is the local left maximum point of $x_i(t)$ and

(2.20)
$$\begin{cases} t_{im} \uparrow \infty, \ |x_i(t_{im})| \to U_i \text{ as } m \to \infty, \quad |x_i(t_{im})| > U_i - \varepsilon, \\ |\dot{x}_i(t_{im})| = 0, \ |x_i(t)| < U_i + \varepsilon \quad \text{for } t \geq t_1, \ m = 1, 2, \ldots \text{ and } i = 1, 2, \ldots, k \end{cases}$$

where $t_1 = \min\{t_{i1} : i = 1, 2, \ldots, k\}$. We can assume that $|x_i(t_{im})| = x_i(t_{im})$ (if necessary, we use $-x_i(t)$ instead of $x_i(t)$ and $-a_{ij}(t)$ instead of $a_{ij}(t)$ for $j \neq i$). Then as in (2.11), we can prove

$$x_i(t_{im} - \tau_{ii}(t_{im})) \leq \sum_{j \neq i}^{n} b_{ij}(U_j + \varepsilon), \qquad i = 1, 2, \ldots, k. \tag{2.21}$$

The proof will be omitted. Now set

$$\beta_i = \sum_{j \neq i}^{n} b_{ij}(U_j + \varepsilon), \qquad i = 1, 2, \ldots, k. \tag{2.22}$$

In what follows, we show that

$$x_i(t_{im}) + \sum_{j\neq i}^{n} \tilde{b}_{ij}(\varepsilon)(U_j+\varepsilon) \tag{2.23}$$
$$\leq \begin{cases} \frac{2\varepsilon(d_i+\varepsilon)^2}{2-(d_i+\varepsilon)^2}, & \text{if } d_i+\varepsilon<1, \\ \frac{2\varepsilon[2(d_i+\varepsilon)-1]}{3-2(d_i+\varepsilon)}, & \text{if } d_i+\varepsilon\geq 1, \end{cases} \quad i=1,2,\dots,k,$$

where $\tilde{b}_{ij}(\varepsilon)$ is defined as in the proof of Theorem 2.1. If $x_i(t_{im}) \leq \beta_i$, then (2.23) obviously holds. If $x_i(t_{im}) > \beta_i$, then by (2.21) there exists $\xi_{im} \in [t_{im} - \tau_{ii}(t_{im}), t_{im})$ such that $x_i(\xi_{im}) = \beta_i$. From (1.1) we have

$$\dot{x}_i(t) \leq a_{ii}(t)[-x_i(t-\tau_{ii}(t)) + \beta_i] \leq a_{ii}(t)[(U_i+\varepsilon)+\beta_i], \qquad t_1 \leq t \leq t_{im}. \tag{2.24}$$

For $t \in [\xi_{im}, t_{im}]$, integrating (2.24) from $t-\tau_{ii}(t)$ to ξ_{im}, we have

$$\beta_i - x_i(t-\tau_{ii}(t)) \leq [(U_i+\varepsilon)+\beta_i]\int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds, \qquad \xi_{im} \leq t \leq t_{im}.$$

Substituting this into the first inequality in (2.24), we obtain

$$\dot{x}_i(t) \leq [(U_i+\varepsilon)+\beta_i]a_{ii}(t)\int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds, \qquad \xi_{im} \leq t \leq t_{im}.$$

Combining this with (2.24), we have

$$\dot{x}_i(t) \leq [(U_i+\varepsilon)+\beta_i]a_{ii}(t)\min\left\{1, \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)ds\right\}, \qquad \xi_{im} \leq t \leq t_{im}. \tag{2.25}$$

We consider the following three cases:

Case 2.1. $d_i+\varepsilon<1$. In this case, by (2.17) and (2.25) we have

$$\begin{aligned} x_i(t_{im}) - x_i(\xi_{im}) &\leq [(U_i+\varepsilon)+\beta_i]\int_{\xi_{im}}^{t_{im}} a_{ii}(t)\int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \\ &= [(U_i+\varepsilon)+\beta_i]\int_{\xi_{im}}^{t_{im}} a_{ii}(t)\left(\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds - \int_{\xi_{im}}^{t} a_{ii}(s)ds\right)dt \\ &\leq [(U_i+\varepsilon)+\beta_i]\left[(d_i+\varepsilon)\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds - \int_{\xi_{im}}^{t_{im}} a_{ii}(t)\int_{\xi_{im}}^{t} a_{ii}(s)dsdt\right] \\ &= [(U_i+\varepsilon)+\beta_i]\left[(d_i+\varepsilon)\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds - \frac{1}{2}\left(\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds\right)^2\right] \\ &\leq \frac{1}{2}(d_i+\varepsilon)^2[(U_i+\varepsilon)+\beta_i] \\ &\leq \frac{1}{2}(d_i+\varepsilon)^2[x_i(t_{im})+\beta_i+2\varepsilon]. \end{aligned}$$

Case 2.2. $d_i + \varepsilon \geq 1$ and $\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds \leq 1$. In this case, by (2.17) and (2.25) we have

$$\begin{aligned}
x_i(t_{im}) - x_i(\xi_{im}) &\leq [(U_i + \varepsilon) + \beta_i] \int_{\xi_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \\
&\leq [(U_i + \varepsilon) + \beta_i] \left[(d_i + \varepsilon) \int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds - \frac{1}{2} \left(\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds \right)^2 \right] \\
&\leq [(U_i + \varepsilon) + \beta_i] \left[(d_i + \varepsilon) \cdot 1 - \frac{1}{2} \cdot 1^2 \right] \\
&= \frac{1}{2} [2(d_i + \varepsilon) - 1][(U_i + \varepsilon) + \beta_i] \\
&\leq \frac{1}{2} [2(d_i + \varepsilon) - 1][x_i(t_{im}) + \beta_i + 2\varepsilon].
\end{aligned}$$

Case 2.3. $d_i + \varepsilon \geq 1$ and $\int_{\xi_{im}}^{t_{im}} a_{ii}(s)ds > 1$. In this case, let $\eta_{im} \in [\xi_{im}, t_{im}]$ be such that $\int_{\eta_{im}}^{t_{im}} a_{ii}(s)ds = 1$. Then by (2.17) and (2.25) we have

$$\begin{aligned}
&x_i(t_{im}) - x_i(\xi_{im}) \\
&\leq [(U_i + \varepsilon) + \beta_i] \left[\int_{\xi_{im}}^{\eta_{im}} a_{ii}(s)ds + \int_{\eta_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\xi_{im}} a_{ii}(s)dsdt \right] \\
&= [(U_i + \varepsilon) + \beta_i] \left[\left(1 - \int_{\eta_{im}}^{t_{im}} a_{ii}(s)ds \right) \int_{\xi_{im}}^{\eta_{im}} a_{ii}(s)ds + \int_{\eta_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\eta_{im}} a_{ii}(s)dsdt \right] \\
&= [(U_i + \varepsilon) + \beta_i] \left[\int_{\eta_{im}}^{t_{im}} a_{ii}(t) \int_{t-\tau_{ii}(t)}^{\eta_{im}} a_{ii}(s)dsdt \right] \\
&= [(U_i + \varepsilon) + \beta_i] \left[\int_{\eta_{im}}^{t_{im}} a_{ii}(t) \left(\int_{t-\tau_{ii}(t)}^{t} a_{ii}(s)ds - \int_{\eta_{im}}^{t} a_{ii}(s)ds \right) dt \right] \\
&\leq [(U_i + \varepsilon) + \beta_i] \left[(d_i + \varepsilon) \int_{\eta_{im}}^{t_{im}} a_{ii}(t)dt - \int_{\eta_{im}}^{t_{im}} a_{ii}(t) \int_{\eta_{im}}^{t} a_{ii}(s)dsdt \right] \\
&= [(U_i + \varepsilon) + \beta_i] \left[(d_i + \varepsilon) \int_{\eta_{im}}^{t_{im}} a_{ii}(t)dt - \frac{1}{2} \left(\int_{\eta_{im}}^{t_{im}} a_{ii}(t)dt \right)^2 \right] \\
&= \frac{1}{2} [2(d_i + \varepsilon) - 1][(U_i + \varepsilon) + \beta_i] \\
&\leq \frac{1}{2} [2(d_i + \varepsilon) - 1][x_i(t_{im}) + \beta_i + 2\varepsilon].
\end{aligned}$$

Combining Cases 2.1, 2.2 and 2.3, we have for $i = 1, 2, \ldots, k$

$$x_i(t_{im}) \le \frac{2 + (d_i + \varepsilon)^2}{2 - (d_i + \varepsilon)^2} \sum_{j \ne i}^{n} b_{ij}(U_i + \varepsilon) + \frac{2\varepsilon(d_i + \varepsilon)^2}{2 - (d_i + \varepsilon)^2}, \qquad \text{if } d_i + \varepsilon < 1,$$

or

$$x_i(t_{im}) \le \frac{1 + 2(d_i + \varepsilon)}{3 - 2(d_i + \varepsilon)} \sum_{j \ne i}^{n} b_{ij}(U_i + \varepsilon) + \frac{2\varepsilon[2(d_i + \varepsilon) - 1]}{3 - 2(d_i + \varepsilon)}, \qquad \text{if } d_i + \varepsilon \ge 1.$$

This shows that (2.23) is true. Let $m \to \infty$ and $\varepsilon \to 0$ in (2.23), we obtain

$$U_i + \sum_{j \ne i}^{n} \tilde{b}_{ij} U_j \le 0, \qquad i = 1, 2, \ldots, k. \tag{2.26}$$

On the other hand, from (1.1) and (2.19), as in the proof of (2.18), we have

$$\begin{aligned} |U_i - x_i(t)| &= \left| \int_t^{\infty} \sum_{j=1}^{n} a_{ij}(s) x_j(s - \tau_{ij}(s)) ds \right| \\ &\ge \int_t^{\infty} a_{ii}(s) \left[|x_i(s - \tau_{ii}(s))| - \sum_{j \ne i}^{n} b_{ij} |x_j(s - \tau_{ij}(s))| \right] ds, \\ &\qquad\qquad\qquad\qquad i = k+1, \ldots, n, \end{aligned}$$

and

$$\liminf_{t \to \infty} \left[|x_i(t - \tau_{ii}(t))| - \sum_{j \ne i}^{n} b_{ij} |x_j(t - \tau_{ij}(t))| \right] \ge U_i - \sum_{j \ne i}^{n} b_{ij} U_j, \qquad i = k+1, \ldots, n.$$

It follows from the above and (2.15) that

$$U_i - \sum_{j \ne i} b_{ij} U_j \le 0, \qquad i = k+1, \ldots, n,$$

and so

$$U_i + \sum_{j \ne i} \tilde{b}_{ij} U_j \le 0, \qquad i = k+1, \ldots, n. \tag{2.27}$$

By (2.26) and (2.27), and using the fact that $\tilde{B}$ is a non-singular M-matrix, we have $U_1 = U_2 = \cdots = U_n = 0$. Hence, the proof is complete.

Using Theorem 2.2, we have immediately

Corollary 2.1. *Assume that* $\tau_{ii} \equiv 0$ *for* $i = 1, 2, \dots, n$, *and that* (2.1) *and* (2.15) *hold. Let* $\hat{B} = (\hat{b}_{ij})$ *be the matrix with entries* $\hat{b}_{ii} = 1$ *for* $i = 1, 2, \dots, n$ *and* $\hat{b}_{ij} = -|b_{ij}|$ *for* $i \neq j$. *If* $\hat{B}$ *is a non-singular M-matrix, then every solution* $x(t)$ *of* (1.1) *tends to* 0 *as* $t \to \infty$.

From Theorem 2.2 and [1, Theorem 5.1], we immediately have

Corollary 2.2. *Let* $\tilde{B}$ *be defined by* (2.3) *and* (2.4). *Assume that* (2.1), (2.2) *and* (2.15) *hold, and that one of the following conditions holds*:

(i) *There exists a vector* $x > 0$ *such that* $\tilde{B}x > 0$:

(ii) *Every real eigenvalue of the matrix* $\tilde{B}$ *is positive*:

(iii) $\tilde{B}$ *is non-singular and* $\tilde{B}^{-1} \geq 0$;

(iv) *The real part of any eigenvalue of* $\tilde{B}$ *is positive*;

(v) *The leading principal minors of* $\tilde{B}$ *are positive*;

(vi) $\tilde{B}x \geq 0$ *implies* $x \geq 0$.

Then every solution $x(t)$ *of* (1.1) *tends to* 0 *as* $t \to \infty$.

For the coupled system of *two* non-autonomous delay differential equations

$$\begin{aligned} \dot{x}_1(t) &= -[a_{11}(t)x_1(t - \tau_{11}(t)) + a_{12}(t)x_2(t - \tau_{12}(t))], \\ \dot{x}_2(t) &= -[a_{21}(t)x_1(t - \tau_{21}(t)) + a_{22}(t)x_2(t - \tau_{22}(t))], \end{aligned} \tag{2.28}$$

where $a_{ij}(t), \tau_{ij}(t),\ i, j = 1, 2$, are the same as in (1.1), we can derive the following simple criterion.

Corollary 2.3. *Assume that* (2.1), (2.2) *and* (2.15) *hold for* $n = 2$, *and that*

$$\frac{(1 - D_1)(1 - D_2)}{(1 + D_1)(1 + D_2)} > |b_{12}b_{21}|, \tag{2.29}$$

where

$$D_i = \begin{cases} d_i^2/2, & \text{if } d_i < 1, \\ d_i - 1/2, & \text{if } d_i \geq 1, \end{cases} \quad i = 1, 2.$$

Then every nontrivial solution of (2.28) *satisfies*

$$\lim_{t \to \infty} [x_1^2(t) + x_2^2(t)] = 0. \tag{2.30}$$

3. Two examples

In the last section, we give two examples to illustrate the applications of our theorems.

Example 3.1. Consider the system of two non-autonomous delay differential equations

$$\dot{x}_1(t) = -\left[\frac{2}{3}\sin^2 t x_1(t-\pi) + a\sin^3 t x_2(t-2\pi)\right], \tag{3.1}$$

$$\dot{x}_2(t) = -\left[b\cos^3 t x_1(t-3\pi) + \frac{2}{3}\cos^2 t x_2(t-\pi)\right].$$

Since $a_{11}(t) = \frac{2}{3}\sin^2 t$, $a_{12}(t) = a\sin^3 t$, $a_{21}(t) = b\cos^3 t$ and $a_{22}(t) = \frac{2}{3}\cos^2 t$, by a direct calculation, we have

$$d_1 = \limsup_{t\to\infty}\int_{t-\pi}^{t} a_{11}(s)ds = \limsup_{t\to\infty}\left(\frac{2}{3}\int_{t-\pi}^{t}\sin^2 s\, ds\right) = \frac{\pi}{3},$$

$$d_2 = \limsup_{t\to\infty}\left(\frac{2}{3}\int_{t-\pi}^{t}\cos^2 s\, ds\right) = \frac{\pi}{3}$$

and thus,

$$\frac{(1-D_1)(1-D_2)}{(1+D_1)(1+D_2)} = \left(\frac{9-2\pi}{3+2\pi}\right)^2.$$

Hence, in view of Corollary 2.3, if

$$|ab| < \frac{4}{9}\left(\frac{9-2\pi}{3+2\pi}\right)^2,$$

then every nontrivial solution of (3.1) satisfies

$$\lim_{t\to\infty}[x_1^2(t) + x_2^2(t)] = 0.$$

Example 3.2. Consider the system of three non-autonomous delay differential equations

$$\dot{x}_1(t) = -[x_1(t-0.6) - 0.2\sin t x_2(t-\pi) + 0.1\cos t x_3(t-2\pi)], \tag{3.2}$$

$$\dot{x}_2(t) = -[0.4\cos 2t x_1(t-\pi) + x_2(t-0.5) - 0.3\cos t x_3(t-\pi)],$$

$$\dot{x}_3(t) = -[-0.8\cos^2 t x_1(t-3\pi) + 0.2\sin t x_2(t-\pi) + 0.5x_3(t-1)].$$

Here

$$a_{11}(t) = a_{22}(t) = 1, \quad a_{33}(t) = 0.5; \quad b_{12} = 0.2, \quad b_{13} = 0.1;$$

$$b_{21} = 0.4, \quad b_{23} = 0.3; \quad b_{31} = 1.6; \quad b_{32} = 0.4.$$

By a direct calculation, we have

$$d_1 = 0.6, \qquad d_2 = d_3 = 0.5; \qquad \tilde{b}_{12} = -0.288, \qquad \tilde{b}_{13} = -0.144;$$
$$\tilde{b}_{21} = -0.514, \qquad \tilde{b}_{23} = -0.386; \qquad \tilde{b}_{31} = -2.057; \qquad \tilde{b}_{32} = -0.514,$$

and the successive principal minors of the matrix $\tilde{B}$ are as follows:

$$\tilde{b}_{11} = 1 > 0, \qquad \det\begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} \\ \tilde{b}_{21} & \tilde{b}_{22} \end{pmatrix} = 0.819 > 0 \quad \text{and}$$
$$\det\begin{pmatrix} \tilde{b}_{11} & \tilde{b}_{12} & \tilde{b}_{13} \\ \tilde{b}_{21} & \tilde{b}_{22} & \tilde{b}_{23} \\ \tilde{b}_{31} & \tilde{b}_{32} & \tilde{b}_{33} \end{pmatrix} = 0.091 > 0.$$

In view of Corollary 2.2 (v), every solution of (3.2) satisfies

$$\lim_{t\to\infty}[x_1^2(t) + x_2^2(t) + x_3^2(t)] = 0.$$

References

[1] Fiedler, M., *Special matrices and their applications in numerical mathematics*, Martinus Nijhoff Publ. (Kluwer), Dordrecht, 1986.

[2] Gopalsamy, K., *Stability and oscillations in delay differential equations of population dynamics*, Kluwer Academic Publishers, Boston, 1992.

[3] Gopalsamy, K., Stability criteria for a linear system $\dot{X}(t) + A(t)X(t-\tau) = 0$ and an application to a non-linear system, Int. J. Systems Sci., **21** (1990), 1841–1853.

[4] Györi, I., Stability in a class of integrodifferential systems, Agarwal, R. P. (ed.), Recent trends in differential equations, World Sci. Ser. Appl. Anal. 1, Singapore: World Scientific Publishing, 1992, 269–284.

[5] Hale, J. K. and Lunel, S. M. Verduyn, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

[6] Hofbauer, J. and So, J. W.-H., Diagonal dominance and harmless off-diagonal delays, Proc. Amer. Math. Soc., **128** (2000), 2575–2682.

[7] Horn, R. A. and Johnson, C. R., Topics in matrix analysis, Cambridge University Press, Cambridge, New York, 1991.

[8] Kuang, Y., *Delay differential equations with applications in population dynamics*, Academic Press, Boston, 1993.

[9] So, J. W.-H., Tang, X. H. and Zou, X., Stability in a linear delay system without instantaneous negative feedback, SIAM J. Math. Anal., **33** (2002), 1297–1304.

[10] So, J. W.-H. and Yu, J. S., Global attractivity for a population model with time delay, Proc. Amer. Math. Soc., **123** (1995), 2687–2694.

[11] Yoneyama, T., The 3/2 stability theorem for one-dimensional delay-differential equations with unbounded delay, J. Math. Anal. Appl., **165** (1992), 133–143.

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