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Stability Regions for Linear Differential Equations with Two Kinds of Time Lags

By

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Dedicated to Professor Kusuo Ise on his seventieth birthday

Abstract. Our purpose is to give some stability regions for linear differential equations with two kinds of time lags, namely, a discrete time lag and a distributed time lag. We show that there are various stability regions which depend on the choice of the time lags and also some region is the union of two adjoining domain.

Key Words and Phrases. Linear delay differential equations, Stability regions, Discrete time lag, Distributed time lag.

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1. Introduction

There are many results concerning asymptotic stability for differential equations with two time lags. See e.g. [1] through [10]. Most of them are the results for differential equations with two discrete time lags. A few results are known for differential equations with a distributed time lag. In [2:XI.4], Diekmann et al. studied a prey-predator system with a distributed time lag. Their method to analyze the corresponding characteristic equation is also useful for proving our theorems. In this paper, we shall discuss asymptotic stability for linear differential equation with a discrete time lag and a distributed time lag

$$\dot{x}(t) = ax(t-\tau) + b\int_{t-h}^{t} x(s)ds, \tag{1}$$

where $\tau \geq 0$, $h > 0$ and a and b are both real. In case $b = 0$ and $\tau > 0$, this equation becomes

$$\dot{x}(t) = ax(t-\tau) \tag{2}$$

for which with $\tau > 0$, the following result is well known.

Theorem A. *The zero solution of* (2) *is asymptotically stable if and only if* $-\pi/(2\tau) < a < 0$.

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On the other hand, in case $a = 0$, the following result is known.

Theorem B ([3]). *The zero solution of the equation*

$$\dot{x}(t) = b\int_{t-h}^{t} x(s)ds$$

is asymptotically stable if and only if $-\pi^2/(2h^2) < b < 0$.

In what follows, stability region for (1) means the set of all (a, b) for which the zero solution of (1) is asymptotically stable. Then Theorems A and B assert that the stability region for (1) contains the segment $\{(a, 0) \mid -\pi/(2\tau) < a < 0\}$ on a-axis and the segment $\{(0, b) \mid -\pi^2/(2h^2) < b < 0\}$ on b-axis. So, it is of great interest to find the stability region for (1). Our purpose in this paper is to find such regions when h/τ or τ/h are some integers.

Now we shall consider the characteristic equation for (1)

$$\lambda = ae^{-\lambda\tau} + b\int_{-h}^{0} e^{\lambda s}\, ds. \tag{3}$$

It is easy to see that (3) is reduced to

$$a + bh = 0 \qquad \text{when } \lambda = 0, \tag{4}$$

$$\lambda = ae^{-\lambda\tau} + \frac{b}{\lambda}(1 - e^{-\lambda h}) \qquad \text{when } \lambda \neq 0. \tag{5}$$

We denote the straight line $a + bh = 0$ in the ab-plane by l_0. The following two results will be employed to find the stability regions.

Theorem C ([6]). *The zero solution of* (1) *is asymptotically stable if and only if any root of* (3) *has negative real part.*

Proposition D. *If* $a + bh \geq 0$, *then* (3) *has a nonnegative real root.*

Proof of Proposition D. Define

$$f(\alpha) = \alpha - ae^{-\alpha\tau} - \frac{b}{\alpha}(1 - e^{-\alpha h})$$

for real $\alpha \neq 0$. Then $\lim_{\alpha\to+0} f(\alpha) = -(a + bh) \leq 0$ and $\lim_{\alpha\to\infty} f(\alpha) = \infty$. Hence there is a positive α such that $f(\alpha) = 0$, and so we arrive at the conclusion of this proposition.

Proposition D asserts that the stability region is contained in the half-plane $a + bh < 0$. We divide this half-plane into an infinite number of regions D_k by curves Γ_n or straight lines l_n which will be given in the next section. Then the following proposition is valid.

Proposition E. *Let* (a, b) *belong to a region* D_k. *Then the number of roots of* (3) *whose real parts are positive, depends on* D_k *but not on the choice of* (a, b).

The proof of this proposition is analogous to that of Theorem 2.1 in [8], and so it is omitted. In what follows, for D_k containing (a, b), $\nu(D_k)$ denotes the number of roots of (3) whose real parts are positive.

2. Curves to divide the ab-plane

In this section, we shall give an infinite number of curves or straight lines. For the point (a, b) on these curves or straight lines, the characteristic equation (3) has some purely imaginary roots $\pm i\omega$, $\omega > 0$. We shall prove that the stability region for (1) is enclosed by some of them.

Suppose (3) has the roots $\pm i\omega$, $\omega > 0$. Then (5) implies

$$a\omega \cos \omega\tau + b \sin \omega h = 0 \tag{6}$$

and

$$a\omega \sin \omega\tau + b(1 - \cos \omega h) = -\omega^2. \tag{7}$$

There are four cases for us to discuss.

Case I: $\tau > 0$, $h = m\tau$ and $m = 2p$ for some positive integer p.

Case II: $\tau > 0$, $h = m\tau$ and $m = 2p - 1$ for some positive integer p.

Case III: $\tau > 0$, $\tau = mh$ for some positive integer m.

Case IV: $\tau = 0$ and $h > 0$.

We first consider Case I. Since $\omega h = 2p\omega\tau$, there exists a polynomial $P(x, y)$ such that

$$\sin \omega h = \cos \omega\tau \cdot P(\cos \omega\tau, \sin \omega\tau),$$

and hence (6) becomes

$$\cos \omega\tau\{a\omega + b \cdot P(\cos \omega\tau, \sin \omega\tau)\} = 0.$$

Suppose $\cos \omega\tau = 0$. Then

$$\omega\tau = \frac{(2n-1)\pi}{2}$$

for some positive integer n, and so

$$\sin \omega\tau = (-1)^{n-1},$$

$$\cos \omega h = \cos(2n-1)p\pi = (-1)^p, \qquad \sin \omega h = 0.$$

Hence for each n, (7) means the straight line:

$$a = (-1)^n \frac{(2n-1)\pi}{2\tau} \tag{8}$$

if p is even, or

$$b = (-1)^n \frac{(2n-1)\pi}{4\tau} a - \frac{(2n-1)^2\pi^2}{8\tau^2} \tag{9}$$

if p is odd. On the other hand, in case $\cos \omega\tau \neq 0$, we have from (6) and (7), the curve with parametric representation:

$$a = \frac{\omega \sin \omega h}{\cos \omega\tau - \cos \omega(\tau - h)}, \qquad b = -\frac{\omega^2 \cos \omega\tau}{\cos \omega\tau - \cos \omega(\tau - h)} \tag{10}$$

for $\omega_{n-1} < \omega < \omega_n$, where $\{\omega_n\}$ is the increasing sequence which satisfies $\omega_0 = 0$ and

$$\cos \omega_n\tau - \cos \omega_n(\tau - h) = 0.$$

Note that each ω_n is expressed as

$$\omega_n = \frac{2k\pi}{m\tau} \tag{11}$$

or

$$\omega_n = \frac{2k\pi}{(m-2)\tau} \tag{12}$$

for some positive integer k, and also if either

$$\omega = \frac{2k\pi}{m\tau} \qquad \text{or} \qquad \omega = \frac{2k\pi}{(m-2)\tau}$$

holds, then there are no pairs of a and b satisfying both (6) and (7), whenever $\cos \omega\tau \neq 0$. Moreover in case $h = 2\tau$ $(m = 2)$, we can find only the straight lines (9), because

$$\cos \omega\tau - \cos \omega(\tau - h) = 0$$

for all $\omega > 0$.

In Case II or in Case III, we have the curves defined by (10), but no straight lines are found. Besides, (10) may be valid even if $\cos \omega\tau = 0$. In Case II, the increasing sequence $\{\omega_n\}$ satisfies (11), and in Case III, $\{\omega_n\}$ satisfies

$$\omega_n = \frac{2k\pi}{h} \qquad \text{or} \qquad \omega_n = \frac{2k\pi}{(2m-1)h}$$

for some positive integer k.

Finally, in Case IV, we have only the curves defined by (10), that is

$$a = \frac{\omega \sin \omega h}{1 - \cos \omega h}, \qquad b = -\frac{\omega^2}{1 - \cos \omega h} \tag{13}$$

for $\omega_{n-1} < \omega < \omega_n$, where $\omega_n = 2n\pi/h$.

3. Theorems and Proofs

In this section, we shall give the stability regions for the cases $h = \tau$, $h = 2\tau$, $h = 4\tau$ and $\tau = 0$. We denote the line (8) or (9) by l_n and the curve (10) defined on the interval (ω_{n-1}, ω_n) by Γ_n. In case $h = \tau$, the half-plane $a + bh < 0$ is divided into regions D_k by the curves Γ_n:

$$a = \frac{\omega \sin \omega\tau}{\cos \omega\tau - 1}, \qquad b = -\frac{\omega^2 \cos \omega\tau}{\cos \omega\tau - 1} \tag{14}$$

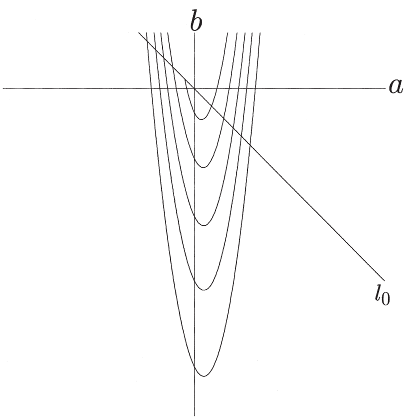

Fig. 1. Curves Γ_n

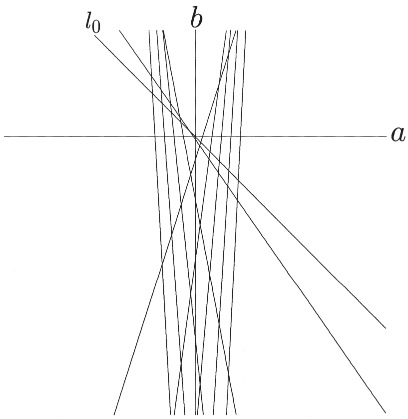

Fig. 2. Lines l_n

for $\omega_{n-1} < \omega < \omega_n$, where $\omega_n = 2n\pi$, and the following is valid.

Theorem 1. *Let $h = \tau$ and let D_1 be the region enclosed by l_0 and Γ_1. Then D_1 is the stability region for* (1).

Proof. Since (14) implies $a^2 - 2b = \omega^2$, the inequalities $\omega_{n-1}^2 < a^2 - 2b < \omega_n^2$ hold for $(a, b) \in \Gamma_n$, and hence Γ_i and Γ_j are disjoint if $i \neq j$. (See Figure 1.) So, let D_n be the region enclosed by l_0, Γ_{n-1} and Γ_n for each $n \geq 2$. Suppose Γ_n intersects the a-axis at a point (a, b). Then it follows from (14) that

$$\cos \omega\tau = 0 \qquad \text{and} \qquad \sin \omega\tau = -\frac{a}{\omega} > 0,$$

so that $\sin \omega\tau = 1$ and $a = -\omega$. Then we have $a\tau = -\pi/2 - 2(n-1)\pi$. Hence the intersection of D_n and the a-axis is the segment whose end points are $(-\pi/2 - 2(n-1)\pi, 0)$ and $(-\pi/2 - 2(n-2)\pi, 0)$. Thus Theorems A and E assert that

$$v(D_1) = 0 < v(D_n)$$

for $n \geq 2$. This completes the proof.

In case $h = 2\tau$, the equation (3) has roots $\pm i\omega$, $\omega > 0$ only if $\omega\tau = (2n-1)\pi/2$ and the point (a, b) lies on some line l_n:

$$b = (-1)^n \frac{(2n-1)\pi}{4\tau} a - \frac{(2n-1)^2\pi^2}{8\tau^2}.$$

Then also the equality

$$(-1)^n \cdot a\omega = 2b + \omega^2 \tag{15}$$

holds. Each line l_{2m} intersects any l_{2k-1} in the half-plane $a + bh < 0$. (See Figure 2.) But each l_n does not any of l_{n+2k} with $k \neq 0$ in the half-plane $a + bh < 0$, because the intersection of those is the point $((-1)^n(2n+2k-1)\pi/\tau, \{(2n-1)^2 + 4k(2n-1)\}\pi^2/8\tau^2)$ and so lies on the half-plane $a + bh > 0$. The lines l_n divide the half-plane $a + bh < 0$ into regions D_k. It is easily seen that each region D_k approaches the origin with shrinking as $\tau \to \infty$. This shows the following result.

Theorem 2. *Let $h = 2\tau$ and let D_1 be the triangular region enclosed by l_0, l_1 and l_2. Then D_1 is the stability region for* (1).

Proof. In the same way as the proof of Theorem 1, we can easily see from Theorems A and B that $v(D_1) = 0 < v(D_k)$ for $k \geq 2$, whenever D_k intersects the a-axis or the b-axis. Therefore we consider D_k which does not so. Suppose D_k is adjoining to D_p and suppose (a, b) lies on the common boundary of D_k and D_p at $\tau = \tau^*$, where $\omega\tau^* = (2n-1)\pi/2$. We may assume that D_p is closer to the origin than D_k and that the point (a, b) moves from D_p into D_k as τ grows, but actually the regions D_k and D_p move and the point (a, b) does not. Let $\lambda = \lambda(\tau)$ be the root of (3) satisfying $\lambda = i\omega$, $\omega\tau^* = (2n-1)\pi/2$ at $\tau = \tau^*$. Put

$$f(\lambda, \tau) = \lambda^2 - a\lambda e^{-\lambda\tau} - b(1 - e^{-2\lambda\tau}).$$

Then (5) implies $f(i\omega, \tau^*) = 0$. Since $e^{-i\omega\tau^*} = (-1)^n i$, we have from (15) that

$$\begin{aligned}\frac{\partial f}{\partial \lambda}(i\omega, \tau^*) &= i2\omega + a(i\omega\tau^* - 1)e^{-i\omega\tau^*} - 2b\tau^* e^{-i2\omega\tau^*}\\ &= (-1)^{n+1} a\omega\tau^* + 2b\tau^* + i\{2\omega - (-1)^n a\}\\ &= -\omega^2\tau^* + i\{2\omega - (-1)^n a\}\\ &\neq 0.\end{aligned}$$

Hence the implicit function theorem guarantees

$$\begin{aligned}\Re\frac{d\lambda}{d\tau}(\tau^*) &= -\Re\frac{\partial f/\partial\tau(i\omega,\tau^*)\overline{\partial f/\partial\lambda(i\omega,\tau^*)}}{|\partial f/\partial\lambda(i\omega,\tau^*)|^2}\\ &= -\frac{-i\omega^3\{-i(2\omega - (-1)^n a)\}}{|\partial f/\partial\lambda(i\omega,\tau^*)|^2}\\ &= \frac{\omega^2(\omega^2 - 2b)}{|\partial f/\partial\lambda(i\omega,\tau^*)|^2}\end{aligned}$$

If $b < 0$, then it is clear that $\Re(d\lambda/d\tau(\tau^*)) > 0$, and hence we assume $b \geq 0$. Since (a, b) lies on some line l_n, let (a_n, b_n) be the intersection of l_n and the line $a + 2b\tau^* = 0$. Then it follows from (15) that

$$2b_n = \frac{\omega^2}{(-1)^{n+1}\omega\tau^* - 1} \leq \frac{\omega^2}{\omega\tau^* - 1} < \omega^2$$

for $n \geq 2$, because $\omega\tau^* \geq 3\pi/2$. And so the inequality $\omega^2 - 2b > 0$ holds for any $(a, b) \in l_n \cap \{a + 2b\tau^* < 0\}$ for $n \geq 2$. On the other hand, in case $n = 1$, we choose $(a, b) = (-\omega, 0)$. Then this point lies on l_1 and also we have $\omega^2 - 2b > 0$. Thus there exists a point (a, b) which lies on the common boundary of D_k and D_p, and for this point (a, b) the inequality

$$\Re\frac{d\lambda}{d\tau}(\tau^*) > 0$$

is valid. Since (a, b) moves from D_p into D_k as τ grows near τ^*, we arrive at the conclusion that

$$v(D_k) > v(D_p)$$

and hence

$$v(D_k) > 0$$

for all $k \geq 2$. This completes the proof.

In case $h = 4\tau$, the half-plane $a + bh < 0$ is divided into regions D_k by the lines (8) and the curves (10). Since $h = 4\tau$, each line (8) means l_n:

$$a = (-1)^n \omega, \qquad \omega = \frac{(2n-1)\pi}{2\tau} \tag{16}$$

and each (10) means Γ_n:

$$a = \frac{\omega \cos 2\omega\tau}{\sin \omega\tau}, \qquad b = -\frac{\omega^2}{4 \sin^2 \omega\tau} \tag{17}$$

for $\omega_{n-1} < \omega < \omega_n$, where $\omega_n = n\pi/\tau$. Then all (a, b) lying on Γ_n satisfy

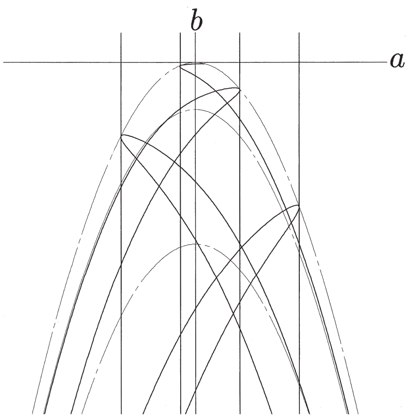

Fig. 3. Γ_n and l_n

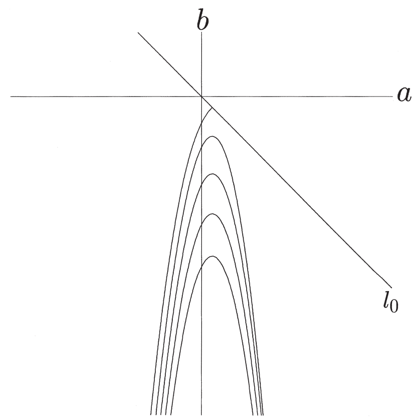

Fig. 4. Curves Γ_n

$$a^2 + 4b = -4\omega^2 \cos^2 \omega\tau.$$

Now put

$$p(\omega) = -4\omega^2 \cos^2 \omega\tau.$$

Then $p(\omega)$ attains its maximum 0 at $\omega = (2n-1)\pi/2\tau$. This shows that each curve Γ_n comes in contact with the parabola $a^2 + 4b = 0$ at the point P_n $((-1)^n(2n-1)\pi/2\tau, -(2n-1)^2\pi^2/16\tau^2)$. Hence it follows from (16) that Γ_n intersects the line l_n at P_n, and also P_n approaches the origin along the parabola $a^2 + 4b = 0$ as $\tau \to \infty$. (See Figure 3.) Since $p(\omega)$ tends to $-4n^2\pi^2/\tau^2$ as $\omega \to n\pi/\tau - 0$, the curve Γ_n is asymptotic to the parabola

$$a^2 + 4b = -\frac{4n^2\pi^2}{\tau^2}$$

as $\omega \to n\pi/\tau - 0$, for each $n \geq 1$. Furthermore, the curve Γ_n is asymptotic to the parabola

$$a^2 + 4b = -\frac{4(n-1)^2\pi^2}{\tau^2}$$

as $\omega \to (n-1)\pi/\tau + 0$, for $n \geq 2$. In case $n = 1$, the point (a, b) lying on Γ_1 approaches the point $(1/\tau, -1/4\tau^2)$ as $\omega \to +0$. Thus we obtain the following theorem.

Theorem 3. *Let $h = 4\tau$ and let D_1 be the region enclosed by Γ_1, l_0 and l_1, and D_2 the region enclosed by Γ_1 and l_1 only. Then the union of D_1 and D_2 is the stability region for* (1).

Proof. The regions D_1 and D_2 touch each other at $P_1(-\pi/2\tau, -\pi^2/16\tau^2)$. Let D_3 be the region enclosed by the curves Γ_1, Γ_2 and the lines l_0, l_1 and l_2. Then the region D_3 adjoins both the regions D_1 and D_2. Theorems A and B imply that

$$v(D_1) = 0 < v(D_k)$$

for $k \geq 3$, whenever D_k intersects the a-axis or the b-axis. Therefore it is easily seen that $v(D_3) > 0$. Similarly to the proof of Theorem 2, we need to consider the behavior of $\lambda(\tau)$ corresponding to (a, b) which lies on the boundary of D_k at $\tau = \tau^*$, where $(n-1)\pi < \omega\tau^* < n\pi$ for some $n \geq 1$. Suppose the point (a, b) lies on the curve Γ_n or on the line l_n at $\tau = \tau^*$. Put

$$g(\lambda, \tau) = \lambda^2 - a\lambda e^{-\lambda\tau} - b(1 - e^{-4\lambda\tau}).$$

Then (5) implies $g(i\omega, \tau^*) = 0$ for some ω, where $(n-1)\pi < \omega\tau^* < n\pi$. We consider first the case that $(a, b) \in l_n$. Since $e^{-i\omega\tau^*} = (-1)^n i$, it follows from (16) that

$$\begin{aligned}\frac{\partial g}{\partial \lambda}(i\omega, \tau^*) &= i2\omega + a(i\omega\tau^* - 1)e^{-i\omega\tau^*} - 4b\tau^* e^{-i4\omega\tau^*} \\ &= -(a^2 + 4b)\tau^* + i\omega \\ &\neq 0,\end{aligned}$$

and so

$$\begin{aligned}\Re\frac{d\lambda}{d\tau}(\tau^*) &= -\Re\frac{-a\omega^2 e^{-i\omega\tau^*} - i4b\omega e^{-i4\omega\tau^*}}{\partial g/\partial\lambda(i\omega, \tau^*)} \\ &= \frac{a^2(a^2 + 4b)}{|\partial g/\partial\lambda(i\omega, \tau^*)|^2}.\end{aligned}$$

Hence, if $a^2 + 4b > 0$, then we have

$$\Re \frac{d\lambda}{d\tau}(\tau^*) > 0, \tag{18}$$

because $a^2 = \omega^2 > 0$. Next, we consider the case that $(a, b) \in \Gamma_n$. Since the function

$$h(z) = \frac{i2z}{\tau^*} + \frac{z \cos 2z(iz - 1)e^{-iz}}{\tau^* \sin z} + \frac{z^2 e^{-i4z}}{\tau^* \sin^2 z}$$

is holomorphic in the strip region $\{z \mid (n-1)\pi < \Re z < n\pi\}$ of the complex plane and since $h(z)$ satisfies $h(\omega\tau^*) = \partial g/\partial\lambda(i\omega, \tau^*)$, the unicity theorem shows that $\partial g/\partial\lambda(i\omega, \tau^*) \neq 0$ except for at most a finite number of ω in $((n-1)\pi/\tau^*, n\pi/\tau^*)$. Then, from (17), we have

$$\Re \frac{d\lambda}{d\tau}(\tau^*) = -\frac{2a\omega^3 \sin \omega\tau^* + a^2\omega^2 - 8b\omega^2 \cos 4\omega\tau^* + 4ab\omega \sin 3\omega\tau^*}{|\partial g/\partial\lambda(i\omega, \tau^*)|^2}.$$

A computation implies that

$$\Re \frac{d\lambda}{d\tau}(\tau^*) = \frac{2\omega^4 \sin^2 \omega\tau^*}{|\partial g/\partial\lambda(i\omega, \tau^*)|^2} > 0 \tag{19}$$

except for the case of $(a, b) = P_n$. Note that the point $(a, b) = (0, -\pi^2/32\tau^{*2})$ lies on the common boundary of D_1 and D_3. For this point (a, b), (3) has a pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. As τ grows near τ^*, the point (a, b) moves from D_1 into D_3 across the curve Γ_1. Since $\nu(D_1) = 0$, (19) implies $\nu(D_3) = 2$. On the other hand, the common boundary of D_2 and D_3 is a part of the line l_1 which approaches the b-axis as τ grows, and also each point (a, b) lying on the common boundary satisfies $a = -\pi/2\tau$ and $b < -\pi^2/16\tau^2$, and so $a^2 + 4b < 0$. Hence (18) shows that $\nu(D_2) = 0$, because such a point (a, b) moves from D_3 into D_2 across the line l_1. Since each parabola $a^2 + 4b = -4n^2\pi^2/\tau^2$ approaches the parabola $a^2 + 4b = 0$ as $\tau \to \infty$, each curve Γ_n also approaches the parabola $a^2 + 4b = 0$. Especially, the point P_n on Γ_n approaches the origin as $\tau \to \infty$. It is clear that each line l_n approaches the b-axis as $\tau \to \infty$. Therefore we arrive at the conclusion that

$$\nu(D_1) = \nu(D_2) = 0 < \nu(D_k)$$

for all $k \geq 3$. This completes the proof.

In case $\tau = 0$, the characteristic equation (3) has the purely imaginary roots $\pm i\omega$, $\omega > 0$, only if the point (a, b) lies on Γ_n defined by (13). Furthermore, the curves Γ_n divide the half-plane $a + bh < 0$ into regions D_k. Then we arrive at the following theorem.

Theorem 4. *Let $\tau = 0$ and let D_1 be the region enclosed by Γ_1 and the line l_0. Then D_1 is the stability region for* (1).

Proof. Since (13) implies that the equality

$$a^2 + 2b = -\omega^2$$

with $2(n-1)\pi/h < \omega < 2n\pi/h$ for any (a, b) lying on Γ_n, each Γ_n does not intersect another Γ_k. For each $n \geq 2$, Γ_n is the curve like a parabola and intersects the b-axis only at the point $(0, -(2n-1)^2\pi^2/2h^2)$. This implies that each region D_k intersects the b-axis. (See Figure 4.) Therefore Theorem B shows that

$$v(D_1) = 0 < v(D_k)$$

for all $k \geq 2$. This completes the proof.

Remark. To our regret, we can not yet find the stability region exactly for all cases. But Theorems A and B lead us to the result that in each of Figures 5 through 27, the shaded portion is contained in the stability region and the others are not so, whenever they are contained in the half-plane $a + bh > 0$ or intersect the a-axis or the b-axis. Our conjecture is that the following figures illustrate the stability regions. Furthermore Theorems 1 through 4 assert that the shaded portions in Figures 5, 6, 8 and 16 are stability regions.

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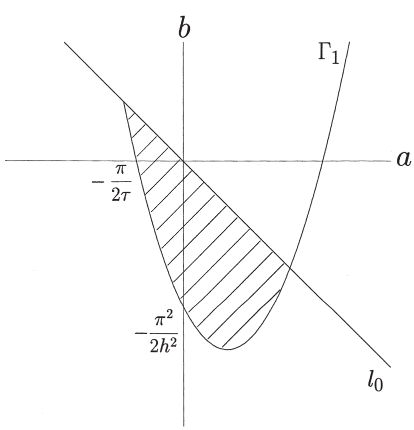

Fig. 5. S-region for $h = \tau$

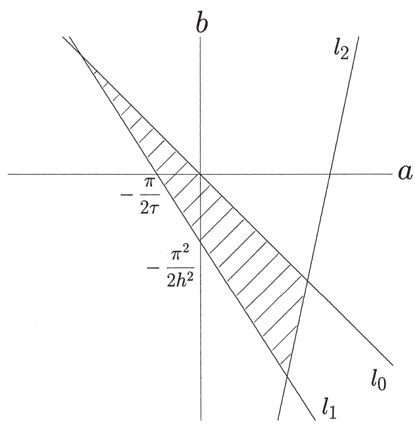

Fig. 6. S-region for $h = 2\tau$

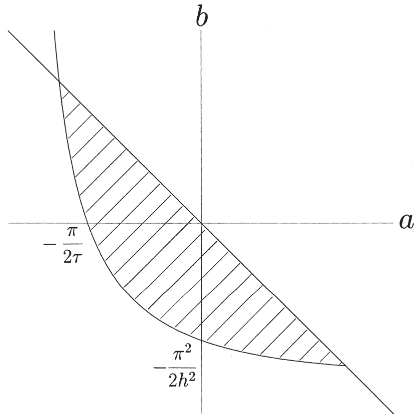

Fig. 7. S-region for $h = 3\tau$

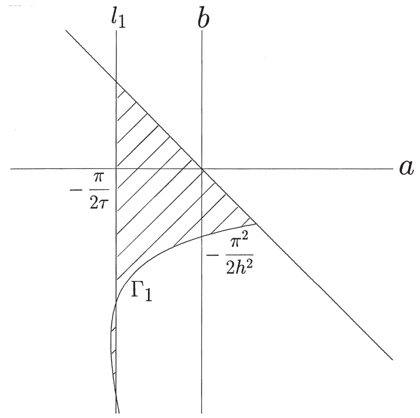

Fig. 8. S-region for $h = 4\tau$

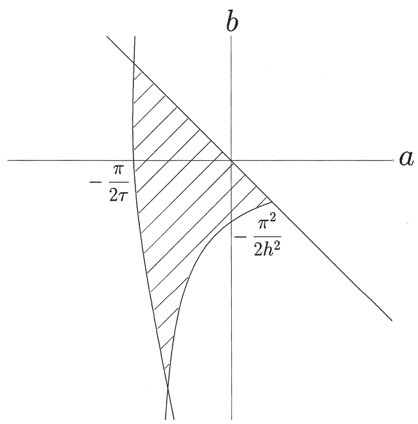

Fig. 9. S-region for $h = 5\tau$

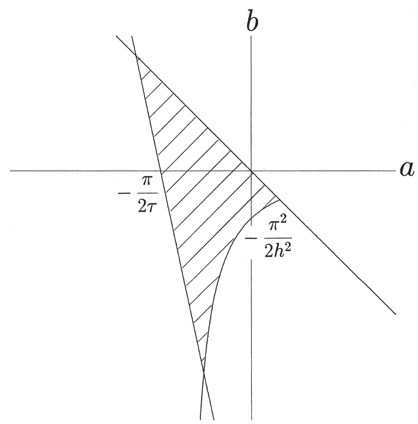

Fig. 10. S-region for $h = 6\tau$

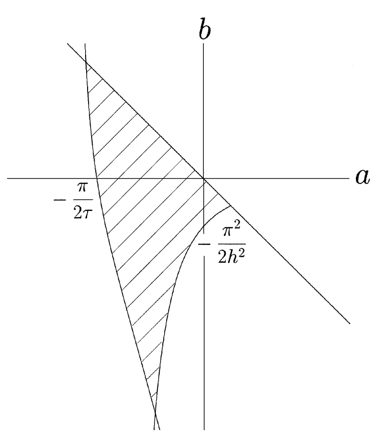

Fig. 11. S-region for $h = 7\tau$

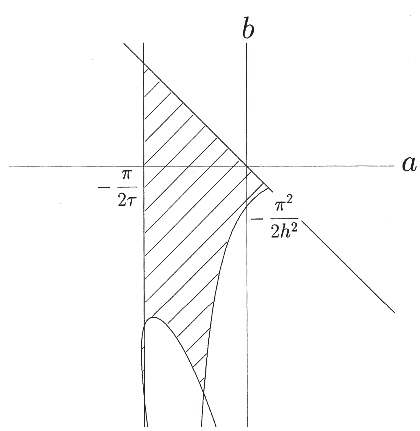

Fig. 12. S-region for $h = 8\tau$

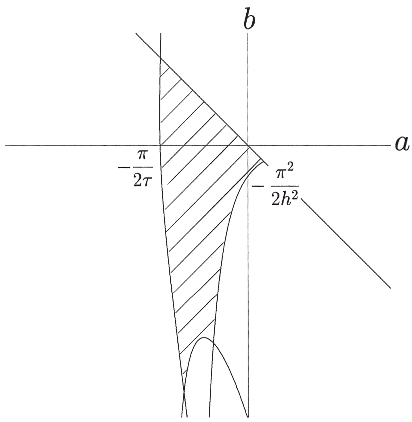

Fig. 13. S-region for $h = 9\tau$

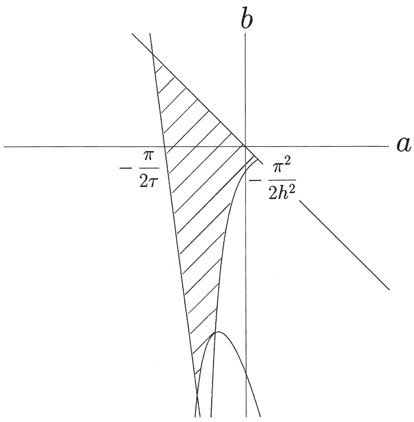

Fig. 14. S-region for $h = 10\tau$

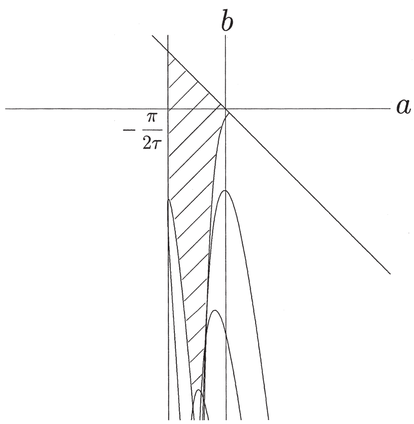

Fig. 15. S-region for $h = 20\tau$

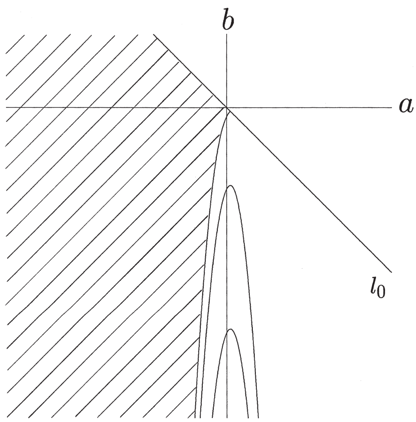

Fig. 16. S-region for $\tau = 0$

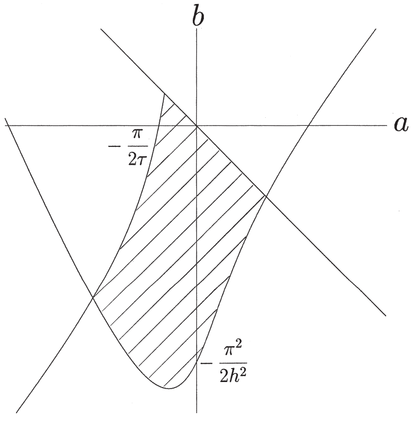

Fig. 17. S-region for $\tau = 2h$

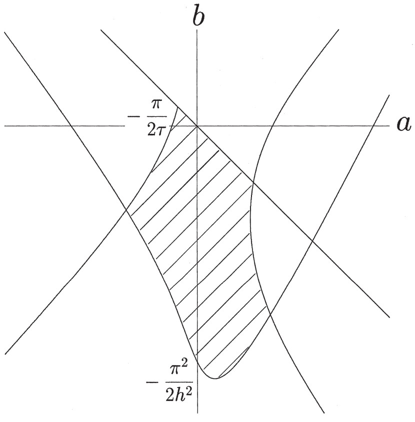

Fig. 18. S-region for $\tau = 3h$

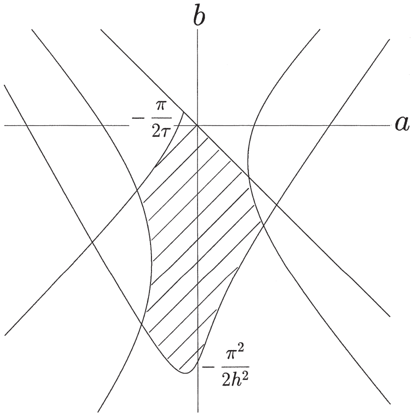

Fig. 19. S-region for $\tau = 4h$

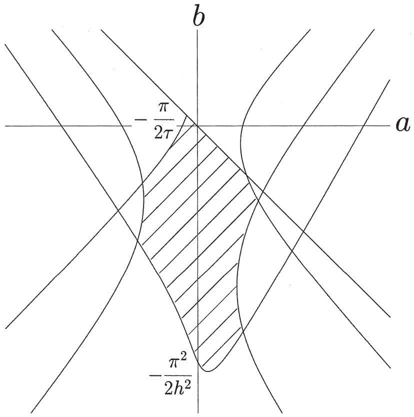

Fig. 20. S-region for $\tau = 5h$

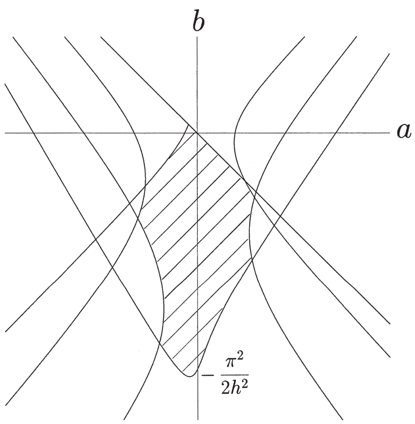

Fig. 21. S-region for $\tau = 6h$

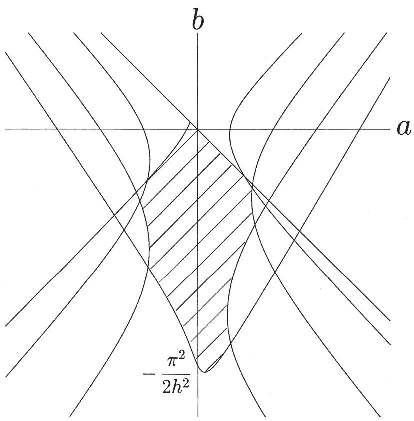

Fig. 22. S-region for $\tau = 7h$

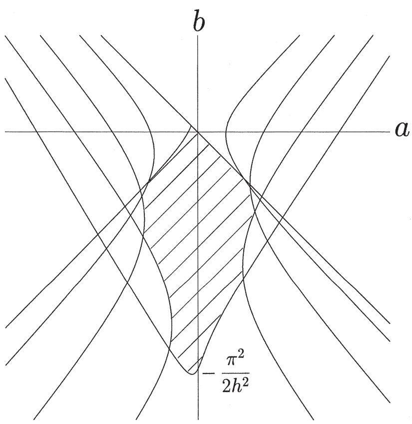

Fig. 23. S-region for $\tau = 8h$

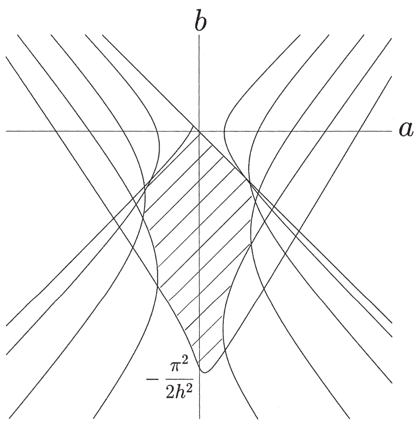

Fig. 24. S-region for $\tau = 9h$

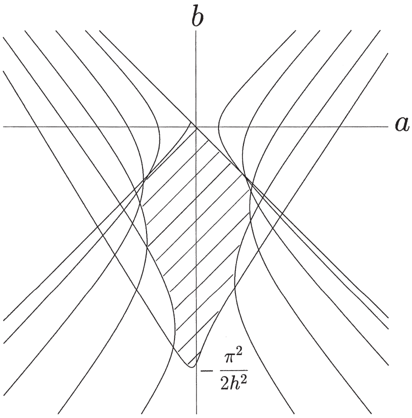

Fig. 25. S-region for $\tau = 10h$

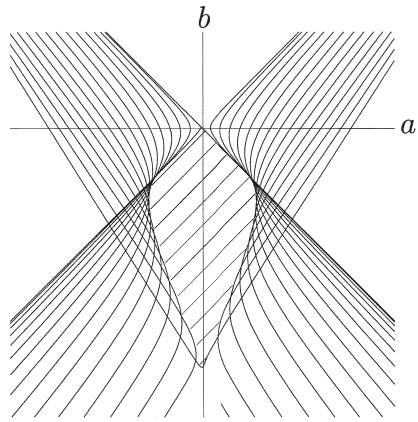

Fig. 26. S-region for $\tau = 30h$

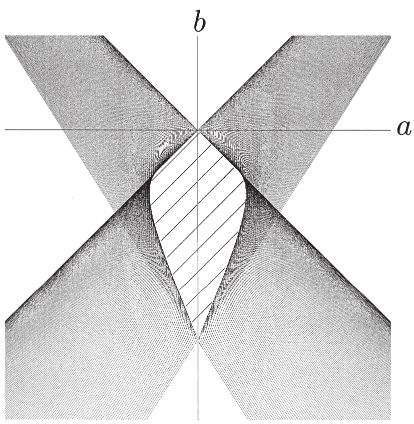

Fig. 27. S-region for $\tau = 300h$

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