Funkcialaj Ekvacioj, **47** (2004) 119–127

Uniform Multisummability and Convergence of a Power Series

By

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Abstract. The purpose of this paper is to give a sufficient condition for convergence of a multisummable power series. Previously, similar results had been obtained for formal solutions of ordinary differential equations and also of a completely integrable Pfaffian system. In those cases, proof was based heavily on the form of equations. But, in relatively recent developement, it was shown that formal solutions of ordinary differential equations at an irregular singular point are generally multisummable. In this paper, proof of convergence is totally independent of any equations and based only on multisummability of power series.

Key Words and Phrases. Power series, Asymptotic expansions, Multisummability, Ordinary differential equations, Pfaffian systems, Singularity.

2000 *Mathematics Subject Classification Numbers.* 34A25, 34M30, 35C10.

§ 1. Main theorem

Let $\vec{p} = \sum_{m=0}^{+\infty} x^m \vec{\alpha}_m(\vec{\varepsilon})$ be a formal power series in a complex variable x whose coefficients $\vec{\alpha}_m(\vec{\varepsilon})$ are $\boldsymbol{C}^n$-valued functions of some other complex parameters $\vec{\varepsilon} \in \boldsymbol{C}^\mu$, where n and μ are positive integers.

We assume that

(1) there exists a positive number r such that the $\vec{\alpha}_m(\vec{\varepsilon})$ are holomorphic in the domain $\mathscr{D}(r) = \{\vec{\varepsilon}; |\vec{\varepsilon}| < r\}$, where $|\vec{\varepsilon}| = \max_{h=1}^{\mu} |\varepsilon_h|$ for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_\mu) \in \boldsymbol{C}^\mu$,

(2) the power sereis $\vec{p}$ is of the Gevrey order $1/k$ uniformly with repect to $\vec{\varepsilon}$ in $\mathscr{D}(r)$, i.e.

$$|\vec{\alpha}_m(\vec{\varepsilon})| \leq K_0 (m!)^{1/k} B_0^m \qquad (m = 0, 1, \dots) \tag{1.1}$$

in the domain $\mathscr{D}(r)$ for some positive numbers K_0 and B_0, where k is a positive integer.

A power series $\vec{p} = \sum_{m=0}^{+\infty} x^m \vec{\alpha}_m(\vec{\varepsilon})$ of the Gevrey order $1/k$ uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r)$ is said to be k-summable in the direction d uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r_1)$ if there exists a $\boldsymbol{C}^n$-valued function $\vec{\eta}_d(x, \vec{\varepsilon})$ such that

(3) $\vec{\eta}_d(x,\vec{\varepsilon})$ is holomorphic in a domain:

$$(1.2) \qquad |\arg x - d| < \frac{\pi}{2k} + \delta_d, \qquad 0 < |x| < r_d, \qquad \vec{\varepsilon} \in \mathscr{D}(r_1),$$

where $\delta_d, r_d,$ and r_1 are some positive numbers,

(4) there exist positive numbers K_1 and B_1 such that

$$(1.3) \qquad \left|\vec{\eta}_d(x,\vec{\varepsilon}) - \sum_{m=0}^{M-1} x^m \vec{\alpha}_m(\vec{\varepsilon})\right| \le K_1(M!)^{1/k} B_1^M |x|^M \qquad (M = 0, 1, \ldots)$$

in domain (1.2).

The function $\vec{\eta}_d(x,\vec{\varepsilon})$ is called the k-sum of the power series $\vec{p}$ in the direction d uniformly with repect to $\vec{\varepsilon} \in \mathscr{D}(r_1)$. Note that the direction d is independent of $\vec{\varepsilon}$ in $\mathscr{D}(r_1)$. The k-sum of $\vec{p}$ in the direction d is uniquely determined.

Let $\vec{k} = (k_1, k_2, \ldots, k_\nu)$, where k_j $(j = 1, 2, \ldots, \nu)$ are positive integers such that

$$(1.4) \qquad 0 < k_1 < k_2 < \cdots < k_\nu.$$

A formal power series $\vec{p} = \sum_{m=0}^{+\infty} x^m \vec{\alpha}_m(\vec{\varepsilon})$ is said to be $\vec{k}$-multisummable in a direction d uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r)$, if, for each $j = 1, 2, \ldots, \nu$, there exists a power series $\vec{p}_j = \sum_{m=0}^{+\infty} x^m \vec{\alpha}_{j,m}(\vec{\varepsilon})$ such that

(5) $\vec{p}_j$ is a k_j-summable in the direction d uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r)$,

and

(6) we have

$$(1.5) \qquad \vec{p} = \sum_{j=1}^{\nu} \vec{p}_j.$$

Let $\vec{\eta}_{j,d}(x,\vec{\varepsilon})$ be the k_j-sum of the power series $\vec{p}_j$ in the direction d uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r)$. Then

$$(1.6) \qquad \vec{\eta}_d(x,\vec{\varepsilon}) = \sum_{j=1}^{\nu} \vec{\eta}_{j,d}(x,\vec{\varepsilon})$$

is called the $\vec{k}$-multisum of the power series $\vec{p}$ in the direction d uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r)$. The decomposition (1.5) of $\vec{p}$ is not unique, but the $\vec{k}$-multisum $\vec{\eta}_d(x,\vec{\varepsilon})$ of $\vec{p}$ is unique. (For Gevrey series, summable series, and multisummable series, see [Bal].)

Now, let us consider a formal power series

$$\vec{q} = \sum_{m+|\wp|\geq 1} x^m \vec{\varepsilon}^{\wp} \vec{a}_{m,\wp}, \tag{1.7}$$

where $\wp = (p_1, p_2, \dots, p_\mu)$ and the p_h are nonnegative integers, $|\wp| = p_1 + p_2 + \cdots + p_\mu$, $\vec{\varepsilon}^{\wp} = \varepsilon_1^{p_1} \varepsilon_2^{p_2} \dots \varepsilon_\mu^{p_\mu}$, and the $\vec{a}_{m,\wp}$ are in $\boldsymbol{C}^n$. Set, for each fixed nonnegative integer m,

$$\vec{a}_m(\vec{\varepsilon}) = \sum_{m+|\wp|\geq 1} \vec{\varepsilon}^{\wp} \vec{a}_{m,\wp}, \tag{1.8}$$

and set, for each $\wp$,

$$\vec{b}_{\wp}(x) = \sum_{m+|\wp|\geq 1} x^m \vec{a}_{m,\wp}. \tag{1.9}$$

Then,

$$\vec{q} = \sum_{m\geq 0} x^m \vec{a}_m(\vec{\varepsilon}) = \sum_{|\wp|\geq 0} \vec{\varepsilon}^{\wp} \vec{b}_{\wp}(x). \tag{1.10}$$

The main result of this paper is the following theorem in the statement of which we use the notaions given above concerning the formal power series $\vec{q}$ (see (1.7), (1.8), (1.9), and (1.10)).

Theorem 1.1. *A formal power series $\vec{q}$ given by* (1.7) *is convergent in* $(x, \vec{\varepsilon})$ *if the following three conditions are satisfied*:

(a) *there exists a positive number r_0 such that all formal series $\vec{a}_m(\vec{\varepsilon})$ given by* (1.8) *are convergent for* $|\vec{\varepsilon}| < r_0$,

(b) *there exists a positive number ρ_0 such that all formal series $\vec{b}_{\wp}(x)$ given by* (1.9) *are convergent for* $|x| < \rho_0$,

(c) *we denote by $\vec{\alpha}_m(\vec{\varepsilon})$ the sum of $\vec{a}_m(\vec{\varepsilon})$ for each m; then, there exists $\vec{k} = (k_1, k_2, \dots, k_\nu)$ and directions $\{d_1, d_2, \dots, d_\ell\}$ such that*

   (c-1) $0 < k_1 < k_2 < \cdots < k_\nu$,

   (c-2) $\{x \in \boldsymbol{C} : 0 < |x| < \rho_0\} = \bigcup_{i=1}^{\ell} \{x \in \boldsymbol{C} : 0 < |x| < \rho_0, |\arg x - d_i| < \pi/(2k_\nu) + \delta_i\}$, *where the δ_i are some positive numbers*,

   (c-3) *the formal power series $\vec{p} = \sum_{m=0}^{+\infty} x^m \vec{\alpha}_m(\vec{\varepsilon})$ is $\vec{k}$-multisummable in the direction d_i uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r_0)$ for each i.*

Remark 1.2. (i) In each of §2 and §3 we will present an application of Theorem 1.1. The application in §2 is for a system of ordinary differential equations. The application in §3 is for an overdetermined system of partial differential equations. Both of them concern convergence of formal solutions at irregular singularity.

(ii) The power series $q(x, \varepsilon) = \sum_{m=0}^{\infty} m!(\varepsilon x)^m$ does not satisfy condition (c-3), although it satisfies conditions (a) and (b) of Theorem 1.1. Note that $q(x, \varepsilon)$ is not convergent.

(iii) A proof of Theorem 1.1 will be given in §4.

§ 2. Application I

For $j = 1, 2, \ldots, \nu$, let $\vec{u}_j$ be in $\boldsymbol{C}^{n_j}$, where the n_j are positive integers. Set $n = n_1 + n_2 + \cdots + n_\nu$. Consider a system of differential equations

(2.1)

$$x^{1+k_j} \frac{d\vec{u}_j}{dx} = \vec{F}_j(x, \vec{\varepsilon}) + A_j \vec{u}_j + \sum_{h=1}^{\nu} B_{j,h}(x, \vec{\varepsilon}) \vec{u}_h + \vec{G}_j(x, \vec{u}, \vec{\varepsilon}), \qquad (j = 1, 2, \ldots, \nu),$$

where

(i) $\vec{u} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_\nu) \in \boldsymbol{C}^n$ and the k_j are positive integers such that $0 < k_1 < k_2 < \cdots < k_\nu$,

(ii) for each j, $\vec{F}_j$ is a $\boldsymbol{C}^{n_j}$-valued function of $(x, \vec{\varepsilon})$,

(iii) for each j, A_j is an $n_j \times n_j$ matrix with $\boldsymbol{C}$ entries,

(iv) for each (j, h), $B_{j,h}$ is an $n_j \times n_h$ matrix whose entries are $\boldsymbol{C}$-valued functions of $(x, \vec{\varepsilon})$,

(v) for each j, $\vec{G}_j$ is a $\boldsymbol{C}^{n_j}$-valued function of $(x, \vec{u}, \vec{\varepsilon})$.

In the statement of the following Theorem 2.1, we assume that there exist positive numbers ρ_0, r_0, and v_0 such that

(a′) the function $\vec{F}_j$ are holomorphic for $|x| < \rho_0$, $|\vec{\varepsilon}| < r_0$, and $\vec{F}_j(0, \vec{0}) = \vec{0}$,

(b′) the determinants $\det A_j$ are not equal to zero,

(c′) the entries of the matrices $B_{j,h}$ are holomorphic for $|x| < \rho_0$, $|\vec{\varepsilon}| < r_0$, and $B_{j,h}(0, \vec{0}) = O$,

(d′) the functions $\vec{G}_j$ are holomorphic for $|x| < \rho_0$, $|\vec{\varepsilon}| < r_0$, $|\vec{u}| < v_0$, and $G_j(x, \vec{0}, \vec{\varepsilon}) = \vec{0}$ and $\partial \vec{G}_j / \partial \vec{u}(x, \vec{0}, \vec{\varepsilon}) = O$, where, for each j, $\partial \vec{G}_j / \partial \vec{u}$ is the Jacobian matrix of $\vec{G}_j$ with respect to $\vec{u}$.

Also, we use the notations given by (1.8) and (1.9) for a formal power series (1.7).

Theorem 2.1. *Under assumptions* (a′), (b′), (c′), *and* (d′), *a formal solution* $\vec{u} = \vec{q}(x, \varepsilon)$ *given by* (1.7) *of system* (2.1) *is convergent in* $(x, \vec{\varepsilon})$ *if all of the power series* $\vec{b}_\wp$ *are convergent for* $|x| < r_1$, *where* r_1 *is a positive number.*

Sketch of Proof of Theorem 2.1. We check three conditions (a), (b), and (c) of Theorem 1.1. In Theorem 2.1, condition (b) is assumed. Condition (a) can be checked algebraically in a straight forward manner. To verify (c), first we apply the method of [Bra] to derive multisummability of the power series with respect x in the sense of J. Ecalle. Then, by using [Bal; § 10.4, pp 164–166] we derive the multisummability in the sense of § 1 of our paper. Note that in [Bal] coefficients of power series are in a Banach algebra. On the other hand, even though coefficients of power series are in $\boldsymbol{C}^{\ell}$ in [Bra], the method of Braaksma can be applied to cases where those coefficients are in a Banach algebra. Thus we achieve multisummability of the formal solution in x uniformly with respect to $\vec{\varepsilon}$.

Remark 2.2. Previously (cf. [Sib 1] and [Sib 2]) the following result was obtained: Let us consider a system of differential equations:

$$x^{k+1}\frac{d\vec{u}}{dx} = x\vec{f}_0(x,\vec{\varepsilon}) + A(x,\vec{\varepsilon})\vec{u} + \sum_{\wp\geq 2}\vec{u}^{\wp}\vec{f}_{\wp}(x,\vec{\varepsilon}), \tag{2.2}$$

where

(i) $\vec{u}\in\boldsymbol{C}^n$,

(ii) $\vec{f}_0(x,\vec{\varepsilon})$ and $\vec{f}_{\wp}(x,\vec{\varepsilon})$ are $\boldsymbol{C}^n$-valued functions and $A(x,\vec{\varepsilon})$ is an $n\times n$ matrix,

(iii) the entries of $\vec{f}_0, \vec{f}_{\wp}$, and A are holomorphic in a domain $\triangle(r) = \{(x,\vec{\varepsilon}); |x| + |\vec{\varepsilon}| < r\}$,

(iv) the power series $\sum_{|\wp|\geq 2}\vec{u}^{\wp}\vec{f}_{\wp}(x,\vec{\varepsilon})$ is uniformly convergent on each compact subset of the domain $|x| + |\vec{\varepsilon}| < r$, $|\vec{u}| < \rho$.

If the following conditions are satisfied:

(1) k is a positive integer,

(2) $\vec{f}_0(0,\vec{0}) = \vec{0}$,

(3) the matrix $A(0,\vec{0})$ is invertible,

then (2.2) has a unique formal solution:

$$\vec{u} = \vec{q}(x,\vec{\varepsilon}) = \sum_{m+|\wp|\geq 1,\, m\geq 0,\, |\wp|\geq 0} x^m\vec{\varepsilon}^{\wp}\vec{c}_{m,\wp}, \tag{2.3}$$

where the coefficients $\vec{c}_{m,\wp}$ are in $\boldsymbol{C}^n$. Furthermore, if all poser series $\sum_{m=0}^{+\infty} x^m\vec{c}_{m,\wp}$ are convergent in x, then $\vec{q}$ is a convergent power series in $(x,\vec{\varepsilon})$.

Theorem 2.1 is a generalization of this result. The proof of Theorem 2.1 which was mentioned above is similar to the proof in [Sib 2]. The proof in [Sib 1] does not use any asymptotic method. Such a proof of Theorem 2.1 would be also possible.

§ 3. Application II

For $j = 1,2,\ldots,\nu$, let $\vec{u}_j$ be in $\boldsymbol{C}^{n_j}$, where the n_j are positive integers. Set $n = n_1 + n_2 + \cdots + n_\nu$. Also, for $i = 1,2,\ldots,\mu$, let $\vec{v}_i$ be in $\boldsymbol{C}^{m_i}$, where the m_i are positive integers such that $n = m_1 + m_2 + \cdots + m_\mu$. Let $P(x,y)$ be an $n\times n$

matrix whose entries are $\boldsymbol{C}$-valued function of two variables $(x, y) \in \boldsymbol{C}^2$ such that $\vec{v} = P(x, y)\vec{u}$, where $\vec{u} = (\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_\nu) \in \boldsymbol{C}^n$ and $\vec{v} = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_\mu) \in \boldsymbol{C}^n$.

Consider a system of partial differential equations

$$
(3.1) \qquad \begin{cases} x^{1+k_j}\dfrac{\partial \vec{u}_j}{\partial x} = \vec{F}_j(x, y) + A_j\vec{u}_j + \displaystyle\sum_{h=1}^{\nu} B_{j,h}(x, y)\vec{u}_h + \vec{G}_j(x, y, \vec{u}), \\ \qquad (j = 1, 2, \ldots, \nu), \\ y^{1+\ell_i}\dfrac{\partial \vec{v}_i}{\partial y} = \vec{\hat{F}}_i(x, y) + \hat{A}_i\vec{v}_i + \displaystyle\sum_{h=1}^{\mu} \hat{B}_{i,h}(x, y)\vec{v}_h + \vec{\hat{G}}_i(x, y, \vec{u}), \\ \qquad (i = 1, 2, \ldots, \mu), \end{cases}
$$

where

(i) the k_j and ℓ_i are positive integers such that $0 < k_1 < k_2 < \cdots < k_\nu$ and $0 < \ell_1 < \ell_2 < \cdots < \ell_\mu$,

(ii) for each j, $\vec{F}_j$ is a $\boldsymbol{C}^{n_j}$-valued function of (x, y), and, for each i, $\vec{\hat{F}}_i$ is a $\boldsymbol{C}^{m_i}$-valued function of (x, y),

(iii) for each j, A_j is an $n_j \times n_j$ matrix with $\boldsymbol{C}$ entries, and for each i, $\hat{A}_i$ is an $m_i \times m_i$ matrix with $\boldsymbol{C}$ entries,

(iv) for each (j, h), $B_{j,h}$ is an $n_j \times n_h$ matrix whose entries are $\boldsymbol{C}$-valued functions of (x, y), and, for each (i, h), $\hat{B}_{i,h}$ is an $m_i \times m_h$ matrix whose entries are $\boldsymbol{C}$-valued functions of (x, y),

(v) for each j, $\vec{G}_j$ is a $\boldsymbol{C}^{n_j}$-valued function of $(x, y, \vec{u})$, and, for each i, $\vec{\hat{G}}_i$ is a $\boldsymbol{C}^{m_i}$-valued function of $(x, y, \vec{u})$.

In the statement of the following Theorem 3.1, we assume that there exist positive numbers ρ_0, r_0, and v_0 such that

(I) the function $\vec{F}_j$ and $\vec{\hat{F}}_i$ are holomorphic for $|x| < \rho_0$, $|y| < r_0$, and $\vec{F}_j(0,0) = \vec{0}$ and $\vec{\hat{F}}_i(0,0) = \vec{0}$,

(II) the determinants $\det A_j$ and $\det \hat{A}_i$ are not equal to zero,

(III) the entries of the matrices $B_{j,h}$ and $\hat{B}_{i,h}$ are holomorphic for $|x| < \rho_0$, $|y| < r_0$, and $B_{j,h}(0,0) = O$ and $\hat{B}_{i,h}(0,0) = O$,

(IV) the functions $\vec{G}_j$ and $\vec{\hat{G}}_i$ are holomorphic for $|x| < \rho_0$, $|y| < r_0$, $|\vec{u}| < v_0$, and $\vec{G}_j(x, y, \vec{0}) = \vec{0}$ and $\vec{\hat{G}}_i(x, y, \vec{0}) = \vec{0}$, and $\partial\vec{G}_j/\partial\vec{u}(x, y, \vec{0}) = O$ and $\partial\vec{\hat{G}}_i/\partial\vec{u}(x, y, \vec{0}) = O$,

(V) the entries of the matrix $P(x, y)$ are holomorphic for $|x| < \rho_0$, $|y| < r_0$, and $\det P(0,0)$ is not equal to zero.

Theorem 3.1. *Under assumptions* (I), (II), (III), (IV), *and* (V), *if a formal power series*

$$
(3.2) \qquad \vec{u} = \vec{q}(x, y) = \sum_{p+q \geq 1, p \geq 0, q \geq 0}^{+\infty} x^p y^q \vec{c}_{p,q}, \qquad \vec{c}_{p,q} \in \boldsymbol{C}^n
$$

is a formal solution of system (3.1), *then the series* $\vec{q}$ *is convergent in* (x, y).

Sketch of Proof of Theorem 3.1. Let us set

$$\vec{b}_q(x) = \sum_{p=0}^{+\infty} x^p \vec{c}_{p,q} \qquad (q = 0, 1, \ldots). \tag{3.3}$$

Then, convergence of all of $\vec{b}_q$ can be proved by using the system:

$$y^{1+\ell_i} \frac{d\vec{v}_i}{dy} = \hat{\vec{F}}_i(x, y) + \hat{A}_i \vec{v}_i + \sum_{h=1}^{\mu} \hat{B}_{i,h}(x, y)\vec{v}_h + \hat{\vec{G}}_i(x, y, \vec{u}), \qquad (i = 1, 2, \ldots, \mu). \tag{3.4}$$

Hence applying Theorem 2.1 to the system:

$$x^{1+k_j} \frac{d\vec{u}_j}{dx} = \vec{F}_j(x, y) + A_j \vec{u}_j + \sum_{h=1}^{\nu} B_{j,h}(x, y)\vec{u}_h + \vec{G}_j(x, y, \vec{u}), \qquad (j = 1, 2, \ldots, \nu), \tag{3.5}$$

we can finish proof of Theorem 3.1.

Remark 3.2. Previously (cf. [Ger-Sib], [Sib 1], [Sib 2], [Sib 3], and [Sib 4]) the following result was obtained:

Let us consider a Pfaffian system:

$$\begin{cases} x^{k+1} \dfrac{\partial \vec{u}}{\partial x} = \vec{f}_0(x, y) + A(x, y)\vec{u} + \displaystyle\sum_{|\wp| \ge 2} \vec{u}^{\wp} \vec{f}_{\wp}(x, y), \\ y^{h+1} \dfrac{\partial \vec{u}}{\partial y} = \vec{g}_0(x, y) + B(x, y)\vec{u} + \displaystyle\sum_{|\wp| \ge 2} \vec{u}^{\wp} \vec{g}_{\wp}(x, y), \end{cases} \tag{3.6}$$

where

(i) $\vec{u} \in \boldsymbol{C}^n$,

(ii) $\vec{f}_0(x, y), \vec{f}_{\wp}(x, y), \vec{g}_0(x, y)$, and $\vec{g}_{\wp}(x, y)$ are $\boldsymbol{C}^n$-valued functions, $A(x, y)$ and $B(x, y)$ are $n \times n$ matrix,

(iii) the entries of $\vec{f}_0, \vec{f}_{\wp}, \vec{g}_0, \vec{g}_{\wp}, A$, and B are holomorphic in a domain $\triangle(r) = \{(x, y); |x| + |y| < r\}$,

(iv) the power series $\sum_{|\wp| \ge 2} \vec{u}^{\wp} \vec{f}_{\wp}(x, y)$ and $\sum_{|\wp| \ge 2} \vec{u}^{\wp} \vec{g}_{\wp}(x, y)$ are uniformly convergent on each compact subset of the domain $|x| + |y| < r$, $|\vec{u}| < \rho$.

If the following conditions are satisfied:

(1) k and h are positive integers,

(2) $\vec{f}_0(0, 0) = \vec{0}$ and $\vec{g}_0(0, 0) = \vec{0}$,

(3) the matrices $A(0,0)$ and $B(0,0)$ are invertible,

(4) Pfaffian system (3.6) is completely integrable,

then (3.5) has a unique formal solution:

$$\vec{u} = \vec{q}(x,y) = \sum_{m+\ell\geq 1, m\geq 0, \ell\geq 0} x^m y^\ell \vec{c}_{m,\ell}, \tag{3.7}$$

where the coefficients $\vec{c}_{m,\ell}$ are in $\boldsymbol{C}^n$. Furthermore, formal solution (3.7) is a convergent power series in (x,y).

Theorem 3.1 is a generalization of this result. In Theorem 3.1 we assume existence of a formal solution (3.2), instead of assuming complete integrability of (3.1).

§4. Proof of main theorem

We shall use the definitions and notations of §1. Let $\vec{\eta}_{d_i}(x,\vec{\varepsilon})$ be the $\vec{k}$-multisum of the power series $\vec{p}(x,\vec{\varepsilon})$ of (c-3) of Theorem 1.1 in the direction d_i uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r_0)$. Set

$$\vec{\eta}_{d_i}(x,\vec{\varepsilon}) = \sum_{j=1}^{\nu} \vec{\eta}_{j,d_i}(x,\vec{\varepsilon}), \tag{4.1}$$

where $\vec{\eta}_{j,d_i}(x,\vec{\varepsilon})$ is the k_j-sum of a k_j-summable power series in the direction d_i uniformly with respect to $\vec{\varepsilon} \in \mathscr{D}(r_0)$. We shall look into the Stokes phenomena of $\{\vec{\eta}_{d_1}, \vec{\eta}_{d_2}, \dots, \vec{\eta}_{d_\ell}\}$.

Let us expand the functions $\vec{\eta}_{d_i}(x,\vec{\varepsilon})$ and the functions $\vec{\eta}_{j,d_i}(x,\vec{\varepsilon})$ in powers of $\vec{\varepsilon}$, i.e.

$$\begin{cases} \vec{\eta}_{d_i}(x,\vec{\varepsilon}) = \displaystyle\sum_{|\wp|\geq 0} \vec{\varepsilon}^{\wp} \vec{\eta}_{d_i,\wp}(x), \\ \vec{\eta}_{j,d_i}(x,\vec{\varepsilon}) = \displaystyle\sum_{|\wp|\geq 0} \vec{\varepsilon}^{\wp} \vec{\eta}_{j,d_i,\wp}(x). \end{cases} \tag{4.2}$$

These functions are holomorphic in $(x,\vec{\varepsilon})$ in the respective domains:

$$|\arg x - d_i| < \frac{\pi}{2k_\nu} + \delta_i, \qquad 0 < |x| < \rho_0, \qquad |\vec{\varepsilon}| < r_0,$$

and

$$|\arg x - d_i| < \frac{\pi}{2k_j} + \delta_i, \qquad 0 < |x| < \rho_0, \qquad |\vec{\varepsilon}| < r_0.$$

Furthermore,

$$\vec{\eta}_{d_i,\wp}(x) = \sum_{j=1}^{\nu} \vec{\eta}_{j,d_i,\wp}(x).$$

Therefore, each function $\vec{\eta}_{d_i,\wp}(x)$ is the $\vec{k}$-multisum of a $\vec{k}$-multisummable power series in the direction d_i respectively. On the other hand, note that the asymptotic expansion of $\vec{\eta}_{d_i,\wp}(x)$ as $x \to 0$ is the formal power series $\vec{b}_\wp(x)$ respectively. Since $\vec{b}_\wp(x)$ is convergent for $|x| < \rho_0$, the uniqueness of the $\vec{k}$-multisum implies that $\vec{\eta}_{d_i,\wp}(x)$ is equal to the sum of the convegent power series $\vec{b}_\wp(x)$. Hence, the set $\{\vec{\eta}_{d_1}, \vec{\eta}_{d_2}, \dots, \vec{\eta}_{d_\ell}\}$ defines a single-valued and holomorphic function in $(x, \vec{\varepsilon})$ for $|x| < \rho_0$ and $|\vec{\varepsilon}| < r_0$. This porves the convergence of the formal power series $\vec{q}(x, \vec{\varepsilon})$. □

Acknowledgement. The author wishes to express his appreciation to Professor B. L. J. Braaksma for his useful comments after haveing read the original manuscript.

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(Ricevita la 30-an de oktobro, 2002)
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