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Periodic Solutions for Second Order Equations with the Scalar p-Laplacian and Nonsmooth Potential

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Abstract. In this paper we examine a scalar equation driven by the p-Laplacian and having periodic boundary conditions and a nonsmooth potential $j(t,x)$. We assume that asymptotically at $\pm\infty$, the quantity $pj(t,x)/|x|^p$ lies between the first two eigenvalues $\lambda_0 = 0$ and λ_1, with possible interaction (resonance) with $\lambda_0 = 0$. We show that the equation has a solution. The method of proof uses the nonsmooth Critical Point Theory and in particular a recently established version of the Linking Theorem.

Key Words and Phrases. Nonsmooth critical point theory, Locally Lipschitz function, Subdifferential, Linking sets, Linking theorem, Nonsmooth C-condition, p-Laplacian, Eigenvalues.

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1. Introduction

The purpose of this paper is to prove an existence theorem for a nonlinear differential equation driven by the one dimensional p-Laplacian, with nonsmooth potential and periodic boundary conditions.

Recently there has been an increasing interest for differential equations involving the one dimensional p-Laplacian. We mention the works of Dang-Oppenheimer [4], Del Pino-Manasevich-Murua [5], Fabry-Fayyad [6], Guo [7] and the references therein. In all the aforementioned works the method of proof is based on degree theory and the right hand side nonlinearity is continuous (i.e. the corresponding potential function is C^1). We should mention that in Dang-Oppenheimer [4] and Guo [7] the nonlinearity depends also on x' and consequently the hypotheses are stronger. In contrast here the nonlinear potential $j(t,x)$ is measurable in the time variable $t \in T$ and in the space variable $x \in \boldsymbol{R}$ it is not C^1, but only locally Lipschitz. Our approach is variational and it is based on the nonsmooth Critical Point Theory as this was formulated originally by Chang [2] and generalized recently by Kourogenis-Papageorgiou [10]. For the convenience of the reader, in the next section we recall the basic aspects of this theory that we shall need in what follows.

2. Preliminaries

Let X be a Banach space and X^* its topological dual. In what follows by $\langle \cdot , \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \boldsymbol{R}$ is said to be locally Lipschitz, if for every $z \in X$, there is $U \ni z$ and $k_U > 0$ such that $|\varphi(x) - \varphi(y)| \le k_U \|x - y\|$ for all $x, y \in U$. From Convex Analysis we know that a convex function $\psi : X \to \bar{\boldsymbol{R}} = \boldsymbol{R} \cup \{+\infty\}$ which is proper (i.e. $\operatorname{dom} \psi = \{x \in X : \psi(x) < +\infty\} \neq \emptyset$) and lower semicontinuous (i.e. $\psi \in \Gamma_0(X)$, see Hu-Papageorgiou [8], p. 341), then $\psi|_{\operatorname{intdom} \psi}$ is locally Lipschitz. If X is finite dimensional, a convex $\boldsymbol{R}$-valued function is locally Lipschitz.

Given a locally Lipschitz function $\varphi : X \to \boldsymbol{R}$ for every $x, h \in X$ we define

$$\varphi^0(x; h) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{\varphi(y + \lambda h) - \varphi(y)}{\lambda}$$

the so called generalized directional derivative of φ at x in the direction of h. It is easy to verify that $h \to \varphi^0(x; h)$ is sublinear, continuous. Therefore by the Hahn-Banach Theorem, $\varphi^0(x; \cdot)$ is the support function of a nonempty, w^*-compact and convex set given by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x; h), \text{ for all } h \in X\}.$$

The multifunction $\partial\varphi : X \to 2^{X^*} \backslash \{\emptyset\}$ is called the (generalized or Clarke) subdifferential of φ. It has a graph $\operatorname{Gr} \partial\varphi = \{(x, x^*) \in X \times X^* : x^* \in \partial\varphi(x)\}$ which is sequentially closed in $X \times X^*_{w^*}$, where by $X^*_{w^*}$ we denote the Banach space X^* furnished with the w^*-topology. If $\varphi, \psi : X \to \boldsymbol{R}$ are locally Lipschitz functions, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \qquad \text{and} \qquad \partial(\lambda\varphi)(x) = \lambda\partial\varphi(x)$$

for all $x \in X$ and all $\lambda \in \boldsymbol{R}$. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$.

In analogy with the smooth case, we say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. As it is also true in the smooth case, to develop a critical point theory, we need a compactness-type condition on φ. We employ the weak form of this condition, which is the nonsmooth counterpart of the Cerami condition (or C-condition for short, see Struwe [15], p. 70). In the present nonsmooth setting this condition takes the following form:

"A locally Lipschitz function $\varphi : X \to \boldsymbol{R}$ satisfies the nonsmooth C-condition, if any sequence $\{x_n\}_{n \ge 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n \ge 1}$ is bounded and $(1 + \|x_n\|) m(x_n) = (1 + \|x_n\|) \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$, has a strongly convergent subsequence."

We recall the following geometric notion, which plays a central role in Critical Point Theory (smooth and nonsmooth).

Definition 1. Suppose that X is a topological space and E_1 and D are subsets of X. We say that E_1 and D link (homotopically) in X, if
(a) $E_1 \cap D = \emptyset$ and
(b) there exists a set $E \subseteq X$ such that $E_1 \subseteq E$ and for any continuous function $\eta : E \to X$ such that $\eta|_{E_1} = id$, we have $\eta(E) \cap D \neq \emptyset$.

Kourogenis-Papageorgiou [10] used this notion to prove the following nonsmooth minimax characterization of critical points (nonsmooth Linking Theorem).

Theorem 1. *If X is a reflexive Banach space, $\varphi : X \to \boldsymbol{R}$ is locally Lipschitz, satisfies the nonsmooth C-condition, $E_1 \subseteq E$ and D are nonempty, closed subsets of X such that* (i) *E_1 and D link in X,* (ii) *$\inf_D \varphi > \sup_{E_1} \varphi$ and $c_0 = \inf_{\gamma \in \Gamma} \sup_{x \in E} \varphi(\gamma(x))$ with $\Gamma = \{\gamma \in C(E, X) : \gamma|_{E_1} = id\}$, then c_0 is a critical value of φ and $c_0 \geq \inf_D \varphi$.*

Recall that $\lambda \in \boldsymbol{R}$ is an eigenvalue of minus the scalar p-Laplacian with periodic boundary conditions if the problem

$$-(|x'(t)|^{p-2}x'(t))' = \lambda |x(t)|^{p-2}x(t) \quad \text{a.e. on } T, \tag{1}$$
$$x(0) = x(b),\ x'(0) = x'(b)$$

has a nontrivial solution x, known as the eigenfunction corresponding to λ. We know that the eigenvalues of minus the scalar p-Laplacian with periodic boundary conditions, are $\lambda_n = (2n\pi_p/b)^p$, $n \geq 0$, where $\pi_p = 2(p-1)^{1/p} \cdot (\pi/p)/\sin(\pi/p)$ (see Mawhin [14]).

3. Existence theorem

The problem under consideration is the following

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t, x(t)) \text{ a.e. on } T, \\ x(0) = x(b),\ x'(0) = x'(b),\ 1 < p < \infty. \end{array} \right\} \tag{2}$$

Here $j(t, \cdot)$ is a nonsmooth, locally Lipschitz potential function and $\partial j(t, x)$ is the subdifferential of $j(t, \cdot)$ (see section 2). The precise hypotheses on $j(t, x)$ are the following:

H(j): $j : T \times \boldsymbol{R} \to \boldsymbol{R}$ is a function such that $j(\cdot, 0) \in L^1(T)$ and
(i) for every $x \in \boldsymbol{R}$, $t \to j(t, x)$ is measurable;
(ii) for almost all $t \in T$, $x \to j(t, x)$ is locally Lipschitz;
(iii) for almost all $t \in T$, all $x \in \boldsymbol{R}$ and all $u \in \partial j(t, x)$ we have

$$|u| \leq a(t) + c(t)|x|^{p-1}$$

with $a \in L^q(T)_+$, $1/q + 1/p = 1$ and $c \in L^\infty(T)$;

(iv) $\lim_{|x|\to\infty}(xu - pj(t,x)) = -\infty$ uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$;

(v) $0 \le \liminf_{|x|\to\infty} pj(t,x)/|x|^p \le \limsup_{|x|\to\infty} pj(t,x)/|x|^p \le \theta < \lambda_1$ uniformly for almost all $t \in T$.

Remark 1. Hypothesis H(j) implies that asymptotically the nonsmooth potential lies between the first two eigenvalues $\lambda_0 = 0$ and $\lambda_1 = (2\pi_p/b)^p$. Also it allows interaction (resonance) with $\lambda_0 = 0$.

Let $W^{1,p}_{\text{per}}(T) = \{x \in W^{1,p}(T) : x(0) = x(b)\}$ (since $W^{1,p}(T) \hookrightarrow C(T)$, the pointwise evaluations at $t = 0$ and $t = b$ make sense) and let

$$\varphi : W^{1,p}_{\text{per}}(T) \to \boldsymbol{R}$$

be the functional defined by

$$\varphi(x) = \frac{1}{p}\|x'\|_p^p - \int_0^b j(t, x(t))dt.$$

We know that φ is locally Lipschitz (see Hu-Papageorgiou [9], p. 313).

Proposition 1. *If hypotheses* H(j) *hold, then* φ *satisfies the nonsmooth C-condition.*

Proof. Let $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{\text{per}}(T)$ be a sequence such that

(3) $$\varphi(x_n) \to c_0 \quad \text{and} \quad (1 + \|x_n\|)m(x_n) \to 0 \quad \text{as } n \to \infty.$$

We can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \ge 1$. The existence of such elements follows from the weak compactness of $\partial\varphi(x_n)$ and the weak lower semicontinuity of the norm. Let $A : W^{1,p}_{\text{per}}(T) \to W^{1,p}_{\text{per}}(T)^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt \qquad \text{for all } x, y \in W^{1,p}_{\text{per}}(T).$$

It is easy to check that A is monotone, demicontinuous, hence A is maximal monotone (see Hu-Papageorgiou [8], p. 309). Consider the integral functional $J : L^p(T) \to \boldsymbol{R}$ defined by $J(x) = \int_0^b j(t, x(t))dt$. We know that J is locally Lipschitz (see Hu-Papageorgiou [9], p. 313), hence $\hat{J} = J|_{W^{1,p}_{\text{per}}(T)}$ is locally Lipschitz too. Then from Clarke [3], p. 47, we know that

$$\partial\hat{J}(x) \subseteq L^q(T) \qquad \left(\frac{1}{q} + \frac{1}{p} = 1\right)$$

and moreover, if $u \in \partial\hat{J}(x)$, then $u(t) \in \partial j(t, x(t))$, a.e. on T. For all $n \ge 1$ we have

$$x_n^* = A(x_n) - u_n, \qquad u_n \in \partial \hat{J}(x_n).$$

From (3) we have

$$\begin{aligned} |\langle x_n^*, x_n\rangle - p\varphi(x_n) + pc_0| &\le \|x_n^*\|\,\|x_n\| + |p\varphi(x_n) - pc_0| \\ &\le (1 + \|x_n\|)m(x_n) + |p\varphi(x_n) - pc_0| \to 0 \end{aligned}$$

as $n \to \infty$. We have

$$\langle x_n^*, x_n\rangle = \langle A(x_n) - u_n, x_n\rangle = \|x'\|_p^p - \int_0^b u_n(t)x_n(t)dt$$

and as $n \to \infty$

$$\Rightarrow p\varphi(x_n) - \langle x_n^*, x_n\rangle = \int_0^b (u_n(t)x_n(t) - pj(t, x_n(t)))dt \to pc_0. \tag{4}$$

We claim that $\{x_n\}_{n\ge 1} \subseteq W_{\text{per}}^{1,p}(T)$ is bounded. Suppose that the claim is not true. By passing to a subsequence if necessary, we may assume that $\|x_n\| \to \infty$. Set $y_n = x_n/\|x_n\|$, $n \ge 1$. We may assume that $y_n \xrightarrow{w} y$ in $W_{\text{per}}^{1,p}(T)$ and $y_n \to y$ in $C(T)$ (recall that $W_{\text{per}}^{1,p}(T)$ is embedded compactly in $C(T)$). From Lebourg's Mean Value Theorem (see Lebourg [12] or Clarke [3], p. 41), we have $|j(x,t) - j(x,0)| = |vx|$ with $v \in \partial j(t, \eta x)$ where $\eta \in (0,1)$ and so $|j(x,t)| \le a_1(t) + c_1(t)|x|^p$ a.e. on T with $a_1 \in L^q(T)$, $c_1 \in L^\infty(T)$ (see hypothesis H(j)(iii)). Then we have

$$\frac{|j(t, x_n(t))|}{\|x_n\|^p} \le \frac{a_1(t)}{\|x_n\|^p} + c_1(t)|y_n(t)|^p \qquad \text{a.e. on } T. \tag{5}$$

Thus by the Dunford-Pettis Theorem and by passing to a subsequence if necessary, we may say that $j(\cdot, x_n(\cdot))/\|x_n\|^p \xrightarrow{w} h$ in $L^1(T)$. Given $\varepsilon \in (0, \lambda_1 - \theta)$ (see hypothesis H(j)(v)), let

$$E_{\varepsilon,n} = \left\{ t \in T : x_n(t) \ne 0, -\varepsilon \le \frac{j(t, x_n(t))}{|x_n(t)|^p} \le \frac{1}{p}\theta + \varepsilon \right\},$$

$n \ge 1$. Because of hypothesis H(j)(v), we have that $\chi_{E_{\varepsilon,n}}(t) \to 1$ a.e on $\{y \ne 0\}$. So

$$\begin{aligned} \chi_{E_{\varepsilon,n}}(t)\frac{j(t, x_n(t))}{\|x_n\|^p} &= \chi_{E_{\varepsilon,n}}(t)\frac{j(t, x_n(t))}{|x_n(t)|^p}|y_n(t)|^p \\ \Rightarrow -\varepsilon|y_n(t)|^p &\le \chi_{E_{\varepsilon,n}}(t)\frac{j(t, x_n(t))}{\|x_n\|^p} \le \left(\frac{1}{p}\theta + \varepsilon\right)|y_n(t)|^p. \end{aligned}$$

Passing to the weak limit in $L^1(\{y \ne 0\})$ as $n \to \infty$ and using Proposition 7.3.9, p. 694, of Hu-Papageorgiou [8], we obtain

$$-\varepsilon|y(t)|^p \leq h(t) \leq \left(\frac{1}{p}\theta + \varepsilon\right)|y(t)|^p \qquad \text{a.e. on } \{y \neq 0\},$$

$$\Rightarrow 0 \leq h(t) \leq \frac{1}{p}\theta|y(t)|^p \qquad \text{a.e. on } \{y \neq 0\}.$$

On the other hand it is clear from (5) that $h(t) = 0$ a.e on $\{y = 0\}$. Therefore finally we can say that $0 \leq h(t) \leq \frac{1}{p}\theta|y(t)|^p$ a.e on T. From (3) we have that for all $n \geq 1$ we have

$$\varphi(x_n) \leq M_1 \qquad \text{with } M_1 > 0$$

$$\Rightarrow \frac{\varphi(x_n)}{\|x_n\|^p} \leq \frac{M_1}{\|x_n\|^p}$$

$$\Rightarrow \frac{1}{p}\|y_n'\|_p^p - \int_0^b \frac{j(t, x_n(t))}{\|x_n\|^p}\, dt \leq \frac{M_1}{\|x_n\|^p}$$

$$\Rightarrow \frac{1}{p}\|y'\|_p^p \leq \int_0^b h(t)dt \leq \int_0^b \frac{1}{p}\theta|y(t)|^p dt.$$

If $y = 0$, then $\|y_n'\|_p \to 0$, hence $y_n \to 0$ in $W^{1,p}_{\text{per}}(T)$, a contradiction since $\|y_n\| = 1$ for all $n \geq 1$. So if $T_0 = \{y \neq 0\} = \{t \in T : y(t) \neq 0\}$, then T_0 is open and we have just seen that $|T_0| > 0$ (here by $|\cdot|$ we denote the Lebesgue measure on $\boldsymbol{R}$). Moreover, note that for all $t \in T_0$ we have $|x_n(t)| \to \infty$ as $n \to \infty$. We have

$$\int_0^b (u_n(t)x_n(t) - pj(t, x_n(t)))dt = \int_{T_0} (u_n(t)x_n(t) - pj(t, x_n(t)))dt$$
$$+ \int_{T_0^c} (u_n(t)x_n(t) - pj(t, x_n(t)))dt.$$

By virtue of hypothesis H(j)(iv), we can find $M_2 > 0$ such that for almost all $t \in T$, all $|x| \geq M_2$ and all $v \in \partial j(t, x)$, we have

$$vx - pj(t, x) \leq -1.$$

On the other hand from hypothesis H(j)(iii) and the growth condition on j established earlier in the proof, we see that for almost all $t \in T$, all $|x| < M_2$ and all $v \in \partial j(t, x)$, we have

$$vx - pj(t, x) \leq \beta_1(t) \qquad \text{for some } \beta_1 > 0, \quad \beta_1 \in L^1(T).$$

Therefore it follows that for almost all $t \in T$, all $x \in \boldsymbol{R}$ and all $v \in \partial j(t, x)$ we have

$$vx - pj(t, x) \leq \beta_1(t). \tag{6}$$

Using Fatou's Lemma and hypothesis H(j)(iv), we obtain

$$\lim_{n\to\infty}\int_{T_0}(u_n(t)x_n(t) - pj(t, x_n(t)))dt = -\infty.$$

Also from (6), we have that

$$\int_{T_0^c}(u_n(t)x_n(t) - pj(t, x_n(t)))dt \le \|\beta_1\|_1.$$

Therefore we deduce that

$$\lim_{n\to\infty}\int_0^b (u_n(t)x_n(t) - pj(t, x_n(t)))dt = -\infty. \tag{7}$$

Comparing (4) and (7), we reach a contradiction. This contradiction implies that $\{x_n\}_{n\ge 1} \subseteq W_{\text{per}}^{1,p}(T)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W_{\text{per}}^{1,p}(T)$ and $x_n \to x$ in $C(T)$. From (3) we have

$$\langle A(x_n), x_n - x\rangle - \int_0^b u_n(t)(x_n - x)(t)dt \le \varepsilon_n \qquad \text{with } \varepsilon_n \downarrow 0.$$
$$\Rightarrow \limsup\langle A(x_n), x_n - x\rangle \le 0.$$

But A being maximal monotone, it is generalized pseudomonotone and so $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \Rightarrow \|x_n'\|_p \to \|x'\|_p$. Because $x_n' \xrightarrow{w} x'$ in $L^p(T)$ and $L^p(T)$ is uniformly convex (hence it has the Kadec-Klee property), it follows that $x_n \to x$ in $W_{\text{per}}^{1,p}(T)$. Therefore φ satisfies the nonsmooth C-condition. □

The next two propositions aim at showing that our problem satisfies the Linking Theorem geometry (see Theorem 1).

Proposition 2. *If hypotheses* H(j) *hold, then* $\varphi(\xi) \to -\infty$ *as* $|\xi| \to +\infty$, $\xi \in \boldsymbol{R}$.

Proof. From Clarke [3], p. 48, we know that for almost all $t \in T$ and all $x > 0$ we have that $x \to j(t,x)/|x|^p$ is locally Lipschitz and we have

$$\partial\left(\frac{j(t,x)}{|x|^p}\right) = \frac{|x|^p\partial j(t,x) - p|x|^{p-2}xj(t,x)}{|x|^{2p}}$$
$$= |x|^{p-1}\frac{x\partial j(t,x) - pj(t,x)}{|x|^{2p}}$$
$$= \frac{x\partial j(t,x) - pj(t,x)}{|x|^{p+1}}.$$

By virtue of hypothesis H(j)(iv), we see that given $\beta_2 > 0$ we can find $M_3 > 0$ such that for almost all $t \in T$, all $x > M_3$ and all $v \in \partial j(t,x)$, we have

$$vx - pj(t,x) \le -\beta_2$$
$$\Rightarrow w \le -\frac{\beta_2}{|x|^{p+1}} \qquad \text{for all } w \in \partial\left(\frac{j(t,x)}{|x|^p}\right).$$

For $t \in T \setminus N$ with $|N| = 0$, the function $x \to j(t,x)/|x|^p$ is locally Lipschitz on $(M_3, +\infty)$, thus by Rademacher's Theorem it is differentiable at every $x \in (M_3, +\infty) \setminus N_1(t)$ with $|N_1(t)| = 0$. We set

$$\xi_0(t,x) = \begin{cases} \frac{d}{dx}\frac{j(t,x)}{|x|^p} & \text{if } x \in (M_3, +\infty)\setminus N_1(t), \\ 0 & \text{if } x \in N_1(t). \end{cases}$$

We know (see Clarke [3], p. 63) that for all $t \in T \setminus N$ and all $x \in (M_3, +\infty)\setminus N_1(t)$, we have $\xi_0(t,x) \in \partial(j(t,x)/|x|^p)$ and so $\xi_0(t,x) \le -\beta_2/|x|^{p+1} = d(\beta_2/p|x|^p)/dx$. We integrate this inequality on the interval $[z, y] \subseteq (M_3, +\infty)$, $z < y$ and so we obtain

$$\int_z^y \xi_0(t,x)dx = \int_z^y \frac{d}{dx}\left(\frac{j(t,x)}{x^p}\right)dx \le \frac{\beta_2}{p}\int_z^y \frac{d}{dx}\left(\frac{1}{x^p}\right)dx$$
$$\Rightarrow \frac{j(t,y)}{y^p} - \frac{j(t,z)}{z^p} \le \frac{\beta_2}{p}\left(\frac{1}{y^p} - \frac{1}{z^p}\right).$$

Let $y \to \infty$. By virtue of hypothesis H(j)(v), we have that

$$\frac{j(t,z)}{z^p} \ge \frac{\beta_2}{p}\frac{1}{z^p}$$
$$\Rightarrow j(t,z) \ge \frac{\beta_2}{p} \qquad \text{for all } t \in T\setminus N \text{ and all } z \in (M_3, +\infty).$$

So if $\xi \in \boldsymbol{R}$, $\xi > M_3$, we have

$$\varphi(\xi) = -\int_0^b j(t,\xi)dt \le -\frac{\beta_2}{p}h.$$

Because $\beta_2 > 0$ was arbitrary, we conclude that $\varphi(\xi) \to -\infty$ as $\xi \to +\infty$. Similarly we show that $\varphi(\xi) \to -\infty$ as $\xi \to -\infty$. □

Let

$$D = \left\{ v \in W^{1,p}_{\text{per}}(T) : \int_0^b |v(t)|^{p-2}v(t)dt = 0 \right\}.$$

Evidently this is a closed cone.

Proposition 3. *If hypotheses* H(j) *hold, then* $\varphi(v) \to +\infty$ *as* $\|v\| \to \infty$ *with* $v \in D$.

Proof. By virtue of hypothesis H(j)(v), given $\varepsilon \in (0, \lambda_1 - \theta)$, we can find $M_4 > 0$ such that for almost all $t \in T$ and all $|x| \geq M_4$, we have

$$j(t,x) \leq \frac{1}{p}(\theta + \varepsilon)|x|^p.$$

Also from the growth condition for $j(t,\cdot)$ that we obtained in the proof of Proposition 1 via Lebourg's Mean Value Theorem, for almost all $t \in T$ and all $|x| < M_4$, we have

$$|j(t,x)| \leq k(t) \qquad \text{with } k \in L^1(T)_+.$$

Therefore for almost all $t \in T$ and all $x \in \boldsymbol{R}$, we have

$$j(t,x) \leq \frac{1}{p}(\theta + \varepsilon)|x|^p + k(t).$$

If $v \in D$, we have

$$\begin{aligned} \varphi(v) &= \frac{1}{p}\|v'\|_p^p - \int_0^b j(t, v(t))dt \\ &\geq \frac{1}{p}\|v'\|_p^p - \frac{\theta}{p}\|v\|_p^p - \frac{\varepsilon}{p}\|v\|_p^p - \beta_3 \qquad \text{for some } \beta_3 > 0. \end{aligned}$$

From Mawhin [14] we know that $\lambda_1\|v\|_p^p \leq \|v'\|_p^p$ for all $v \in D$. Therefore we have

$$\varphi(v) \geq \frac{1}{p}\left(1 - \frac{\theta + \varepsilon}{\lambda_1}\right)\|v'\|_p^p - \beta_3.$$

From the choice of $\varepsilon > 0$, we have that $(\theta + \varepsilon)/\lambda_1 < 1$. Since on D we have $\lambda_1\|v\|_p^p \leq \|v'\|_p^p$ (see Mawhin [14]), from the last inequality it follows that $\varphi|_V$ is coercive, as claimed by the proposition. □

Using the auxiliary results established in Propositions 1, 2 and 3, we can have the following existence theorem for problem (2).

Theorem 2. *If hypotheses* H(j) *hold, then problem* (2) *has a solution* $x \in C^1(T)$.

Proof. Because of Propositions 2 and 3, we can find $\xi \in \boldsymbol{R}$, $\xi > 0$ such that $\varphi(\pm\xi) < \inf_D \varphi$. Let $E = [-\xi, \xi] = \{x \in W^{1,p}_{\text{per}}(T) : -\xi \leq x(t) \leq \xi$ for all $t \in T\}$ and $E_1 = \{-\xi, \xi\}$. Evidently $E_1 \cap D = \emptyset$. Moreover, if $\theta \in C(E, W^{1,p}_{\text{per}}(T))$ and $\theta(\pm\xi) = \pm\xi$, then an easy application of the Intermediate Value Theorem, implies that $\theta(E) \cap D \neq \emptyset$. Therefore E_1 and D link in $W^{1,p}_{\text{per}}(T)$ (see Section 2). Because of this fact and of Proposition 1, we can apply Theorem 1 (the nonsmooth Linking Theorem) and obtain $x \in W^{1,p}_{\text{per}}(T)$ such that $0 \in \partial\varphi(x)$. We have

$$(8) \qquad A(x) = u \quad \text{with } u \in L^q(T), \quad u(t) \in \partial j(t, x(t)) \text{ a.e. on } T.$$

We know that $(|x'|^{p-2}x')' \in W^{-1,q}(T) = W_0^{1,p}(T)^*$ (see Adams [1], p. 50). So for all $\eta \in C_0^\infty(0,b)$ we have

$$\langle -(|x'|^{p-2}x')', \eta \rangle = \langle A(x), \eta \rangle = \int_0^b u(t)\eta(t)dt.$$

Since $C_0^\infty(0,b)$ is dense in $W_0^{1,p}(T)$ (the predual of $W^{-1,q}(T)$), it follows that

$$(9) \qquad -(|x'(t)|^{p-2}x'(t))' = u(t) \text{ a.e. on } T, \qquad x(0) = x(b).$$

Therefore $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)$. For every $y \in W^{1,p}_{\text{per}}(T)$, from (8) and Green's identity we have

$$\langle A(x), y \rangle = \int_0^b u(t)y(t)dt$$

$$\Rightarrow |x'(b)|^{p-2}x'(b)y(b) - |x'(0)|^{p-2}x'(0)y(0) - \int_0^b (|x'(t)|^{p-2}x'(t))'y(t)dt$$

$$= \int_0^b u(t)y(t)dt$$

$$\Rightarrow |x'(0)|^{p-2}x'(0)y(0) = |x'(b)|^{p-2}x'(b)y(b) \qquad \text{(see (9))}.$$

Since $y \in W^{1,p}_{\text{per}}(T)$ was arbitrary we can take $y \in W^{1,p}_{\text{per}}(T) \subseteq C(T)$ such that $y(0) = y(b) = 1$. So

$$|x'(0)|^{p-2}x'(0) = |x'(b)|^{p-2}x'(b),$$

$$x'(0) = x'(b).$$

Finally since $|z|^{p-2}z$ is a homeomorphism on $\boldsymbol{R}$ and $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T) \subseteq C(T)$, we see that $x' \in C(T)$, hence $x \in C^1(T)$. □

Remark 2. It will be very interesting to have a corresponding existence result for systems. Our proof fails in the vectorial case. More precisely the proof of Proposition 2 does not extend to systems. Periodic systems involving p-Laplace like operators, were studied recently by Manasevich-Mawhin [13] and Kyritsi-Matzakos-Papageorgiou [11].

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