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Optimal Control Problems for Nonlinear Hyperbolic Distributed Parameter Systems with Damping Terms

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Abstract. In this paper we study the quadratic optimal control problems for the nonlinear damped second order evolution equations in Hilbert spaces of Gelfand fivefolds. We prove the existence of optimal controls, and establish the necessary conditions of optimality according to various types of observations by using the transposition method.

Key Words and Phrases. Nonlinear hyperbolic system, Optimal control, Optimality condition, Transposition method.

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1. Introduction

In this paper we study quadratic optimal control problems for the control systems described by the nonlinear damped second order evolution equations of the form

$$\frac{d^2y}{dt^2} + A_2(t)\frac{dy}{dt} + A_1(t)y = f(t, y) + Bv \qquad \text{in } (0, T), \tag{1.1}$$

where $A_1(t), A_2(t)$ are time varying operators on appropriate Hilbert spaces embedded in a pivot Hilbert space H, B is a controller, v is a control and $f(t, y)$ is a nonlinear forcing function. This type of equations covers various damped hyperbolic distributed parameter control systems. The initial condition attached to (1.1) is given by

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = y_1. \tag{1.2}$$

For the system (1.1), (1.2) we introduce another Hilbert space V_2 corresponding to the damping term $A_2(t)dy/dt$ beside the Hilbert space V to $A_1(t)y$, and we present the basic results on existence, uniqueness and regularity of solutions in the setting of a Gelfand fivefold $V \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2' \hookrightarrow V$. The introduction of this V_2 admits us to choose an appropriate space of differential operators according to the types of damping effects, so that our treatment provides a convenient and unified way of obtaining the solutions to the nonlinear control system (1.1), (1.2). The basic results for the free system with $Bv = 0$, $y_0 \in V$, $y_1 \in H$ are established in Ha and Nakagiri [9] due to the variational formulation of Dautray-Lions [8].

Our aim of this paper is to study the optimal control problems as studied in Lions [10] and Ahmed and Teo [3] for the nonlinear system (1.1), (1.2) based on the results of [9]. The related optimal control and parameter estimation problems for linear systems, i.e., $f(t, y) = f(t)$, are studied extensively by Ahmed [1, 2], Banks et al. [5, 6] and Omatsu and Seinfeld [13] in the framework of Gelfand triple or fivefold. We also intend to extend some of their results to our nonlinear system.

We shall explain the main feature of this paper. Let $\mathcal{U}$ be a Hilbert space of control variables and $\mathcal{U}_{ad} \subset \mathcal{U}$ be an admissible set. We study the optimal control problem:

$$\text{Minimize } J(v) \text{ over } \mathcal{U}_{ad} \tag{1.3}$$

with the quadratic cost function

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Rv, v)_{\mathcal{U}}, \qquad v \in \mathcal{U}, \tag{1.4}$$

where $y = y(v)$ is the solution of (1.1), (1.2). Here in (1.4), M is an observation space, z_d is a desired element in M and C is an observer, and R is a regulator on $\mathcal{U}$. The first part of (1.3) is the existence of the optimal control u such that $\inf_{v \in \mathcal{U}_{ad}} J(v) = J(u)$. In general, the existence problem of optimal controls for nonlinear systems is difficult to solve. In proving the existence, it is often required to show the strong convergence of $y(v)$ in v with respect to the weak topology of $\mathcal{U}$. In almost all cases (cf. [7]) this convergence is verified provided that underlying spaces have the structure of compact embeddings. Noting this, we give an existence result of optimal controls by assuming that $V \hookrightarrow V_2$ is compact.

Our main concern as the second part of (1.3) is to characterize the necessary optimality conditions for nonlinear system (1.1) on the optimal control u, according to specific types of observations C in time and space. For the purpose we have to show the weak Gâteaux differentiability of the nonlinear map $v \to y(v)$, which is used to define the associate adjoint system. The differentiability is proved by assuming the Fréchet differentiability of $f(t, y)$ in y. Using the system and adjoint system equations, the necessary optimality conditions are completely characterized. These conditions extend the former results in [10, 2, 3, 13] to two directions, one is nonlinear equations and the other is damped equations.

This paper is composed of four sections. In Section 2, after giving the assumptions on operators $A_i(t)$, $i = 1, 2$, and nonlinear Lipschitz continuous term $f(t, y)$, we state the existence, uniqueness and regularity results to the solutions of the system (1.1), (1.2). Based on the results of Section 2 we study in detail the optimal control problems for the nonlinear system (1.1), (1.2) with the quadratic cost (1.3) to be minimized on any admissible set $\mathscr{U}_{ad}$ in Sections 3 and 4. That is, in the first part of Section 3 we give an existence theorem of optimal controls, and in the rest part of Section 3 the necessary conditions of optimality are established according to two types of observations under some additional assumptions on operators and observations. In Section 4 the similar necessary conditions are established without additional assumptions by using the method of transposition (cf. [10], [12]).

2. Existence, uniqueness and energy inequality

Let H be a real pivot Hilbert space, and the inner product and the norm is denoted by $(\cdot,\cdot)_H$ and $|\cdot|_H$, respectively. For $i=1,2$, let V_i be a real separable Hilbert space with the norm $\|\cdot\|_{V_i}$. Assume that each pair (V_i, H) is a Gelfand triple space and that V_1 is continuously embedded in V_2 (for details see Ha and Nakagiri [9]). We are given a family of symmmetric bilinear forms on $V_i \times V_i$ $a_i(t;\phi,\varphi)$, $t\in[0,T]$, $i=1,2$. We assume that there exists a $c_{i1}>0$ such that

$$(2.1)\qquad |a_i(t;\phi,\varphi)| \le c_{i1}\|\phi\|_{V_i}\|\varphi\|_{V_i} \qquad \text{for all } \phi,\psi\in V_i \text{ and } t\in[0,T];$$

and there exist $\alpha_i>0$ and $\lambda_i \in \boldsymbol{R}$ such that

$$(2.2)\qquad a_i(t;\phi,\phi)+\lambda_i|\phi|_H^2 \ge \alpha_i\|\phi\|_{V_i}^2 \qquad \text{for all } \phi\in V_i \text{ and } t\in[0,T].$$

Further, $a_1(t;\phi,\varphi)$, $t\in[0,T]$ is assumed to satisfy that the function $t\to a_1(t;\phi,\varphi)$ is continuously differentiable in $[0,T]$ and there exists a $c_{12}>0$ such that

$$(2.3)\qquad |a_1'(t;\phi,\varphi)| \le c_{12}\|\phi\|_{V_1}\|\varphi\|_{V_1} \qquad \text{for all } \phi,\psi\in V \text{ and } t\in[0,T].$$

Then we can define the operator $A_i(t)\in\mathscr{L}(V_i,V_i')$ defined by the relation $a_i(t;\phi,\varphi)=\langle A_i(t)\phi,\varphi\rangle_{V_i',V_i}$.

From now on, we shall write $V_1=V$ for notational simplicity. We consider the following nonlinear damped second order evolution equation

$$(2.4)\qquad \begin{cases} \dfrac{d^2y}{dt^2}+A_2(t)\dfrac{dy}{dt}+A_1(t)y=f(t,y) \quad \text{in } (0,T), \\ y(0)=y_0\in V, \quad \dfrac{dy}{dt}(0)=y_1\in H, \end{cases}$$

where $f:[0,T]\times V_2\to V_2'$. The solution Hilbert space $W(0,T)$ of (2.4) is defined by

$$W(0,T)=\{g \mid g\in L^2(0,T;V), g'\in L^2(0,T;V_2), g''\in L^2(0,T;V')\},$$

endowed with the norm

$$\|g\|_{W(0,T)} = (\|g\|^2_{L^2(0,T;V)} + \|g'\|^2_{L^2(0,T;V_2)} + \|g''\|^2_{L^2(0,T;V')})^{1/2}.$$

We denote by $\mathscr{D}'(0,T)$ the space of distributions on $(0,T)$. A function $y = y(\cdot)$ is said to be a weak solution of (2.4) if $y \in W(0,T)$ and y satisfies

$$\left\langle \frac{d^2y}{dt^2}(\cdot), \phi \right\rangle_{V',V} + a_2(\cdot\,; y'(\cdot), \phi) + a_1(\cdot\,; y(\cdot), \phi) = \langle f(\cdot\,, y(\cdot)), \phi \rangle_{V_2',V_2}$$

for all $\phi \in V$ in the sense of $\mathscr{D}'(0,T)$

$$y(0) = y_0 \in V, \qquad \frac{dy}{dt}(0) = y_1 \in H.$$

As in [9] we impose the following assumptions on the nonlinear term $f : [0,T] \times V_2 \to V_2'$ in (2.4):
(A1) $t \to f(t,y)$ is strongly measurable in V_2' for all $y \in V_2$,
(A2) there exists a $\beta \in L^2(0,T;\boldsymbol{R}^+)$ such that

$$\|f(t,y) - f(t,z)\|_{V_2'} \le \beta(t)\|y - z\|_{V_2} \qquad \text{a.e. } t \text{ for } y,z \in V_2,$$

(A3) there exists a $\gamma \in L^2(0,T;\boldsymbol{R}^+)$ such that $\|f(t,0)\|_{V_2'} \le \gamma(t)$ a.e. t.

The following theorem on existence, uniqueness, regularity and energy inequality of solutions to (2.4) holds (see [9]).

Theorem 2.1. *Assume that both a_i, $i = 1,2$ satisfy* (2.1)–(2.3) *and $f(t,y)$ satisfy* (A1)–(A3). *Then there exists a unique weak solution $y \in W(0,T) \cap C([0,T];V) \cap C^1([0,T];H)$ of* (2.4). *Moreover, for each $t \in [0,T]$, y satisfies the energy inequality*

$$\|y(t)\|_V^2 + |y'(t)|_H^2 + \int_0^t \|y'(\sigma)\|_{V_2}^2 d\sigma \le c(\|y_0\|_V^2 + |y_1|_H^2 + \|\gamma\|^2_{L^2(0,T;\boldsymbol{R}^+)}), \tag{2.5}$$

where c is a proper constant depending only on β in (A.2).

The energy inequality (2.5) follows from the assumptions (A.1)–(A.3) and the following energy equality in [9, Theorem 4.1]:

$$\begin{aligned} &a_1(t; y(t), y(t)) + |y'(t)|_H^2 + 2\int_0^t a_2(\sigma; y'(\sigma), y'(\sigma))d\sigma \\ &\quad = a_1(0; y_0, y_0) + |y_1|_H^2 + \int_0^t a_1'(\sigma; y(\sigma), y(\sigma))d\sigma \\ &\qquad + 2\int_0^t \langle f(\sigma, y(\sigma)), y'(\sigma)\rangle_{V_2',V_2} d\sigma. \end{aligned} \tag{2.6}$$

After this section we assume that (2.1)–(2.3) and (A1)–(A3) hold without any indication.

3. Quadratic optimal control problems

Let $\mathscr{U}$ be a Hilbert space of control variables, and let B be an operator,

$$B \in \mathscr{L}(\mathscr{U}, L^2(0,T; V_2')), \tag{3.1}$$

called a controller and $v \in \mathscr{U}$ be a control. We consider the following controlled nonlinear damped second order system:

$$\begin{cases} \dfrac{d^2 y(v)}{dt^2} + A_2(t)\dfrac{dy(v)}{dt} + A_1(t)y(v) = f(t, y(v)) + Bv \quad \text{in } (0,T), \\ y(v;0) = y_0 \in V, \quad \dfrac{dy(v;0)}{dt} = y_1 \in H, \end{cases} \tag{3.2}$$

Here in (3.2), $A_1(t), A_2(t)$ and $f(t,y)$ are differential operators and the nonlinear function satisfying the assumptions given in Section 2. By virtue of Theorem 2.1 and (3.1), we can define uniquely the solution map $v \to y(v)$ of $\mathscr{U}$ into $W(0,T)$, because $f(t,\xi) \equiv f(t,\xi) + Bv(t)$ satisfies the assumptions (A1)–(A3). We shall call the weak solution $y(v)$ of (3.2) the state of the control system (3.2). The observation of the state is assumed to be given by

$$z(v) = Cy(v), \qquad C \in \mathscr{L}(W(0,T), M), \tag{3.3}$$

where C is an operator called the observer, and M is a Hilbert space of observation variables. The quadratic cost function associated with the control system (3.2) is given by

$$J(v) = \|Cy(v) - z_d\|_M^2 + (Rv, v)_{\mathscr{U}} \qquad \text{for } v \in \mathscr{U}, \tag{3.4}$$

where $z_d \in M$ is a desired value of $z(v)$ and $R \in \mathscr{L}(\mathscr{U}, \mathscr{U})$ is symmetric and positive, i.e.,

$$(Rv, v)_{\mathscr{U}} = (v, Rv)_{\mathscr{U}} \geq d\|v\|_{\mathscr{U}}^2 \tag{3.5}$$

for some $d > 0$. Let $\mathscr{U}_{ad}$ be a closed convex subset of $\mathscr{U}$, which is called the admissible set. The quadratic cost optimal control problem is usually derived into two problems:

i) Find an element $u \in \mathscr{U}_{ad}$ such that

$$\inf_{v \in \mathscr{U}_{ad}} J(v) = J(u). \tag{3.6}$$

ii) Give a characterization of such a u.

We shall call such the u the optimal control for the cost problem (3.6) with (3.4) subject to (3.2). Since the control system (3.2) includes a nonlinear term, it is not easy task to solve the problem i), the existence of optimal control u for (3.6) with (3.4), and also the problem ii), the necessary condition for optimal control u. For the problem i) we have no general answer under any natural condition. In solving ii) it is necessary to write down formally the optimality condition

$$DJ(u)(v-u) \geq 0 \qquad \text{for all } v \in \mathcal{U}_{ad} \tag{3.7}$$

and to analyze this in view of the proper adjoint state system, where $DJ(u)$ denotes the Gâteaux derivative of $J(v)$ at $v = u$.

For i) as mentioned in Introduction we require giving additional conditions on the underlying spaces or the admissible sets. Here we propose two sufficient cases which affirm the existence of optimal controls as follows:
(B1) $\mathcal{U}_{ad}$ is compact in $\mathcal{U}$.
(B2) The embedding $V \hookrightarrow V_2$ is compact.
The case (B1) is intended for the finite dimensional control space $\mathcal{U}_{ad}$. For the case (B2), we utilize the Aubin-Lions-Temam's compact imbedding theorem (cf. Temam [15; p. 274]).

Proposition 3.1. *Let X_0, X and X_1 be Banach spaces such that the embedings $X_0 \hookrightarrow X \hookrightarrow X_1$ are continuous and dense. Assume that X_0 and X_1 are reflexive spaces and that $\alpha_0, \alpha_1 > 1$. Let $W^{\alpha_0,\alpha_1}(0,T)$ be the space defined by*

$$W^{\alpha_0,\alpha_1}(0,T) = \{y \mid y \in L^{\alpha_0}(0,T;X_0), y' \in L^{\alpha_1}(0,T;X_1)\}.$$

If the embeding $X_0 \hookrightarrow X$ is compact, then any bounded set of $W^{\alpha_0,\alpha_1}(0,T)$ is precompact in $L^{\alpha_0}(0,T;X)$.

By applying Proposition 3.1 in the special case where $X_0 = V$, $X = X_1 = V_2$, $\alpha_0 = \alpha_1 = 2$, we can obtain the following Corollary.

Corollary 3.1. *Assume that the embeding $V \hookrightarrow V_2$ is compact. If a sequence $\{y_n\}$ is bounded in $W^{2,2}(0,T)$, then there is a subsequence $\{y_{n_k}\} \subset \{y_n\}$ and a $z \in L^2(0,T;V_2)$ such that y_{n_k} converges to z strongly in $L^2(0,T;V_2)$.*

Theorem 3.1. *Assume that either* (B1) *or* (B2) *is satisfied. Then there exists at least one optimal control u for the cost* (3.4) *subject to the system* (3.2).

Proof. For the case (B1) this theorem is readily shown if we refer the proof of this theorem and the fact that the map $y(v) : \mathcal{U} \to L^2(0,T;V)$ is strongly continuous which follows from the proof of Lemma 3.1 given below. Hence we prove this theorem for the case (B2).

Let $\inf_{v \in \mathcal{U}_{ad}} J(v) = J$. Then we can extract a sequence $\{v_n\} \subset \mathcal{U}_{ad}$ satisfying $J(v_n) \to J$. Hence $\{J(v_n)\}$ is a bounded sequence in $\boldsymbol{R}^+$. It follows from (3.4) and (3.5) that

$$K_0 \geq J(v_n) \geq (Rv_n, v_n)_{\mathscr{U}} \geq d\|v_n\|_{\mathscr{U}}^2, \tag{3.8}$$

where K_0 is an upper bound of $\{J(v_n)\}$. These inequalities means that $\{v_n\}$ remains in a bounded sequence in $\mathscr{U}$. Since $\mathscr{U}$ is a Hilbert space, there are u and a subsequence, again denoted by $\{v_n\}$, such that $v_n \to u \in \mathscr{U}$ weakly. And $u \in \mathscr{U}_{ad}$ on account of $\mathscr{U}_{ad}$ being convex and closed. Each state $y_n = y(v_n) \in W(0,T)$ corresponding to v_n is the weak solution of

$$\begin{cases} y_n'' + A_2(t)y_n' + A_1(t)y_n = f(t, y_n) + Bv_n \quad \text{in } (0,T), \\ y_n(0) = y_0, \quad y_n'(0) = y_1. \end{cases} \tag{3.9}$$

By (3.8) the term Bv_n is estimated as

$$\begin{aligned} \|Bv_n\|_{L^2(0,T;V_2')} &\leq \|B\|_{\mathscr{L}(\mathscr{U},L^2(0,T;V_2'))}\|v_n\|_{\mathscr{U}} \\ &\leq \|B\|_{\mathscr{L}(\mathscr{U},L^2(0,T;V_2'))}\sqrt{K_0 d^{-1}} \equiv K_1, \qquad n = 1,2,\ldots. \end{aligned} \tag{3.10}$$

Now if we take $f(t, y_n) \equiv f(t, y_n) + (Bv_n)(t)$ in (2.4), then the function γ in (A3) is replaced by $\gamma(t) + \|Bv_n(t)\|_{V_2'}$ and the function β in (A2) is not changed. By (3.10) $\gamma(t) + \|Bv_n(t)\|_{V_2'}$ is estimated as

$$\int_0^T \{\gamma(t) + \|Bv_n(t)\|_{V_2'}\}^2 dt \leq 2\|\gamma\|_{L^2(0,T;\boldsymbol{R}^+)}^2 + 2K_1.$$

Hence it follows from (2.5) that

$$\begin{aligned} &\|y_n(t)\|_V^2 + |y_n'(t)|_H^2 + \int_0^t \|y_n'(\sigma)\|_{V_2}^2 d\sigma \\ &\qquad \leq c(\|y_0\|_V^2 + |y_1|_H^2 + 2\|\gamma\|_{L^2(0,T;\boldsymbol{R}^+)}^2 + 2K_1). \end{aligned}$$

This inequality means that

$$\{y_n\} \text{ is bounded in } L^\infty(0,T;V) \qquad \text{and}$$

$$\{y_n'\} \text{ is bounded in } L^2(0,T;V_2) \cap L^\infty(0,T;H).$$

Also the boundedness of $\{y_n''\} \subset L^2(0,T;V')$ follows from the above boundednesses, (3.10) and the equality $y_n'' = -A_2(t)y_n' - A_1(t)y_n + f(t, y_n) + Bv_n$. Therefore, by the extraction theorem of Rellich, we can find a subsequence, say $\{y_n\}$, of $\{y_n\}$ and find $z \in L^\infty(0,T;V) \subset L^2(0,T;V)$, $z' \in L^2(0,T;V_2) \cap L^\infty(0,T;H)$, $z'' \in L^2(0,T;V')$ such that

(3.11) $y_n \to z$ weakly star in $L^\infty(0,T;V)$ and weakly in $L^2(0,T;V)$, (3.12) $$y_n' \to z' \text{ weakly in } L^2(0,T;V_2) \quad \text{and}$$

(3.13) $$y_n'' \to z'' \text{ weakly in } L^2(0,T;V').$$

It is verified by the standard manupulation as in Dautray and Lions [8] that the limit z satisfies the initial conditions $z(0) = y_0$ and $z'(0) = y_1$.

Now we use the condition (B2). By Corollary 3.1, there is a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that

$$y_{n_k} \to z \text{ strongly in } L^2(0,T;V_2).$$

Then we can choose a subsequence of $\{y_{n_k}\}$, we denote again by $\{y_{n_k}\}$, such that

(3.14) $$y_{n_k}(t) \to z(t) \text{ strongly in } V_2 \text{ for a.e. } t \in [0,T].$$

So that by the assumption on (A2), we have by (3.14) and the Lebesgue dominated convergence theorem that

(3.15) $$\int_0^T \|f(t,y_{n_k}) - f(t,z)\|_{V_2'}^2 dt \le \int_0^T \beta^2(t)\|y_{n_k}(t) - z(t)\|_{V_2}^2 dt \to 0.$$

Here we note that the integrands in (3.15) are bounded by a L^1-integrable function by $\beta \in L^2(0,T;\boldsymbol{R}^+)$ and the boundedness of $\{y_{n_k}, z\}$ in $L^\infty(0,T;V)$.

Hence replacing y_n with y_{n_k} and taking $k \to \infty$ in (3.9) by using (3.11), (3.12), (3.13), (3.15) and continuity of $A_i(t), B$, the limit z satisfies

(3.16) $$\begin{cases} z'' + A_2(t)z' + A_1(t)z = f(t,z) + Bu \quad \text{in } (0,T), \\ z(0) = y_0, \quad z'(0) = y_1 \end{cases}$$

in the weak sense. Also since the equation (3.16) has a unique weak solution $z \in W(0,T)$ by Theorem 2.1, we conclude that $z = y(u)$ in $W(0,T)$ by the uniqueness of solutions, which implies $y(v_n) \to y(u)$ weakly in $W(0,T)$. Since C is continuous on $W(0,T)$ and $\|\cdot\|_M$ is continuous, it follows that

$$\|Cy(u) - z_d\|_M \le \liminf_{n\to\infty} \|Cy(v_n) - z_d\|_M.$$

It is also clear from $\liminf_{k\to\infty}\|R^{1/2}v_n\|_{\mathscr{U}} \ge \|R^{1/2}v\|_{\mathscr{U}}$ that

$$\liminf_{k\to\infty}(Rv_n, v_n)_{\mathscr{U}} \ge (Ru,u)_{\mathscr{U}}.$$

Hence

$$J = \liminf_{n\to\infty} J(v_n) \ge J(u).$$

But since $J(u) \ge J$ by definition, we conclude that $J(u) = \inf_{v\in\mathscr{U}_{ad}} J(v)$. This proves Theorem 3.1.

For the problem ii) we need to solve the question that under what conditions on $f(t, y)$ is the map $v \to y(v)$ of $\mathscr{U}$ into $W(0, T)$ Gâteaux differentiable at u. In order to give an answer to the question, we need the following lemma.

Lemma 3.1. *Let $w \in \mathscr{U}$ be arbitrarily fixed. Then*

$$\lim_{\lambda \to 0} y(u + \lambda w) = y(u) \qquad \textit{in } C([0, T]; V).$$

In particular, $\lim_{\lambda \to 0} y(u + \lambda w) = y(u)$ *in* $L^2(0, T; V)$.

Proof. Let $\lambda \in (0, 1)$ and let $y(u + \lambda w)$ and $y(u)$ be the weak solutions of (3.2) corresponding to $u + \lambda w$ and u, respectively. If we set $y_\lambda = y(u + \lambda w) - y(u)$ for each λ, then it satisfies

$$(3.17) \qquad \begin{cases} y_\lambda'' + A_2(t) y_\lambda' + A_1(t) y_\lambda \\ \qquad = f(t, y(u + \lambda w)) - f(t, y(u)) + \lambda B w \quad \text{in } (0, T), \\ y_\lambda(0) = 0 \in V, \quad y_\lambda'(0) = 0 \in H. \end{cases}$$

The equation (3.17) admits a unique weak solution, because $f(t, \xi) \equiv f(t, \xi + y(u; t)) - f(t, y(u; t)) + \lambda (Bw)(t)$ satisfies (A1)–(A3) with same β and $\gamma(t) = \|\lambda (Bw)(t)\|_{V_2'}$. Hence, applying the energy inequality (2.5) to (3.17) we have

$$(3.18) \qquad \|y_\lambda(t)\|_V^2 + |y_\lambda'(t)|_H^2 + \int_0^t \|y_\lambda'(\sigma)\|_{V_2}^2 d\sigma \le c\lambda^2 \|Bw\|_{L^2(0, T; V_2')}^2.$$

Therefore $y_\lambda \to 0$ in $C([0, T]; V)$ as $\lambda \to 0$. In particular, $y_\lambda \to 0$ in $L^2(0, T; V)$. This completes the proof.

Let us denote by $C^1(V_2, V_2')$ the set of continuously Fréchet differentiable functions. Let $t \in [0, T]$ and assume $f(t, y) \in C^1(V_2, V_2')$ for a.e. t. For the fixed t we denote by $f_y(t, \xi)$ the Fréchet derivative of $f(t, y)$ with respect to y at ξ. If $f(t, y) \in C^1(V_2, V_2')$ for fixed t, then we have the useful equality

$$(3.19) \qquad f(t, y) - f(t, z) = \int_0^1 f_y(t, \theta y + (1 - \theta) z) d\theta (y - z).$$

The solution map $v \to y(v)$ of $\mathscr{U}$ into $W(0, T)$ is said to be Gâteaux differentiable at $v = u$ if for any $w \in \mathscr{U}$ there exists a $Dy(u) \in \mathscr{L}(\mathscr{U}, W(0, T))$ such that

$$\left\langle \frac{1}{\lambda}(y(u + \lambda w) - y(u)) - Dy(u) w, \phi \right\rangle_{W(0, T), W(0, T)'} \to 0 \qquad \text{as } \lambda \to 0$$

for each $\phi \in W(0, T)'$. The operator $Dy(u)$ is called the Gâteaux derivative of

$y(u)$ at $v = u$ and the function $Dy(u)w \in W(0,T)$ is called the Gâteaux derivative in the direction $w \in \mathscr{U}$.

Now, in order to obtain the Gâteaux differentiability of the solution map $v \to y(v)$, we impose a further assumption to the nonlinear term $f(t,y)$.

$$\text{(A4)} \qquad \begin{cases} f(t,y) \in C^1(V_2, V_2') \quad \text{a.e. } t \in [0,T] \\ \text{and there is } \beta_1 \in L^2(0,T;\boldsymbol{R}^+) \text{ such that} \\ \|f_y(t,y)\|_{\mathscr{L}(V_2,V_2')} \le \beta_1(t)(\|y\|_{V_2} + 1) \quad \text{a.e. } t \in [0,T]. \end{cases}$$

Theorem 3.2. *Assume that* (A4) *holds. Then the map $v \to y(v)$ of $\mathscr{U}$ into $W(0,T)$ is weakly Gâteaux differentiable at $v = u$ and such the Gâteaux derivative of $y(v)$ at $v = u$ in the direction $v - u \in \mathscr{U}$, say $z = Dy(u)(v-u)$, is a unique weak solution satisfying the following equation*

$$\text{(3.20)} \qquad \begin{cases} z'' + A_2(t)z' + A_1(t)z = f_y(t, y(u;t))z + B(v-u) \quad \text{in } (0,T), \\ z(0) = z'(0) = 0. \end{cases}$$

Proof. Let $\lambda \in (0,1)$, and let $y(u + \lambda(v-u))$ and $y(u)$ be the weak solutions of (3.2) corresponding to $u + \lambda(v-u)$ and u, respectively. We set $z_\lambda = \lambda^{-1}(y(u+\lambda(u-v)) - y(u))$. Then z_λ is a unique weak solution of

$$\text{(3.21)} \qquad \begin{cases} z_\lambda'' + A_2(t)z_\lambda' + A_1(t)z_\lambda \\ \qquad = \dfrac{1}{\lambda}\{f(t, y(u+\lambda(v-u))) - f(t, y(u))\} + B(v-u) \quad \text{in } (0,T), \\ z_\lambda(0) = 0, \quad z_\lambda'(0) = 0. \end{cases}$$

Let $w = v - u$. By the energy equality (2.6), z_λ satisfies

$$\text{(3.22)} \quad a_1(t; z_\lambda(t), z_\lambda(t)) + |z_\lambda'(t)|_H^2 + 2\int_0^t a_2(\sigma; z_\lambda'(\sigma), z_\lambda'(\sigma))d\sigma$$
$$= \int_0^t a_1'(\sigma; z_\lambda(\sigma), z_\lambda(\sigma))d\sigma + 2\int_0^t \langle Bw(\sigma), z_\lambda'(\sigma)\rangle_{V_2',V_2} d\sigma$$
$$+ 2\int_0^t \langle \lambda^{-1}[f(\sigma, y(u+\lambda(v-u);\sigma)) - f(\sigma, y(u;\sigma))], z_\lambda'(\sigma)\rangle_{V_2',V_2} d\sigma.$$

Note that the last term of (3.22) can be estimated as

$$\text{(3.23)} \qquad \left| 2\int_0^t \langle \lambda^{-1}[f(\sigma, y(u+\lambda(v-u);\sigma)) - f(\sigma, y(u;\sigma))], z_\lambda'(\sigma)\rangle_{V_2',V_2} d\sigma \right|$$
$$\le \frac{1}{\varepsilon}\int_0^t \beta^2(\sigma)\|z_\lambda(\sigma)\|_{V_2}^2 d\sigma + \varepsilon\int_0^t \|z_\lambda'\|_{V_2}^2 d\sigma$$

for any $\varepsilon > 0$. Then, by taking $\varepsilon > 0$ sufficiently small, as well as in the proof of Theorem 2.1 in [9], we can verify that (3.22), (3.23) imply

(3.24) $$\|z_\lambda(t)\|_V^2 \le \|z_\lambda(t)\|_V^2 + |z_\lambda'(t)|_H^2 + \int_0^t \|z_\lambda'(\sigma)\|_{V_2}^2 d\sigma$$
$$\le K\int_0^t \beta(\sigma)^2 \|z_\lambda(\sigma)\|_V^2 d\sigma + K\|Bw\|^2_{L^2(0,T;V_2')},$$

for some $K > 0$. Hence by applying Gronwall's inequality to (3.24), we have

(3.25) $$\|z_\lambda(t)\|_V^2 \le K\|Bw\|^2_{L^2(0,T;V_2')} \exp(K\|\beta\|^2_{L^2(0,T;\boldsymbol{R}^+)}).$$

It follows from substituting (3.25) into (3.24) that

(3.26) $$\|z_\lambda(t)\|_V^2 + |z_\lambda'(t)|_H^2 + \int_0^t \|z_\lambda'(\sigma)\|_{V_2}^2 d\sigma$$
$$\le K[1 + K\|\beta\|^2_{L^2(0,T;\boldsymbol{R}^+)} \exp(\|\beta\|^2_{L^2(0,T;\boldsymbol{R}^+)})]\|Bw\|^2_{L^2(0,T;V_2')}.$$

It is easily verified that the above inequality yields the boundedness of $\{z_\lambda''\}$ in $L^2(0,T;V')$. Therefore there exists a $z \in W(0,T)$ and a sequence $\{\lambda_k\} \subset (0,1)$ tending to 0 such that

(3.27) $$\begin{cases} z_{\lambda_k} \to z \text{ weakly in } L^2(0,T;V) \text{ as } k \to \infty, \\ z_{\lambda_k} \to z \text{ weakly star in } L^\infty(0,T;V) \text{ as } k \to \infty, \\ z_{\lambda_k}' \to z' \text{ weakly in } L^2(0,T;V_2) \text{ as } k \to \infty, \\ z_{\lambda_k}'' \to z'' \text{ weakly in } L^2(0,T;V') \text{ as } k \to \infty, \\ z(0) = 0, \quad z'(0) = 0. \end{cases}$$

Let us prove that

(3.28) $$\frac{1}{\lambda_k}\{f(t, y(u + \lambda_k(v-u))) - f(t, y(u))\} \to f_y(t, y(u))z$$

weakly in $L^2(0,T;V_2')$ as $k \to \infty$. It follows from $f(t,y) \in C^1(V_2, V_2')$ for a.e. $t \in [0,T]$ that

$$\frac{1}{\lambda_k}\{f(t, y(u + \lambda_k(v-u))) - f(t, y(u))\}$$
$$= \int_0^1 f_y(t, \theta y(u + \lambda_k(v-u)) + (1-\theta)y(u))d\theta z_{\lambda_k} \qquad \text{a.e. } t \in [0,T].$$

Since $f_y(t,y)$ is continuous on y in $\mathscr{L}(V_2, V_2')$ and $y(u + \lambda_k(v-u)) \to y(u)$ strongly in $C(0,T;V)$ as $k \to \infty$ by Lemma 3.1, we have

(3.29)
$$f_y(t, \theta y(u + \lambda_k(v-u)) + (1-\theta)y(u)) \to f_y(t, y(u)) \qquad \text{a.e. } t \text{ in } \mathscr{L}(V_2, V_2')$$

uniformly in $\theta \in [0,1]$. Again by Lemma 3.1, there is a k_0 independent of t such that

$$\|y(u+\lambda_k(v-u);t)\|_{V_2} \le (\|y(u;t)\|_{V_2}+1), \qquad \forall k \ge k_0. \tag{3.30}$$

It is verified by using (A4) and (3.30) that

$$\begin{aligned} &\|f_y(t,\theta y(u+\lambda_k(v-u);t)+(1-\theta)y(u;t))\|_{\mathscr{L}(V_2,V_2')} \\ &\quad \le \beta_1(t)(\|y(u;t)\|_{V_2}+2) \\ &\quad \le \beta_1(t)(\|y(u)\|_{L^\infty(0,T;V_2)}+2) < +\infty, \qquad \forall \theta \in [0,1],\ \forall k \ge k_0. \end{aligned} \tag{3.31}$$

Since $\beta_1 \in L^2(0,T;\boldsymbol{R}^+) \subset L^1(0,T;\boldsymbol{R}^+)$, by the Lebesgue dominated convergence theorem, we have from (3.29) the strong convergence in $\mathscr{L}(V_2,V_2')$, i.e.,

$$\int_0^1 f_y(t,\theta y(u+\lambda_k(v-u))+(1-\theta)y(u))d\theta \to f_y(t,y(u)) \quad \text{a.e. } t \in [0,T]. \tag{3.32}$$

For simplicity of calculations we set

$$F_k(t) = \int_0^1 f_y(t,\theta y(u+\lambda_k(v-u))+(1-\theta)y(u))d\theta.$$

Then for each $\phi \in L^2(0,T;V_2)$, we have

$$\begin{aligned} &\left|\int_0^T \langle (F_k(t)-f_y(t,y(u)))z_{\lambda_k}(t),\phi(t)\rangle_{V_2',V_2}dt\right| \\ &\quad \le \int_0^T \|F_k(t)-f_y(t,y(u))\|_{\mathscr{L}(V_2,V_2')}\|z_{\lambda_k}(t)\|_{V_2}\|\phi(t)\|_{V_2}dt \\ &\quad \le \|z_{\lambda_k}\|_{L^\infty(0,T;V_2)}\int_0^T \|F_k(t)-f_y(t,y(u))\|_{\mathscr{L}(V_2,V_2')}\|\phi(t)\|_{V_2}dt. \end{aligned} \tag{3.33}$$

Here we note that the integrand in the last term of (3.33) is bounded by an L^1-integrable function due to (3.31). By using the Lebesgue dominated convergence theorem thanks to (3.32) once again, we see that the last term of (3.33) converges to 0. On the other hand, by (3.27) and (A4) it is evident that

$$\int_0^T \langle f_y(t,y(u))(z(t)-z_{\lambda_k}(t)),\phi(t)\rangle_{V_2',V_2}dt \to 0. \tag{3.34}$$

Hence we deduce from (3.33) and (3.34) the desired convergence (3.28).

Now let us take $k \to \infty$ in (3.21) with $\lambda = \lambda_k$ by using (3.27) and (3.28).

Then the element $z \in W(0,T)$ satisfies the equation (3.20) in the weak sense. If we take $f(t,\xi) \equiv f_y(t, y(u;t))\xi + B(v-u)(t) \in L^2(0,T;V_2')$ in Theorem 2.1, then this equation has a unique weak solution $z \in W(0,T)$. In fact, by (A4) the term $f_y(t, y(u;t))\xi$ satisfies

$$\begin{aligned}\|f_y(t, y(u;t))\xi\|_{V_2'} &\le \|f_y(t, y(u;t))\|_{\mathscr{L}(V_2,V_2')}\|\xi\|_{V_2}\\ &\le \beta_1(t)(\|y(u)\|_{L^\infty(0,T;V_2)}+1)\|\xi\|_{V_2},\end{aligned}$$

so that (A2) is satisfied. This means that z is a weak solution of (3.20). Hence by (3.27), (3.27) and (3.20) we see that $z_\lambda \to z = Dy(u)(v-u)$ weakly in $W(0,T)$ as $\lambda \to 0$. This completes the proof.

Theorem 3.2 means that the cost $J(v)$ is Gâteaux differentiable at u in the direction $v-u$ and the optimality condition (3.4) is rewritten by

$$\begin{aligned}(3.35)\qquad &(Cy(u)-z_d, C(Dy(u)(v-u)))_M + (Ru, v-u)_{\mathscr{U}}\\ &= \langle C^*\Lambda_M(Cy(u)-z_d), Dy(u)(v-u)\rangle_{W(0,T)', W(0,T)}\\ &\quad + (Ru, v-u)_{\mathscr{U}} \ge 0, \qquad \forall v \in \mathscr{U}_{ad},\end{aligned}$$

where Λ_M is the canonical isomorphism M onto M'.

For each $t \in [0,T]$, let us define the adjoint operator $f_y(t,y)^*$ of $f_y(t,y)$ as follows:

$$(3.36)\qquad \langle f_y(t,y)\phi, \varphi\rangle_{V_2',V_2} = \langle \phi, f_y^*(t,y)\varphi\rangle_{V_2,V_2'}$$

for all $\phi, \varphi \in V_2$. Then we have $f_y(t, y(u;t))^* \in L^2(0,T;\mathscr{L}(V_2,V_2'))$ provided that (A4) hold.

In order to avoid the complexity of setting up observation spaces, we consider the following two usual types of distributive and terminal value observations (see other types of observations see Lions [4]). That is, the following two cases:

1. We take $C_1 \in \mathscr{L}(L^2(0,T;V), M)$ and observe $z(v) = C_1 y(v)$.
2. We take $C_2 \in \mathscr{L}(V, M)$ and observe $z(v) = C_2 y(T;v)$.

Since $y \in W(0,T) \cap C([0,T];V)$ by Theorem 2.1, the above observations are meaningful.

Now we assume that (A4) holds in each subsections. Further we suppose the following differentiability of $a_2(t;\phi,\varphi)$ in Subsections 3.1 and 3.2:

The function $t \to a_2(t;\phi,\varphi)$ is continuously differentiable in $[0,T]$ and there exists $c_{22} > 0$ such that

$$(3.37)\quad |a_2'(t;\phi,\varphi)| \le c_{22}\|\phi\|_{V_2}\|\varphi\|_{V_2} \qquad \text{for all } \phi,\psi \in V_2 \text{ and } t \in [0,T].$$

By the above assumption we can define the operator $A_2'(t)$ for $t \in [0,T]$ which belongs to $L^\infty(0,T;\mathscr{L}(V_2,V_2'))$.

3.1. Case of $C_1 \in \mathscr{L}(L^2(0,T;V),M)$

In this case the cost functional (3.4) is expressed by

$$(3.38)\qquad J(v) = \|C_1 y(v) - z_d\|_M^2 + (Rv,v)_{\mathscr{U}}, \qquad \forall v \in \mathscr{U}.$$

Let u be the optimal control subject to (3.2) and (3.38). Then the optimality condition (3.35) is rewritten as

$$(3.39)\quad \int_0^T \langle C_1^*\Lambda_M(C_1 y(u) - z_d)(t), z(t)\rangle_{V',V} dt + (Ru, v-u)_{\mathscr{U}} \ge 0, \qquad \forall v \in \mathscr{U}_{ad},$$

where z is the weak solution of the equation (3.20). Let us introduce the adjoint state $p(u)$ defined by

$$(3.40)\quad \begin{cases} p''(u) - A_2(t)p'(u) + (A_1(t) - A_2'(t))p(u) \\ \qquad = f_y^*(t, y(u;t))p(u) + C_1^*\Lambda_M(C_1 y(u) - z_d) \quad \text{in } (0,T), \\ p(u;T) = 0, \quad p'(u;T) = 0. \end{cases}$$

By the time reversion $t \to T - t$, it is verified by Theorem 2.1 via (A4) and (3.37) that (3.40) permits a unique solution $p(u)$ in $W(0,T)$ provided that $C_1^*\Lambda_M(C_1 y(u) - z_d) \in L^2(0,T;V_2')$. In fact, let us take for $\xi \in V_2$

$$f(t,\xi) = -A_2'(t)\xi + f_y^*(t, y(u;t))\xi + C^*\Lambda_M(Cy(u) - z_d)(t).$$

Since $\sup_{t\in[0,T]}\|A_2'(t)\|_{\mathscr{L}(V_2,V_2')} \le c_{22} < \infty$ by (3.37) and

$$\|f_y(\cdot\,, y(u;\cdot))^*\|_{L^2(0,T;\mathscr{L}(V_2,V_2'))} < \infty$$

by (A4), $f(t,\xi)$ satisfies the conditions (A1)–(A3). But, $C_1^*\Lambda_M(C_1 y(u) - z_d) \notin L^2(0,T;V_2')$ in general, and we cannot apply Theorem 2.1 directly. This difficulty is overcome by the method of transpositions, which gives us a 'weak' solution to (3.40) under present conditions, in the next section. Thus in this section we confine ourselves to the formal calculations of optimality conditions and we will justify these under restricted conditions. In what follows $(\cdot\,,\cdot)$ denotes the appropriate inner products or duality pairings (which may have no meanings). Now we proceed the formal calculations. Multiply both sides of the equation in (3.40) by z and integrate them on $[0,T]$. Then we have

$$(3.41)\quad \int_0^T (p''(u;t), z(t))dt - \int_0^T (A_2(t)p'(u;t), z(t))dt + \int_0^T (A_1(t)p(u;t), z(t))dt$$
$$- \int_0^T (A_2'(t)p(u;t), z(t))dt - \int_0^T (f_y^*(t, y(u;t))p(u;t), z(t))dt$$
$$= \int_0^T (C_1^* \Lambda_M (C_1 y(u) - z_d)(t), z(t))dt$$
$$= (C_1 y(u) - z_d, C_1 z)_M.$$

If we integrate the left hand side of (3.41) by parts by using the symmetricity of $A_1(t)$ and $A_2(t)$ and the fact that $p(u;T) = p'(u;T) = z(0) = z'(0) = 0$, it is calculated formally as

$$(3.42)\quad \int_0^T (p(u;t), z''(t))dt - \int_0^T (p(u;t), A_2(t)z'(t))dt$$
$$+ \int_0^T (p(u;t), A_1(t)z(t))dt - \int_0^T (f_y^*(t, y(u;t))p(u;t), z(t))dt$$
$$= \int_0^T \left(p(u;t), \left(\frac{d^2}{dt^2} + A_2(t)\frac{d}{dt} + A_1(t) - f_y(t; y(u;t)) \right) z(t) \right) dt$$
$$= \int_0^T (p(u;t), B(v-u)(t))dt$$
$$= \langle B^* p(u), v - u \rangle_{\mathscr{U}', \mathscr{U}} = (\Lambda_{\mathscr{U}}^{-1} B^* p(u), v - u)_{\mathscr{U}}.$$

Thus, by (3.41) and (3.42), the condition (3.39) is rewritten formally as

$$(3.43)\qquad (\Lambda_{\mathscr{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}.$$

Here in (3.43) we do not know whether B^* can apply to $p(u)$, i.e., $p(u) \in L^2(0,T;V_2)$ or not. The above calculations suggest us that if

$$(3.44)\qquad C_1 \in \mathscr{L}(L^2(0,T;V_2), M),$$

then by $C_1^* \Lambda_M (Cy(u) - z_d) \in L^2(0,T;V_2')$, we can apply Theorem 2.1 and all calculations above are meaningful for the unique weak solution $p(u)$ of (3.40). Hence we have the following theorem.

Theorem 3.3. *Assume that* (3.44) *holds. The optimal control u is then characterized by the following system of equations and inequality*:

$$\begin{cases} y''(u) + A_2(t)y'(u) + A_1(t)y(u) = f(t, y(u)) + Bu, \\ y(u;0) = y_0 \in V, \quad y'(u;0) = y_1 \in H, \end{cases}$$

$$\begin{cases} p''(u) - A_2(t)p'(u) + (A_1(t) - A_2'(t))p(u) \\ \qquad = f_y(t, y(u))^* p(u) + C_1^* \Lambda_M (C_1 y(u) - z_d), \\ p(u;T) = 0, \quad p'(u;T) = 0, \end{cases}$$

$$(\Lambda_{\mathscr{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}$$

with $y(u), p(u) \in W(0,T)$.

3.2. Case of $C_2 \in \mathscr{L}(V, M)$

If we take $z(v) = C_2 y(T; v)$ as the terminal value observation, then the cost function is expressed as

$$J(v) = \|C_2 y(v;T) - z_d\|_M^2 + (Rv, v)_{\mathscr{U}}, \qquad \forall v \in \mathscr{U}. \tag{3.45}$$

In this occasion, the optimal control u subject to (3.2) and (3.45) is characterized by

$$\langle C_2^* \Lambda_M (C_2 y(u;T) - z_d), z(T) \rangle_{V',V} + (Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}, \tag{3.46}$$

where z is the solution of (3.20). According to the optimality condition (3.46), we introduce the adjoint system defined by

$$\begin{cases} p''(u) - A_2(t)p'(u) + (A_1(t) - A_2'(t))p(u) - f_y^*(t, y(u;t))p(u) = 0, \\ p(u;T) = 0, \quad p'(u;T) = C_2^* \Lambda_M (C_2 y(u;T) - z_d). \end{cases} \tag{3.47}$$

Since $C_2^* \Lambda (C_2 y(u;T) - z_d) \notin H$ by assumption, we cannot apply Theorem 2.1 to obtain existence and uniqueness of solutions for (3.47). Hence we further assume

$$C_2 \in \mathscr{L}(H, M), \tag{3.48}$$

so that $C_2^* \Lambda_M (C_2 y(u;T) - z_d) \in H$. If we take $f(t, \xi) = A_2'(t)\xi + f_y^*(t, y(u;t))\xi$, then $f(t, \xi)$ satisfies the assumptions (A1)–(A3). Hence, the system (3.47) is well-posed and permits a unique solution $p(u)$ in $W(0,T)$ if one consider the change of time variables $t \to T - t$. This completes the following theorem.

Theorem 3.4. *Assume that* (3.48) *holds. Then the optimal control u is characterized by the following system of equations and inequality*:

$$\begin{cases} y''(u) + A_2(t)y'(u) + A_1(t)y(u) = Bu + f(t, y(u)), \\ y(u;0) = y_0 \in V, \quad y'(u;0) = y_1 \in H, \end{cases}$$

$$\begin{cases} p''(u) - A_2(t)p'(u) + (A_1(t) - A_2'(t))p(u) - f_y(t, y(u))^* p(u) = 0, \\ p(u;T) = 0, \quad p'(u;T) = C_2^* \Lambda (C_2 y(u;T) - z_d), \end{cases}$$

$$(-\Lambda_{\mathscr{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}$$

with $y(u), p(u) \in W(0,T)$.

Proof. Multiplying both sides of the equation in (3.47) by z and integrating it on $(0,T)$, we have

$$(3.49)\qquad \int_0^T \langle p''(u;t), z(t)\rangle_{V',V} - \int_0^T \left\langle \frac{d}{dt}(A_2(t)p'(u;t)), z(t)\right\rangle_{V_2',V_2}$$
$$+ \int_0^T \langle A_1(t)p(u;t), z(t)\rangle_{V',V}dt$$
$$- \int_0^T \langle f_y^*(t, y(u;t))p(u;t), z(t)\rangle_{V_2',V_2}dt$$
$$= (p'(u,t), z(t))_H|_0^T - \int_0^T (p'(u;t), z'(t))_H dt$$
$$+ \int_0^T \langle A_2(t)p(u;t), z'(t)\rangle_{V_2',V_2}dt + \int_0^T \langle p(u;t), A_1(t)z(t)\rangle_{V,V'}dt$$
$$- \int_0^T \langle p(u;t), f_y(t, y(u;t))z(t)\rangle_{V_2',V_2}dt$$
$$= (p'(u;T), z(T))_H$$
$$+ \int_0^T \left\langle p(u;t), \left(\frac{d^2}{dt^2} + A_2(t)\frac{d}{dt} + A_1(t) - f_y(t, y(u;t))\right)z(t)\right\rangle_{V_2',V_2} dt$$
$$= \int_0^T \langle p(u;t), B(v-u)(t)\rangle_{V_2,V_2'}dt + (p'(u;T), z(T))_H = 0.$$

Here we have used integration by parts and the fact that $\langle\phi,\psi\rangle_{V',V} = \langle\phi,\psi\rangle_{V_2',V_2} = (\phi,\psi)_H$ for ϕ,ψ in respective spaces. Since $p'(u;T) = C_2^*\Lambda_M\cdot(C_2y(u;T) - z_d)$, by (3.49) the optimal condition (3.46) can be written as

$$(3.50)\qquad (-\Lambda_{\mathscr{U}}^{-1}B^*p(u) + Ru, v-u)_{\mathscr{U}} \ge 0, \qquad \forall v\in\mathscr{U}_{ad}.$$

This completes the theorem.

4. Transposition and necessary conditions

Throughout this section we assume (A4) holds. Let $g\in L^2(0,T;V_2')$. Then we have a unique weak solution $\psi\in W(0,T)$ of the following equation

$$(4.1)\qquad \begin{cases} \psi'' + A_2(t)\psi' + A_1(t)\psi = f_y(t, y(u;t))\psi + g \quad \text{in } (0,T),\\ \psi(0) = 0, \quad \psi'(0) = 0,\end{cases}$$

because $f(t,\xi) = f_y(t, y(u;t))\xi + g(t)$ satisfies the assumptions (A1)–(A3) by (A4). Let us define the space

$$X \equiv \{\psi \mid \psi \text{ satisfies (4.1) with } g \in L^2(0,T;V_2')\}.$$

It is seen in Theorem 2.1 that $X \subset W(0,T) \cap C([0,T];V) \cap C^1([0,T];H)$. We give a inner product $(\cdot,\cdot)_X$ on X by $(\psi_1,\psi_2)_X = (g_1,g_2)_{L^2(0,T;V_2')}$, where ψ_1,ψ_2 are the weak solutions of (4.1) for given $g = g_1, g_2 \in L^2(0,T;V_2')$, respectively. We see easily that $(X,(\cdot,\cdot)_X)$ is a Hilbert space. And, we can see that the map $\psi \to \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t,y(u;t))\psi$ of X onto $L^2(0,T;V_2')$ is an isomorphism. Hence for each continuous linear functional $L : X \to \boldsymbol{R}$, there exists uniquely a $p = p_L \in L^2(0,T;V_2)$ such that

$$\int_0^T \langle p(t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t,y(u;t))\psi(t)\rangle_{V_2,V_2'}dt = L(\psi), \qquad \forall \psi \in X. \tag{4.2}$$

For $g \in L^1(0,T;V')$, $p_0 \in H$ and $p_1 \in V'$, let us define the functional $L = L(g,p_0,p_1)$ by

$$L(\psi) = \int_0^T \langle g(t),\psi(t)\rangle_{V',V}dt + \langle p_1,\psi(T)\rangle_{V',V} - (p_0,\psi'(T))_H. \tag{4.3}$$

Then this L is linear on X. Next we shall show the boundedness of L. It is easily checked from the fact $\psi \in X \subset C([0,T];V) \cap C^1([0,T];H)$ that

$$|L(\psi)| \le (\|g\|_{L^1(0,T;V')} + \|p_1\|_{V'} + |p_0|_H) \times (\|\psi\|_{C([0,T];V)} + \|\psi(T)\|_V + |\psi'(T)|_H). \tag{4.4}$$

Since $\psi \in X$, by the definition of X there exists a $g(\psi) \in L^2(0,T;V_2')$ such that (4.1) holds with $g = g(\psi)$. If we take $f(t,\psi) = f_y(t,y(u;t))\psi + g(\psi)(t)$, then by (A.4) this f satisfies (A.2) and (A.3) in which $\beta(t)$ and $\gamma(t)$ are given by $\beta(t) = \beta_1(t)(\|y(u)\|_{L^\infty(0,T;V_2)} + 1)$ and $\gamma(t) = \|g(\psi;t)\|_{V_2'}$, respectively. Hence by the energy inequality in Theorem 2.1 we see that ψ satisfies

$$\|\psi(t)\|_V^2 + |\psi'(t)|_H^2 + \int_0^t \|\psi'(s)\|_{V_2}^2 dt \le c(\|g(\psi)\|^2_{L^2(0,T;V_2')}), \tag{4.5}$$

where c is a constant depending only on the above β. Since $\psi \in C([0,T];V)$ and $\psi' \in C([0,T];H)$, we have from (4.5)

$$\|\psi\|_{C([0,T];V)} + \|\psi'\|_{C([0,T];H)} + \|\psi'\|_{L^2(0,T;V_2)} \le K_1\|g(\psi)\|_{L^2(0,T;V_2')}, \tag{4.6}$$

where $K_1 > 0$ is some constant independent of ψ. Consequently, it follows by (4.4) and (4.6) that

$$|L(\psi)| \le K_2\|g(\psi)\|_{L^2(0,T;V_2')} = K_2\|\psi\|_X,$$

which proves that L is a continuous linear functional on X, where K_2 is a positive constant depending on (g, p_0, p_1). In particular, if $p_0 \in V_2$, then we can define another linear functional given by

$$L(\psi) = \int_0^T \langle g(t), \psi(t)\rangle_{V',V} dt + \langle p_1, \psi(T)\rangle_{V',V} + \langle A_2(T)p_0, \psi(T)\rangle_{V_2',V_2} - (p_0, \psi'(T))_H. \tag{4.7}$$

It is easily verified by $|\langle A_2(T)p_0, \psi(T)\rangle_{V_2',V_2}| \le c_{21}k\|p_0\|_{V_2}\|\psi(T)\|_V$ that the linear functional defined by (4.7) is bounded on X, where k is a positive constant such that $\|\xi\|_{V_2} \le k\|\xi\|_V$ for $\xi \in V$. We remark that $L(\psi) + \int_0^T \langle h(t), \psi'(t)\rangle_{V_2',V_2} dt$ is also a bounded linear functional on X for $h \in L^2(0, T; V_2)$.

Theorem 4.1. *For $g \in L^1(0, T; V')$ and $p_0 \in H$, $p_1 \in V'$, there is a unique solution $p \in L^2(0, T; V_2)$ such that*

$$\begin{cases} \int_0^T \langle p(t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t, y(u;t))\psi(t)\rangle_{V_2,V_2'} dt \\ \quad = \int_0^T \langle g(t), \psi(t)\rangle_{V',V} dt + \langle p_1, \psi(T)\rangle_{V',V} - (p_0, \psi'(T))_H, \quad \forall \psi \in X. \end{cases} \tag{4.8}$$

Proof. It is easily followed from applying $L(\psi)$ given by (4.3) to the variational equation (4.1).

Corollary 4.1. *For $g \in L^1(0, T; V')$ and $p_0 \in V_2$, $p_1 \in V'$, there is a unique solution $p \in L^2(0, T; V_2)$ such that*

$$\begin{cases} \int_0^T \langle p(t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t, y(u;t))\psi(t)\rangle_{V_2,V_2'} dt \\ \quad = \int_0^T \langle g(t), \psi(t)\rangle_{V',V} dt + \langle p_1, \psi(T)\rangle_{V',V} \\ \quad + \langle A_2(T)p_0, \psi(T)\rangle_{V_2',V_2} - (p_0, \psi'(T))_H \end{cases} \tag{4.9}$$

for any $\psi \in X$.

Proof. By applying $L(\psi)$ given by (4.7) to the variational equation (4.1), we have this corollary.

Remark 4.1. By formal calculations, the weak solution $p(u)$ of (4.8) is verified to satisfy formally

$$\begin{cases} p''(u) - A_2(t)p'(u) + [A_1(t) - A_2'(t) - f_y^*(t, y(u;t))]p(u) = g \quad \text{in } (0, T), \\ p(T) = p_0, \quad p'(T) = A_2(T)p_0 + p_1, \end{cases}$$

and the weak solution $p(u)$ of (4.9) satisfies formally

$$\begin{cases} p''(u) - A_2(t)p'(u) + [A_1(t) - A_2'(t) - f_y^*(t, y(u;t))]p(u) = g \quad \text{in } (0, T), \\ p(T) = p_0, \quad p'(T) = p_1. \end{cases}$$

We have two kinds of extended adjoint systems (4.8) and (4.9). In fact, the adjoint systems (4.8) in Theorem 4.1 extend those given in Section 3 for the both cases of C_1 and C_2.

4.1. Application to the case of C_1

We assume that $C_1 \in \mathscr{L}(L^2(0, T; V); M)$. We can now formulate the adjoint system to describe the optimality condition by applying Theorem 4.1. Since $C_1^* \Lambda_M(C_1 y(u) - z_d) \in L^2(0, T; V') \subset L^1(0, T; V')$, there exists a $p(u) \in L^2(0, T; V_2)$ satisfying

$$\begin{cases} \int_0^T \langle p(u;t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t, y(u))\psi(t) \rangle_{V_2, V_2'} dt \\ \qquad = \int_0^T \langle C_1^* \Lambda_M(C_1 y(u) - z_d)(t), \psi(t) \rangle_{V', V} dt, \\ \forall \psi \text{ such that} \\ \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t, y(u;t))\psi \in L^2(0, T; V_2'), \\ \psi(0) = 0, \quad \psi'(0) = 0. \end{cases} \tag{4.10}$$

The Gâteaux derivative $\psi = z = Dy(u)(v - u)$ satisfies

$$\left(\frac{d^2}{dt^2} + A_2(t) \frac{d}{dt} + A_1(t) - f_y(t, y(u;t)) \right) \psi = B(v - u) \in L^2(0, T; V_2')$$

and $\psi(0) = \psi'(0) = 0$. Therefore, if we take $\psi = z = Dy(u)(v - u)$ in (4.10), then we have

$$\begin{aligned} &(C_1 y(u) - z_d, C_1 z)_M \\ &\quad = \int_0^T \langle C_1^* \Lambda_M(C_1 y(u) - z_d)(t), z(t) \rangle_{V', V} dt \\ &\quad = \int_0^T \left\langle p(u;t), \left(\frac{d^2}{dt^2} + A_2(t) \frac{d}{dt} + A_1(t) - f_y(t, y(u)) \right) z(t) \right\rangle_{V_2, V_2'} dt \\ &\quad = \int_0^T \langle p(u;t), B(v - u)(t) \rangle_{V_2', V_2} dt = (\Lambda_{\mathscr{U}}^{-1} B^* p(u), v - u)_{\mathscr{U}}. \end{aligned}$$

Therefore we conclude that the optimality condition (3.39) is equivalent to

$$(\Lambda_{\mathscr{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}.$$

Hence, we show the following theorem.

Theorem 4.2. *The optimal control u for* (3.38) *is characterized by the following system of equations and inequality*:

$$\begin{cases} y''(u) + A_2(t)y'(u) + A_1(t)y(u) = Bu + f(t, y(u)) \quad \textit{in } (0, T), \\ y(u;0) = y_0 \in V, \quad y'(u;0) = y_1 \in H, \end{cases}$$

$$\begin{cases} \displaystyle\int_0^T \langle p(u;t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t, y(u))\psi(t)\rangle_{V_2,V_2'} dt \\ \qquad = (C_1 y(u) - z_d, C_1\psi)_M \\ \forall\psi \textit{ such that} \\ \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t, y(u))\psi \in L^2(0, T; V_2'), \\ \psi(0) = 0, \quad \psi'(0) = 0, \end{cases}$$

$$(\Lambda_{\mathscr{U}}^{-1} B^* p(u) + Ru, v - u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}.$$

4.2. Application to the case of C_2

Let $C_2 \in \mathscr{L}(V, M)$. Then $C_2^* \Lambda_M(C_2 y(u;T) - z_d) \in V'$. Now we can apply Theorem 4.1 to formulate the adjoint system in order to obtain the optimality condition (3.46). We define $p(u)$ as a unique solution in $L^2(0, T; V_2)$ of the adjoint system

$$\begin{cases} \displaystyle\int_0^T \langle p(u;t), \psi''(t) + A_2(t)\psi'(t) + A_1(t)\psi(t) - f_y(t, y(u;t))\psi(t)\rangle_{V_2,V_2'} dt \\ \qquad = -\langle C_2^* \Lambda_M(C_2 y(u;T) - z_d), \psi(T)\rangle_{V',V} \\ \forall\psi \text{ such that} \\ \psi'' + A_2(t)\psi' + A_1(t)\psi - f_y(t, y(u;t))\psi \in L^2(0, T; V_2'), \\ \psi(0) = 0, \quad \psi'(0) = 0. \end{cases} \tag{4.11}$$

We can take $\psi = z = Dy(u)(v - u)$ in (4.11) as the same reason as for the case C_1. Hence by direct calculations we have

$$\begin{aligned} &-(C_2 y(u;T) - z_d, C_2 z(T))_M \\ &\quad = -\langle C_2^* \Lambda_M(C_2 y'(u;T) - z_d), z(T)\rangle_{V',V} \\ &\quad = \int_0^T \left\langle p(u;t), \left(\frac{d^2}{dt^2} + A_2(t)\frac{d}{dt} + A_1(t) - f_y(t, y(u;t))\right) z(t) \right\rangle_{V_2,V_2'} dt \\ &\quad = \int_0^T \langle p(u;t), B(v - u)(t)\rangle_{V_2',V_2} dt = (\Lambda_{\mathscr{U}}^{-1} B^* p(u), v - u)_{\mathscr{U}}. \end{aligned}$$

Thus we conclude that the optimality condition (3.46) is equivalent to

$$(-\Lambda_{\mathscr{U}}^{-1}B^*p(u)+Ru, v-u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}.$$

Therefore, we have proved the following theorem.

Theorem 4.3. *The optimal control u for* (3.45) *is characterized by the following system of equations and inequality*:

$$\begin{cases} y''(u)+A_2(t)y'(u)+A_1(t)y(u)=Bu+f(t,y(u)) \quad in \ (0,T), \\ y(u;0)=y_0 \in V, \quad y'(u;0)=y_1 \in H, \end{cases}$$

$$\begin{cases} \int_0^T \langle p(u;t), \psi''(t)+A_2(t)\psi'(t)+A_1(t)\psi(t)-f_y(t,y(u))\psi(t)\rangle_{V_2,V_2'}dt \\ \qquad = -(C_2y(u;T)-z_d, C_2\psi(T))_M \\ \forall \psi \ \ such \ that \\ \psi''+A_2(t)\psi'+A_1(t)\psi-f_y(t,y(u))\psi \in L^2(0,T;V_2'), \\ \psi(0)=0, \quad \psi'(0)=0, \end{cases}$$

$$(-\Lambda_{\mathscr{U}}^{-1}B^*p(u)+Ru, v-u)_{\mathscr{U}} \geq 0, \qquad \forall v \in \mathscr{U}_{ad}.$$

Finally we note that there are many applications of our results to the optimal control problem involving various types of nonlinear partial differential equations having damping terms, for example, the damped sine-Gordon equations and the structural damped Euler-Bernoulli beam equations among others.

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