Counterexample to global existence for systems of nonlinear wave equations with different propagation speeds

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1 Introduction and main result

We consider the Cauchy problem for systems of semilinear wave equations with different propagation speeds in three space dimensions of the form

$$\Box_{c_i} u_i = F_i(u, \partial_t u), \quad (x,t) \in \mathbb{R}^3 \times [0,\infty),\ i=1,2, \tag{1.1}$$

$$u_i(x,0) = \varepsilon\varphi_i(x), \quad \partial_t u_i(x,0) = \varepsilon\psi_i(x), \quad x \in \mathbb{R}^3,\ i=1,2, \tag{1.2}$$

where $\Box_c = \partial_t^2 - c^2\Delta$, c_1, c_2, ε are positive constants, $c_1 \neq c_2$, and $u=(u_1,u_2)$ is an $\mathbb{R}^2$-valued unknown function of (x,t). We assume that the nonlinear functions F_1 and F_2 are quadratic with respect to $(u, \partial_t u)$, and study the small data global existence and blowup for (1.1). Here, we say that the small data global existence holds for (1.1) if for any φ_i, $\psi_i \in C_0^\infty(\mathbb{R}^3)$ $(i=1,2)$ there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the Cauchy problem (1.1)–(1.2) admits a unique global classical solution $u \in C^\infty(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^2)$. Moreover, we say that the small data blowup occurs if the small data global existence does not hold. In the present paper, we do not consider the case where the nonlinear terms F_i depend only on u (for that case, see Kubo and Ohta [10]), and we put

$$F_i(u, \partial_t u) = \sum_{j,k=1,2} (A_i^{j,k} u_j \partial_t u_k + B_i^{j,k} \partial_t u_j \partial_t u_k), \tag{1.3}$$

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where $A_i^{j,k}$, $B_i^{j,k} \in \mathbb{R}$, $i = 1, 2$. In what follows, we always assume that

$$A_i^{i,i} = B_i^{i,i} = 0, \quad i = 1, 2, \tag{1.4}$$

because it is proved by F. John [4] that the small data blowup occurs for the single equations $\Box u = u\partial_t u$ and $\Box u = (\partial_t u)^2$ in three space dimensions (see Klainerman [8] and Christodoulou [2] for the small data global existence when $c_1 = c_2$).

For the case $c_1 \neq c_2$ and (1.4), the small data global existence for (1.1) has been studied by many authors (see, e.g., [1, 3, 5, 6, 7, 9, 11, 12, 13]). Yokoyama [13] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,k} = B_i^{i,i} = 0$ for $i, j, k = 1, 2$. For the case where both F_1 and F_2 can be written in the divergent form

$$F_i = \partial_t \left(\sum_{j,k=1,2} D_i^{j,k} u_j u_k \right), \quad D_i^{j,k} \in \mathbb{R},\ i = 1, 2,$$

it is proved in [5] that the small data global existence holds for (1.1) if $c_1 \neq c_2$ and $D_i^{i,i} = 0$ for $i = 1, 2$. Moreover, Katayama [7] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,j} = B_i^{j,j} = 0$ for $i, j = 1, 2$.

However, to our knowledge, no results on the small data blowup have been obtained for (1.1) with (1.3) when $c_1 \neq c_2$ and (1.4). The purpose in the present paper is to show that the condition (1.4) is not sufficient to prove the small data global existence for (1.1) with (1.3) when $c_1 \neq c_2$. More precisely, we consider

$$\begin{cases} \Box_{c_1} u_1 = u_2 \partial_t u_1, & (x,t) \in \mathbb{R}^3 \times [0,\infty), \\ \Box_{c_2} u_2 = (\partial_t u_1)^2, & (x,t) \in \mathbb{R}^3 \times [0,\infty), \\ u_1(x,0) = 0,\ \partial_t u_1(x,0) = \varepsilon \psi_1(|x|), & x \in \mathbb{R}^3, \\ u_2(x,0) = 0,\ \partial_t u_2(x,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

The main result in the present paper is as follows.

Theorem 1.1 *Let $0 < c_1 < c_2$, $\varepsilon \in (0,1]$, $\psi_1(|x|) \in C_0^\infty(\mathbb{R}^3)$, and we assume that there exists a constant $\delta > 0$ such that*

$$\psi_1(r) > 0 \ \text{for}\ r \in [0,\delta), \quad \psi_1(r) = 0 \ \text{for}\ r \in [\delta,\infty). \tag{1.6}$$

Then, the classical solution (u_1, u_2) of (1.5) blows up in a finite time $T^*(\varepsilon)$. Moreover, there exists a positive constant C^*, which is independent of ε, such that

$$T^*(\varepsilon) \le \exp(C^* \varepsilon^{-2}), \quad \varepsilon \in (0,1].$$

In the next section, we will give the proof of Theorem 1.1.

2 Proof of Theorem 1.1

Since the equations and initial data in (1.5) are radially symmetric, by the uniqueness of classical solutions, the classical solution (u_1, u_2) of (1.5) is radially symmetric. So, in what follows, we put $r = |x|$, and write $u_i(r,t)$ for $u_i(x,t)$ $(i = 1, 2)$. Moreover, for $v = v(r,t)$ and $c > 0$, we denote

$$\dot{v} = \partial_t v, \quad \square_c v = r^{-1}\{\partial_t^2(rv) - c^2\partial_r^2(rv)\}.$$

Then, (1.5) is written in the following form:

$$(2.1)\qquad \begin{cases} \square_{c_1} u_1 = \dot{u}_1 u_2, & (r,t) \in [0,\infty)^2, \\ \square_{c_2} u_2 = (\dot{u}_1)^2, & (r,t) \in [0,\infty)^2, \\ u_1(r,0) = 0, \ \dot{u}_1(r,0) = \varepsilon\psi_1(r), & r \in [0,\infty), \\ u_2(r,0) = 0, \ \dot{u}_2(r,0) = 0, & r \in [0,\infty). \end{cases}$$

The following lemma is well known.

Lemma 2.1 *Let $v(r,t)$ be the classical solution of*

$$(2.2)\qquad \begin{cases} \square_c v = f(r,t), & (r,t) \in [0,\infty) \times [0,T), \\ v(r,0) = 0, \ \dot{v}(r,0) = g(r), & r \in [0,\infty). \end{cases}$$

Then, for $(r,t) \in [0,\infty) \times [0,T)$, we have

$$rv(r,t) = \frac{1}{2c}\int_{|r-ct|}^{r+ct} \rho g(\rho)\, d\rho + \frac{1}{2c}\int_0^t \left(\int_{|r-c(t-\tau)|}^{r+c(t-\tau)} \rho f(\rho,\tau)\, d\rho \right) d\tau.$$

Moreover, if $r \geq ct \geq 0$, we have

$$\begin{aligned} r\dot{v}(r,t) = &\frac{1}{2}\{(r+ct)g(r+ct) + (r-ct)g(r-ct)\} \\ &+\frac{1}{2}\int_0^t \{(r+c(t-\tau))f(r+c(t-\tau),\tau) \\ &\qquad +(r-c(t-\tau))f(r-c(t-\tau),\tau)\}\, d\tau. \end{aligned}$$

Lemma 2.2 *Assume (1.6). Let $T^*(\varepsilon) \in (0,\infty]$ be the life span of the classical solution (u_1,u_2) of (2.1). Then, for any $(r,t) \in [0,\infty) \times [0,T^*(\varepsilon))$, we have $u_2(r,t) \geq 0$. Moreover, we have $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$, and $\dot{u}_1(r,t) = 0$ if $r - c_1 t \geq \delta$.*

Proof. By Lemma 2.1, for $(r,t) \in [0,\infty) \times [0,T^*(\varepsilon))$, we have

$$ru_2(r,t) = \frac{1}{2c_2}\int_0^t \left(\int_{|r-c_2(t-\tau)|}^{r+c_2(t-\tau)} \rho \dot{u}_1(\rho,\tau)^2\, d\rho \right) d\tau \geq 0.$$

Moreover, since $u_1(r,0) = \dot{u}_1(r,0) = 0$ for $r \geq \delta$, by the uniqueness of classical solutions of (2.1), we have $\dot{u}_1(r,t) = 0$ for $r - c_1 t \geq \delta$. Finally, we show that $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$. We put

$$\Omega = \{(\rho,\tau) \in [0,\infty) \times (0, T^*(\varepsilon)) : \ 0 < \rho - c_1\tau < \delta\}.$$

Moreover, for $(r,t) \in \Omega$, we put

$$D(r,t) = \{(\rho,\tau) \in [0,\infty)^2 : \ 0 \leq \tau \leq t, \ |\rho - r| \leq c_1(t-\tau)\}.$$

By (1.6) and the continuity of $\dot{u}_1$, there exists $(r_1,t_1) \in \Omega$ such that $\dot{u}_1(r,t) > 0$ for $(r,t) \in D(r_1,t_1) \cap \Omega$. Suppose that there exists $(r_0,t_0) \in \Omega$ such that $\dot{u}_1(r_0,t_0) \leq 0$. Then, by the continuity of $\dot{u}_1$, there exists $(r_2,t_2) \in \Omega$ such that $\dot{u}_1(r_2,t_2) = 0$ and $\dot{u}_1(r,t) \geq 0$ for $(r,t) \in D(r_2,t_2)$. Since $\dot{u}_1(r,t) \geq 0$ and $u_2(r,t) \geq 0$ for $(r,t) \in D(r_2,t_2)$, by Lemma 2.1, we have

$$0 = r_2\dot{u}_1(r_2,t_2) \geq \frac{\varepsilon}{2}(r_2 - c_1t_2)\psi_1(r_2 - c_1t_2).$$

On the other hand, since $(r_2,t_2) \in \Omega$, by (1.6), we have $(r_2 - c_1t_2)\psi_1(r_2 - c_1t_2) > 0$. This is a contradiction. Hence, we obtain that $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$. □

Let $0 < \delta_1 < \delta_2 < \delta$, and we put

$$\Sigma = \{(r,t) \in [0,\infty)^2 : \ \delta_1 \leq r - c_1 t \leq \delta_2\}, \quad \Sigma(t) = \{r \in [0,\infty) : \ (r,t) \in \Sigma\}.$$

Moreover, for the classical solution (u_1,u_2) of (2.1), we define

$$U_1(t) = \inf\{r\dot{u}_1(r,t) : \ r \in \Sigma(t)\}, \quad U_2(t) = \inf\{ru_2(r,t) : \ r \in \Sigma(t)\}.$$

Then, by Lemma 2.2, we have $U_1(t) \geq 0$ and $U_2(t) \geq 0$ for $t \in [0, T^*(\varepsilon))$.

Lemma 2.3 *Under the assumptions in Theorem 1.1, there exist positive constants C_1, C_2, C_3 such that*

$$U_1(t) \geq C_1\varepsilon + C_2 \int_1^t \frac{U_1(\tau)U_2(\tau)}{\tau}\, d\tau, \quad t \geq 1, \tag{2.3}$$

$$U_2(t) \geq C_3 \int_{(c_2-c_1)t/c_2}^t \left(1 - \frac{\tau}{t}\right) \frac{U_1(\tau)^2}{\tau}\, d\tau, \quad t \geq \frac{\delta_2}{c_2 - c_1}. \tag{2.4}$$

Proof. First, we show (2.3). Let $(r,t) \in \Sigma$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} r\dot{u}_1(r,t) &\geq \frac{\varepsilon}{2}(r - c_1t)\psi_1(r - c_1t) \\ &\quad + \frac{1}{2}\int_0^t (r - c_1t + c_1\tau)\dot{u}_1(r - c_1t + c_1\tau, \tau)u_2(r - c_1t + c_1\tau, \tau)\, d\tau \\ &\geq C_1\varepsilon + \frac{1}{2}\int_0^t \frac{U_1(\tau)U_2(\tau)}{r - c_1t + c_1\tau}\, d\tau \geq C_1\varepsilon + \frac{1}{2}\int_0^t \frac{U_1(\tau)U_2(\tau)}{c_1\tau + \delta_2}\, d\tau, \end{aligned}$$

where $C_1 = \inf\{\rho\psi_1(\rho)/2 : \delta_1 \le \rho \le \delta_2\} > 0$ by (1.6). This implies (2.3). Next, we show (2.4). Let $(r,t) \in \Sigma$ with $t \ge \delta_2/(c_2 - c_1)$. Then, we have $0 \le (c_2 t - r)/c_2 \le (c_2 - c_1)t/c_2$ and $c_2 t + r \ge \delta_2$. By Lemma 2.1, we have

$$
\begin{aligned}
(2.5)\quad r u_2(r,t) &\ge \int_{(c_2 t - r)/c_2}^{t} \left(\int_{|r - c_2(t-\tau)|}^{r + c_2(t-\tau)} \frac{(\rho \dot{u}_1(\rho,\tau))^2}{\rho} \chi_{\Sigma(\tau)}(\rho)\, d\rho \right) d\tau \\
&\ge \int_{(c_2 - c_1)t/c_2}^{t} \bar{\ell}(t,\tau) \frac{U_1(\tau)^2}{c_1 \tau + \delta_2}\, d\tau,
\end{aligned}
$$

where we put

$$\bar{\ell}(t,\tau) = \inf\{\ell(r,t,\tau) :\ r \in \Sigma(t)\}, \quad \ell(r,t,\tau) = \int_{|r - c_2(t-\tau)|}^{r + c_2(t-\tau)} \chi_{\Sigma(\tau)}(\rho)\, d\rho.$$

Then, we have $\bar{\ell}(t,\tau) \ge (\delta_2 - \delta_1)(1 - \tau/t)$ for $(c_2 - c_1)t/c_2 \le \tau \le t$. By (2.5), for any $(r,t) \in \Sigma$ with $t \ge \delta_2/(c_2 - c_1)$, we have

$$r u_2(r,t) \ge (\delta_2 - \delta_1) \int_{(c_2 - c_1)t/c_2}^{t} \left(1 - \frac{\tau}{t}\right) \frac{U_1(\tau)^2}{c_1 \tau + \delta_2}\, d\tau,$$

which implies (2.4). □

Lemma 2.4 *Let $C_1,\ C_2,\ C_3 > 0$, $\alpha \ge \beta > 1$, $\varepsilon \in (0,1]$. Assume that $(f(t), g(t))$ satisfies*

$$(2.6)\qquad f(t) \ge C_1 \varepsilon,\ f(t) \ge C_2 \int_1^t \frac{f(\tau) g(\tau)}{\tau}\, d\tau, \quad t \ge 1,$$

$$(2.7)\qquad g(t) \ge C_3 \int_{t/\beta}^{t} \left(1 - \frac{\tau}{t}\right) \frac{f(\tau)^2}{\tau}\, d\tau, \quad t \ge \alpha.$$

Then, $(f(t), g(t))$ blows up in a finite time $T_*(\varepsilon)$. Moreover, there exists a positive constant C_*, which is independent of ε, such that $T_*(\varepsilon) \le \exp(C_* \varepsilon^{-2})$.

Proof. We define

$$F(s) = \varepsilon^{-1} f(\exp(\varepsilon^{-2} s)), \quad G(s) = \varepsilon^{-2} g(\exp(\varepsilon^{-2} s)).$$

Then, we have

$$(2.8)\quad F(s) \ge C_1,\ F(s) \ge C_2 \int_0^s F(\sigma) G(\sigma)\, d\sigma, \ s \ge 0,$$

$$(2.9)\quad G(s) \ge C_3 \varepsilon^{-2} \int_{s - \varepsilon^2 \log\beta}^{s} \left\{1 - \exp(-\varepsilon^{-2}(s - \sigma))\right\} F(\sigma)^2\, d\sigma, \ s \ge \log\alpha.$$

Here, we assume that $F(s) \geq A > 0$ for $s \geq S \geq 0$, and let $h \in (0,1]$. Then, by (2.9), we have for $s \geq \max\{S + h\log\beta, \log\alpha\}$

$$
\begin{aligned}
(2.10) \quad G(s) &\geq C_3\varepsilon^{-2}A^2 \int_{s-\varepsilon^2 h\log\beta}^{s} \left\{1 - \exp(-\varepsilon^{-2}(s-\sigma))\right\} d\sigma \\
&= C_3A^2 \int_0^{h\log\beta} (1 - e^{-\sigma})\, d\sigma \geq \frac{C_3(\beta-1)\log\beta}{2\beta} h^2A^2,
\end{aligned}
$$

where we used the fact that

$$
1 - e^{-\sigma} \geq \frac{1 - \exp(-\log\beta)}{\log\beta}\sigma = \frac{\beta - 1}{\beta\log\beta}\sigma, \quad 0 \leq \sigma \leq \log\beta.
$$

By (2.8) and (2.10), we have for $s \geq \max\{S + h(1+\log\beta), \log\alpha + h\}$

$$
F(s) \geq C_1C_2 \int_{s-h}^{s} G(\sigma)\, d\sigma \geq \frac{C_1C_2C_3(\beta-1)\log\beta}{2\beta} h^3A^2.
$$

Moreover, since $F(s) \geq C_1$ for $s \geq 0$, we have

$$
\begin{aligned}
(2.11) \qquad F(s) &\geq C_1C_2 \int_{\log\alpha}^{s} G(\sigma)\, d\sigma \\
&\geq \frac{C_1^3C_2C_3(\beta-1)\log\beta}{2\beta}(s - \log\alpha), \quad s \geq \log\alpha.
\end{aligned}
$$

We define constants γ and A_1 by

$$
\gamma = \max\{1, \frac{2\beta}{C_1C_2C_3(\beta-1)\log\beta}\}, \quad A_1 = \gamma\exp\left(1 + 6\sum_{k=1}^{\infty} 2^{-k}\log k\right).
$$

Then, by (2.11), there exists a constant $S_1 \geq \log\alpha$ such that $F(s) \geq A_1$ for $s \geq S_1$. Furthermore, we define sequences $\{A_n\}$ and $\{S_n\}$ by

$$
A_{n+1} = \frac{A_n^2}{\gamma n^6}, \quad S_{n+1} = S_n + \frac{1+\log\beta}{n^2}, \quad n \in \mathbb{N}.
$$

Then, for any $n \in \mathbb{N}$, we have $F(s) \geq A_n$ for $s \geq S_n$, and

$$
\begin{aligned}
S_n &= S_1 + (1+\log\beta)\sum_{k=1}^{n-1}\frac{1}{k^2}, \\
\log A_{n+1} &= 2^n\left(\log A_1 - (1-2^{-n})\log\gamma - 6\sum_{k=1}^{n} 2^{-k}\log k\right) \geq 2^n.
\end{aligned}
$$

Therefore, $(F(s), G(s))$ blows up at some $s = S_*$ satisfying $S_* \leq \lim_{n\to\infty} S_n < \infty$. This completes the proof. $\square$

Theorem 1.1 follows from Lemmas 2.3 and 2.4.

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