Counterexample to global existence for systems of nonlinear wave equations with different propagation speeds

Masahito Ohta *
Department of Mathematics
Faculty of Science, Saitama University
Saitama 338-8570, Japan
E-mail: mohta@rimath.saitama-u.ac.jp

1 Introduction and main result

We consider the Cauchy problem for systems of semilinear wave equations with different propagation speeds in three space dimensions of the form

$$\Box_{c_i} u_i = F_i(u, \partial_t u), \quad (x,t) \in \mathbb{R}^3 \times [0,\infty),\ i=1,2, \tag{1.1}$$

$$u_i(x,0) = \varepsilon\varphi_i(x), \quad \partial_t u_i(x,0) = \varepsilon\psi_i(x), \quad x \in \mathbb{R}^3,\ i=1,2, \tag{1.2}$$

where $\Box_c = \partial_t^2 - c^2\Delta$, c_1, c_2, ε are positive constants, $c_1 \neq c_2$, and $u = (u_1, u_2)$ is an $\mathbb{R}^2$-valued unknown function of (x,t). We assume that the nonlinear functions F_1 and F_2 are quadratic with respect to $(u, \partial_t u)$, and study the small data global existence and blowup for (1.1). Here, we say that the small data global existence holds for (1.1) if for any φ_i, $\psi_i \in C_0^\infty(\mathbb{R}^3)$ $(i=1,2)$ there exists a constant $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ the Cauchy problem (1.1)–(1.2) admits a unique global classical solution $u \in C^\infty(\mathbb{R}^3 \times [0,\infty), \mathbb{R}^2)$. Moreover, we say that the small data blowup occurs if the small data global existence does not hold. In the present paper, we do not consider the case where the nonlinear terms F_i depend only on u (for that case, see Kubo and Ohta [10]), and we put

$$F_i(u, \partial_t u) = \sum_{j,k=1,2} (A_i^{j,k} u_j \partial_t u_k + B_i^{j,k} \partial_t u_j \partial_t u_k), \tag{1.3}$$

*Supported in part by Grant-in-Aid for Science Research (14740099), JSPS

where $A_i^{j,k}$, $B_i^{j,k} \in \mathbb{R}$, $i = 1, 2$. In what follows, we always assume that

$$A_i^{i,i} = B_i^{i,i} = 0, \quad i = 1, 2, \tag{1.4}$$

because it is proved by F. John [4] that the small data blowup occurs for the single equations $\Box u = u\partial_t u$ and $\Box u = (\partial_t u)^2$ in three space dimensions (see Klainerman [8] and Christodoulou [2] for the small data global existence when $c_1 = c_2$).

For the case $c_1 \neq c_2$ and (1.4), the small data global existence for (1.1) has been studied by many authors (see, e.g., [1, 3, 5, 6, 7, 9, 11, 12, 13]). Yokoyama [13] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,k} = B_i^{i,i} = 0$ for $i, j, k = 1, 2$. For the case where both F_1 and F_2 can be written in the divergent form

$$F_i = \partial_t \left(\sum_{j,k=1,2} D_i^{j,k} u_j u_k \right), \quad D_i^{j,k} \in \mathbb{R}, \; i = 1, 2,$$

it is proved in [5] that the small data global existence holds for (1.1) if $c_1 \neq c_2$ and $D_i^{i,i} = 0$ for $i = 1, 2$. Moreover, Katayama [7] proved that the small data global existence holds for (1.1) with (1.3) if $c_1 \neq c_2$ and $A_i^{j,j} = B_i^{j,j} = 0$ for $i, j = 1, 2$.

However, to our knowledge, no results on the small data blowup have been obtained for (1.1) with (1.3) when $c_1 \neq c_2$ and (1.4). The purpose in the present paper is to show that the condition (1.4) is not sufficient to prove the small data global existence for (1.1) with (1.3) when $c_1 \neq c_2$. More precisely, we consider

$$\begin{cases} \Box_{c_1} u_1 = u_2 \partial_t u_1, & (x,t) \in \mathbb{R}^3 \times [0,\infty), \\ \Box_{c_2} u_2 = (\partial_t u_1)^2, & (x,t) \in \mathbb{R}^3 \times [0,\infty), \\ u_1(x,0) = 0, \; \partial_t u_1(x,0) = \varepsilon \psi_1(|x|), & x \in \mathbb{R}^3, \\ u_2(x,0) = 0, \; \partial_t u_2(x,0) = 0, & x \in \mathbb{R}^3. \end{cases} \tag{1.5}$$

The main result in the present paper is as follows.

Theorem 1.1 *Let* $0 < c_1 < c_2$, $\varepsilon \in (0,1]$, $\psi_1(|x|) \in C_0^\infty(\mathbb{R}^3)$, *and we assume that there exists a constant* $\delta > 0$ *such that*

$$\psi_1(r) > 0 \text{ for } r \in [0,\delta), \quad \psi_1(r) = 0 \text{ for } r \in [\delta,\infty). \tag{1.6}$$

Then, the classical solution (u_1, u_2) *of* (1.5) *blows up in a finite time* $T^*(\varepsilon)$. *Moreover, there exists a positive constant* C^*, *which is independent of* ε, *such that*

$$T^*(\varepsilon) \leq \exp(C^* \varepsilon^{-2}), \quad \varepsilon \in (0,1].$$

In the next section, we will give the proof of Theorem 1.1.

2 Proof of Theorem 1.1

Since the equations and initial data in (1.5) are radially symmetric, by the uniqueness of classical solutions, the classical solution (u_1, u_2) of (1.5) is radially symmetric. So, in what follows, we put $r = |x|$, and write $u_i(r,t)$ for $u_i(x,t)$ $(i = 1, 2)$. Moreover, for $v = v(r,t)$ and $c > 0$, we denote

$$\dot{v} = \partial_t v, \quad \Box_c v = r^{-1}\{\partial_t^2(rv) - c^2\partial_r^2(rv)\}.$$

Then, (1.5) is written in the following form:

$$(2.1) \qquad \begin{cases} \Box_{c_1} u_1 = \dot{u}_1 u_2, & (r,t) \in [0,\infty)^2, \\ \Box_{c_2} u_2 = (\dot{u}_1)^2, & (r,t) \in [0,\infty)^2, \\ u_1(r,0) = 0, \ \dot{u}_1(r,0) = \varepsilon\psi_1(r), & r \in [0,\infty), \\ u_2(r,0) = 0, \ \dot{u}_2(r,0) = 0, & r \in [0,\infty). \end{cases}$$

The following lemma is well known.

Lemma 2.1 *Let $v(r,t)$ be the classical solution of*

$$(2.2) \qquad \begin{cases} \Box_c v = f(r,t), & (r,t) \in [0,\infty) \times [0,T), \\ v(r,0) = 0, \ \dot{v}(r,0) = g(r), & r \in [0,\infty). \end{cases}$$

Then, for $(r,t) \in [0,\infty) \times [0,T)$, we have

$$rv(r,t) = \frac{1}{2c}\int_{|r-ct|}^{r+ct} \rho g(\rho)\,d\rho + \frac{1}{2c}\int_0^t \left(\int_{|r-c(t-\tau)|}^{r+c(t-\tau)} \rho f(\rho,\tau)\,d\rho\right) d\tau.$$

Moreover, if $r \geq ct \geq 0$, we have

$$\begin{aligned} r\dot{v}(r,t) = &\frac{1}{2}\{(r+ct)g(r+ct) + (r-ct)g(r-ct)\} \\ &+\frac{1}{2}\int_0^t \{(r+c(t-\tau))f(r+c(t-\tau),\tau) \\ &\qquad +(r-c(t-\tau))f(r-c(t-\tau),\tau)\}\,d\tau. \end{aligned}$$

Lemma 2.2 *Assume (1.6). Let $T^*(\varepsilon) \in (0,\infty]$ be the life span of the classical solution (u_1, u_2) of (2.1). Then, for any $(r,t) \in [0,\infty) \times [0,T^*(\varepsilon))$, we have $u_2(r,t) \geq 0$. Moreover, we have $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$, and $\dot{u}_1(r,t) = 0$ if $r - c_1 t \geq \delta$.*

Proof. By Lemma 2.1, for $(r,t) \in [0,\infty) \times [0,T^*(\varepsilon))$, we have

$$ru_2(r,t) = \frac{1}{2c_2}\int_0^t \left(\int_{|r-c_2(t-\tau)|}^{r+c_2(t-\tau)} \rho\dot{u}_1(\rho,\tau)^2\,d\rho\right) d\tau \geq 0.$$

Moreover, since $u_1(r,0) = \dot{u}_1(r,0) = 0$ for $r \geq \delta$, by the uniqueness of classical solutions of (2.1), we have $\dot{u}_1(r,t) = 0$ for $r - c_1 t \geq \delta$. Finally, we show that $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$. We put

$$\Omega = \{(\rho,\tau) \in [0,\infty) \times (0, T^*(\varepsilon)) : \ 0 < \rho - c_1\tau < \delta\}.$$

Moreover, for $(r,t) \in \Omega$, we put

$$D(r,t) = \{(\rho,\tau) \in [0,\infty)^2 : \ 0 \leq \tau \leq t, \ |\rho - r| \leq c_1(t-\tau)\}.$$

By (1.6) and the continuity of $\dot{u}_1$, there exists $(r_1,t_1) \in \Omega$ such that $\dot{u}_1(r,t) > 0$ for $(r,t) \in D(r_1,t_1) \cap \Omega$. Suppose that there exists $(r_0,t_0) \in \Omega$ such that $\dot{u}_1(r_0,t_0) \leq 0$. Then, by the continuity of $\dot{u}_1$, there exists $(r_2,t_2) \in \Omega$ such that $\dot{u}_1(r_2,t_2) = 0$ and $\dot{u}_1(r,t) \geq 0$ for $(r,t) \in D(r_2,t_2)$. Since $\dot{u}_1(r,t) \geq 0$ and $u_2(r,t) \geq 0$ for $(r,t) \in D(r_2,t_2)$, by Lemma 2.1, we have

$$0 = r_2 \dot{u}_1(r_2,t_2) \geq \frac{\varepsilon}{2}(r_2 - c_1 t_2)\psi_1(r_2 - c_1 t_2).$$

On the other hand, since $(r_2,t_2) \in \Omega$, by (1.6), we have $(r_2 - c_1t_2)\psi_1(r_2 - c_1t_2) > 0$. This is a contradiction. Hence, we obtain that $\dot{u}_1(r,t) > 0$ if $0 < r - c_1 t < \delta$. □

Let $0 < \delta_1 < \delta_2 < \delta$, and we put

$$\Sigma = \{(r,t) \in [0,\infty)^2 : \ \delta_1 \leq r - c_1 t \leq \delta_2\}, \quad \Sigma(t) = \{r \in [0,\infty) : \ (r,t) \in \Sigma\}.$$

Moreover, for the classical solution (u_1,u_2) of (2.1), we define

$$U_1(t) = \inf\{r\dot{u}_1(r,t) : \ r \in \Sigma(t)\}, \quad U_2(t) = \inf\{r u_2(r,t) : \ r \in \Sigma(t)\}.$$

Then, by Lemma 2.2, we have $U_1(t) \geq 0$ and $U_2(t) \geq 0$ for $t \in [0, T^*(\varepsilon))$.

Lemma 2.3 *Under the assumptions in Theorem 1.1, there exist positive constants C_1, C_2, C_3 such that*

$$U_1(t) \geq C_1\varepsilon + C_2 \int_1^t \frac{U_1(\tau)U_2(\tau)}{\tau}\, d\tau, \quad t \geq 1, \tag{2.3}$$

$$U_2(t) \geq C_3 \int_{(c_2-c_1)t/c_2}^t \left(1 - \frac{\tau}{t}\right) \frac{U_1(\tau)^2}{\tau}\, d\tau, \quad t \geq \frac{\delta_2}{c_2 - c_1}. \tag{2.4}$$

Proof. First, we show (2.3). Let $(r,t) \in \Sigma$. By Lemmas 2.1 and 2.2, we have

$$\begin{aligned} r\dot{u}_1(r,t) \ &\geq \ \frac{\varepsilon}{2}(r - c_1 t)\psi_1(r - c_1 t) \\ &\quad + \frac{1}{2}\int_0^t (r - c_1 t + c_1\tau)\dot{u}_1(r - c_1 t + c_1\tau, \tau) u_2(r - c_1 t + c_1 \tau, \tau)\, d\tau \\ &\geq \ C_1\varepsilon + \frac{1}{2}\int_0^t \frac{U_1(\tau)U_2(\tau)}{r - c_1 t + c_1\tau}\, d\tau \geq C_1\varepsilon + \frac{1}{2}\int_0^t \frac{U_1(\tau)U_2(\tau)}{c_1\tau + \delta_2}\, d\tau, \end{aligned}$$

where $C_1 = \inf\{\rho\psi_1(\rho)/2 : \delta_1 \le \rho \le \delta_2\} > 0$ by (1.6). This implies (2.3). Next, we show (2.4). Let $(r,t) \in \Sigma$ with $t \ge \delta_2/(c_2 - c_1)$. Then, we have $0 \le (c_2 t - r)/c_2 \le (c_2 - c_1)t/c_2$ and $c_2 t + r \ge \delta_2$. By Lemma 2.1, we have

$$
\begin{aligned}
(2.5)\quad ru_2(r,t) &\ge \int_{(c_2 t - r)/c_2}^{t} \left(\int_{|r - c_2(t-\tau)|}^{r + c_2(t-\tau)} \frac{(\rho \dot{u}_1(\rho,\tau))^2}{\rho} \chi_{\Sigma(\tau)}(\rho)\, d\rho \right) d\tau \\
&\ge \int_{(c_2 - c_1)t/c_2}^{t} \bar{\ell}(t,\tau) \frac{U_1(\tau)^2}{c_1 \tau + \delta_2}\, d\tau,
\end{aligned}
$$

where we put

$$
\bar{\ell}(t,\tau) = \inf\{\ell(r,t,\tau) :\ r \in \Sigma(t)\}, \quad \ell(r,t,\tau) = \int_{|r - c_2(t-\tau)|}^{r + c_2(t-\tau)} \chi_{\Sigma(\tau)}(\rho)\, d\rho.
$$

Then, we have $\bar{\ell}(t,\tau) \ge (\delta_2 - \delta_1)(1 - \tau/t)$ for $(c_2 - c_1)t/c_2 \le \tau \le t$. By (2.5), for any $(r,t) \in \Sigma$ with $t \ge \delta_2/(c_2 - c_1)$, we have

$$
ru_2(r,t) \ge (\delta_2 - \delta_1) \int_{(c_2 - c_1)t/c_2}^{t} \left(1 - \frac{\tau}{t}\right) \frac{U_1(\tau)^2}{c_1 \tau + \delta_2}\, d\tau,
$$

which implies (2.4). $\square$

Lemma 2.4 *Let $C_1,\ C_2,\ C_3 > 0,\ \alpha \ge \beta > 1,\ \varepsilon \in (0,1]$. Assume that $(f(t), g(t))$ satisfies*

$$
(2.6)\qquad f(t) \ge C_1 \varepsilon,\ \ f(t) \ge C_2 \int_1^t \frac{f(\tau) g(\tau)}{\tau}\, d\tau, \quad t \ge 1,
$$

$$
(2.7)\qquad g(t) \ge C_3 \int_{t/\beta}^{t} \left(1 - \frac{\tau}{t}\right) \frac{f(\tau)^2}{\tau}\, d\tau, \quad t \ge \alpha.
$$

Then, $(f(t), g(t))$ blows up in a finite time $T_*(\varepsilon)$. Moreover, there exists a positive constant C_*, which is independent of ε, such that $T_*(\varepsilon) \le \exp(C_* \varepsilon^{-2})$.

Proof. We define

$$
F(s) = \varepsilon^{-1} f(\exp(\varepsilon^{-2} s)), \quad G(s) = \varepsilon^{-2} g(\exp(\varepsilon^{-2} s)).
$$

Then, we have

$$
(2.8)\quad F(s) \ge C_1,\ \ F(s) \ge C_2 \int_0^s F(\sigma) G(\sigma)\, d\sigma, \ \ s \ge 0,
$$

$$
(2.9)\quad G(s) \ge C_3 \varepsilon^{-2} \int_{s - \varepsilon^2 \log\beta}^{s} \left\{1 - \exp(-\varepsilon^{-2}(s - \sigma))\right\} F(\sigma)^2\, d\sigma, \ \ s \ge \log\alpha.
$$

Here, we assume that $F(s) \geq A > 0$ for $s \geq S \geq 0$, and let $h \in (0,1]$. Then, by (2.9), we have for $s \geq \max\{S + h\log\beta, \log\alpha\}$

$$\begin{aligned} G(s) &\geq C_3\varepsilon^{-2}A^2 \int_{s-\varepsilon^2 h\log\beta}^{s} \left\{1 - \exp(-\varepsilon^{-2}(s-\sigma))\right\} d\sigma \\ &= C_3A^2 \int_0^{h\log\beta} (1-e^{-\sigma})\, d\sigma \geq \frac{C_3(\beta-1)\log\beta}{2\beta} h^2A^2, \end{aligned} \tag{2.10}$$

where we used the fact that

$$1 - e^{-\sigma} \geq \frac{1-\exp(-\log\beta)}{\log\beta}\sigma = \frac{\beta-1}{\beta\log\beta}\sigma, \quad 0 \leq \sigma \leq \log\beta.$$

By (2.8) and (2.10), we have for $s \geq \max\{S + h(1+\log\beta), \log\alpha + h\}$

$$F(s) \geq C_1C_2 \int_{s-h}^{s} G(\sigma)\, d\sigma \geq \frac{C_1C_2C_3(\beta-1)\log\beta}{2\beta} h^3A^2.$$

Moreover, since $F(s) \geq C_1$ for $s \geq 0$, we have

$$\begin{aligned} F(s) &\geq C_1C_2 \int_{\log\alpha}^{s} G(\sigma)\, d\sigma \\ &\geq \frac{C_1^3C_2C_3(\beta-1)\log\beta}{2\beta}(s-\log\alpha), \quad s \geq \log\alpha. \end{aligned} \tag{2.11}$$

We define constants γ and A_1 by

$$\gamma = \max\{1, \frac{2\beta}{C_1C_2C_3(\beta-1)\log\beta}\}, \quad A_1 = \gamma\exp\left(1 + 6\sum_{k=1}^{\infty} 2^{-k}\log k\right).$$

Then, by (2.11), there exists a constant $S_1 \geq \log\alpha$ such that $F(s) \geq A_1$ for $s \geq S_1$. Furthermore, we define sequences $\{A_n\}$ and $\{S_n\}$ by

$$A_{n+1} = \frac{A_n^2}{\gamma n^6}, \quad S_{n+1} = S_n + \frac{1+\log\beta}{n^2}, \quad n \in \mathbb{N}.$$

Then, for any $n \in \mathbb{N}$, we have $F(s) \geq A_n$ for $s \geq S_n$, and

$$\begin{aligned} &S_n = S_1 + (1+\log\beta)\sum_{k=1}^{n-1}\frac{1}{k^2}, \\ &\log A_{n+1} = 2^n\left(\log A_1 - (1-2^{-n})\log\gamma - 6\sum_{k=1}^{n} 2^{-k}\log k\right) \geq 2^n. \end{aligned}$$

Therefore, $(F(s), G(s))$ blows up at some $s = S_*$ satisfying $S_* \leq \lim_{n\to\infty} S_n < \infty$. This completes the proof. □

Theorem 1.1 follows from Lemmas 2.3 and 2.4.

Acknowledgements. The author would like to thank Professors Soichiro Katayama, Hideo Kubo and Kazuyoshi Yokoyama for their valuable discussions.

References

[1] R. Agemi and K. Yokoyama, The null conditions and global existence of solutions to systems of wave equations with different propagation speeds, "Advances in nonlinear partial differential equations and stochastics" (S. Kawashima and T. Yanagisawa ed.), Series on Adv. in Math. for Appl. Sci., Vol. 48, 43–86, World Scientific, Singapore, 1998.

[2] D. Christodoulou, Global solutions of nonlinear hyperbolic equations for small initial data, *Comm. Pure Appl. Math.* **39** (1986), 267–282.

[3] A. Hoshiga and H. Kubo, Global small amplitude solutions of nonlinear hyperbolic systems with a critical exponent under the null condition, *SIAM J. Math. Anal.* **31** (2000), 486–513.

[4] F. John, Blow-up of solutions for quasi-linear wave equations in three space dimensions, *Comm. Pure Appl. Math.* **34** (1981), 29–51.

[5] S. Katayama, Global existence for a class of systems of nonlinear wave equations in three space dimensions, *Preprint.*

[6] S. Katayama, Global and almost global existence for systems of nonlinear wave equations with different propagation speeds, *Preprint.*

[7] S. Katayama, Global existence for systems of wave equations with nonresonant nonlinearities and null forms, *Preprint.*

[8] S. Klainerman, The null condition and global existence to nonlinear wave equations, *Lectures in Appl. Math.* **23** (1986), 293–326.

[9] M. Kovalyov, Resonance-type behaviour in a system of nonlinear wave equations, *J. Differential Equations* **77** (1989), 73–83.

[10] H. Kubo and M. Ohta, On systems of semilinear wave equations with unequal propagation speeds in three space dimensions, *Preprint.*

[11] K. Kubota and K. Yokoyama, Global existence of classical solutions to systems of nonlinear wave equations with different speeds of propagation, *Japanese J. Math.* **27** (2001), 113–202.

[12] T. C. Sideris and S.-Y. Tu, Global existence for systems of nonlinear wave equations in 3D with multiple speeds, *SIAM J. Math. Anal.* **33** (2001), 477–488.

[13] K. Yokoyama, Global existence of classical solutions to systems of wave equations with critical nonlinearity in three space dimensions, *J. Math. Soc. Japan* **52** (2000), 609–632.