One-dimensional Shape Memory Alloy Problems Including a Hysteresis Operator

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1 Introduction

In this paper we are concerned with the global existence and uniqueness of a solution to a one-dimensional model of thermomechanical evolution of shape memory alloys. First, the following two differential equations are derived from the conservation laws of linear momentum and energy:

$$u_{tt} + \gamma u_{xxxx} = \hat{\sigma}_x \quad \text{in } Q(T) := (0,T) \times (0,1), \tag{1.1}$$

$$U_t + q_x = \hat{\sigma}\varepsilon_t \quad \text{in } Q(T), \tag{1.2}$$

where u denotes the displacement, $\varepsilon := u_x$ is the strain, $\hat{\sigma}$ is the stress, U is the internal energy, q is the heat flux and γ is a positive constant. Here, we refer Brokate-Sprekels [4, Section 5] and Pawlow [10] for the physical background of these laws. Now, we use the classical Fourier law and an elementary approximation $U_t = \theta_t$ where $\theta := \theta(t,x)$ is the temperature field. Therefore, (1.2) can be written by

$$\theta_t - \kappa\theta_{xx} = \hat{\sigma}\varepsilon_t \quad \text{in } Q(T), \tag{1.3}$$

where κ is a positive constant depending on the specific heat and the heat conductivity. By some mathematical reasons we assume that there are interior frictions in the form of viscous stresses in the material. Then we can apply Hooke's-like law so that we have

$$\hat{\sigma} = \sigma + \mu\varepsilon_t, \tag{1.4}$$

where $\mu > 0$ is the constant viscosity. The above composition of the stress was investigated by many mathematicians (cf. [9, 6, 4]). Falk's model [5] is well known as the system describing the dynamics of one-dimensional shape memory alloys. Falk's model is based on the Landau-Devonshire theory. This means that σ is decided by the derivative of the Helmholtz energy $\Psi := \Psi(\theta, \varepsilon)$, that is,

$$\sigma = \frac{\partial \Psi(\theta, \varepsilon)}{\partial \varepsilon}.$$

However, by some experiments we know that the relationship between the stress and the strain is described by the hysteresis loop depending on the temperature. In our previous

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works [1, 2] we have already pointed out that the relationship can be represented by the ordinary differential equations including the subdifferentials of the indicator function of the closed interval as follows:

$$\sigma_t + \partial I(\theta, \varepsilon; \sigma) \ni c\varepsilon_t, \tag{1.5}$$

where c is a positive constant depending on the hysteresis loops and I is the indicator function of the closed interval $[f_a(\theta, \varepsilon), f_d(\theta, \varepsilon)]$ for given continuous functions f_a and f_d on $R \times R$ with $f_a \le f_d$ on $R \times R$, that is,

$$I(\theta, \varepsilon; \sigma) = \begin{cases} 0 & \text{if } f_a(\theta, \varepsilon) \le \sigma \le f_d(\theta, \varepsilon), \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore the following system $\mathrm{P}_0 := \mathrm{P}_0(u_0, v_0, \theta_0, \sigma_0)$ is derived from (1.1) and (1.3) $\sim$ (1.5).

$$u_{tt} + \gamma u_{xxxx} - \mu u_{xxt} - \sigma_x = 0 \quad \text{in } Q(T), \tag{1.6}$$

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} + \mu |u_{xt}|^2 \quad \text{in } Q(T), \tag{1.7}$$

$$\sigma_t + \partial I(\theta, \varepsilon; \sigma) \ni c u_{xt} \quad \text{in } Q(T), \tag{1.8}$$

$$u(t, 0) = u(t, 1) = u_{xx}(t, 0) = u_{xx}(t, 1) = 0 \quad \text{for } 0 < t < T, \tag{1.9}$$

$$\theta_x(t, 0) = \theta_x(t, 1) = 0 \quad \text{for } 0 < t < T, \tag{1.10}$$

$$\sigma_x(t, 0) = \sigma_x(t, 1) = 0 \quad \text{for } 0 < t < T, \tag{1.11}$$

$$u(0) = u_0, u_t(0) = v_0, \theta(0) = \theta_0, \sigma(0) = \sigma_0 \quad \text{on } (0, 1), \tag{1.12}$$

where u_0, v_0, θ_0 and σ_0 are given initial functions.

Our formulation does not require the monotonicity for f_a and f_d, but needs the boundedness of f_a and f_d on $R \times R$. Moreover, it can cover the special case where

$$f_a(\theta, \varepsilon) = f_d(\theta, \varepsilon) = \frac{\partial \Psi(\theta, \varepsilon)}{\partial \varepsilon},$$

which gives a Falk-type model with σ bounded. The main purpose of this paper is to give the existence and uniqueness theorem for P_0. In [1] we have already proved the wellposedness for P_0 with (1.13) and (1.14) instead of (1.7) and (1.8), respectively.

$$\theta_t - \kappa\theta_{xx} = \sigma u_{xt} \quad \text{in } Q(T), \tag{1.13}$$

$$\sigma_t - \nu\sigma_{xx} + \partial I(\theta, \varepsilon; \sigma) \ni c u_{xt} \quad \text{in } Q(T), \tag{1.14}$$

where $\nu > 0$ is a positive constant. Also, in [2] we studied P_0 with (1.14) instead of (1.8), which is denoted by $\mathrm{P}_\nu = \mathrm{P}_\nu(u_0, v_0, \theta_0, \sigma_0)$ for $\nu > 0$.

In this paper we refer the book [3] and [7] for the theory on maximal monotone operators and subdifferentials of convex functions in a Hilbert space.

2 Main result

Throughout this paper we shall use the following notations. $H := L^2(0, 1)$, $V := H_0^1(0, 1)$, V^* is the dual space of V, and $< \cdot, \cdot >$ is the duality pair on $V \times V^*$. Also, for given $\theta \in H$

and $\varepsilon \in H$ we denote by $I(\theta, \varepsilon; \cdot)$ the function on H defined by

$$I(\theta, \varepsilon; \sigma) = \begin{cases} 0 & \text{if } \sigma \in K(\theta, \varepsilon), \\ +\infty & \text{otherwise,} \end{cases}$$

where $K(\theta, \varepsilon) = \{\sigma \in H : f_a(\theta, \varepsilon) \leq \sigma \leq f_d(\theta, \varepsilon) \text{ a.e. on } (0,1)\}$.

Clearly, $I(\theta, \varepsilon; \cdot)$ is proper, l.s.c. and convex on H, the effective domain $D(I(\theta, \varepsilon; \cdot)) = K(\theta, \varepsilon)$, and its subdifferential $\partial I(\theta, \varepsilon; \cdot)$ is a multivalued operator in H which has the following property: $\xi \in \partial I(\theta, \varepsilon; \sigma)$ if and only if $\sigma \in H$ with $f_a(\theta, \varepsilon) \leq \sigma \leq f_d(\theta, \varepsilon)$ a.e. on $(0,1)$ and $\xi \in H$ satisfying

$$\int_0^1 \xi(z - \sigma)dx \leq 0 \quad \text{for any } z \in K(\theta, \varepsilon).$$

We begin with the precise assumptions for data.
(A1) γ, μ, κ and c are positive constants.
(A2) $f_a, f_d \in C^2(R \times R) \cap W^{2,\infty}(R \times R)$ and $f_a \leq f_d$ on $R \times R$. Here, we put

$$L = \max\{|f_a|_{W^{2,\infty}(R\times R)}, |f_d|_{W^{2,\infty}(R\times R)}\}.$$

(A3) $u_0 \in H^4(0,1)$ with $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$, $v_0 \in V \cap H^2(0,1)$, $\theta_0 \in H^1(0,1)$ and $\sigma_0 \in H^1(0,1)$. Moreover, $f_a(\theta_0, \varepsilon_0) \leq \sigma_0 \leq f_d(\theta_0, \varepsilon_0)$ on $(0,1)$ where $\varepsilon_0 = u_{0x}$.

Definition 2.1. For any positive number T let $\{u, \theta, \sigma\}$ be a triplet of functions u, θ and σ on $Q(T)$. If the following conditions (S1) $\sim$ (S4) are satisfied, then we call that $\{u, \theta, \sigma\}$ is a solution of P_0 on $Q(T)$:
(S1) $u \in W^{2,\infty}(0,T; V^*) \cap W^{1,\infty}(0,T; V) \cap W^{1,2}(0,T; H^2(0,1)) \cap L^\infty(0,T; H^3(0,1))$, $\theta \in W^{1,2}(0,T; H) \cap L^\infty(0,T; H^1(0,1))$, $\sigma \in W^{1,2}(0,T; H) \cap L^\infty(0,T; H^1(0,1))$.
(S2) $u(0,x) = u_0(x)$ and $u_t(0,x) = v_0(x)$ for $x \in (0,1)$ and

$$< u_{tt}(t), \eta > -\gamma \int_0^1 u_{xxx}(t)\eta_x dx - \mu \int_0^1 u_{txx}(t)\eta dx$$

$$= \int_0^1 \sigma_x(t)\eta dx \qquad \text{for any } \eta \in V \text{ and a.e. } t \in [0,T].$$

(S3) (1.7) holds in the usual sense, (1.10) and the initial condition for θ hold.
(S4) There exists $\xi \in L^2(Q(T))$ such that $\xi(t) \in \partial I(\theta(t), \varepsilon(t); \sigma(t))$ for a.e. $t \in [0,T]$, and (1.8) holds for a.e. $(t,x) \in Q(T)$, and $\sigma(0,x) = \sigma_0(x)$ for $x \in (0,1)$.

By the above definition we can easily get the following remark.
Remark 2.1. If $\{u, \theta, \sigma\}$ is a solution of P_0 on $[0,T]$, then $\theta \in L^\infty(Q(T))$, $\theta_{xx} \in L^2(Q(T))$, $\sigma \in L^\infty(Q(T))$, $u_{tx} \in L^\infty(Q(T))$, and $u_t(t,0) = u_t(t,1) = 0$ for a.e. $t \in [0,T]$.

The next theorem is the main result of this paper.
Theorem 2.1. *Suppose that (A1) $\sim$ (A3) hold, and $\mu^2 > 4\gamma$. Then:*
(1) (Uniqueness) For any $T > 0$ let $\{u_i, \theta_i, \sigma_i\}$ be a solution of P_0 on $[0,T]$, $i = 1,2$. If

$u_{itx} \in L^\infty(Q(T))$ *for* $i = 1, 2$, *then* $u_1 = u_2$, $\theta_1 = \theta_2$ *and* $\sigma_1 = \sigma_2$ *on* $Q(T)$.
(2) (Exisistence) For any $T > 0$ *there exists a solution* $\{u, \theta, \sigma\}$ *of* P_0 *on* $[0, T]$ *satisfying* $u \in W^{2,\infty}(0, T; H) \cap W^{2,2}(0, T; H^1(0, 1)) \cap W^{1,\infty}(0, T; H^2(0, 1))$.

We prove Theorem 2.1 in the following way. In the next section we show the uniqueness of a solution in the similar way to those of [1, 2]. On the existence we already obtained a solution of P_ν for $\nu > 0$ in [2]. By using some classical theory for parabolic equations we can get the uniform estimates for solutions of P_ν with respect to ν so that these estimates lead to the existence of a solution. The assumption $\mu^2 > 4\gamma$ in Therorem 2.1 is necessary to prove the following useful inequality.

Lemma 2.1. *(cf. [2, Lemma 3.1.]) Let* $T > 0$, γ *and* μ *are positive constants,* $\sigma \in L^2(0, T; H^2(0, 1))$, $u_0 \in H^4(0, 1)$ *with* $u_0(0) = u_0(1) = u_{0xx}(0) = u_{0xx}(1) = 0$ *and* $v_0 \in V \cap H^2(0, 1)$. *Then there exists one and only one* $u \in L^\infty(0, T; H^4(0, 1)) \cap W^{1,2}(0, T; H^3(0, 1)) \cap W^{1,\infty}(0, T; H^2(0, 1))$ *satisfying (1.6), (1.9) and* $u(0) = u_0$ *and* $u_t(0) = v_0$ *on* $(0, 1)$. *Moreover, if* $\mu^2 > 4\gamma$, $p > 1$ *and* $\sigma \in L^p(Q(T))$, *then there exists a positive constant* K_p *depending only on* μ, γ, p *and* T *such that*

$$(\int_0^T |u_{tx}(t)|^p_{L^p(0,1)} dt)^{1/p} \le K_p(|\sigma|_{L^p(Q(T))} + |u_{0xx}|_{W^{2-2/p,p}(0,1)} + |v_0|_{W^{2-2/p,p}(0,1)}).$$

3 Uniqueness

The aim of this section is to prove the uniqueness of a solution to P_0. First, we show the following lemma concerned with the regularity of a solution.

Lemma 3.1. *Let* $T > 0$. *If* $u \in W^{2,\infty}(0, T; V^*) \cap W^{1,\infty}(0, T; V)$, *then the function* $t \to \frac{1}{2}|u_t(t)|^2_H$ *is absolutely continuous on* $[0, T]$, *and*

$$< u_{tt}(t), u_t(t) >= \frac{1}{2}\frac{d}{dt}|u_t(t)|^2_H \quad \text{for a.e. } t \in [0, T].$$

This lemma is quite standard, so we omit its proof.

We begin to give a proof of the uniqueness. The proof is rather long so that we provide several lemmas. From now on we assume (A1) $\sim$ (A3) and $\mu^2 > 4\gamma$ and denote by $\{u_1, \theta_1, \sigma_1\}$ and $\{u_2, \theta_2, \sigma_2\}$ two solutions of $P_0(u_0, v_0, \theta_0, \sigma_0)$ on $[0, T]$ for $T > 0$. Moreover, we suppose that $u_{itx} \in L^\infty(Q(T))$ for $i = 1, 2$. Here, for simplicity we put $\theta = \theta_1 - \theta_2$, $u = u_1 - u_2$, $\sigma = \sigma_1 - \sigma_2$, $\varepsilon = \varepsilon_1 - \varepsilon_2$ and

$$M(s) = \max\{|f_a(\theta_1, \varepsilon_1) - f_a(\theta_2, \varepsilon_2)|_{L^\infty(Q(s))}, |f_d(\theta_1, \varepsilon_1) - f_d(\theta_2, \varepsilon_2)|_{L^\infty(Q(s))}\} \text{ for } 0 < s \le T,$$

where $\varepsilon_i = u_{ix}$ for $i = 1, 2$.

Lemma 3.2. *It holds that*

$$\frac{1}{2}\frac{d}{dt}(|[\sigma(t) - M(s)]^+|^2_H + |[-\sigma(t) - M(s)]^+|^2_H)$$

$$\le c\int_0^1 ([\sigma(t) - M(s)]^+ - [-\sigma(t) - M(s)]^+)\varepsilon_t(t)dx \quad \text{for } 0 < t \le s \le T. \tag{3.1}$$

Proof. Let $s \in (0,T]$ be fixed. By Definition 2.1 for $i = 1, 2$ there exists a function ξ_i such that $\xi_i \in L^2(Q(T))$, $\xi_i(t) \in \partial I(\theta_i(t), \varepsilon_i(t); \sigma_i(t))$ for a.e. $t \in [0,T]$ and $\sigma_{it} + \xi_i = c\varepsilon_{it}$ a.e. on $Q(T)$. We put $z_1 = \sigma_1 - [\sigma_1 - \sigma_2 - M(s)]^+$ and $z_2 = \sigma_2 + [\sigma_1 - \sigma_2 - M(s)]^+$ on $Q(s)$. Clearly, $z_i(t) \in K(\theta_i(t), \varepsilon_i(t))$ for a.e. $t \in [0, s]$ and $i = 1, 2$. As is easily seen, we have

$$\left.\begin{aligned} \int_0^1 \sigma_{1t}(t)(\sigma_1(t) - z_1(t))dx \le c\int_0^1 \varepsilon_{1t}(t)(\sigma_1(t) - z_1(t))dx \\ \int_0^1 \sigma_{2t}(t)(\sigma_2(t) - z_2(t))dx \le c\int_0^1 \varepsilon_{2t}(t)(\sigma_2(t) - z_2(t))dx \end{aligned}\right\} \text{for a.e. } t \in [0, s].$$

Thus we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |[\sigma_1(t) - \sigma_2(t) - M(s)]^+|^2 dx \\ = &\int_0^1 (\sigma_{1t}(t) - \sigma_{2t}(t))[\sigma_1(t) - \sigma_2(t) - M(s)]^+ dx \\ \le &\; c\int_0^1 (\varepsilon_1(t) - \varepsilon_2(t))[\sigma_1(t) - \sigma_2(t) - M(s)]^+ dx \quad \text{for a.e. } t \in [0, s]. \end{aligned}$$

Similarly, we can get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 |[\sigma_2(t) - \sigma_1(t) - M(s)]^+|^2 dx \\ \le &\; c\int_0^1 (\varepsilon_2(t) - \varepsilon_1(t))[\sigma_2(t) - \sigma_1(t) - M(s)]^+ dx \quad \text{for a.e. } t \in [0, s]. \end{aligned}$$

Hence, (3.1) holds. □

Lemma 3.3. *It holds that*

$$\frac{1}{2}\frac{d}{dt}(|u_t(t)|_H^2 + \gamma|u_{xx}(t)|_H^2) + \mu|u_{tx}(t)|_H^2 = -\int_0^1 \sigma(t)\varepsilon_t(t)dx \quad \textit{for a.e. } t \in [0,T]. \tag{3.2}$$

Proof. By substituting u_t as η in Definition 2.1 (S2) we have

$$\begin{aligned} &< u_{tt}(t), u_t(t) > -\gamma\int_0^1 u_{xxx}(t)u_{tx}(t)dx + \mu\int_0^1 |u_{tx}(t)|^2 dx \\ = &\; -\int_0^1 \sigma(t)u_{tx}(t)dx \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

It follows from Lemma 3.1 that (3.2) is true. □

Lemma 3.4. *For each $s \in (0,T]$ the following inequality holds:*

$$\frac{1}{2}\frac{d}{dt}(|[\sigma(t)-M(s)]^+|_H^2 + |[-\sigma(t)-M(s)]^+|_H^2)$$

$$+\frac{c}{2}\frac{d}{dt}(|u_t(t)|_H^2 + \gamma|u_{xx}(t)|_H^2) + \frac{c\mu}{2}|u_{tx}(t)|_H^2$$

$$\leq \frac{c}{2\mu}M(s)^2 \quad \textit{for a.e. } t \in [0,s]. \tag{3.3}$$

Proof. Let $s \in (0,T]$ be fixed. Lemmas 3.2 and 3.3 imply that

$$\frac{1}{2}\frac{d}{dt}(|[\sigma(t)-M(s)]^+|_H^2 + |[-\sigma(t)-M(s)]^+|_H^2)$$

$$+\frac{c}{2}\frac{d}{dt}(|u_t(t)|_H^2 + \gamma|u_{xx}(t)|_H^2) + \frac{c\mu}{2}|u_{tx}(t)|_H^2$$

$$\leq c\int_0^1 u_{tx}(t)([\sigma(t)-M(s)]^+ - [-\sigma(t)-M(s)]^+ - \sigma(t))dx \quad \text{for a.e. } t \in [0,s].$$

It is easy to see that

$$|[\sigma(t)-M(s)]^+ - [-\sigma(t)-M(s)]^+ - \sigma(t)| \leq M(s) \quad \text{on } (0,1) \text{ for a.e. } t \in [0,s].$$

Hence, we get (3.3). $\square$

Lemma 3.5 *There exists a positive constant L_1 depending only on μ, κ, c, $|\sigma_2|_{L^\infty(Q(T))}$, $|u_{1tx}|_{L^\infty(0,T;H))}$ and $|u_{2tx}|_{L^\infty(0,T;H)}$ such that*

$$\frac{1}{2}\frac{d}{dt}|\theta(t)|_H^2 + \frac{\kappa}{4}|\theta_x(t)|_H^2 \leq \frac{7c\mu}{16}|u_{tx}(t)|_H^2 + L_1(|\theta(t)|_H^2 + |\sigma(t)|_H^2) \textit{ for a.e. } t \in [0,T].$$

Proof. First, we have

$$\frac{1}{2}\frac{d}{dt}|\theta(t)|_H^2 + \kappa|\theta_x(t)|_H^2$$

$$= \int_0^1 (\sigma_1(t)u_{1tx}(t) - \sigma_2(t)u_{2tx}(t))\theta(t)dx$$

$$+\mu\int_0^1 (|u_{1tx}(t)|^2 - |u_{2tx}(t)|^2)\theta(t)dx \quad \text{for a.e. } t \in [0,T].$$

Now, we recall the Gagliardo-Nirenberg inequality:

$$|w|_{L^\infty(0,1)} \leq 2(|w_x|_H^{1/2}|w|_H^{1/2} + |w|_H) \quad \text{for any } w \in H^1(0,1). \tag{3.4}$$

By using (3.4) we observe that

$$|\int_0^1 (\sigma_1(t)u_{1tx}(t) - \sigma_2(t)u_{2tx}(t))\theta(t)dx|$$

$$\begin{aligned}
&\le \ |\int_0^1 \sigma(t)u_{1tx}(t)\theta(t)dx| + |\int_0^1 \sigma_2(t)u_{tx}(t)\theta(t)dx| \\
&\le \ |\sigma(t)|_H|u_{1tx}(t)|_H|\theta(t)|_{L^\infty(0,1)} + |\sigma_2(t)|_{L^\infty(0,1)}|u_{tx}(t)|_H|\theta(t)|_H \\
&\le \ 2|\sigma(t)|_H|u_{1tx}(t)|_H(|\theta_x(t)|_H^{1/2}|\theta(t)|_H^{1/2} + |\theta(t)|_H) + \frac{c\mu}{4}|u_{tx}(t)|_H^2 + \frac{1}{c\mu}|\sigma_2(t)|^2_{L^\infty(0,1)}|\theta(t)|_H^2 \\
&\le \ \frac{\kappa}{2}|\theta_x(t)|_H^2 + C_1|u_{1tx}|^2_{L^\infty(0,T;H)}|\sigma(t)|_H^2 + C_1|\theta(t)|_H^2 \\
&\quad +\frac{c\mu}{4}|u_{tx}(t)|_H^2 + \frac{1}{c\mu}|\sigma_2|^2_{L^\infty(Q(T))}|\theta(t)|_H^2 \quad \text{for a.e. } t \in [0,T];
\end{aligned}$$

$$\begin{aligned}
&\mu\int_0^1 (|u_{1tx}(t)|^2 - |u_{2tx}(t)|^2)\theta(t)dx \\
&\le \ \mu(|u_{1tx}(t)|_H + |u_{2tx}(t)|_H)|u_{tx}(t)|_H|\theta(t)|_{L^\infty(0,1)} \\
&\le \ 2\mu(|u_{1tx}|_{L^\infty(0,T;H)} + |u_{2tx}|_{L^\infty(0,T;H)})|u_{tx}(t)|_H(|\theta_x(t)|_H^{1/2}|\theta(t)|_H^{1/2} + |\theta(t)|_H) \\
&\le \ \frac{\kappa}{4}|\theta_x(t)|_H^2 + \frac{3c\mu}{16}|u_{tx}(t)|_H^2 \\
&\quad +C_1(|u_{1tx}|^4_{L^\infty(0,T;H)} + |u_{2tx}|^4_{L^\infty(0,T;H)} + 1)|\theta(t)|_H^2 \quad \text{for a.e. } t \in [0,T],
\end{aligned}$$

where C_1 is a positive constant depending only on κ, c and μ.

Therefore, we can conclude that this lemma is valid. □

Here, we introduce the following notations:

$$\begin{aligned}
E_0(t) \ &:= \ \frac{1}{2}(|[\sigma(t) - M(s)]^+|_H^2 + |[-\sigma(t) - M(s)]^+|_H^2) \\
&\quad +\frac{c}{2}|u_t(t)|_H^2 + \frac{c\gamma}{2}|u_{xx}(t)|_H^2 + \frac{1}{2}|\theta(t)|_H^2;
\end{aligned}$$

$$E_1(t) := \frac{c\mu}{16}|u_{tx}(t)|_H^2 + \frac{\kappa}{4}|\theta_x(t)|_H^2 \quad \text{for } t \in [0,s] \text{ and } 0 < s \le T.$$

Lemma 3.6. *There exists a positive constant L_2 such that*

$$\frac{d}{dt}E_0(t) + E_1(t) \le L_2 M(s)^2 + L_2 E_0(t) \quad \textit{for } t \in [0,s] \textit{ and } 0 < s \le T. \tag{3.5}$$

Proof. Let $s \in (0,T]$. Then we have

$$\frac{d}{dt}E_0(t) + E_1(t) \le \frac{c}{2\mu}M(s)^2 + L_1|\theta(t)|_H^2 + L_1|\sigma(t)|_H^2 \text{ for } t \in [0,s] \text{ and } 0 < s \le T.$$

This is a direct consequence of Lemmas 3.4 and 3.5. Immediately, we have

$$|\sigma| \le [\sigma - M(s)]^+ + [-\sigma - M(s)]^+ + M(s) \quad \text{on } Q(s). \tag{3.6}$$

Hence, by putting $L_2 = \frac{c}{2\mu} + 18L_1$ we obtain the assertion of this lemma. □

Lemma 3.7. *There exists a positive constant L_3 depending only on κ, μ, $|\sigma_2|_{L^\infty(Q(T))}$, $|u_{1tx}|_{L^\infty(Q(T))}$ and $|u_{2tx}|_{L^\infty(Q(T))}$ such that*

$$\frac{1}{2}\frac{d}{dt}|\theta_x(t)|_H^2 + \frac{\kappa}{4}|\theta_{xx}(t)|_H^2 \le L_3(|u_{tx}(t)|_H^2 + |\sigma(t)|_H^2) \quad \text{for a.e. } t \in [0,T], \tag{3.7}$$

$$|\theta|^2_{L^\infty(Q(s))} \le L_3 \sup_{0\le t\le s} E_0(t) + L_3\int_0^s (|\sigma(t)|_H^2 + E_1(t))dt \text{ for any } 0 < s \le T.$$

Proof. First, we have:

$$\theta_t - \kappa\theta_{xx} = \sigma_1 u_{1tx} - \sigma_2 u_{2tx} + \mu|u_{1tx}|^2 - \mu|u_{2tx}|^2 \quad \text{on } Q(T).$$

Multiplying it by $-\theta_{xx}$ and integrating it over $(0,1)$, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta_x(t)|_H^2 + \kappa|\theta_{xx}(t)|_H^2 \\
= & -\int_0^1 (\sigma_1(t)u_{1tx}(t) - \sigma_2(t)u_{2tx}(t))\theta_{xx}(t)dx - \mu\int_0^1 (|u_{1tx}(t)|^2 - |u_{2tx}(t)|^2)\theta_{xx}(t)dx \\
\le & |\sigma_2(t)|_{L^\infty(0,1)}|u_{tx}(t)|_H|\theta_{xx}(t)|_H + |\sigma(t)|_H|u_{1tx}(t)|_{L^\infty(0,1)}|\theta_{xx}(t)|_H \\
& +\mu(|u_{1tx}(t)|_{L^\infty(0,1)} + |u_{2tx}(t)|_{L^\infty(0,1)})|u_{tx}(t)|_H|\theta_{xx}(t)|_H \\
\le & \frac{3\kappa}{4}|\theta_{xx}(t)|_H^2 + \frac{1}{2\kappa}|\sigma_2(t)|^2_{L^\infty(0,1)}|u_{tx}(t)|_H^2 + \frac{2}{\kappa}|u_{1tx}(t)|^2_{L^\infty(0,1)}|\sigma(t)|_H^2 \\
& +\frac{4\mu^2}{\kappa}(|u_{1tx}(t)|^2_{L^\infty(0,1)} + |u_{2tx}(t)|^2_{L^\infty(0,1)})|u_{tx}(t)|_H^2 \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

By using the assumption $u_{itx} \in L^\infty(Q(T))$ for $i = 1, 2$ we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\theta_x(t)|_H^2 + \frac{\kappa}{4}|\theta_{xx}(t)|_H^2 \\
\le & (\frac{1}{2\kappa} + \frac{4\mu^2}{\kappa})(|\sigma_2|^2_{L^\infty(Q(T))} + |u_{1tx}|^2_{L^\infty(Q(T))} + |u_{2tx}|^2_{L^\infty(Q(T))})|u_{tx}(t)|_H^2 \\
& +\frac{2}{\kappa}|u_{1tx}|^2_{L^\infty(Q(T))}|\sigma(t)|_H^2 \quad \text{for a.e. } t \in [0,T].
\end{aligned} \tag{3.8}$$

Thus we get (3.7).

Next, integrating (3.8) on $[0,\tau]$, $0 < \tau \le T$, implies that

$$\frac{1}{2}|\theta_x(\tau)|_H^2 + \frac{\kappa}{4}\int_0^\tau |\theta_{xx}(t)|_H^2 dt \le L_4\int_0^\tau (|u_{tx}(t)|_H^2 + |\sigma(t)|_H^2)dt,$$

where $L_4 = (\frac{1}{2\kappa} + \frac{4\mu^2}{\kappa})(|\sigma_2|^2_{L^\infty(Q(T)} + |u_{1tx}|^2_{L^\infty(Q(T))} + |u_{2tx}|^2_{L^\infty(Q(T))}) + \frac{2}{\kappa}|u_{1tx}|^2_{L^\infty(Q(T))}$. Therefore, we see that

$$|\theta|^2_{L^\infty(Q(s))}$$

$$
\begin{aligned}
&\leq 12(|\theta|^2_{L^\infty(0,s;H)} + |\theta_x|^2_{L^\infty(0,s;H)}) \\
&\leq 24 \sup_{0\leq t\leq s} E_0(t) + 24L_4 \int_0^s (|u_{tx}(t)|^2_H + |\sigma(t)|^2_H)dt \\
&\leq L_3 \sup_{0\leq t\leq s} E_0(t) + L_3 \int_0^s |\sigma(t)|^2_H dt + L_3 \int_0^s E_1(t)dt \quad \text{for } 0 < s \leq T,
\end{aligned}
$$

where L_3 is a suitable positive constant. This gives the conclusion of this lemma. □

Lemma 3.8. *There exists a positive constant L_5 such that*

$$
|\varepsilon|^2_{L^\infty(Q(s))} \leq L_5 \sup_{0\leq t\leq s} E_0(t) \quad \textit{for any } 0 < s \leq T.
$$

Proof. It follows from (3.4) that

$$
\begin{aligned}
|\varepsilon(t)|^2_{L^\infty(0,1)} &\leq 8(|u_{xx}(t)|_H |u_x(t)|_H + |u_x(t)|^2_H) \\
&\leq C_2(|u_{xx}(t)|^2_H + |u(t)|^2_H) \\
&\leq C_2(\frac{2}{c\gamma} \sup_{0\leq\tau\leq s} E_0(\tau) + \frac{2T^2}{c} \sup_{0\leq\tau\leq s} E_0(\tau)) \quad \text{for } 0 \leq t \leq s \leq T,
\end{aligned}
$$

where C_2 is a positive constant, since it holds that

$$
|w_x|^2_H \leq |w_{xx}|_H |w|_H \quad \text{for } w \in V.
$$

Hence, we can show that Lemma 3.8 holds. □

Lemma 3.9. *There exists a positive constant L_6 such that*

$$
M(s)^2 \leq L_6 \sup_{0\leq t\leq s} E_0(t) + L_6 \int_0^s (|\sigma(t)|^2_H + E_1(t))dt \quad \textit{for } 0 \leq t \leq s \leq T,
$$

$$
|\sigma(t)|^2_H \leq L_6 \left(\sup_{0\leq\tau\leq s} E_0(\tau) + \int_0^s (|\sigma(\tau)|^2_H + E_1(\tau))d\tau \right) \quad \textit{for } 0 \leq t \leq s \leq T.
$$

Proof. On account of Lemmas 3.7 and 3.8 we show that

$$
\begin{aligned}
&M(s)^2 \\
\leq\ & 2L^2(|\theta|^2_{L^\infty(Q(s))} + |\varepsilon|^2_{L^\infty(Q(s))}) \\
\leq\ & 2L^2 \left((L_3 + L_5) \sup_{0\leq t\leq s} E_0(t) + L_3 (\int_0^s |\sigma(t)|^2_H dt + \int_0^s E_1(t)dt) \right) \quad \text{for } 0 \leq s \leq T.
\end{aligned}
$$

Then (3.6) implies the second assertion of this lemma. □

Now, we arrive at just before the end of the proof of the uniqueness.

Proof of Theorem 2.1 (1). By applying the Gronwall inequality to (3.5) we have

$$E_0(t) + \int_0^t E_1(\tau)d\tau \le L_2 e^{L_2 t} M(s)^2 s \quad \text{for any } 0 \le t \le s \le T.$$

Hence, by putting $L_7 = L_2 e^{L_2 T}$ Lemma 3.9 guarantees that

$$\begin{aligned} & E_0(t) + \int_0^t E_1(\tau)d\tau \\ \le\ & L_6 L_7 s\Big(\sup_{0\le\tau\le s} E_0(\tau) + \int_0^s (|\sigma(\tau)|_H^2 + E_1(\tau))d\tau\Big) \quad \text{for } 0 \le t \le s \le T. \end{aligned}$$

Now, we choose $0 < s_1 \le T$ satisfying $L_6 L_7 s_1 \le \frac{1}{2}$. Then it follows from Lemma 3.9 that

$$\begin{aligned} A(s) &:= \sup_{0\le t\le s} E_0(t) + \int_0^s E_1(\tau)d\tau \\ &\le L_8 s \int_0^s |\sigma(\tau)|_H^2 d\tau \\ &\le L_8 L_6 s \int_0^s (A(s) + \int_0^s |\sigma(t)|_H^2 dt)d\tau \\ &\le L_8 L_6 s^2 (A(s) + B(s)) \quad \text{for } 0 \le s \le s_1, \end{aligned}$$

where $L_8 = 2L_6 L_7$ and $B(s) := \int_0^s |\sigma(\tau)|_H^2 d\tau$, since $B(s) \le L_6 s(A(s) + B(s))$ for $0 \le s \le T$. Hence, by putting $s_2 = \min\{s_1, \frac{1}{2L_6}\}$ one can get $B(s) \le 2L_6 s A(s) \le A(s)$ and $A(s) \le 2L_8 L_6 s^2 A(s)$ for $0 \le s \le s_2$. Therefore, $A(s) \le \frac{1}{2}A(s)$ for $0 \le s \le s_3$ where $s_3 = \min\{s_2, \frac{1}{2\sqrt{L_8 L_6}}\}$. Thus

$$A(s) = 0 \text{ and } B(s) = 0 \text{ for } 0 \le s \le s_3, \text{ and } \theta = 0, u = 0, \sigma = 0 \text{ on } Q(s_3).$$

Since s_3 does not depend on initial values, we have proved the uniqueness of a solution to P_0. □

4 Some auxiliary inequalities

The first lemma of this section is concerned with the wellposedness for P_ν for $\nu > 0$.

Lemma 4.1. *(cf. [2, Theorem 2.1]) Let $\nu > 0$. If (A1) $\sim$ (A3) hold and $\mu^2 > \gamma$, then there exists one and only one solution $\{u, \theta, \sigma\}$ of $P_\nu(u_0, v_0, \theta_0, \sigma_0)$ on $[0, T]$ in the following sense: $u \in L^\infty(0, T; H^4(0, 1)) \cap W^{1,2}(0, T; H^3(0, 1)) \cap W^{1,\infty}(0, T; H^2(0, 1))$, $\theta \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1))$, $\sigma \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1))$, (1.6) and (1.7) hold, there exists $\xi \in L^2(Q(T))$ such that*

(4.1) $\xi(t) \in \partial I(\theta(t), \varepsilon(t); \sigma(t)), \sigma_t(t) - \nu\sigma_{xx}(t) + \xi(t) = cu_{xt}(t)$ *in H for a.e. $t \in [0, T]$,*

and (1.9) $\sim$ (1.12) hold.

One of the purposes of this section is to obtain the following inequalities which are used in the proof of the existence of a solution to P_0.

Proposition 4.1. *Let $\nu > 0$, $T > 0$ and suppose (A1) $\sim$ (A3) and $\mu^2 > \gamma$. If $\{u, \theta, \sigma\}$ is a solution of P_ν on $[0,T]$ and $\xi \in L^2(Q(T))$ satisfying (4.1), then*

$$\begin{aligned}
&\frac{1}{2}\int_0^t |\sigma_\tau(\tau)|_H^2 d\tau + 2I(\theta(t), \varepsilon(t); \sigma(t)) + \frac{1}{16}\int_0^t |\xi(\tau)|_H^2 d\tau \\
&\le 2I(\theta_0, \varepsilon_0; \sigma_0) + \frac{\nu}{2}|\sigma_{0x}|_H^2 + 4\nu^2 \int_0^t (|f_a(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2) d\tau \\
&\quad + c^2 \int_0^t |u_{x\tau}(\tau)|_H^2 d\tau + 10 \int_0^t (|f_a(\theta(\tau), \varepsilon(\tau))_\tau|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_\tau|_H^2) d\tau;
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\frac{1}{2}|\sigma_x(t)|_H^2 + I(\theta(t), \varepsilon(t); \sigma(t)) + \frac{1}{16}\int_0^t |\xi(\tau)|_H^2 d\tau \\
&\le \frac{e^T}{2}|\sigma_{0x}|_H^2 + \frac{e^T}{2} I(\theta_0, \varepsilon_0; \sigma_0) + \frac{c^2 e^T}{2}\int_0^t (|u_{x\tau}(\tau)|_H^2 + |u_{xx\tau}(\tau)|_H^2) d\tau \\
&\quad + (4\nu^2 + 8)e^T \int_0^t (|f_a(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2) d\tau \\
&\quad + 2e^T \int_0^t (|f_a(\theta(\tau), \varepsilon(\tau))_\tau|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_\tau|_H^2) d\tau \quad \textit{for } 0 \le t \le T.
\end{aligned} \tag{4.3}$$

In order to prove Proposition 4.1 we introduce the approximate problem for (4.1). To do so for each $\lambda > 0$ let $I_\lambda(\theta, \varepsilon; \cdot)$ be the Yosida approximation for $I(\theta, \varepsilon; \cdot)$. We have already known the certain expression of I_λ and ∂I_λ.

Lemma 4.2. *(cf. [8, Section 4]) For each $\lambda > 0$ it holds that*

$$I_\lambda(\theta, \varepsilon; \sigma) = \frac{1}{2\lambda}\{|[\sigma - f_d(\theta, \varepsilon)]^+|_H^2 + |[f_a(\theta, \varepsilon) - \sigma]^+|_H^2\} \textit{ for } \theta, \varepsilon, \sigma \in H,$$

$$\partial I_\lambda(\theta, \varepsilon; \sigma) = \frac{1}{\lambda}\{[\sigma - f_d(\theta, \varepsilon)]^+ - [f_a(\theta, \varepsilon) - \sigma]^+\} \textit{ for } \theta, \varepsilon, \sigma \in L^2(0,1).$$

In order to prove Proposition 4.1 it is sufficient only to consider the following problem $\mathrm{CP}_{\nu,\lambda}(\theta, \varepsilon)$.

$$\sigma_{\lambda t} - \nu\sigma_{\lambda xx} + \partial I_\lambda(\theta, \varepsilon; \sigma_\lambda) = c\varepsilon_t \quad \text{in } Q(T), \tag{4.4}$$

$$\sigma_{\lambda x}(t, 0) = \sigma_{\lambda x}(t, 1) = 0 \quad \text{for } 0 < t < T, \tag{4.5}$$

$$\sigma_\lambda(0) = \sigma_0 \quad \text{on } (0,1). \tag{4.6}$$

The next lemma guarantees the wellposedness of $\mathrm{CP}_{\nu,\lambda}(\theta, \varepsilon)$ for $\lambda > 0$ and $\nu > 0$.

Lemma 4.3. *Assume that $T > 0$, $\nu > 0$, $\lambda > 0$, $\theta \in W^{1,2}(0,T; H) \cap L^2(0,T; H^2(0,1))$,*

$\varepsilon \in W^{1,2}(0,T;H) \cap L^2(0,T;H^2(0,1))$ *and* $\sigma_0 \in H^1(0,1)$. *Then there exists a unique solution* σ_λ *of* $CP_{\nu,\lambda}(\theta,\varepsilon)$ *on* $[0,T]$, *that is,* $\sigma_\lambda \in W^{1,2}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^1(0,1))$ *satisfy (4.4)* $\sim$ *(4.6) in the usual sense.*

The last term in the left hand of (4.4) is Lipshitz continuous with respect to σ_λ so that it is easy to prove Lemma 4.3.

The proof of Proposition 4.1 is rather long so that we provide several lemmas. For $\nu > 0$ there is a solution $\{u_\lambda, \varepsilon_\nu, \sigma_\nu\}$ of P_ν on $[0,T]$, $0 < T < +\infty$ because of Lemma 4.1. Also, Lemma 4.3 implies that $\mathrm{CP}_{\nu,\lambda}(\theta_\nu, \varepsilon_\nu)$ admits a unique solution $\sigma_{\nu\lambda}$ on $[0,T]$, where $\varepsilon_\nu = u_{\nu x}$. From now on, for simplicity we omit the index ν.

Lemma 4.4. *It holds that*

$$\begin{aligned}
&\frac{1}{2}|\sigma_{\lambda t}(t)|_H^2 + \frac{\nu}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2 + 2\frac{d}{dt}I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t)) \\
&+\lambda\nu|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x|_H^2 + \frac{1}{16}|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|_H^2 \\
\le\ & c^2|\varepsilon_t(t)|_H^2 + 4\nu^2(|f_d(\theta(t),\varepsilon(t))_{xx}|_H^2 + |f_a(\theta(t),\varepsilon(t))_{xx}|_H^2) \\
&+10(|f_d(\theta(t),\varepsilon(t))_t|_H^2 + |f_a(\theta(t),\varepsilon(t))_t|_H^2) \quad \textit{for a.e. } t \in [0,T].
\end{aligned} \tag{4.7}$$

Proof. We multiply (4.4) by $\sigma_{\lambda t}$ and integrate it over $(0,1)$. Then we have

$$\begin{aligned}
&|\sigma_{\lambda t}(t)|_H^2 + \frac{\nu}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2 + \int_0^1 \partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))\sigma_{\lambda t}(t)dx \\
=\ & c\int_0^1 \varepsilon_t(t)\sigma_{\lambda t}(t)dx \\
\le\ & \frac{c^2}{2}|\varepsilon_t(t)|_H^2 + \frac{1}{2}|\sigma_{\lambda t}(t)|_H^2 \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

We note that

$$\begin{aligned}
&\frac{d}{dt}I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t)) \\
=\ & \frac{d}{dt}\int_0^1 \frac{1}{2\lambda}(|[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]^+|^2 + |[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]^+|^2)dx \\
=\ & \frac{1}{\lambda}\int_0^1 \sigma_{\lambda t}(t)([\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]^+ - [f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]^+)dx \\
&-\frac{1}{\lambda}\int_0^1 f_d(\theta(t),\varepsilon(t))_t[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]^+dx \\
&+\frac{1}{\lambda}\int_0^1 f_a(\theta(t),\varepsilon(t))_t[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]^+dx
\end{aligned}$$

$$= \int_0^1 \sigma_{\lambda t}(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))dx \tag{4.8}$$

$$-\frac{1}{\lambda}\int_0^1 f_d(\theta(t),\varepsilon(t))_t[\sigma_\lambda(t)-f_d(\theta(t),\varepsilon(t))]^+dx$$

$$+\frac{1}{\lambda}\int_0^1 f_a(\theta(t),\varepsilon(t))_t[f_a(\theta(t),\varepsilon(t))-\sigma_\lambda(t)]^+dx \quad \text{for a.e. } t\in[0,T].$$

Now, it is easy to see that

$$\left.\begin{array}{l}|[\sigma_\lambda(t)-f_d(\theta(t),\varepsilon(t))]^+| \\ |[f_a(\theta(t),\varepsilon(t))-\sigma_\lambda(t)]^+|\end{array}\right\} \le \lambda|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))| \text{ a.e. on } [0,1] \text{ for } t\in[0,T]. \tag{4.9}$$

Accordingly, we obtain

$$\begin{aligned}&\frac{1}{2}|\sigma_{\lambda t}(t)|_H^2+\frac{\nu}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2+\frac{d}{dt}I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))\\ \le\ &\int_0^1(|f_d(\theta(t),\varepsilon(t))_t|+|f_a(\theta(t),\varepsilon(t))_t|)|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|dx\\ &+\frac{c^2}{2}|\varepsilon_t(t)|_H^2 \quad \text{for a.e. } t\in[0,T].\end{aligned} \tag{4.10}$$

Next, multiplying (4.4) by $\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))$ and integrating it over $(0,1)$, we get

$$\begin{aligned}&\int_0^1\sigma_{\lambda t}(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))dx+\nu\int_0^1\sigma_{\lambda x}(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x dx\\ &+\int_0^1|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|^2dx\\ =\ &c\int_0^1\varepsilon_t(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))dx \quad \text{for a.e. } t\in[0,T].\end{aligned}$$

According to (4.8) and (4.9) the above equation guarantees that

$$\begin{aligned}&\frac{d}{dt}I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))+\nu\int_0^1\sigma_{\lambda x}(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x dx\\ &+\frac{1}{2}|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|_H^2\\ \le\ &\int_0^1(|f_d(\theta(t),\varepsilon(t))_t|+|f_a(\theta(t),\varepsilon(t))_t|)|\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|dx\\ &+\frac{c^2}{2}|\varepsilon_t(t)|_H^2 \quad \text{for a.e. } t\in[0,T].\end{aligned} \tag{4.11}$$

By using Lemma 4.2 we infer that

$$\nu\int_0^1\sigma_{\lambda x}(t)\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x dx$$

$$
\begin{aligned}
&= \lambda\nu \int_0^1 \left| \frac{[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]_x^+}{\lambda} \right|^2 dx + \lambda\nu \int_0^1 \left| \frac{[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]_x^+}{\lambda} \right|^2 dx \\
&\quad + \nu \int_0^1 f_d(\theta(t),\varepsilon(t))_x \frac{[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]_x^+}{\lambda} dx \\
&\quad - \nu \int_0^1 f_a(\theta(t),\varepsilon(t))_x \frac{[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]_x^+}{\lambda} dx \\
&= \lambda\nu \int_0^1 \left(\left| \frac{[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]_x^+}{\lambda} \right|^2 + \left| \frac{[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]_x^+}{\lambda} \right|^2 \right) dx \qquad (4.12) \\
&\quad - \nu \int_0^1 f_d(\theta(t),\varepsilon(t))_{xx} \frac{[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]^+}{\lambda} dx \\
&\quad + \nu \int_0^1 f_a(\theta(t),\varepsilon(t))_{xx} \frac{[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]^+}{\lambda} dx \quad \text{for a.e. } t \in [0,T].
\end{aligned}
$$

Obviously, we get

$$
\begin{aligned}
&\int_0^1 \left| \frac{[\sigma_\lambda(t) - f_d(\theta(t),\varepsilon(t))]_x^+}{\lambda} \right|^2 dx + \int_0^1 \left| \frac{[f_a(\theta(t),\varepsilon(t)) - \sigma_\lambda(t)]_x^+}{\lambda} \right|^2 dx \\
&= \int_0^1 |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x|^2 dx \quad \text{for a.e. } t \in [0,T]. \qquad (4.13)
\end{aligned}
$$

Therefore, it follows from (4.11) $\sim$ (4.13) and (4.9) that

$$
\begin{aligned}
&\frac{d}{dt} I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t)) + \frac{1}{2} |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|_H^2 \\
&+ \lambda\nu |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x|_H^2 \\
&\le 4\nu^2 (|f_a(\theta(t),\varepsilon(t))_{xx}|_H^2 + |f_d(\theta(t),\varepsilon(t))_{xx}|_H^2) + \frac{3}{8} |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|_H^2 \qquad (4.14) \\
&\quad + \frac{c^2}{2} |\varepsilon_t(t)|_H^2 + 2(|f_a(\theta(t),\varepsilon(t))_t|_H^2 + |f_d(\theta(t),\varepsilon(t))_t|_H^2) \text{ for a.e. } t \in [0,T].
\end{aligned}
$$

By combining (4.10) and (4.14) we conclude that

$$
\begin{aligned}
&\frac{1}{2} |\sigma_{\lambda t}(t)|_H^2 + \frac{\nu}{2} \frac{d}{dt} |\sigma_{\lambda x}(t)|_H^2 + 2 \frac{d}{dt} I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t)) \\
&+ \lambda\nu |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))_x|_H^2 + \frac{1}{16} |\partial I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t))|_H^2 \\
&\le c^2 |\varepsilon_t(t)|_H^2 + 4\nu^2 (|f_d(\theta(t),\varepsilon(t))_{xx}|_H^2 + |f_a(\theta(t),\varepsilon(t))_{xx}|_H^2) \\
&\quad + 10(|f_d(\theta(t),\varepsilon(t))_t|_H^2 + |f_a(\theta(t),\varepsilon(t))_t|_H^2) \quad \text{for a.e. } t \in [0,T].
\end{aligned}
$$

Thus we have proved this lemma. □

Lemma 4.5. *It holds that*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2 + \nu|\sigma_{\lambda xx}(t)|_H^2 + \frac{1}{16}|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))|_H^2 \\ & + \lambda(1+\nu)|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))_x|_H^2 + \frac{d}{dt} I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t)) \\ \leq \ & (4\nu^2 + 8)(|f_a(\theta(t), \varepsilon(t))_{xx}|_H^2 + |f_d(\theta(t), \varepsilon(t))_{xx}|_H^2) \\ & + 2(|f_a(\theta(t), \varepsilon(t))_t|_H^2 + |f_d(\theta(t), \varepsilon(t))_t|_H^2) \\ & + \frac{c^2}{2}(|\varepsilon_t(t)|_H^2 + |\varepsilon_{tx}(t)|_H^2) + \frac{1}{2}|\sigma_{\lambda x}(t)|_H^2 \quad \textit{for a.e. } t \in [0, T]. \end{aligned}$$

Proof. We multiply (4.4) by $-\sigma_{\lambda xx}$ and integrate it over $(0, 1)$. Then we see that

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2 + \nu|\sigma_{\lambda xx}(t)|_H^2 \\ = \ & \int_0^1 \partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))\sigma_{\lambda xx}(t)dx - c\int_0^1 \varepsilon_t(t)\sigma_{\lambda xx}(t)dx \\ =: \ & J_1(t) - J_2(t) \quad \text{for a.e. } t \in [0, T]. \end{aligned} \tag{4.15}$$

It is easy to see that

$$|J_2(t)| \leq \frac{c^2}{2}|\varepsilon_{tx}(t)|_H^2 + \frac{1}{2}|\sigma_{\lambda x}(t)|_H^2 \quad \text{for a.e. } t \in [0, T].$$

Similarly to (4.12), we have

$$\begin{aligned} & J_1(t) \\ \leq \ & -\lambda|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))_x|^2 \\ & + \int_0^1 (|f_d(\theta(t), \varepsilon(t))_{xx}| + |f_a(\theta(t), \varepsilon(t))_{xx}|)|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))|dx \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

Combining this with (4.15), we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|\sigma_{\lambda x}(t)|_H^2 + \nu|\sigma_{\lambda xx}(t)|_H^2 + \lambda|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))_x|_H^2 \\ \leq \ & \int_0^1 (|f_d(\theta(t), \varepsilon(t))_{xx}| + |f_a(\theta(t), \varepsilon(t))_{xx}|)|\partial I_\lambda(\theta(t), \varepsilon(t); \sigma_\lambda(t))|dx \\ & + \frac{c^2}{2}|\varepsilon_{tx}(t)|_H^2 + \frac{1}{2}|\sigma_{\lambda x}(t)|_H^2 \quad \text{for a.e. } t \in [0, T]. \end{aligned} \tag{4.16}$$

Moreover, by adding (4.16) to (4.14) we obtain the assertion of this lemma. □

Proof of Proposition 4.1. Integrating (4.7) over $[0,t]$, $0<t<T$, implies

$$\begin{aligned}
&\frac{1}{2}\int_0^t |\sigma_{\lambda\tau}(\tau)|_H^2 d\tau + \frac{\nu}{2}|\sigma_{\lambda x}(t)|_H^2 + 2I_\lambda(\theta(t),\varepsilon(t);\sigma_\lambda(t)) \\
&+\lambda\nu\int_0^t |\partial I_\lambda(\theta(\tau),\varepsilon(\tau);\sigma_\lambda(\tau))_x|_H^2 d\tau + \frac{1}{16}\int_0^t |\partial I_\lambda(\theta(\tau),\varepsilon(\tau);\sigma_\lambda(\tau))|_H^2 d\tau \\
\le\ & \frac{\nu}{2}|\sigma_{0x}|_H^2 + 2I(\theta_0,\varepsilon_0;\sigma_0) + c^2\int_0^t |\varepsilon_\tau(\tau)|_H^2 d\tau \\
&+4\nu^2\int_0^t (|f_d(\theta(\tau),\varepsilon(\tau))_{xx}|_H^2 + |f_a(\theta(\tau),\varepsilon(\tau))_{xx}|_H^2)d\tau \\
&+10\int_0^t (|f_d(\theta(\tau),\varepsilon(\tau))_\tau|_H^2 + |f_a(\theta(\tau),\varepsilon(\tau))_\tau|_H^2)d\tau \quad \text{for any } t\in[0,T].
\end{aligned} \tag{4.17}$$

It yields that $\{\sigma_\lambda\}$ is bounded in $W^{1,2}(0,T;H)$ and $L^\infty(0,T;H^1(0,1))$, $\{I_\lambda(\theta(\cdot),\varepsilon(\cdot);\sigma_\lambda(\cdot))\}$ is bounded in $L^\infty(0,T)$ and $\{\partial I_\lambda(\theta,\varepsilon;\sigma_\lambda)\}$ is bounded in $L^2(Q(T))$. Then we can take a subsequence $\{\lambda_j\}$ of $\{\lambda\}$ such that

$$\left.\begin{aligned}
&\sigma_j \to \hat\sigma \text{ weakly in } W^{1,2}(0,T;H),\ \text{weakly* in } L^\infty(0,T;H^1(0,1)) \\
&\quad\text{and in } C(\overline{Q(T)}), \\
&I_{\lambda_j}(\theta,\varepsilon;\sigma_j) \to \hat I \text{ weakly* in } L^\infty(0,T), \\
&\partial I_{\lambda_j}(\theta,\varepsilon;\sigma_\lambda) \to \hat\xi \text{ weakly in } L^2(Q(T))
\end{aligned}\right\} \text{as } j\to\infty, \tag{4.18}$$

where $\sigma_j=\sigma_{\lambda_j}$, $\hat\sigma\in W^{1,2}(0,T;H)\cap L^\infty(0,T;H^1(0,1))$, $\hat I\in L^\infty(0,T)$ and $\hat\xi\in L^2(Q(T))$.

Now, we show that $\{u,\theta,\hat\sigma\}$ is a solution of P_ν on $[0,T]$. It follows from the above convergences that $\hat\sigma_t-\nu\hat\sigma_{xx}+\hat\xi=c\varepsilon_t$ in $Q(T)$. Hence, it is sufficient to prove that $\hat\xi\in\partial I(\theta,\varepsilon;\hat\sigma)$ a.e. on $Q(T)$. In fact, Lemma 4.2 guarantees that

$$[\sigma_j-f_d(\theta,\varepsilon)]^+ - [f_a(\theta,\varepsilon)-\sigma_j]^+ = \lambda_j\partial I_{\lambda_j}(\theta,\varepsilon;\sigma_j)\to 0 \text{ in } L^2(Q(T)) \text{ as } j\to\infty.$$

This convergence shows that $f_a(\theta,\varepsilon)\le\hat\sigma\le f_d(\theta,\varepsilon)$ on $Q(T)$. Next, let z be any function in $L^2(Q(T))$ with $f_a(\theta,\varepsilon)\le z\le f_d(\theta,\varepsilon)$ a.e. on $Q(T)$. Clearly,

$$\int_{Q(T)}\partial I_{\lambda_j}(\theta,\varepsilon;\sigma_j)(z-\sigma_j)dxdt \to \int_{Q(T)}\hat\xi(z-\hat\sigma)dxdt \quad \text{as } j\to\infty,$$

and

$$\int_{Q(T)}\partial I_{\lambda_j}(\theta,\varepsilon;\sigma_j)(z-\sigma_j)dxdt\le 0 \quad \text{for each } j.$$

Therefore, we have

$$\int_{Q(T)}\hat\xi(z-\sigma)dxdt\le 0$$

so that $\hat\xi(t)\in\partial I(\theta(t),\varepsilon(t);\hat\sigma(t))$ for a.e. $t\in[0,T]$. Noting that $\{u,\theta,\sigma\}$ satisfies (S4), we easily deduce by the standard monotonicity argument that $\sigma=\hat\sigma$ a.e. on $Q(T)$. Moreover, by letting $j \to \infty$ in (4.17) we get (4.2). Here, we use properties of subdifferential operators (cf. [7, Proposition 0.3.5.]).

To accomplish the proof of Proposition 4.1 we shall show (4.3). By applying the Gronwall inequality to Lemma 4.5 we see that

$$\begin{aligned}
&\frac{1}{2}|\sigma_{\lambda_j x}(t)|_H^2 + \frac{1}{16}\int_0^t |\partial I_{\lambda_j}(\theta(\tau), \varepsilon(\tau); \sigma_j(\tau))|_H^2 d\tau + I_{\lambda_j}(\theta(t), \varepsilon(t); \sigma_j(t)) \\
\leq\ & \frac{e^T}{2}|\sigma_{0x}|_H^2 + e^T I(\theta_0, \varepsilon_0; \sigma_0) \\
&+(4\nu^2+8)e^T \int_0^T (|f_a(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_{xx}|_H^2) d\tau \\
&+2e^T \int_0^T (|f_a(\theta(\tau), \varepsilon(\tau))_\tau|_H^2 + |f_d(\theta(\tau), \varepsilon(\tau))_\tau|_H^2) d\tau \\
&+\frac{c^2 e^T}{2} \int_0^T (|\varepsilon_\tau(\tau)|_H^2 + |\varepsilon_{\tau x}(\tau)|_H^2) d\tau \quad \text{for } t \in [0, T].
\end{aligned}$$

Letting $j \to \infty$ and (4.18) imply (4.3). □

In the rest of this section we consider the following problem SP(g, v_0, w_0):

(4.19) $$v_{tt} + \gamma v_{xxxx} - \mu v_{txx} = g_{xt} \quad \text{in } Q(T),$$

(4.20) $$v(t,0) = v(t,1) = v_{xx}(t,0) = v_{xx}(t,1) = 0 \quad \text{for } t \in [0,T],$$

(4.21) $$v(0,\cdot) = v_0, v_t(0,\cdot) = w_0 \quad \text{on } (0,1),$$

where g is a given function on $Q(T)$ and v_0 and w_0 are initial functions.

Now, we give a definition of a weak solution of SP.

Definition 4.1. We say that v is a weak solution of SP(g, v_0, w_0) on $[0,T]$ if the following conditions hold.

(D1) $v \in W^{1,2}(0,T;H) \cap L^2(0,T;H^2(0,1)) \cap L^2(0,T;V)$;

(D2) It holds that

$$\begin{aligned}
&-\int_{Q(T)} v_t \eta_t dxdt + \gamma \int_{Q(T)} v_{xx}\eta_{xx} dxdt + \mu \int_{Q(T)} v_{xx}\eta_t dxdt \\
=\ & -\int_{Q(T)} g_t \eta_x dxdt + \int_0^1 w_0 \eta(0,\cdot) dx - \mu \int_0^1 v_{0xx}\eta(0,\cdot) dx \\
&\text{for } \eta \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \cap L^2(0,T;H^2(0,1)) \text{ with } \eta(T) = 0;
\end{aligned}$$

(D3) $v(0,x) = v_0(x)$ for $x \in (0,1)$.

The following lemma is concerned with the wellposedness of SP.

Lemma 4.6. *If $g \in W^{1,2}(0,T;H)$, $v_0 \in H^2(0,1)$ and $w_0 \in H$, then SP(g, v_0, w_0) admits* *a unique weak solution v on $[0,T]$. Moreover, there exists a positive constant K_* depending only on μ and γ such that*

$$\begin{aligned}&|v_{xx}(t)|_H^2+|v_t(t)|_H^2+\int_0^t|v_{\tau x}(\tau)|_H^2d\tau\\ \le\ & K_*(|v_{0xx}|_H^2+|w_0|_H^2+|g_t|_{L^2(0,T;H)}^2)\quad \textit{for } t\in[0,T].\end{aligned}$$

The proof of this lemma is quite standard so we omit it.

5 Proof of the existence

The aim of this section is to prove Theorem 2.1 (2). Throughout this section we suppose all the assumptions in Theorem 2.1. The proof is so long that we divide it into several lemmas. First, for each $\nu>0$ there exists a unique solution $\{u_\nu,\theta_\nu,\sigma_\nu\}$ of P_ν on $[0,T]$ because of Lemma 4.1, and we write $\varepsilon_\nu=u_{\nu x}$.

Lemma 5.1. *It holds that*

$$|\sigma_\nu|_{L^\infty(Q(T))}\le L\quad \textit{for } \nu>0.$$

Proof. By the definition of the subdifferential and (A2) we obtain $-L\le f_a(\theta_\nu,\varepsilon_\nu)\le\sigma_\nu\le f_d(\theta_\nu,\varepsilon_\nu)\le L$ on $Q(T)$. □

Lemma 5.2. *There exists a positive constant K_1 such that*

$$|u_{\nu t}(t)|_H^2\le K_1, |u_{\nu xx}(t)|_H^2\le K_1 \textit{ for any } t\in[0,T] \textit{ and } \int_0^T|u_{\nu xt}(t)|_H^2dt\le K_1,\ \textit{for } \nu>0.$$

Proof. By integrating by parts it is easy to show that

$$\frac{1}{2}\frac{d}{dt}|u_{\nu t}(t)|_H^2+\frac{\gamma}{2}\frac{d}{dt}|u_{\nu xx}(t)|_H^2+\mu|u_{\nu xt}(t)|_H^2=-\int_0^1\sigma_\nu(t)u_{\nu xt}(t)dx\quad \text{for a.e. } t\in[0,T].$$

Then we have

$$\frac{1}{2}\frac{d}{dt}|u_{\nu t}(t)|_H^2+\frac{\gamma}{2}\frac{d}{dt}|u_{\nu xx}(t)|_H^2+\frac{\mu}{2}|u_{\nu xt}(t)|_H^2\le\frac{L^2}{2\mu}\quad \text{for a.e. } t\in[0,T]$$

so that the assertion of this lemma is true. □

Lemma 5.3. *There exists a positive constant K_2 such that*

$$|u_{\nu xt}|_{L^4(Q(T))}\le K_2\quad \textit{for } \nu\in(0,1].$$

Proof. By taking $p=4$ in Lemma 2.1 we can prove Lemma 5.3. □

Lemma 5.4. *There exists a positive constant K_3 such that*

$$\int_0^T |\theta_{\nu\tau}(\tau)|_H^2 dt + \int_0^T |\theta_{\nu xx}(\tau)|_H^2 dt + |\theta_{\nu x}(t)|_H^2 \leq K_3 \text{ for } t \in [0,T] \text{ and } \nu \in (0,1].$$

Proof. Immediately, we obtain

$$\begin{aligned}
& |\theta_{\nu t}(t)|_H^2 + \frac{\kappa}{2}\frac{d}{dt}|\theta_{\nu x}(t)|_H^2 \\
= & \int_0^1 (\sigma_\nu(t)u_{\nu xt}(t) + \mu|u_{\nu xt}(t)|^2)\theta_{\nu t}(t)dx \\
\leq & \frac{1}{2}|\theta_{\nu t}(t)|_H^2 + L^2|u_{\nu xt}(t)|_H^2 + \mu^2\int_0^1 |u_{\nu xt}(t)|^4 dx \quad \text{for a.e. } t \in [0,T].
\end{aligned}$$

Hence, we have

$$|\theta_{\nu t}(t)|_H^2 + \kappa\frac{d}{dt}|\theta_{\nu x}(t)|_H^2 \leq 2L^2|u_{\nu xt}(t)|_H^2 + 2\mu^2\int_0^1 |u_{\nu xt}(t)|^4 dx \quad \text{for a.e. } t \in [0,T].$$

Lemmas 5.2 and 5.3 imply the conclusion of this lemma. □

From now on, for simplicity we put

$$\left.\begin{aligned}
F_{1\nu}(t) &= |f_a(\theta_\nu(t),\varepsilon_\nu(t))_{xx}|_H^2 + |f_d(\theta_\nu(t),\varepsilon_\nu(t))_{xx}|_H^2 \\
F_{2\nu}(t) &= |f_a(\theta_\nu(t),\varepsilon_\nu(t))_t|_H^2 + |f_d(\theta_\nu(t),\varepsilon_\nu(t))_t|_H^2
\end{aligned}\right\} \text{ for } t \in [0,T] \text{ and } \nu > 0.$$

Lemma 5.5. *For $\nu \in (0,1]$ let $\xi_\nu \in L^2(Q(T))$ be a function defined by (4.1). Then there exists a positive constant K_4 independent of ν such that*

$$\begin{aligned}
&\int_0^T |\sigma_{\nu t}(t)|_H^2 dt \leq K_4, \int_0^T |u_{\nu txx}(t)|_H^2 dt \leq K_4, \int_0^T |\xi_\nu(t)|_H^2 dt \leq K_4, \\
&I(\theta_\nu(t),\varepsilon_\nu(t);\sigma_\nu(t)) \leq K_4, |u_{\nu tx}(t)|_H^2 \leq K_4, |u_{\nu xxx}(t)|_H^2 \leq K_4 \quad \text{for any } t \in [0,T].
\end{aligned}$$

Proof. We multiply (1.6) by $-u_{\nu txx}(t)$ and integrate it over $(0,1)$. Then we see that

$$\begin{aligned}
& \frac{d}{dt}|u_{\nu tx}(t)|_H^2 + \frac{\gamma}{2}\frac{d}{dt}|u_{\nu xxx}(t)|_H^2 + \mu|u_{\nu txx}(t)|_H^2 \\
= & -\int_0^1 \sigma_{\nu x}(t)u_{\nu txx}(t)dx \\
= & \int_0^1 \sigma_\nu(t)u_{\nu txxx}(t)dx \\
= & \frac{d}{dt}\int_0^1 \sigma_\nu(t)u_{\nu xxx}(t)dx - \int_0^1 \sigma_{\nu t}(t)u_{\nu xxx}(t)dx \quad \text{for a.e. } t \in [0,T].
\end{aligned} \tag{5.1}$$

Next, we integrate (5.1) over $[0,t]$, $0<t<T$, and get

$$\begin{aligned} & |u_{\nu tx}(t)|_H^2 + \frac{\gamma}{2}|u_{\nu xxx}(t)|_H^2 + \mu\int_0^t |u_{\nu\tau xx}(\tau)|_H^2 d\tau \\ = \ & |v_{0x}|_H^2 + \frac{\gamma}{2}|u_{0xxx}|_H^2 + \int_0^1 \sigma_\nu(t)u_{\nu xxx}(t)dx \\ & -\int_0^1 \sigma_0 u_{0xxx}dx - \int_0^t\int_0^1 \sigma_{\nu\tau}(\tau)u_{\nu xxx}(\tau)dxd\tau \quad \text{for } t\in[0,T]. \end{aligned} \tag{5.2}$$

Here, by using Proposition 4.1 the following inequality is obtained.

$$\begin{aligned} & \frac{1}{2}\int_0^t |\sigma_{\nu\tau}(\tau)|_H^2 d\tau + 2I(\theta_\nu(t),\varepsilon_\nu(t);\sigma_\nu(t)) + \frac{1}{16}\int_0^t |\xi_\nu(\tau)|_H^2 d\tau \\ \le \ & 2I(\theta_0,\varepsilon_0;\sigma_0) + \frac{\nu}{2}|\sigma_{0x}|_H^2 + 4\nu^2\int_0^t F_{1\nu}(\tau)d\tau \\ & +c^2\int_0^t |u_{\nu x\tau}(\tau)|_H^2 d\tau + 10\int_0^t F_{2\nu}(\tau)d\tau \quad \text{for } t\in[0,T]. \end{aligned} \tag{5.3}$$

Adding (5.2) and (5.3) gives

$$\begin{aligned} & \frac{1}{2}\int_0^t |\sigma_{\nu\tau}(\tau)|_H^2 d\tau + |u_{\nu tx}(t)|_H^2 + \frac{\gamma}{2}|u_{\nu xxx}(t)|_H^2 + \mu\int_0^t |u_{\nu\tau xx}(\tau)|_H^2 d\tau \\ & +2I(\theta_\nu(t),\varepsilon_\nu(t);\sigma_\nu(t)) + \frac{1}{16}\int_0^t |\xi_\nu(\tau)|_H^2 d\tau \\ \le \ & |v_{0x}|_H^2 + \frac{\gamma}{2}|u_{0xxx}|_H^2 + 2I(\theta_0,\varepsilon_0;\sigma_0) + \frac{\nu}{2}|\sigma_{0x}|_H^2 - \int_0^1 \sigma_0 u_{0xxx}dx \\ & +\frac{\gamma}{4}|u_{\nu xxx}(t)|_H^2 + \frac{L^2}{\gamma} + \frac{1}{4}\int_0^t |\sigma_{\nu\tau}(\tau)|_H^2 d\tau + \int_0^t |u_{\nu xxx}(\tau)|_H^2 d\tau \\ & +4\nu^2\int_0^t F_{1\nu}(\tau)d\tau + c^2\int_0^t |u_{\nu x\tau}(\tau)|_H^2 d\tau + 10\int_0^t F_{2\nu}(\tau)d\tau \text{ for } t\in[0,T]. \end{aligned} \tag{5.4}$$

Now, we note that

$$\begin{aligned} & \int_0^t (|f_a(\theta_\nu(\tau),\varepsilon_\nu(\tau))_{xx}|_H^2 + |f_d(\theta_\nu(\tau),\varepsilon_\nu(\tau))_{xx}|_H^2)d\tau \\ \le \ & 8L^2\int_0^t\int_0^1 (|\theta_{\nu xx}(\tau)| + |\theta_{\nu x}(\tau)||\varepsilon_{\nu x}(\tau)| + |\varepsilon_{\nu xx}(\tau)| + |\theta_{\nu x}(\tau)|^2 + |\varepsilon_{\nu x}(\tau)|^2)^2 dxd\tau \\ \le \ & C_3\int_0^t (|\theta_{\nu xx}(\tau)|_H^2 + |\theta_{\nu x}(\tau)|_{L^4(0,1)}^4 + |u_{\nu xx}(\tau)|_{L^4(0,1)}^4 + |u_{\nu xxx}(t)|_H^2)d\tau; \end{aligned}$$

$$\int_0^t (|f_a(\theta_\nu(\tau), \varepsilon_\nu(\tau))_\tau|_H^2 + |f_d(\theta_\nu(\tau), \varepsilon_\nu(\tau))_\tau|_H^2) d\tau$$

$$\leq \; 4L^2 \int_0^t (|\theta_{\nu\tau}(\tau)|_H^2 + |u_{\nu x\tau}(\tau)|_H^2) d\tau \quad \text{for } t \in [0,T],$$

where C_3 is a positive constant depending only on L. We recall the Gagliardo-Nirenberg inequality, again;

$$|w|_{L^4(0,1)} \leq C_0 |w_x|_H^{1/4} |w|_H^{3/4} \quad \text{for } w \in V,$$

where C_0 is a positive constant. Since $\theta_{\nu x}(t) \in V$ and $u_{\nu xx}(t) \in V$ for a.e. $t \in [0,T]$, we can apply the above inequality so that we obtain

$$\int_0^t (|\theta_{\nu x}(\tau)|_{L^4(0,1)}^4 + |u_{\nu xx}(\tau)|_{L^4(0,1)}^4) d\tau$$

$$\leq \; C_0^4 \int_0^t (|\theta_{\nu xx}(\tau)|_H |\theta_{\nu x}(\tau)|_H^3 + |u_{\nu xxx}(\tau)|_H |u_{\nu xx}(\tau)|_H^3) d\tau$$

$$\leq \; C_0^4 K_3^{3/2} \int_0^t |\theta_{\nu xx}(\tau)|_H d\tau + C_0^4 K_1^{3/2} \int_0^t |u_{\nu xxx}(\tau)|_H d\tau$$

$$\leq \; C_0^4 K_3^{3/2} (T + K_3) + C_0^4 K_1^{3/2} (T + \int_0^t |u_{\nu xxx}(\tau)|_H^2 d\tau)$$

$$\leq \; C_4 (1 + \int_0^t |u_{\nu xxx}(\tau)|_H^2 d\tau) \quad \text{for } t \in [0,T],$$

where $C_4 = C_0^4 K_3^{3/2}(T + K_3) + C_0^4 K_1^{3/2}(1 + T)$. From the above arguments we have

$$\frac{1}{4} \int_0^t |\sigma_{\nu\tau}(\tau)|_H^2 d\tau + |u_{\nu tx}(t)|_H^2 + \frac{\gamma}{4} |u_{\nu xxx}(t)|_H^2 + \mu \int_0^t |u_{\nu\tau xx}(\tau)|_H^2 d\tau$$

$$+2I(\theta_\nu(t), \varepsilon_\nu(t); \sigma_\nu(t)) + \frac{1}{16} \int_0^t |\xi_\nu(\tau)|_H^2 d\tau$$

(5.5) $$\leq \; C(u_0, v_0, \theta_0, \sigma_0) + C_5 (1 + \int_0^t |u_{\nu xxx}(\tau)|_H^2 d\tau) \quad \text{for } t \in [0,T],$$

where $C(u_0, v_0, \theta_0, \sigma_0) = |v_{0x}|_H^2 + \frac{\gamma}{2} |u_{0xxx}|_H^2 + 2I(\theta_0, \varepsilon_0; \sigma_0) - \int_0^1 \sigma_0 u_{0xxx} dx + \frac{\nu}{2} |\sigma_{0x}|_H^2$ and C_5 is a positive constant independent of ν. By applying the Gronwall inequality to (5.5) we get the assertion of Lemma 5.5. □

Lemma 5.6. *There exists a positive constant K_5 such that*

$$|\sigma_{\nu x}(t)|_H^2 \leq K_5 \quad \textit{for } t \in [0,T] \textit{ and } \nu \in (0,1].$$

Proof. (4.3) implies that there exists a positive constant C_6 satisfying

$$|\sigma_{\nu x}(t)|_H^2 + I(\theta_\nu(t), \varepsilon_\nu(t); \sigma_\nu(t)) + \int_0^t |\xi_\nu(\tau)|_H^2 d\tau$$

$$\le \; C_6|\sigma_{0x}|_H^2 + C_6 I(\theta_0, \varepsilon_0; \sigma_0) + C_6 \int_0^T (|u_{\nu tx}(t)|_H^2 + |u_{\nu txx}(t)|_H^2) d\tau$$

$$+ C_6 \int_0^T (F_{1\nu}(\tau) + F_{2\nu}(\tau)) dt \quad \text{for } 0 \le t \le T.$$

Similarly to the proof of Lemma 5.5, the estimates for $F_{1\nu}$ and $F_{2\nu}$ hold. Hence, previous lemmas guarantee this lemma. □

Lemma 5.7. *$\{u_{\nu tt}\}$ is bounded in $L^\infty(0, T; V^*)$.*

Proof. Let $\eta \in V$. Immediately, we see that

$$\begin{aligned}
&\int_0^1 u_{\nu tt}(t) \eta dx \\
= \; &\int_0^1 \sigma_{\nu x}(t) \eta dx + \mu \int_0^1 u_{\nu txx}(t) \eta dx - \gamma \int_0^1 u_{\nu xxxx}(t) \eta dx \\
= \; &-\int_0^1 \sigma_\nu(t) \eta_x dx - \mu \int_0^1 u_{\nu tx}(t) \eta_x dx + \gamma \int_0^1 u_{\nu xxx}(t) \eta_x dx \quad \text{for a.e. } t \in [0, T].
\end{aligned}$$

Hence,

$$\left| \int_0^1 u_{\nu tt}(t) \eta dx \right| \le (|\sigma_\nu(t)|_H + \mu |u_{\nu tx}(t)|_H + \gamma |u_{\nu xxx}(t)|_H) |\eta|_V \quad \text{for a.e. } t \in [0, T].$$

This gives the assertion of this lemma. □

Now, we give a proof of the existence of a solution to P_0.

Proposition 5.1. *P_0 admits at least one solution $\{u, \theta, \sigma\}$ on $[0, T]$.*

Proof. By Lemmas 5.1 ∼ 5.7 there exists a subsequence $\{\nu_j\}$ of $\{\nu\}$ such that

$$\begin{aligned}
\theta_j := \theta_{\nu_j} \to \theta \quad &\text{weakly in } W^{1,2}(0, T; H), \text{ weakly* in } L^\infty(0, T; H^1(0, 1)), \\
u_j := u_{\nu_j} \to u \quad &\text{weakly* in } L^\infty(0, T; H^3(0, 1)) \text{ and } W^{2,\infty}(0, T; V^*), \\
u_{jtxx} \to u_{txx} \quad &\text{weakly in } L^2(0, T; H), \\
u_{jtx} \to u_{tx} \quad &\text{weakly* in } L^\infty(0, T; H) \text{ and weakly in } L^4(Q(T)), \\
\sigma_j := \sigma_{\nu_j} \to \sigma \quad &\text{weakly in } W^{1,2}(0, T; H), \text{ weakly* in } L^\infty(Q(T)), \\
&\text{weakly* in } L^\infty(0, T; H^1(0, 1)), \\
\xi_j := \xi_{\nu_j} \to \xi \quad &\text{weakly in } L^2(0, T; H), \text{ as } j \to \infty,
\end{aligned}$$

where $u \in W^{2,2}(0, T; V^*) \cap W^{1,2}(0, T; H^2(0, 1)) \cap W^{1,\infty}(0, T; H^1(0, 1)) \cap L^\infty(0, T; H^3(0, 1))$, $\theta \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1))$, $\sigma \in W^{1,2}(0, T; H) \cap L^\infty(0, T; H^1(0, 1))$ and $\xi \in L^2(0, T; H)$. These convergences imply that

(5.6) $$\theta_j \to \theta, u_{jx} \to u_x \text{ and } \sigma_j \to \sigma \text{ in } C(\overline{Q(T)}) \text{ as } j \to \infty.$$

First, we show that (1.7) holds for a.e. $(t, x) \in Q(T)$. Putting $\eta \in L^2(0, T; H^1(0, 1))$, we have

$$(5.7) \quad \int_{Q(T)} \theta_{jt}\eta dxdt + \kappa \int_{Q(T)} \theta_{jx}\eta_x dxdt = \int_{Q(T)} \sigma_j u_{jxt}\eta dxdt + \mu \int_{Q(T)} |u_{jxt}|^2 \eta dxdt.$$

It is easy to see that

$$(5.8) \int_{Q(T)} \theta_{jt}\eta dxdt \to \int_{Q(T)} \theta_t \eta dxdt \text{ and } \int_{Q(T)} \theta_{jx}\eta_x dxdt \to \int_{Q(T)} \theta_x \eta_x dxdt \text{ as } j \to \infty.$$

It holds that

$$(5.9) \qquad \int_{Q(T)} \sigma_j u_{jxt}\eta dxdt \to \int_{Q(T)} \sigma u_{xt}\eta dxdt \text{ as } j \to \infty.$$

Indeed, $\sigma_j u_{jxt} = (\sigma_j - \sigma)u_{jxt} + \sigma u_{jxt}$, $(\sigma_j - \sigma)u_{jxt} \to 0$ in $L^2(Q(T))$ and $\sigma u_{jxt} \to \sigma u_{xt}$ weakly in $L^2(Q(T))$.

Also, we have

$$|\int_{Q(T)} \sigma_j u_{jxt}\eta dxdt - \int_{Q(T)} \sigma_j u_{jxt}\eta_i dxdt| \le L|u_{jxt}|_{L^2(Q(T))}|\eta_i - \eta|_{L^2(Q(T))}$$

and

$$|\int_{Q(T)} \sigma u_{xt}\eta dxdt - \int_{Q(T)} \sigma u_{xt}\eta_i dxdt| \le L|u_{xt}|_{L^2(Q(T))}|\eta_i - \eta|_{L^2(Q(T))}.$$

These three inequalities imply the convergence (5.9).

Next, we prove

$$(5.10) \qquad \int_{Q(T)} |u_{jxt}|^2 \eta dxdt \to \int_{Q(T)} |u_{xt}|^2 \eta dxdt \text{ as } j \to \infty.$$

In fact, putting $v_j = u_{jt}$ for each j, we have already known that $\{v_{jt}\}$ is bounded in $L^\infty(0, T; V^*)$ and $\{v_j\}$ is bounded in $L^2(0, T; H^2(0, 1))$. Then by Aubin's compact theorem we may write $v_j \to v$ in $L^2(0, T; H^1(0, 1))$ as $j \to \infty$, that is, $u_{jtx} \to u_{tx}$ in $L^2(0, T; H)$ as $j \to \infty$. Now, let $\{\eta_m\}$ be a sequence in $C^\infty(\overline{Q(T)})$ satisfying $\eta_m \to \eta$ in $L^2(Q(T))$ as $m \to \infty$. For each m and j we infer that

$$\begin{aligned}
& |\int_{Q(T)} |u_{jtx}|^2 \eta_m dxdt - \int_{Q(T)} |u_{tx}|^2 \eta_m dxdt| \\
\le & \int_{Q(T)} |u_{jtx} - u_{tx}|^2 \eta_m dxdt + 2|\int_{Q(T)} (u_{jtx} - u_{tx})u_{tx}\eta_m dxdt| \\
(5.11) \quad \le & |\eta_m|_{L^\infty(Q(T))}|u_{jtx} - u_{tx}|^2_{L^2(Q(T))} + |\eta_m|_{L^\infty(Q(T))}|u_{jtx} - u_{tx}|_{L^2(Q(T))}|u_{tx}|_{L^2(Q(T))}.
\end{aligned}$$

Also, it holds that

$$\begin{aligned}
& \int_{Q(T)} (|u_{jtx}|^2 + |u_{tx}|^2)|\eta_m - \eta| dxdt \\
(5.12) \qquad \le & (|u_{jtx}|^2_{L^4(Q(T))} + |u_{tx}|^2_{L^4(Q(T))})|\eta_m - \eta|_{L^2(Q(T))} \quad \text{for } j \text{ and } m.
\end{aligned}$$

Therefore, (5.10) is derived from (5.11) and (5.12) so that

$$\begin{aligned}&\int_{Q(T)} \theta_t \eta dxdt + \kappa \int_{Q(T)} \theta_x \eta_x dxdt \\ = &\int_{Q(T)} \sigma u_{xt} \eta dxdt + \mu \int_{Q(T)} |u_{xt}|^2 \eta dxdt \quad \text{for } \eta \in L^2(0,T;H^1(0,1)).\end{aligned}$$

It follows from this identity that (1.7) is valid. $\theta(0) = \theta_0$ is trivial.

The proofs of (S2) and the following equation are straightforward:

$$\sigma_t + \xi = cu_{xt} \quad \text{a.e. on } Q(T),$$

since by (1.14) the set $\{\sqrt{\nu}\sigma_{\nu xx}\}$ is bounded in $L^2(Q(T))$ and $\nu\sigma_{\nu xx} \to 0$ in $L^2(Q(T))$ as $\nu \to 0$. Consequently, in order to accomplish the proof of this proposition it is sufficient to show that $\xi \in \partial I(\theta, \varepsilon; \sigma)$ a.e. on $Q(T)$ where $\varepsilon = u_x$. By definition we obtain $f_a(\theta_j, \varepsilon_j) \le \sigma_j \le f_d(\theta_j, \varepsilon_j)$ a.e. on $Q(T)$ for each j where $\varepsilon_j = u_{jx}$. Hence, it follows from (5.6) that

$$f_a(\theta, \varepsilon) \le \sigma \le f_d(\theta, \varepsilon) \quad \text{a.e. on } Q(T).$$

Let $z \in L^2(Q(T))$ and $f_a(\theta, \varepsilon) \le z \le f_d(\theta, \varepsilon)$ a.e. on $Q(T)$ and

$$z_j = \max\{\min\{f_d(\theta_j, \varepsilon_j), z\}, f_a(\theta_j, \varepsilon_j)\}.$$

Immediately, we have $f_a(\theta_j, \varepsilon_j) \le z_j \le f_d(\theta_j, \varepsilon_j)$ a.e. on $Q(T)$ and $\int_{Q(T)} \xi_j (z_j - \sigma_j) dxdt \le 0$ for each j. Letting $j \to \infty$ in the inequality, we conclude that

$$\int_{Q(T)} \xi(z - \sigma) dxdt \le 0.$$

Thus we have proved this proposition. □

At the rest of paper we must show the regularity result for u mentioned in the statement of Theorem 2.1.

Proof of Theorem 2.1 (2). It holds that

(5.13) $\quad u_t \in L^\infty(0,T;H^2(0,1)), u \in W^{2,\infty}(0,T;H)$ and $u_t \in W^{1,2}(0,T;H^1(0,1))$.

In fact, we put $v_j = u_{jt}$. Clearly,

$$\int_{Q(T)} (u_{jtt} + \gamma u_{jxxxx} - \mu u_{jtxx}) \eta_t dxdt = \int_{Q(T)} \sigma_{jx} \eta_t dxdt$$

for $\eta \in C^\infty(\overline{Q(T)})$ with $\eta(T) = 0$ and $\eta(t,0) = \eta(t,1) = 0$ for $t \in [0,T]$. Now, we have

$$\int_{Q(T)} u_{jtt} \eta_t dxdt = \int_{Q(T)} v_{jt} \eta_t dxdt;$$

$$
\begin{aligned}
\gamma \int_{Q(T)} u_{jxxxx}\eta_t dxdt &= \gamma \int_{Q(T)} u_{jxx}\eta_{txx} dxdt \\
&= -\gamma \int_{Q(T)} u_{jtxx}\eta_{xx} dxdt - \gamma \int_0^1 u_{0xx}\eta(0,\cdot)_{xx} dx \\
&= -\gamma \int_{Q(T)} v_{jxx}\eta_{xx} dxdt - \gamma \int_0^1 u_{0xxxx}\eta(0,\cdot) dx;
\end{aligned}
$$

$$
\mu \int_{Q(T)} u_{jtxx}\eta_t dxdt = \mu \int_{Q(T)} v_{jxx}\eta_t dxdt;
$$

$$
-\int_{Q(T)} \sigma_{jx}\eta_t dxdt = \int_{Q(T)} \sigma_{jt}\eta_x dxdt - \int_0^1 \sigma_{0x}\eta(0,\cdot) dx.
$$

Hence,

$$
\begin{aligned}
& -\int_{Q(T)} v_{jt}\eta_t dxdt + \gamma \int_{Q(T)} v_{jxx}\eta_{xx} dxdt + \mu \int_{Q(T)} v_{jxx}\eta_t dxdt \\
= & -\int_{Q(T)} \sigma_{jt}\eta_x dxdt + \int_0^1 w_0\eta(0,\cdot) dx - \mu \int_0^1 v_{0xx}\eta(0,\cdot) dx
\end{aligned}
$$

for $\eta \in C^\infty(\overline{Q(T)})$ with $\eta(T) = 0$ and $\eta(t,0) = \eta(t,1) = 0$ for $t \in [0,T]$,

where $w_0 = \mu v_{0xx} + \sigma_{0x} - \gamma u_{0xxxx}$. It implies that for each j v_j is a weak solution of $\mathrm{SP}(\sigma_j, v_0, w_0)$ on $[0,T]$. By Lemma 4.6 we get

$$
\begin{aligned}
& |u_{jtxx}(t)|_H^2 + |u_{jtt}(t)|_H^2 + \int_0^t |u_{j\tau\tau x}(\tau)|_H^2 d\tau \\
\leq & \; K_*(|v_{0xx}|_H^2 + |w_0|_H^2 + |\sigma_{jt}|_{L^2(0,T;H)}^2) \quad \text{for } t \in [0,T].
\end{aligned}
$$

Therefore, (5.13) is easily obtained. Thus we have completely proved Thorem 2.1. □

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