Frequent Oscillation Criteria for a Delay Difference Equation *

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Abstract

This paper is concerned with the linear delay difference equation

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \; n = 0, 1, 2, \cdots,$$

where k is a nonnegative integer and $\{p_n\}_{n=0}^{\infty}$ is a real sequence. Sufficient conditions for this equation to be frequently oscillatory are derived.

AMS Subject Classification : 39A10
Key Words: Difference equation, Frequent oscillation.

1 Introduction

A nontrivial extension of the well known difference equation

$$x_{n+1} = x_n + x_{n-1}, \; n = 0, 1, 2, ...,$$

satisfied by the Fibonacci numbers is the following

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \; n = 0, 1, 2, ... \tag{1}$$

where k is a fixed positive integer. This equation has received much attention. In particular, Erbe and Zhang [1] proved that when $\{p_n\}$ is eventually nonnegative, then every solution of (1) oscillates provided

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1,$$

*Partially supported by the National Natural Science Foundation of China

or

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{2}$$

or

$$\liminf_{n\to\infty} p_n = \delta \geq 0 \text{ and } \limsup_{n\to\infty} p_n > 1-\delta.$$

Ladas et al. in [2] obtained the same conclusion when (2) is replaced by

$$\liminf_{n\to\infty} \frac{1}{k}\sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}.$$

Since then, there have been many improvements (see [1-11]). In particular, a summary of related results can be found in the recent paper [9].

In this paper, we intend to obtain several nonstandard oscillation criteria based on the concept of frequent oscillation. Since frequent oscillation implies oscillation, our results will either be more general than or complementary to some of the results in [1-11].

In order to derive these criteria, we first recall that a real sequence is said to be oscillatory if it is neither eventually positive nor eventually negative. Clearly, such a definition does not capture the fine details of an oscillatory sequence as can be seen from the following two oscillatory sequence $\{1,-1,1,-1,...\}$ and $\{1,1,1,-1,1,1,1,-1,...\}$. For this reason, Tian et al. in [10] introduced the concept of frequent oscillation. For the sake of completeness, its definition and associated information will be briefly sketched as follows. Let $N=\{1,2,3,...\}$, Z the set of integers and D a subset of Z of the form $\{a,a+1,a+2,a+3,...\}$, where a is an integer. The size of a set Ω will be denoted by $|\Omega|$. The union, intersection and difference of two sets A and B will be denoted by $A+B$, $A\cap B$ and $A\backslash B$ respectively. Let Ω be a set of integers. We will denote the set of all integers in Ω which are less than or equal to an integer n by $\Omega^{(n)}$, that is, $\Omega^{(n)}=\Omega\cap\{...,n-1,n\}$, and we will denote the set $\{x+m|x\in\Omega\}$ of translates of the elements in Ω by $E^m\Omega$, where m is an integer. Let α and β be two integers such that $\alpha\leq\beta$. The union

$$\sigma_\alpha^\beta(\Omega)=\sum_{i=\alpha}^{\beta} E^i\Omega,$$

will be called a derived set of Ω. Note that an integer $j\in E^m\Omega$ if and only if $j-m\in\Omega$. Thus

$$j\in Z\backslash\left(\sigma_\alpha^\beta(\Omega)\right)\Leftrightarrow j-k\in Z\backslash\Omega \;\, for\;\, \alpha\leq k\leq\beta. \tag{3}$$

Let Ω be a set of integers. If $\limsup_{n\to\infty}\left|\Omega^{(n)}\right|/n$ exists, then this limit, denoted by $\mu^*(\Omega)$, will be called the upper frequency measure of Ω. Similarly, if $\liminf_{n\to\infty}\left|\Omega^{(n)}\right|/n$ exists, then this limit, denoted by $\mu_*(\Omega)$, will be called the lower frequency measure of Ω. If $\mu^*(\Omega)=\mu_*(\Omega)$, then the common limit, denoted by $\mu(\Omega)$, will be called the frequency measure of Ω.

For the sake of convenience, we will adopt the usual notation for level sets of a sequence, that is, let $x:D\to R$ be a real function, then the set $\{k\in D|x_k\leq c\}$ will be denoted by $(x\leq c)$ or $(x_k\leq c)$. The notations $(x\geq c)$, $(x<c)$, etc. will have similar meanings. Let $x=\{x_k\}_{k=a}^\infty$ be a real sequence. If $\mu^*(x\leq 0)=0$, then the sequence x is said to be frequently positive. If $\mu^*(x\geq 0)=0$, then x is said to be frequently negative. The sequence x is said to be frequently oscillatory if it is neither frequently positive nor frequently negative. Note that if a sequence x is eventually positive, then it is frequently positive; and if x is eventually negative, then it is frequently negative. Thus, if it is frequently oscillatory, then it is oscillatory.

Let $x = \{x_k\}_{k=a}^{\infty}$ be a real sequence. If $\mu^*(x \le 0) \le \omega$, then x is said to be frequently positive of upper degree ω. If $\mu^*(x \ge 0) \le \omega$, then x is said to be frequently negative of upper degree ω. The sequence x is said to be frequently oscillatory of upper degree ω if it is neither frequently positive nor frequently negative of the same upper degree ω. The concepts of frequently positive of lower degree, etc. are similarly defined by means of μ_*. We say that x is frequently positive of the lower degree ω if $\mu_*(x \le 0) \le \omega$, frequently negative of the lower degree ω if $\mu_*(x \ge 0) \le \omega$, and frequently oscillatory of lower degree ω if it is neither frequently positive nor frequently negative of the same lower degree ω. Note that if the sequence x is frequently oscillatory of the lower degree ω, then it is also frequently oscillatory of upper degree ω. Note further that if the sequence x is frequently oscillatory of upper degree ω for some $\omega > 0$, then it is frequently oscillatory.

We first recall three results from [10] needed in the sequel.

LEMMA 1. Let Ω and Γ be subsets of $D = \{a, a+1, ...\}$. Then

$$\mu^*(\Omega + \Gamma) \le \mu^*(\Omega) + \mu^*(\Gamma).$$

Furthermore, if Ω and Γ are disjoint, then

$$\mu_*(\Omega) + \mu_*(\Gamma) \le \mu_*(\Omega + \Gamma) \le \mu_*(\Omega) + \mu^*(\Gamma) \le \mu^*(\Omega + \Gamma) \le \mu^*(\Omega) + \mu^*(\Gamma),$$

so that

$$\mu_*(\Omega) + \mu^*(N \backslash \Omega) = 1.$$

LEMMA 2. Let Ω and Γ be subsets of $D = \{a, a+1, ...\}$ such that $\mu^*(\Omega) + \mu_*(\Gamma) > 1$. Then $\Omega \cap \Gamma$ cannot be a finite set.

LEMMA 3. For any subset Ω of $D = \{a, a+1, ...\}$, we have

$$\mu^*\left(\sigma_\alpha^\beta(\Omega)\right) \le (\beta - \alpha + 1)\mu^*(\Omega),$$

and

$$\mu_*\left(\sigma_\alpha^\beta(\Omega)\right) \le (\beta - \alpha + 1)\mu_*(\Omega).$$

2 Preparatory Lemmas

We will assume throughout this section that k is a positive integer. First of all, we note that

$$\frac{k}{(k+1)^{1+1/k}} \ge \frac{k^{k+1}}{(k+1)^{k+1}}. \quad (4)$$

Indeed, the above inequality holds for $k = 1, 2$. Let

$$h(x) = (x+1)^{1-x^{-2}}, \; H(x) = \frac{h(x)}{x}, \; x \ge 3.$$

Then

$$H'(x) = \frac{h(x)}{x^3}\left(\frac{2\ln(x+1)}{x} - 1\right) \le 0, \; x \ge 3.$$

Furthermore, since $\lim_{x\to\infty} H(x) = 1$, thus $H(x) \ge 1$ for $x \ge 3$. In particular,

$$H(k) = \left(\frac{k}{(k+1)^{1+1/k}} \frac{(k+1)^{k+1}}{k^{k+1}}\right)^{\frac{1}{k}} \ge 1, \; k = 3, 4, ...,$$

as required.

LEMMA 4. For each $\delta \in [0, k^k/(k+1)^{k+1}]$, the sequence $\{r_n\}_{n=1}^{\infty}$ defined by

$$r_1 = \delta/(1-\delta)^k, \ r_{n+1} = \delta/(1-r_n)^k, \ n \geq 1, \tag{5}$$

is increasing and converges to a number in $[\delta, 1/(k+1))$.

PROOF. If $\delta = 0$, then $r_n = 0$ for $n \geq 0$ and our assertion is true. Suppose $\delta > 0$. Note that

$$0 < \delta \leq \alpha = k^k/(k+1)^{k+1} \leq 1/(k+1) < 1,$$

$$\delta < r_1 \leq \frac{k^k}{(k+1)^{k+1}} \frac{1}{(1-\alpha)^k} < \frac{k^k}{(k+1)^{k+1}} \frac{1}{(k/(k+1))^k} = \frac{1}{k+1},$$

and

$$r_1 = \frac{\delta}{(1-\delta)^k} < \frac{\delta}{(1-r_1)^k} = r_2 < \frac{k^k}{(k+1)^{k+1}} \frac{1}{(k/(k+1))^k} = \frac{1}{k+1}.$$

Assume by induction that $\delta < r_1 < r_2 < ... < r_n < 1/(k+1)$. Then

$$\delta < r_{n+1} \leq \frac{k^k}{(k+1)^{k+1}} \frac{1}{(1-r_n)^k} < \frac{k^k}{(k+1)^{k+1}} \frac{1}{(k/(k+1))^k} = \frac{1}{k+1}$$

and

$$\frac{r_{n+1}}{r_n} = \frac{(1-r_{n-1})^k}{(1-r_n)^k} > 1$$

as required. The proof is complete.

Note that the sequence $\{r_n\}_{n=1}^{\infty}$ depends on δ. Therefore in case of confusion, we will write $\{r_n(\delta)\}_{n=1}^{\infty}$ instead of $\{r_n\}_{n=1}^{\infty}$. In view of the above result, its limit also defines a function f on $[0, k^k/(k+1)^{k+1}]$. For instance, when $k = 1$, $k^k/(k+1)^{k+1} = 1/4$ and

$$f(\delta) = \frac{1-\sqrt{1-4\delta}}{2}, \ 0 \leq \delta \leq \frac{1}{4}.$$

Next, note that by taking limits on both sides of $r_{n+1} = \delta/(1-r_n)^k$, $f(\delta)$ is a root of the equation

$$x(1-x)^k = \delta.$$

Since the function

$$g(x) = x(1-x)^k, \tag{6}$$

is strictly increasing on $[0, 1/(k+1))$, decreasing on $(1/(k+1), 1]$ and

$$\max_{x \in [0,1]} g(x) = g\left(\frac{1}{k+1}\right) = \frac{k^k}{(k+1)^{k+1}}.$$

Thus for any constant $\delta \in [0, k^k/(k+1)^{k+1}]$, the equation $g(x) = \delta$ has exactly two roots $\theta_1, \theta_2 \in [0,1]$ such that $\theta_1 \leq \theta_2$, where equality holds only if $\delta = k^k/(k+1)^{k+1}$. Note that $\delta = g(\theta_1) = \theta_1(1-\theta_1)^k \leq \theta_1$, thus $r_n \leq \theta_1$ for $n \geq 1$ and hence $f(\delta) = \theta_1$.

For similar reasons, we may also see that

$$\lambda^k(1-\lambda) = \delta$$

has exactly two roots λ_1 and λ_2 such that $\lambda_1 \leq \lambda_2$ and $1 - f(\delta) = \lambda_2$.

LEMMA 5. For each $\delta \in \left[0, k/(k+1)^{1+1/k}\right]$, the equation

$$x(1 - x^k) = \delta \tag{7}$$

has exactly two roots $\mu_1, \mu_2 \in [0,1]$ such that $\mu_1 \leq \mu_2$, where $\mu_1 = \mu_2$ only if $\delta = k/(k+1)^{1+1/k}$. Furthermore, μ_1 is a continuous and increasing function of δ.

Again, existence of such roots μ_1, μ_2 in $[0,1]$ and monotonicity of μ_1 follows from the fact that the function $G(x) = x(1 - x^k)$ is increasing on $[0, 1/(k+1)^{1/k})$ and decreasing on $(1/(k+1)^{1/k}, 1]$ and

$$\max_{x \in [0,1]} G(x) = G\left(\frac{1}{(k+1)^{1/k}}\right) = \frac{k}{(k+1)^{1+1/k}}.$$

Continuity of μ_1 follows from the implicit function theorem.

LEMMA 6. For any $\delta \in \left[0, k/(k+1)^{1+1/k}\right]$, the sequence $\{R_n\}_{n=1}^{\infty}$ defined by

$$R_1 = \frac{\delta}{(1 - \delta^k)}, \; R_{n+1} = \frac{\delta}{(1 - R_n^k)}, \; n = 1, 2, ... \tag{8}$$

is increasing and converges to the root μ_1 in Lemma 5.

PROOF. If $\delta = 0$, then $\mu_1 = 0$ and $R_n = 0$ for $n \geq 1$, so that $\lim_{n\to\infty} R_n = \mu_1$. If $\delta > 0$, then it is easy to see that $\delta = \mu_1\left(1 - \mu_1^k\right) < \mu_1 \in (0,1)$. Hence $0 < R_1 < \mu_1 < 1$. Assume that $0 < R_n < \mu_1 < 1$, then

$$0 < R_{n+1} = \frac{\delta}{(1 - R_n^k)} < \frac{\delta}{\left(1 - \mu_1^k\right)} = \mu_1 < 1.$$

By induction, we may then see that the sequence $\{R_n\}$ is positive and bounded above. The sequence $\{R_n\}$ is also nondecreasing. Indeed, it is easily checked that $R_1 < R_2$. Assume by induction that $R_{m-1} < R_m$. Then

$$\frac{R_{m+1}}{R_m} = \frac{1 - R_{m-1}^k}{1 - R_m^k} > 1$$

as required. Therefore, $\lim_{n\to\infty} R_n$ exists and equals, say, L. By taking limits on both sides of $R_{n+1} = \delta/(1 - R_n^k)$, we see that $L(1 - L^k) = \delta$. But since $L \leq \mu_1$, we see that $L = \mu_1$. The proof is complete.

Note that the sequence $\{R_n\}_{n=1}^{\infty}$ depends on δ. Therefore in case of confusion, we will write $\{R_n(\delta)\}_{n=1}^{\infty}$ instead of $\{R_n\}_{n=1}^{\infty}$. In view of the above result, its limit also defines a function F on $[0, k/(k+1)^{1+1/k}]$. Furthermore, in view of Lemma 5 and the fact that $\mu_1 = F(\delta)$, F is an increasing and continuous function.

We remark further that if $\delta > k/(k+1)^{1+1/k}$, there is a positive integer M such that $R_M(\delta) > 1$. Indeed, if the contrary holds, then $\lim_{n\to\infty} R_n(\delta)$ exists and belongs to $(0,1]$. If we denote this limit by L, then

$$\frac{k}{(k+1)^{1+1/k}} < \delta = L(1 - L^k) \leq \max_{0 < x \leq 1} x\left(1 - x^k\right) = \frac{k}{(k+1)^{1+1/k}},$$

which is a contradiction.

The final preparatory result is in [6].

LEMMA 7. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1) such that $x_n > 0$ for $n = m-3k, m-3k+1, ..., m+k+1$ and $p_n > 0$ for $n = m-2k, m-2k+1, ..., m+k$, where m is a positive integer. If

$$\sum_{i=n-k}^{n-1} p_i \geq B > 0,\ n = m-k, m-k+1, ..., m+k,$$

then

$$\frac{x_{m-k}}{x_m} \leq \frac{4}{B^2}.$$

3 Main Results

We first recall that $f(\delta)$ and $F(\delta)$ are limits of the sequences (5) and (8) respectively.

THEOREM 1. Suppose there is $\delta \in \left[0, k^k/(k+1)^{k+1}\right]$ and a constant $c > (1-f(\delta))^k (1-F(k\delta))$ such that $\mu(p_n < 0) = 0$, $\mu^*(p_n \geq c) > 0$ and $\mu(q_n < \delta) = 0$ where

$$q_n = \frac{1}{k} \sum_{j=n-k}^{n-1} p_j,\ n = k, k+1, k+2, \cdots. \tag{9}$$

Then every solution $\{x_n\}_{n=-k}^{\infty}$ of (1) is frequently oscillatory (and hence oscillatory).

PROOF. In view of (4), $F(k\delta)$ exists. Suppose to the contrary that $\{x_n\}$ is a frequently positive solution of (1) such that $\mu^*(x_n \leq 0) = 0$. From the definition of f and F, we see that there is a large positive integer M such that

$$c > (1 - r_M(\delta))^k (1 - R_M(k\delta)).$$

In view of Lemma 1 and Lemma 3,

$$\mu_* \left(N \backslash \left\{ \sigma_{-(M+1)k}^{(M+2)k}(p_n < 0) + \sigma_{-(M+1)k}^{(M+2)k}(q_n < \delta) + \sigma_{-(M+1)k-1}^{(M+3)k}(x_n \leq 0) \right\} \right) = 1.$$

Hence by Lemma 2, the intersection

$$N \backslash \left\{ \sigma_{-(M+1)k}^{(M+2)k}(p_n < 0) + \sigma_{-(M+1)k}^{(M+2)k}(q_n < \delta) + \sigma_{-(M+1)k-1}^{(M+3)k}(x_n \leq 0) \right\} \cap (p_n \geq c)$$

must be an infinite subset of N. In view of (3), there exists a positive integer n such that $p_n \geq c$ and

$$p_i \geq 0,\ q_i \geq \delta,\ n-(M+2)k \leq i \leq n+(M+1)k,$$

and

$$x_i > 0,\ n-(M+3)k \leq i \leq n+(M+1)k+1.$$

Since (1) implies $\{x_i\}$ is decreasing for $n-(M+2)k \leq i \leq n+(M+1)k+1$,

$$\frac{x_{i+1}}{x_i} = 1 - p_i \frac{x_{i-k}}{x_i},\ n-(M+3)k \leq i \leq n+(M+1)k+1, \tag{10}$$

and

$$0 < x_{i+1} = x_i - p_i x_{i-k} = \left(1 - p_i \frac{x_{i-k}}{x_i}\right) x_i \leq (1-p_i) x_i \tag{11}$$

for $n-(M+1)k \le i \le n+(M+1)k+1$. Hence

$$\frac{x_i}{x_{i-k}} = \prod_{j=i-k}^{i-1}\left(1-p_j\frac{x_{j-k}}{x_j}\right) \le \left(1-\frac{1}{k}\sum_{j=i-k}^{i-1}p_j\frac{x_{j-k}}{x_j}\right)^k$$
$$\le \left(1-\frac{1}{k}\sum_{j=i-k}^{i-1}p_j\right)^k \le (1-\delta)^k \tag{12}$$

for $n-Mk \le i \le n+(M+1)k$, so that

$$x_{i-k} \ge \frac{x_i}{\prod_{j=i-k}^{i-1}\left(1-p_j\frac{x_{j-k}}{x_j}\right)} \ge \frac{x_i}{\prod_{j=i-k}^{i-1}\left(1-\frac{p_j}{(1-\delta)^k}\right)}$$
$$\ge \frac{x_i}{\left(1-\frac{1}{k(1-\delta)^k}\sum_{j=i-k}^{i-1}p_j\right)^k} \ge \frac{x_i}{\left(1-\frac{\delta}{(1-\delta)^k}\right)^k}$$
$$\ge \frac{x_i}{(1-r_1)^k}$$

for $n-(M-1)k \le i \le n+(M+1)k$. Similarly for $n-(M-2)k \le i \le n+(M+1)k$,

$$x_{i-k} \ge \frac{x_i}{\prod_{j=i-k}^{i-1}\left(1-p_j\frac{x_{j-k}}{x_j}\right)} \ge \frac{x_i}{\prod_{j=i-k}^{i-1}\left(1-\frac{p_j}{(1-r_1)^k}\right)}$$
$$\ge \frac{x_i}{\left(1-\frac{1}{k(1-r_1)^k}\sum_{j=i-k}^{i-1}p_j\right)^k} \ge \frac{x_i}{\left(1-\frac{\delta}{(1-r_1)^k}\right)^k}$$
$$\ge \frac{x_i}{(1-r_2)^k}.$$

By induction, we may then obtain

$$x_{i-k} \ge \frac{x_i}{(1-r_M)^k},\ n \le i \le n+(M+1)k.$$

If we substitute the above inequality into (1), we obtain

$$x_n = x_{n+1} + p_n x_{n-k} \ge x_{n+1} + p_n\frac{x_n}{(1-r_M)^k}, \tag{13}$$

and for $n \le i \le n+(M+1)k$

$$x_{i+1} \ge x_{i+k+1} + \sum_{j=1}^{k}p_{i+j}x_{i+j-k} \ge x_{i+k+1} + \sum_{j=1}^{k}p_{i+j}x_i. \tag{14}$$

Thus

$$x_{i+1} \ge x_{i+k+1} + \left\{\sum_{j=1}^{k}p_{i+j}\right\}x_i \ge x_{i+k+1} + (k\delta)x_i \ge (k\delta)x_i,\ n \le i \le n+Mk.$$

Hence

$$\frac{x_{i+1}}{x_i} \ge k\delta,\ n \le i \le n+Mk,$$

so that

$$\frac{x_{i+k+1}}{x_{i+1}} \geq (k\delta)^k, \ n \leq i \leq n+(M-1)k.$$

Therefore from (14)

$$\frac{x_{i+1}}{x_i} \geq \frac{k\delta}{1-(k\delta)^k} = R_1(k\delta), \ n \leq i \leq n+(M-1)k,$$

and by induction, we have

$$\frac{x_{n+1}}{x_n} \geq \frac{k\delta}{1-R_{M-1}^k(k\delta)} = R_M(k\delta).$$

In view of (13),

$$0 \geq -x_n + x_{n+1} + \frac{p_n x_n}{(1-r_M)^k} \geq \left(-1 + R_M(k\delta) + \frac{p_n}{(1-r_M)^k}\right) x_n.$$

This then leads to

$$p_n \leq (1 - R_M(k\delta))\,(1-r_M(\delta))^k < c,$$

which is contrary to our previous conclusion that $p_n \geq c$.

The case where $\{x_n\}$ is a frequently negative solution of (1) such that $\mu^*(x_n \geq 0) = 0$ is similarly proved. The proof is complete.

As an immediate corollary, if $k = 1$ in (1), then $k^k/(k+1)^{k+1} = 1/4$, $q_n = p_n$ and

$$(1 - f(\delta))^k\,(1 - F(k\delta)) = \left(\frac{1+\sqrt{1-4\delta}}{2}\right)^2.$$

Thus we have the following result.

COROLLARY 1. Suppose $k = 1$ and suppose there are $\delta \in [0, 1/4]$ and c such that $\mu(p_n < \delta) = 0$, $\mu^*(p_n \geq c) > 0$ and

$$c > \left(\frac{1+\sqrt{1-4\delta}}{2}\right)^2.$$

Then every solution of (1) is frequently oscillatory.

As an example, consider the difference equation

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \ n = 0, 1, 2, ..., \qquad (15)$$

where $\{p_n\}_{n=0}^{\infty}$ is defined by

$$p_n = \begin{cases} 1/n & n \in A_1 = \{2^m, 2^m+1, 2^m+2; \ m = 0, 1, 2, ...\} \\ 0.1 & n = 4m, 4m+1, 4m+2 \text{ and } n \notin A_1 \\ 0.385 & n = 4m+3 \ \ and \ \ n \notin A_1 \end{cases}.$$

If we take $\delta = 0.1$ and $c = 0.385$, then we can calculate

$$k^k/(k+1)^{k+1} = 0.10546875, \ f(\delta) \in (0.18, 0.185), \ F(k\delta) \in (0.305, 0.31).$$

Thus

$$(1 - f(\delta))^3\,(1 - F(k\delta)) < 0.385 = c,$$

and

$$\mu\left(p_n < 0\right) = \mu\left(q_n < \delta\right) = 0,\ \mu\left(p_n \geq 0.385\right) = 0.25.$$

By Theorem 1, every solution of (15) is frequently oscillatory. However, since it is easy to see that

$$\liminf_{n\to\infty} \frac{1}{3} \sum_{j=n-3}^{n-1} p_j = 0,$$

Theorem 2.1 or Theorem 2.2 in [9] cannot be applied to conclude the oscillation of (15).

As another example, let $k = 1$ and let $\{p_n\}_{n=0}^{\infty}$ be defined by

$$p_n = \begin{cases} 1/n & n = 2^m;\ m = 1, 2, ... \\ 0.26 & \text{otherwise} \end{cases}.$$

Since $\mu\left(p_n < 0\right) = \mu\left(p_n < 0.25\right) = 0$ and $\mu\left(p_n \geq 0.26\right) = 1$, Corollary 1 implies that (1) is oscillatory. However, since $\liminf_{n\to\infty} p_n = 0$, Theorem 2.1 or Theorem 2.2 in [9] cannot be applied to make the same conclusion.

THEOREM 2. Suppose $\mu\left(p_n < 0\right) = 0$, and there exist $\delta \in \left[0, k^k/(k+1)^{k+1}\right]$ and $\beta \geq \delta$ such that $\mu\left(q_n < \delta\right) = 0$ and $\mu^*\left(Q_n < \beta\right) = 0$, where q_n is defined in (9) and

$$Q_n = \frac{1}{k} \sum_{j=1}^{k} \frac{p_{n+j}}{\prod_{s=n+j-k}^{n-1}(1-p_s)}.$$

Suppose further that $k\beta > k/(k+1)^{1+1/k}$ or there exist a constant $c > (1 - f(\delta))^k\,(1 - F(k\beta))$ such that $\mu^*\left(p_n \geq c\right) > 0$. Then every solution of (1) is frequently oscillatory.

PROOF. Suppose to the contrary that $\{x_n\}$ is a frequently positive solution of (1) such that $\mu^*\left(x_n \leq 0\right) = 0$. If $k\beta \leq k/(k+1)^{1+1/k}$, then the limit $F(k\beta)$ exists. From the definition of f and F, we see that there is a large positive integer M such that

$$c > \left(1 - r_M(\delta)\right)^k \left(1 - R_M(k\beta)\right).$$

As in the proof of Theorem 1, there exists a positive integer n such that $p_n \geq c$ and

$$p_i \geq 0,\ q_i \geq \delta,\ Q_i \geq \beta,\ n - (M+2)k \leq i \leq n + (M+1)k,$$

and

$$x_i > 0,\ n - (M+3)k \leq i \leq n + (M+1)k + 1.$$

Hence

$$x_{i-k} \geq \frac{x_i}{\left(1 - r_M(\delta)\right)^k},\quad n \leq i \leq n + (M+1)\,k,$$

and (13) holds. From (1) and (11), we can obtain

$$\begin{aligned} x_{i+1} &= x_{i+k+1} + \sum_{j=1}^{k} p_{i+j} x_{i+j-k} = x_{i+k+1} + \left(\sum_{j=1}^{k} p_{i+j} \frac{x_{i+j-k}}{x_i}\right) x_i \\ &\geq x_{i+k+1} + \left(\sum_{j=1}^{k} \frac{p_{i+j}}{\prod_{s=i+j-k}^{i-1}(1-p_s)}\right) x_i,\ n \leq i \leq n + Mk, \end{aligned}$$

Thus

$$x_{i+1} \geq x_{i+k+1} + kQ_i x_i \geq x_{i+k+1} + (k\beta)x_i, \; n \leq i \leq n + Mk.$$

Following the same reasoning used in the proof of Theorem 1, we may then conclude that

$$1 \geq \frac{x_{n+1}}{x_n} \geq \frac{k\beta}{1 - R^k_{M-1}(k\beta)} = R_M(k\beta). \tag{16}$$

In view of (13),

$$0 \geq -x_n + x_{n+1} + \frac{p_n x_n}{(1 - r_M(\delta))^k} \geq \left(-1 + R_M(k\beta) + \frac{p_n}{(1 - r_M(\delta))^k}\right) x_n.$$

This then leads to

$$p_n \leq (1 - R_M(k\beta))\,(1 - r_M(\delta))^k < c,$$

which is a contradiction.

If $k\beta > k/(k+1)^{1+1/k}$, then by the remark following Lemma 6, there exists a positive integer M such that $R_M(k\beta) > 1$. As before, (16) holds, which is a contradiction. The proof is complete.

To compare Theorem 1 and Theorem 2, let us take $\beta = \delta$ in Theorem 2. Then the conditions in Thereom 2 include those in Theorem 1 with the additional condition $\mu^*(Q_n < \delta) = 0$. Only when the sequence $\{p_k\}$ satisfies $\mu^*(Q_n < \delta) = 0$ (for instance, if $\mu(Q_n \geq q_n) = 1$, then $\mu(Q_n < \delta) = 0$ in view of the assumption $\mu(q_n < \delta) = 0$), then Theorem 1 will be a special case of Theorem 2.

To illustrate Theorem 2, consider the difference equation

$$x_{n+1} - x_n + p_n x_{n-3} = 0, \; n = 0, 1, 2, \cdots \tag{17}$$

where $\{p_n\}_{n=0}^{\infty}$ is defined by

$$p_n = \begin{cases} -1 & n \in \{1, 2, 2^2, 2^3, ...\}, \\ 0.105 & n = 4m, \; 4m+1, \; 4m+2 \text{ and } n \notin \{1, 2, 2^2, ...\} \\ 0.305 & n = 4m+3 \text{ and } n \notin \{1, 2, 2^2, ...\} \end{cases}.$$

If we take $\delta = 0.105$ and $\beta = 0.117$, then we can check that $\mu(p_n < 0) = \mu(q_n < \delta) = 0$, $k\delta = 0.315$ and $k\beta = 0.351$, and

$$\delta < \frac{k^k}{(k+1)^{k+1}} = 0.10546875 < \frac{k}{(k+1)^{1+1/k}} = \frac{3}{4\sqrt[3]{4}} \in (0.472, 0.473),$$

$$f(\delta) \in (0.229, 0.23), \quad F(k\delta) \in (0.32, 0.33), \quad F(k\beta) \in (0.36, 0.37).$$

Since

$$kQ_n = \sum_{j=1}^{3} \frac{p_{n+j}}{\prod_{s=n+j-3}^{n-1}(1 - p_s)} = \frac{p_{n+1}}{(1 - p_{n-1})(1 - p_{n-2})} + \frac{p_{n+2}}{1 - p_{n-1}} + p_{n+3},$$

then for any n, if $n+3, n+2, n+1, n-1, n-2 \notin \{1, 2, 2^2, 2^3, \cdots\}$, then we have

$$kQ_n \geq 0.3534 > 0.351, \quad (1 - f(\delta))^3(1 - F(k\delta)) \in (0.3058, 0.3117),$$

and

$$(1 - f(\delta))^3(1 - F(k\beta)) \in (0.287, 0.294).$$

Let $c = 0.3$, then $\mu(Q_n < \beta) = 0$ and $\mu(p_n \geq c) = 0.25$. By Theorem 2, every solution of (17) is frequently oscillatory. The same conclusion cannot be obtained from Theorem 1. Indeed, for any $\delta \in [0.105, k^k/(k+1)^{k+1}]$, we have $\mu(q_n < \delta) > 0$ (and not $\mu(q_n < \delta) = 0$). And for any $\delta \in [0, 0.105)$, we have $\mu(p_n < 0) = \mu(q_n < \delta) = 0$. But since

$$(1 - f(\delta))^3(1 - F(3\delta)) \geq (1 - f(0.105))^3(1 - F(3 \times 0.105)) > 0.305,$$

thus there does not exist any $c > (1 - f(\delta))^3(1 - F(3\delta))$ such that $\mu^*(p_n > c) > 0$.

THEOREM 3. Suppose there exist three nonnegative numbers λ, η and θ such that

$$\mu^* \left(p_n < 0\right) = \theta \geq 0,$$

$$\mu_* \left(\sum_{j=n-k}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1 - p_s)} > 1 \right) = \eta > 0$$

and $\eta > (3k+1)\theta + (4k+2)\lambda$. Then every solution of (1) is frequently oscillatory of lower degree λ.

PROOF. Let us set

$$\tilde{p}_n = \sum_{j=n-k}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1 - p_s)}, \ n = 2k, 2k+1, 2k+2, \cdots.$$

Suppose to the contrary that $\{x_n\}$ is a frequently positive solution of lower degree λ such that $\mu_* \left(x_n \leq 0\right) \leq \lambda$. Then in view of Lemma 1 and Lemma 3,

$$\begin{aligned} & \mu^* \left(N \backslash \left(\sigma_0^{3k} \left(p_n < 0\right) + \sigma_{-1}^{4k} \left(x_n \leq 0\right)\right)\right) + \mu_* \left(\tilde{p}_n > 1\right) \\ \geq \ & 1 - \mu_* \left(\sigma_0^{3k} \left(p_n < 0\right) + \sigma_{-1}^{4k} \left(x_n \leq 0\right)\right) + \eta \\ \geq \ & 1 - (3k+1)\theta - (4k+2)\lambda + \eta \\ > \ & 1. \end{aligned}$$

Thus by Lemma 2, the intersection

$$\left\{N \backslash \left(\sigma_0^{3k} \left(p_n < 0\right) + \sigma_{-1}^{4k} \left(x_n \leq 0\right)\right)\right\} \cap \left(\tilde{p}_n > 1\right)$$

must be an infinite subset of N. From (3), there is a natural number n such that $\tilde{p}_n > 1$, $p_i \geq 0$ for $n - 3k \leq i \leq n$ and $x_i > 0$ for $n - 4k \leq i \leq n + 1$. In view of (1) and (11), $\{x_i\}$ is decreasing on $\{n - 3k, n - 3k + 1, ..., n + 1\}$ and

$$0 \geq x_{n+1} - x_{n-k} + \sum_{i=n-k}^{n} p_i x_{i-k} \geq \left(-1 + \sum_{i=n-k}^{n} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1 - p_s)} \right) x_{n-k},$$

which is a contradiction. The proof is complete.

COROLLARY 2. Suppose $\{p_n\}_{n=0}^{\infty}$ is eventually nonnegative and

$$\limsup_{n \to \infty} \sum_{j=n-k}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1 - p_s)} > 1.$$

Then every solution of (1) is oscillatory.

THEOREM 4. Suppose there exist $\delta \in (0, k^k/(k+1)^{k+1}]$ and numbers θ, η, ξ and λ such that $\mu^*(p_n < 0) = \theta \geq 0$,

$$\mu^*\left(\frac{1}{k}\sum_{i=n-k}^{n-1} p_i < \delta\right) = \eta \geq 0,$$

$$\mu_*\left(\sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} > 1 - \frac{(k\delta)^2}{4}\right) = \xi > 0,$$

and

$$(4k+1)\theta + (2k+1)\eta + (5k+2)\lambda < \xi.$$

Then every solution of (1) is frequently oscillatory of lower degree λ.

PROOF. Suppose to the contrary that $\{x_n\}$ is a frequently positive solution of lower degree λ such that $\mu_*(x_n \leq 0) \leq \lambda$. Then in view of Lemmas 1 and 3,

$$\begin{aligned}
& \mu^*\left(N\backslash\left\{\sigma_{-k}^{3k}(p_n<0) + \sigma_{-k}^{k}\left(\sum_{i=n-k}^{n-1} p_i < k\delta\right) + \sigma_{-(k+1)}^{4k}(x_n \leq 0)\right\}\right) \\
& +\mu_*\left(\sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} > 1 - \frac{(k\delta)^2}{4}\right) \\
\geq\ & 1 - \mu_*\left(\sigma_{-k}^{3k}(p_n<0) + \sigma_{-k}^{k}\left(\sum_{i=n-k}^{n-1} p_i < k\delta\right) + \sigma_{-(k+1)}^{4k}(x_n \leq 0)\right) + \xi \\
\geq\ & 1 - \{(4k+1)\theta + (2k+1)\eta + (5k+2)\lambda\} + \xi \\
>\ & 1.
\end{aligned}$$

In view of Lemma 2, the intersection

$$\begin{aligned}
& \left(N\backslash\left\{\sigma_{-k}^{3k}(p_n<0) + \sigma_{-k}^{k}\left(\sum_{i=n-k}^{n-1} p_i < k\delta\right) + \sigma_{-(k+1)}^{4k}(x_n \leq 0)\right\}\right) \\
\cap & \left(\sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} > 1 - \frac{(k\delta)^2}{4}\right)
\end{aligned}$$

must be an infinite subset of N, so that there exists a positive integer n such that

$$\sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} > 1 - \frac{(k\delta)^2}{4},$$

$$p_i \geq 0,\ n-3k \leq i \leq n+k,$$

$$\sum_{j=i-k}^{i-1} p_j \geq k\delta,\ n-k \leq i \leq n+k,$$

and

$$x_i > 0,\ n-4k \leq i \leq n+k+1.$$

In view of (1) and (11), $\{x_i\}$ is decreasing for $i \in \{n-3k, n-3k+1, ..., n+k+1\}$, and

$$\begin{aligned} 0 &= x_n - x_{n-k} + \sum_{i=n-k}^{n-1} p_i x_{i-k} \geq x_n - x_{n-k} + \left(\sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} \right) x_{n-k} \\ &\geq \left(\frac{x_n}{x_{n-k}} - 1 + \sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} \right) x_{n-k}. \end{aligned}$$

By Lemma 7, we have

$$\begin{aligned} 0 &\geq \frac{x_n}{x_{n-k}} - 1 + \sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} \\ &\geq \sum_{i=n-k}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k}(1-p_s)} - 1 + \frac{(k\delta)^2}{4}, \end{aligned}$$

which is a contradiction. The proof is complete.

We now present two examples to show that Theorems 3 and 4 are independent of each other.

First, consider the difference equation

$$x_{n+1} - x_n + p_n x_{n-2} = 0, \ n = 0, 1, 2, \cdots, \tag{18}$$

where $\{p_n\}_{n=0}^{\infty}$ is defined by

$$p_n = \begin{cases} -1 & n = 2^m; \ m = 1, 2, ... \\ 0.25 & \text{otherwise} \end{cases}.$$

Let

$$A = \left\{2, 2^2, 2^3, 2^4, ...\right\}, \tag{19}$$

then $p_n = -1$ for $n \in A$. For any n, if $n, n-1, n-2, n-3, n-4 \notin A$, then we have

$$\Lambda = \sum_{j=n-2}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1-p_s)} = p_n + \frac{p_{n-1}}{1-p_{n-3}} + \frac{p_{n-2}}{(1-p_{n-3})(1-p_{n-4})} = \frac{37}{36} > 1,$$

and

$$\Delta = \sum_{j=n-2}^{n-1} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1-p_s)} = \frac{p_{n-1}}{1-p_{n-3}} + \frac{p_{n-2}}{(1-p_{n-3})(1-p_{n-4})} = \frac{7}{9} < 1,$$

$$\delta = \frac{1}{2} \sum_{j=n-2}^{n-1} p_j = 0.25 \text{ and } 1 - \frac{(k\delta)^2}{4} = \frac{15}{16}.$$

Hence it is easy to see that $\mu^*(p_n < 0) = 0$ and

$$\mu^* \left(\sum_{j=n-2}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1-p_s)} > 1 \right) = 1 \text{ and } \mu^* \left(\frac{1}{2} \sum_{j=n-2}^{n-1} p_j < \delta \right) = 0,$$

and

$$\mu_* \left(\sum_{j=n-2}^{n-1} \frac{p_j}{\prod_{s=j-k}^{n-1-k} (1-p_s)} > 1 - \frac{(k\delta)^2}{4} \right) = 0.$$

Hence by Theorem 3, for any nonnegative constant $\lambda < 0.1$, every solution of (18) is frequently oscillatory of lower degree λ. But it is impossible to obtain the same conclusion from Theorem 4. Indeed, the case where $\delta \in [0, 0.25]$ is impossible since

$$1 - \frac{(2\delta)^2}{4} \geq \frac{15}{16},$$

and

$$\mu_* \left(\sum_{i=n-2}^{n-1} \frac{p_i}{\prod_{s=i-k}^{n-1-k} (1-p_s)} > 1 - \frac{(k\delta)^2}{4} \right) = 0.$$

The case where $\delta \in (0.25, 4/9]$ is also not possible, since

$$\mu^* \left(\frac{1}{2} \sum_{i=n-2}^{n-1} p_i < \delta \right) = 1 = \eta,$$

so that

$$1 \geq \xi > (4k+1)\theta + (2k+1)\eta + (5k+2)\lambda \geq (2 \times 2 + 1) = 5$$

is a contradiction.

As our final example, consider the difference equation (18) where $\{p_n\}_{n=0}^{\infty}$ is defined by

$$p_n = \begin{cases} -0.5 & n \in A, \\ 0 & n = 3m,\ m = 0, 1, 2, ... \\ 0.25 & n = 3m+1,\ n \notin A \text{ and } m = 0, 1, 2, ... \\ 0.5 & n = 3m+2,\ n \notin A \text{ and } m = 0, 1, 2, ... \end{cases},$$

and A is defined by (19). Let $\delta = 0.125$ and

$$\Delta = \sum_{j=n-2}^{n-1} \frac{p_j}{\prod_{s=j-k}^{n-1-k} (1-p_s)} = \frac{p_{n-1}}{1-p_{n-3}} + \frac{p_{n-2}}{(1-p_{n-3})(1-p_{n-4})},$$

and

$$\Lambda = \sum_{j=n-2}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k} (1-p_s)} = p_n + \frac{p_{n-1}}{1-p_{n-3}} + \frac{p_{n-2}}{(1-p_{n-3})(1-p_{n-4})},$$

then $1 - (k\delta)^2/4 = 0.984375$ and for any n, if $n, n-1, n-2, n-3, n-4 \notin A$, then we have

$$\Delta = \begin{cases} 1 & n = 3m \\ 0.67 & n = 3m+1 \\ 0.5 & n = 3m+2 \end{cases}, \quad \Lambda = \begin{cases} 1 & n = 3m \\ 0.917 & n = 3m+1 \\ 1 & n = 3m+2 \end{cases}.$$

Hence it is easy to see that $\mu^*(p_n < 0) = 0$ and

$$\mu^* \left(\frac{1}{2} \sum_{j=n-2}^{n-1} p_j < \delta \right) = 0, \quad \mu_* \left(\sum_{j=n-2}^{n} \frac{p_j}{\prod_{s=j-k}^{n-1-k} (1-p_s)} > 1 \right) = 0,$$

and

$$\mu_* \left(\sum_{j=n-2}^{n-1} \frac{p_j}{\prod_{s=j-k}^{n-1-k}(1-p_s)} > 1 - \frac{(k\delta)^2}{4} \right) = \frac{1}{3}.$$

By Theorem 4, for any constant $\lambda < 1/36$, every solution of (18) is frequently oscillatory of lower degree λ. But it is impossible to obtain the same conclusion from Theorem 3. Indeed, for any $\delta \in [0, 4/9]$, $\mu^*(p_n < 0) = 0$ and

$$\mu_* \left(\sum_{j=n-2}^{n} \frac{p_j}{\prod_{s=i-k}^{n-1-k}(1-p_s)} > 1 \right) = 0.$$

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