Structure of positive solutions to a semilinear initial value problem

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Abstract. We study an initial value problem for a semilinear ordinary differential equation. This problem is closely related to the blow-up behaviour of a semilinear parabolic problem. Under some restrictions, we characterize the structure of solutions and derive the uniqueness of positive global solution of this initial value problem.

Keywords: initial value problem, semilinear, blow-up behavior, global solution

1 Introduction

In this paper, we are concerned with the positive solutions of the initial value problem (P):

$$w'' - \frac{y}{2}w' - \alpha w + w^p = 0, \quad y > 0, \tag{1.1}$$

$$w(0) > 0, w'(0) = -w^q(0), \tag{1.2}$$

where $w = w(y), q = (p+1)/2$, and

$$\alpha = \frac{1}{p-1}. \tag{1.3}$$

Hereafter the prime denotes the differentiation with respect to y. We always assume that $p > 1$.

The problem (P) is closely related to the following semilinear parabolic problem

$$u_t = u_{xx} + u^p, \quad x \in (0,1), t > 0, \tag{1.4}$$

$$u_x(0,t) = 0, u_x(1,t) = u^q(1,t), \quad t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.6}$$

where $u_0(x)$ is a positive smooth function. We say that the solution u of the problem (1.4)–(1.6) blows up if there is a finite time T such that $\max_{x\in[0,1]} u(x,t) \to \infty$ as $t \uparrow T$. It has been shown by Lin and Wang [3] that the solution $u(x,t)$ of the problem (1.4)-(1.6) always blows up, since $p > 1$. Also, under some conditions (for example, $u_0' \geq 0$), $x = 1$ is the only blow-up point. See [3] and [1].

In order to understand the time asymptotic behaviour of $u(x,t)$ as $t \uparrow T$, we make the following well-known Giga-Kohn transformation [2]

$$y = \frac{1-x}{\sqrt{T-t}}, \quad s = -\ln(T-t),$$
$$w(y,s) = (T-t)^\alpha u(x,t).$$

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Then the function w satisfies

$$w_s = w_{yy} - \frac{y}{2}w_y - \alpha w + w^p, \quad 0 < y < e^{s/2}, \quad s > -\ln T,$$
$$w_y(0,s) = -w^q(0,s), \quad w_y(e^{s/2},s) = 0, \quad s > -\ln T,$$
$$w(y,-\ln T) = T^\alpha u_0(1 - y\sqrt{T}), \quad 0 \le y \le 1/\sqrt{T}.$$

It is nature to expect that, as $s \to \infty$ (or, $t \uparrow T$), $w(y,s)$ tends to a global positive solution of (P). Therefore, the existence and uniqueness of global positive (monotone decreasing) solution of (P) plays an important role in studying the time asymptotic behaviour of $u(x,t)$ as $t \uparrow T$. In fact, if $W(y)$ is the unique global positive monotone decreasing solution of (P), then we have

$$(T-t)^\alpha u(1 - y\sqrt{T-t}, t) \to W(y)$$

as $t \uparrow T$ uniformly for $y \in [0,C]$ for any $C > 0$. See [1] for more detail.

It has been proved in [4] (See also [1]) that there is a global positive monotone decreasing solution of (P) for any $p > 1$. Also, the uniqueness of global positive monotone decreasing solutions of (P) for $p \in (1,2]$ was proved in [1]. The main purpose of this paper is to show the following theorem on the structure of positive solutions of (P).

Theorem 1.1 *Suppose that $1 < p \le 2$. Let $w(y;\eta)$ be the solution of* (P) *with the initial value $w(0;\eta) = \eta > 0$. Then there exists a unique $\bar{\eta} > 0$ such that*

(i) if $\eta > \bar{\eta}$, then $w(y;\eta)$ is decreasing to zero at some finite R;

(ii) if $\eta = \bar{\eta}$, then $w(y;\eta)$ is a global positive monotone decreasing solution;

(iii) if $\eta < \bar{\eta}$, then there exist $y_0, y_1, y_2 > 0$ such that $w'(y) < 0$ for $y \in [0,y_0)$, $w'(y) > 0$ for $y \in (y_0,y_1)$, $w'(y) < 0$ for $y \in (y_1,y_2)$, and $w(y_2) = 0$.

We emphasize here that it follows from Theorem 1.1 that any solution of (P) must vanish at some finite R except the solution starting with $\bar{\eta}$.

This paper is organized as follows. We first recall some facts from [1] in §2 and derive the assertions (i) and (ii) of Theorem 1.1. Then in §3 we prove the assertion (iii) of Theorem 1.1. Hence the only global positive solution of (P) is the monotone decreasing solution $w(y;\bar{\eta})$.

2 Preliminary

Let us recall some facts from [1]. Given any $\eta > 0$, there is a unique local solution $w := w(y;\eta)$ of (P) with $w(0;\eta) = \eta$. Let $\rho(y) = \exp(-y^2/4)$ and $f(w) = w^p - \alpha w$. Then w satisfies

$$(\rho w')(y) = -\eta^q - \int_0^y \rho(s) f(w(s))ds. \tag{2.1}$$

Next, we define the energy functional E_w by

$$E_w[y] = \frac{1}{2}[w'(y)]^2 + F(w(y)), \tag{2.2}$$

where $F(w) = \int_\kappa^w f(s)ds$ and κ is the unique positive solution of $f(w) = 0$. Note that

$$E'_w[y] = \frac{1}{2}y[w'(y)]^2$$

which implies that $E_w[y]$ is increasing in y as long as $w(y)$ is defined. Finally, we define

$$I_1 = \{\eta > 0 \mid w(y;\eta) \text{ is decreasing to zero at some finite } R\},$$
$$I_2 = \{\eta > 0 \mid w'(y;\eta) \text{ vanishes before } w(y;\eta) \text{ vanishes }\}.$$

Notice that w and w' cannot vanish at the same time. Hence I_1 and I_2 are disjoint. It is shown in [1] that the sets I_1, I_2 are open. Moreover, $I_1 \supset [\kappa, \infty)$ and $I_2 \supset (0, \eta_*)$ for some positive constant $\eta_* < \kappa$.

Set $\bar{\eta} = \inf I_1$. Then the corresponding solution $\bar{w}(y) := w(y; \bar{\eta})$ is a global positive monotone decreasing solution of (P) satisfying $\bar{w}(y) \to 0$ as $y \to \infty$ (cf. [1]). Indeed, for any $\eta \notin I_1 \cup I_2$, the corresponding solution $w(y; \eta)$ is a global positive monotone decreasing solution of (P) satisfying $w(y; \eta) \to 0$ as $y \to \infty$.

The following lemma is similar to Corollary 4.8 in [1]. For self-containedness, we give a proof here (cf. [5] and [6]).

Lemma 2.1 *Suppose that $w(y)$ is a global positive solution of* (P) *satisfying $w(y) \to 0$ as $y \to \infty$. Then*

$$\int_0^\infty s\rho(s)\left\{\frac{1}{8} - \frac{p-1}{2(p+1)}w^{p-1}(s)\right\}w^2(s)ds = \frac{p+3}{p+1}w^{p+1}(0) + \frac{p-5}{4(p-1)}w^2(0). \tag{2.3}$$

Proof. Consider the quantity

$$J(y) := \rho(y)[w'(y)]^2 - \frac{y}{2}\rho(y)w(y)w'(y) + (\frac{1}{4} - \alpha)\rho(y)w^2(y) + \frac{2}{p+1}\rho(y)w^{p+1}(y).$$

Note that

$$J'(y) = y\rho(y)\left\{\frac{p-1}{2(p+1)}w^{p-1}(y) - \frac{1}{8}\right\}w^2(y). \tag{2.4}$$

Since $w(y)$ is a global positive solution of (P) and $\lim_{y\to\infty} w(y) = 0$, there is a sequence $y_n \to \infty$ such that $\lim_{n\to\infty} w'(y_n) = 0$. Integrating (2.4) from 0 to y_n and noting that

$$J(y_n) \to 0 \quad \text{as} \quad n \to \infty, \quad J(0) = \frac{p+3}{p+1}w^{p+1}(0) + \frac{p-5}{4(p-1)}w^2(0),$$

(2.3) follows by taking $n \to \infty$. □

As a consequence of (2.3) (cf. Lemma 4.9 of [1]), we have

$$\eta < \bar{\kappa} := \left(\frac{\alpha}{2}\right)^\alpha \tag{2.5}$$

for any global positive solution w of (P) with $w(0) = \eta$ such that $w(y) \to 0$ as $y \to \infty$.

Also, the following uniqueness theorem was proved in [1] (cf. Theorem 4.10 in [1]).

Theorem 2.2 *If $1 < p \leq 2$, then there is a unique global positive monotone decreasing solution of* (P).

The main idea of the proof of Theorem 2.2 is as follows. First, we show that any two global positive monotone decreasing solutions of (P) must intersect each other at least once. Then the theorem follows by a contradiction argument using the estimate (2.5).

We remark that, as a by product of Theorem 2.2, I_1 is connected. Hence $I_1 = (\bar{\eta}, \infty)$ and $I_2 = (0, \bar{\eta})$, if $p \in (1, 2]$.

3 Structure of positive solutions

In this section, we shall study the solution behavior for $\eta \in I_2$. In the sequel, a solution of (P) always means a positive solution. First, we show that there is no unbounded solution of (P).

Lemma 3.1 *Any solution of* (P) *cannot be monotone increasing to infinity at finite time.*

Proof. For contradiction, suppose that $w(y)$ is a solution of (P) such that $w' > 0, w > \kappa$ in $[y_0, R)$ and $w(y) \to \infty$ as $y \to R^-$ for some $y_0 > 0$ and $R \in (y_0, \infty)$. From (1.1), it follows that

$$(\rho w')(y) = (\rho w')(y_0) - \int_{y_0}^y \rho(s)f(w(s))ds. \tag{3.1}$$

Since $w(y) > \kappa$ for all $y \in [y_0, R)$, $\int_{y_0}^y \rho(s)f(w(s))ds > 0$ for all $y \in [y_0, R)$. Hence, by (3.1), $w'(y)$ is bounded in $[y_0, R)$, a contradiction. This completes the proof. □

Lemma 3.2 *Any global solution of* (P) *cannot be monotone increasing to infinity.*

Proof. For contradiction, we suppose that $w(y)$ is a global solution of (P) such that $w'(y) > 0$ for all $y \geq y_0$, for some $y_0 > 0$, and $w(y) \to \infty$ as $y \to \infty$.

We claim that there exists $y_1 > y_0$ such that $w''(y) > 0$ for all $y \geq y_1$. Since $\lim_{y\to\infty} w(y) = +\infty$, there exists $y_1 > y_0$ such that

$$\frac{1}{2} + \alpha - pw^{p-1}(y) < 0 \ \text{ for all } \ y \geq y_1. \tag{3.2}$$

For contradiction, we assume that there exists $\bar{y} \geq y_1$ such that $w''(\bar{y}) \leq 0$. By differentiating (1.1), we obtain that

$$w'''(y) = \frac{y}{2}w''(y) + (\frac{1}{2} + \alpha - pw^{p-1}(y))w'(y). \tag{3.3}$$

By (3.2), (3.3), and the definition of $\bar{y}$, $w'''(y) < 0$ for all $y \geq \bar{y}$. Hence there exists $m > 0$ such that $w''(y) \leq -m$ for all $y \geq \bar{y} + 1$. Then $w'(y) \to -\infty$ as $y \to \infty$, a contradiction. Hence $w''(y) > 0$ for all $y \geq y_1$, and so

$$w'(y) \geq w'(y_1) \ \text{ for all } \ y \geq y_1. \tag{3.4}$$

On the other hand, from $w''(y) > 0$ for all $y \geq y_1$, and (1.1), it follows that

$$\frac{y}{2}w'(y) + \alpha w(y) - w^p(y) > 0 \ \text{ for all } \ y \geq y_1. \tag{3.5}$$

Since $\lim_{y\to\infty} w(y) = +\infty$ and $p > 1$, we can choose $y_2 \geq y_1$ such that

$$w^p(y) - \alpha w(y) > \frac{1}{2}w^p(y) \ \text{ for all } \ y \geq y_2. \tag{3.6}$$

From (3.5) and (3.6) we have

$$yw'(y) > w^p(y) \ \text{ for all } \ y \geq y_2,$$

which implies for $\delta = (p-1)/2$ that

$$w^{-p}(y)w'(y) > \frac{1}{y} \geq \frac{1}{y^{1+\delta}} \ \text{ for all } \ y \geq y_3 = \max\{y_2, 1\}.$$

Integrating the above inequality from $y \geq y_3$ to $+\infty$, we obtain that

$$w(y) \leq (\frac{\delta}{p-1})^{\alpha} y^{1/2} \ \text{ for all } \ y \geq y_3,$$

which contradicts with (3.4). This completes the proof. □

Next, we want to prove that $w(y;\eta)$ vanishes at some finite point for all $\eta \in (0, \bar{\eta})$. Define the "initial" energy function by

$$g(\eta) = \frac{1}{2}\eta^{p+1} + \int_{\kappa}^{\eta} f(s)ds, \ \eta > 0. \tag{3.7}$$

Note that $g(\eta) = E_w[0]$ if $w(0) = \eta$. It is easy to check that

$$g'(\eta) < 0 \ \text{ for all } \ \eta \in (0, \kappa_0), \ \text{ and } g'(\eta) > 0 \ \text{ for all } \ \eta \in (\kappa_0, \infty), \ \text{ where } \kappa_0 := (\frac{2\alpha}{p+3})^{\alpha}. \tag{3.8}$$

We derive a lower bound of $\bar{\eta}$ for $1 < p < 3$ as follows.

Lemma 3.3 *For* $1 < p < 3$, *we have* $\bar{\eta} > \kappa_0$.

Proof. By (2.3) and the definition of $\bar{\eta}$, we have

$$\int_0^\infty s\rho(s)\{\frac{1}{8}-\frac{p-1}{2(p+1)}\bar{w}^{p-1}(s)\}\bar{w}^2(s)ds=\frac{p+3}{p+1}(\bar{\eta})^{p+1}+\frac{p-5}{4(p-1)}(\bar{\eta})^2, \tag{3.9}$$

where $\bar{w}(y)=w(y;\bar{\eta})$. By (2.5), we have

$$\bar{w}(y)<\bar{\kappa}\ \text{ for all }\ y\geq 0.$$

Hence

$$\frac{1}{8}-\frac{p-1}{2(p+1)}\bar{w}^{p-1}(y)>0\ \text{ for all }\ y\geq 0.$$

From (3.9) it follows that

$$\frac{p+3}{p+1}(\bar{\eta})^{p+1}+\frac{p-5}{4(p-1)}(\bar{\eta})^2>0$$

and so

$$(\bar{\eta})^{p-1}>\frac{(5-p)(p+1)}{4(p-1)(p+3)}. \tag{3.10}$$

On the other hand, since $(5-p)(p+1)>8$ if and only if $1<p<3$, we have

$$\frac{(5-p)(p+1)}{4(p-1)(p+3)}>\frac{2}{(p+3)(p-1)}\quad\text{for } p\in(1,3).$$

Since

$$\frac{2}{(p+3)(p-1)}=\kappa_0^{p-1},$$

the lemma follows. □

Remark 3.1 *It follows from the same argument as the proof of Lemma 3.3 that $\eta>\kappa_0$ for any global positive solution $w(y;\eta)$ of (P) with $w(y;\eta)\to 0$ as $y\to\infty$, if $1<p<3$.*

Now, we define a positive number $\underline{\eta}$ as follows. By Lemma 3.3 and the properties of $g(\eta)$, there is a unique positive number $\underline{\eta}$ such that $0<\underline{\eta}<\kappa_0$ and $g(\underline{\eta})=g(\bar{\eta})$. We divide $(0,\bar{\eta})$ into two sets $(0,\underline{\eta})$ and $[\underline{\eta},\bar{\eta})$.

Proposition 3.4 *For $1<p\leq 2$ and $\eta\in(0,\underline{\eta})$, there exist $y_0,y_1,y_2>0$ such that $w'(y;\eta)<0$ for $y\in[0,y_0)$, $w'(y;\eta)>0$ for $y\in(y_0,y_1)$, $w'(y;\eta)<0$ for $y\in(y_1,y_2)$, and $w(y_2)=0$.*

Proof. For $\eta\in(0,\underline{\eta})$, from the properties of $g(\eta)$, there is a unique number $\hat{\eta}>\bar{\eta}$ such that $g(\hat{\eta})=g(\eta)$, if $g(\eta)\leq g((\alpha/p)^\alpha)$; $\hat{\eta}=(\alpha/p)^\alpha$, if $g(\eta)>g((\alpha/p)^\alpha)$. Note that $\bar{\eta}<\bar{\kappa}\leq(\alpha/p)^\alpha$ and $\hat{\eta}\leq(\alpha/p)^\alpha<\kappa$ for $p\in(1,2]$.

Since $\eta<\underline{\eta}<\bar{\eta}$, there exists $y_0>0$ such that $w'(y;\eta)<0$ for all $y\in[0,y_0)$, and $w'(y_0;\eta)=0$. By Lemmas 3.1 and 3.2, there exists $y_1>y_0$ such that $w(y_1;\eta)>\kappa$, $w'(y_1;\eta)=0$, and $w'(y;\eta)>0$ for all $y\in(y_0,y_1)$.

Since $E_w[y]$ is increasing in y and $\hat{\eta}>\eta>w(y_0;\eta)$, we can choose $z_1>y_1$ such that $w(z_1;\eta)=\hat{\eta}$ and $w'(y;\eta)<0$ for $y\in(y_1,z_1]$. Otherwise, if there exists $z>y_1$ such that $w(z;\eta)\in[\hat{\eta},\kappa)$, $w'(z;\eta)=0$, and $w'<0$ in (y_1,z), then

$$E_w[z]=F(w(z))<F(w(y_0))=E_w[y_0],$$

a contradiction. Therefore, there exists $z_1>y_1$ such that $w(z_1;\eta)=\hat{\eta}$ and $w'(y;\eta)<0$ for $y\in(y_1,z_1]$. Moreover, since $E_w[z_1]>g(\eta)\geq g(\hat{\eta})$, i.e.,

$$\frac{1}{2}[w'(z_1;\eta)]^2+\int_\kappa^{w(z_1;\eta)}f(s)ds>g(\eta)\geq g(\hat{\eta})=\frac{1}{2}[w'(0;\hat{\eta})]^2+\int_\kappa^{\hat{\eta}}f(s)ds,$$

we have

$$w'(z_1;\eta) < w'(0;\hat{\eta}) < 0. \tag{3.11}$$

Define $\phi(y) := w(y+z_1;\eta)$ for $y \geq 0$ as long as $w(y+z_1;\eta)$ is defined. Then $\phi(y)$ satisfies

$$\phi''(y) = \frac{z_1}{2}\phi'(y) + \frac{y}{2}\phi'(y) + \alpha\phi(y) - \phi^p(y). \tag{3.12}$$

We claim that $\phi''(y) < w''(y;\hat{\eta})$ for $y \geq 0$ as long as $\phi(y)$ and $w(y;\hat{\eta})$ are defined. Indeed, from (1.1), (3.11), and (3.12), it follows that $\phi''(0) < w''(0;\hat{\eta})$, and thus by continuity $\phi''(y) < w''(y;\hat{\eta})$ for all $y \in [0, z_2)$, for some $z_2 > 0$. If there exists $z_3 \geq z_2$ such that

$$\phi''(y) < w''(y;\hat{\eta}) \text{ for all } y \in [0, z_3), \tag{3.13}$$

and $\phi''(z_3) = w''(z_3;\hat{\eta})$, then by (3.11) and (3.13)

$$\phi'(y) < w'(y;\hat{\eta}) < 0 \text{ for all } y \in [0, z_3], \tag{3.14}$$

and so

$$\phi(z_3) < w(z_3;\hat{\eta}). \tag{3.15}$$

Using (3.14) and (3.15) and noting that $f(s)$ is decreasing on $[0, (\alpha/p)^\alpha]$, it follows from (1.1) and (3.12) that

$$\phi''(z_3) < w''(z_3;\hat{\eta}),$$

a contradiction. Hence $\phi''(y) < w''(y;\hat{\eta})$ for $y \geq 0$ as long as $\phi(y)$ and $w(y;\hat{\eta})$ are defined. Therefore,

$$\phi'(y) < w'(y;\hat{\eta}) < 0 \tag{3.16}$$

for $y \geq 0$ as long as $\phi(y)$ and $w(y;\hat{\eta})$ are defined.

Recall that $\hat{\eta} > \bar{\eta}$. Then $w(y;\hat{\eta})$ is monotone decreasing to 0 at some finite point, say, $\hat{R} > 0$. Thus $w(y_2;\eta) = 0$ for some $y_2 \in (z_1, z_1 + \hat{R})$ and, by (3.16), $w'(y;\eta) < 0$ for all $y \in [z_1, y_2)$. Hence $w'(y;\eta) < 0$ for $y \in (y_1, y_2)$. This completes the proof. □

Let us turn to the case for $\eta \in [\underline{\eta}, \bar{\eta})$. From (3.7), we compute that

$$g(\eta) = \frac{p+3}{2(p+1)}\eta^{p+1} - \frac{\alpha}{2}\eta^2 + \frac{1}{p+1}\kappa^2. \tag{3.17}$$

Recall $\bar{\kappa} = (\alpha/2)^\alpha$. Since $g(\kappa_0) < g(\bar{\kappa}) < g(0)$ and $g' < 0$ in $(0, \kappa_0)$, there exists a unique $\underline{\kappa} \in (0, \kappa_0)$ such that $g(\underline{\kappa}) = g(\bar{\kappa})$. Notice that

$$0 < \underline{\kappa} < \underline{\eta} < \kappa_0 < \bar{\eta} < \bar{\kappa} < \kappa \text{ if } p \in (1, 2]. \tag{3.18}$$

Lemma 3.5 *For any $p \in (1, 2]$, we have*

$$\underline{\kappa} > (\frac{\alpha}{4})^\alpha.$$

Proof. By a simple calculation using (3.17), we get

$$g((\frac{\alpha}{4})^\alpha) - g(\underline{\kappa}) = g((\frac{\alpha}{4})^\alpha) - g(\bar{\kappa}) = \frac{\kappa^2}{4^{2\alpha+1}(p+1)}h(p),$$

where

$$h(p) := 4^\alpha - \frac{3p+1}{2(p-1)}.$$

Since

$$h'(p) = \frac{-2}{(p-1)^2}(4^\alpha \ln 2 - 1) \leq \frac{-2}{(p-1)^2}(4\ln 2 - 1) < 0 \quad \text{for all} \quad p \in (1,2]$$

and $h(2) = 1/2 > 0$, it follows that $h(p) > 0$ for all $p \in (1,2]$. Since $(\alpha/4)^\alpha < \bar{\kappa}$, the lemma follows from the property of g in (3.8). □

Lemma 3.6 *For $1 < p \leq 2$, we have*

$$\frac{1}{6}\left(\frac{1}{4}\right)^\alpha \kappa^2 + g(\kappa_0) > g(\bar{\kappa}).$$

Proof. Using (3.17), we obtain that

$$g(\bar{\kappa}) - g(\kappa_0) = \left[\frac{1}{p+1}(\frac{2}{p+3})^{2\alpha} - \frac{2^{-2\alpha}}{2(p+1)}\right]\frac{\kappa^2}{2}.$$

Thus in order to prove the lemma it suffices to prove that

$$\frac{1}{3}\left(\frac{1}{4}\right)^\alpha > \frac{1}{p+1}(\frac{2}{p+3})^{2\alpha} \quad \text{for all} \quad p \in (1,2],$$

which is equivalent to

$$m(p) := \frac{p+1}{3}\left(\frac{p+3}{4}\right)^{2\alpha} > 1 \quad \text{for all} \quad p \in (1,2].$$

Note that

$$m'(p) = (\frac{p+3}{4})^{2\alpha}\frac{1}{3(p+3)(p-1)^2}m_1(p),$$

where

$$m_1(p) := (p+3)(p-1)^2 + 2(p+1)(p-1) - 2(p+1)(p+3)\ln[(p+3)/4].$$

We claim that $m_1(p) > 0$ for all $p \in (1,2]$. We compute that

$$\begin{aligned} m_1'(p) &= (3p+7)(p-1) - 4(p+2)\ln[(p+3)/4], \\ m_1''(p) &= (6p+4) - 4\frac{p+2}{p+3} - 4\ln[(p+3)/4]. \end{aligned}$$

Since $m_1(1) = m_1'(1) = 0$ and $m_1''(p) > 0$ for $p \in [1,2]$, we obtain that $m_1(p) > 0$ for all $p \in (1,2]$. Since $m(1^+) = \frac{2}{3}\sqrt{e} > 1$, the lemma follows. □

We are ready to prove the following proposition.

Proposition 3.7 *For $1 < p \leq 2, \underline{\eta} \leq \eta < \bar{\eta}$, there exist $y_0, y_1, y_2 > 0$ such that $w'(y;\eta) < 0$ for $y \in [0, y_0)$, $w'(y;\eta) > 0$ for $y \in (y_0, y_1)$, $w'(y;\eta) < 0$ for $y \in (y_1, y_2)$, and $w(y_2) = 0$.*

Proof. Since $\eta < \bar{\eta}$ and $1 < p \leq 2$, there exists $y_0 > 0$ such that $w'(y;\eta) < 0$ for all $y \in [0, y_0)$ and $w'(y_0;\eta) = 0$. Then

$$\begin{aligned} 0 - (-\eta^q) &= \int_0^{y_0} w''(y;\eta)dy \\ &= \int_0^{y_0} [\frac{y}{2}w'(y;\eta) + \alpha w(y;\eta) - w^p(y;\eta)]dy \\ &< (\alpha\eta - \eta^p)y_0, \end{aligned}$$

since $\eta < \bar{\kappa}$ and $f(s) = s^p - \alpha s$ is decreasing on $[0, \bar{\kappa}]$ for $p \in (1, 2]$. Hence we obtain that

$$y_0 > \frac{\eta^{q-1}}{\alpha - \eta^{p-1}}. \tag{3.19}$$

Recall (3.18). Using Lemma 3.5, we have

$$\eta > (\alpha/4)^\alpha \quad \text{for all} \quad \eta \in [\underline{\eta}, \bar{\eta}). \tag{3.20}$$

From (3.19) and (3.20), it follows that

$$y_0 > \frac{4}{3\alpha}\eta^{q-1}. \tag{3.21}$$

Using the fact that $w'(y_0; \eta) = 0, w''(y_0; \eta) > 0$, and Lemmas 3.1 and 3.2, we can find $y_1 > y_0$ such that $w(y_1; \eta) > \kappa, w'(y_1; \eta) = 0$, and $w'(y; \eta) > 0$ for all $y \in (y_0, y_1)$. Thus there exist $y_0 < z_1 < z_2 < y_1$ such that $w(z_1; \eta) = \bar{\kappa}$ and $w(z_2; \eta) = \kappa$. Since $E_w[y]$ is increasing and $\int_\kappa^\eta f(s)ds > \int_\kappa^{w(y;\eta)} f(s)ds$ for all $y \in [z_1, z_2]$, we obtain that

$$w'(y; \eta) > \eta^q \quad \text{for all} \quad y \in [z_1, z_2]. \tag{3.22}$$

From (3.20)-(3.22), it follows that

$$\begin{aligned}
E_w[z_2] - E_w[z_1] &= \int_{z_1}^{z_2} \frac{y}{2}[w'(y;\eta)]^2 dy \\
&\geq \frac{y_0}{2}\eta^q \int_{z_1}^{z_2} w'(y;\eta)dy \\
&= \frac{y_0}{2}\eta^q(\kappa - \bar{\kappa}) \\
&\geq \frac{2}{3\alpha}\eta^{2q-1}(\kappa - \bar{\kappa}) \\
&\geq \frac{1}{12}(\frac{1}{4})^\alpha \kappa^2
\end{aligned}$$

where in the last inequality we have used the fact that $\kappa/2 > \bar{\kappa}$ for $p \in (1, 2]$.

Similarly, we can choose $y_1 < z_3 < z_4$ such that $w(z_3; \eta) = \kappa$, $w(z_4; \eta) = \bar{\kappa}$, and $w' < 0$ in $(y_1, z_4]$. Note that as before $w'(y; \eta) < -\eta^q$ for all $y \in [z_3, z_4]$. Thus in the same way, we have

$$E_w[z_4] - E_w[z_3] \geq \frac{1}{12}(\frac{1}{4})^\alpha \kappa^2.$$

Hence from Lemma 3.6, $E_w[0] = g(\eta)$, and $g(\kappa_0) = \min_{\eta \in [0,\infty)} g(\eta)$ it follows that

$$\begin{aligned}
E_w[z_4] &= (E_w[z_4] - E_w[z_3]) + (E_w[z_3] - E_w[z_2]) \\
&\quad + (E_w[z_2] - E_w[z_1]) + (E_w[z_1] - E_w[0]) + E_w[0] \\
&\geq \frac{1}{6}(\frac{1}{4})^\alpha \kappa^2 + g(\kappa_0) \\
&\geq g(\bar{\kappa}).
\end{aligned}$$

Using the above estimate and the fact that $w(y; \bar{\kappa})$ is decreasing to zero at some finite point and proceeding as in the last part of the proof of Proposition 3.4, it follows that $w(y; \eta)$ vanishes at some finite $y_2 > z_4$ and $w'(y; \eta) < 0$ for all $y \in [z_4, y_2)$. This completes the proof. □

Therefore, the assertion (iii) of Theorem 1.1 follows from Propositions 3.4 and 3.7 and the proof of Theorem 1.1 is complete.

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