Decay Properties and Asymptotic Profiles of Bounded Solutions to a Parabolic System of Chemotaxis in $\mathbb{R}^n$

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1 Introduction

In this paper, we study the large time behavior of bounded solutions to the Cauchy problem for the following system of partial differential equations in $\mathbb{R}^n$, $n \geq 2$:

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u\nabla v), \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta v - v + u, \qquad x \in \mathbb{R}^n,\ t > 0, \tag{1.2}$$

$$u(x,0) = u_0, \quad v(x,0) = v_0, \qquad x \in \mathbb{R}^n. \tag{1.3}$$

The system (1.1), (1.2) is a mathematical model describing chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant(see [2, 10, 17]). $u(x,t)$ corresponds to the population of the organism at place x and time t, and $v(x,t)$ to the concentration of the chemical.

Throughout this paper, it is always assumed that

$$u_0,\ v_0,\ \partial_j v_0 \in L^1(\mathbb{R}^n)\cap\mathcal{B}(\mathbb{R}^n)\ (1\le j\le n).$$

Here, we use the notation $\partial_j=\partial/\partial x_j$ for simplicity, and $\mathcal{B}(\mathbb{R}^n)$ is the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^n$ with the usual supremum norm. We write (1.1)–(1.3) in the form of the integral equation:

$$u(t)=e^{t\Delta}u_0-\int_0^t\nabla\cdot e^{(t-s)\Delta}(u\nabla v)(s)ds, \tag{1.4}$$

$$v(t)=e^{-t}e^{t\Delta}v_0+\int_0^t e^{-t+s}e^{(t-s)\Delta}u(s)ds, \tag{1.5}$$

where

$$(e^{t\Delta}f)(x)=\int_{\mathbb{R}^n}G(x-y,t)f(y)dy$$

and $G(x,t)$ is the heat kernel

$$G(x,t)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\exp(-\frac{|x|^2}{4t}).$$

A function $(u(x,t),v(x,t))$ on $\mathbb{R}^n\times[0,T]$ $(0<T<\infty)$ is said to be a solution of (1.1)–(1.3) on $\mathbb{R}^n\times[0,T]$ if

$$u,\ v,\ \partial_j v\in C([0,T]:L^1(\mathbb{R}^n))\cap C([0,T]:\mathcal{B}(\mathbb{R}^n))\quad(1\le j\le n)$$

and (u,v) satisfies (1.4), (1.5) on $[0,T]$. It is also said that (u,v) is a solution of (1.1)–(1.3) on $\mathbb{R}^n\times[0,\infty)$ if (u,v) is a solution of (1.1)–(1.3) on $\mathbb{R}^n\times[0,T]$ for every $0<T<\infty$. Using standard regularity arguments for the heat equation(see Chapter IV of [11]), we see that (u,v) is a classical solution of (1.1)–(1.3), which satisfies

$$u,v\in C((0,T):W^{2,p}(\mathbb{R}^n))\cap C^1((0,T):L^p(\mathbb{R}^n))\quad\text{for every}\quad 1<p<\infty.$$

It is shown in [14] that every bounded solution of (1.1)–(1.3) on $\mathbb{R}^2\times[0,\infty)$ decays to zero as $t\to\infty$ and behaves like the heat kernel. We give the large time behavior for higher dimensional case in the following theorem. In what follows, $\|\cdot\|_p$ represents the usual L^p-norm.

Theorem 1.1 *Let* (u, v) *be a solution of* (1.1)–(1.3) *on* $\mathbb{R}^n \times [0, \infty)$ *and* $n \geq 2$. *Suppose that*

$$\sup_{t>0} \big(\|u(t)\|_p + \|v(t)\|_p \big) < \infty \quad \textit{for } p = 1, \infty. \tag{1.6}$$

Then, for every $1 < p \leq \infty$,

$$\sup_{t>0} (1+t)^{n(1-1/p)/2} \big(\|u(t)\|_p + \|v(t)\|_p \big) < \infty,$$

$$\lim_{t \to \infty} t^{n(1-1/p)/2} \Big\| u(t) - \int_{\mathbb{R}^n} u_0 \, dy G(t) \Big\|_p = 0,$$

$$\lim_{t \to \infty} t^{n(1-1/p)/2} \Big\| v(t) - \int_{\mathbb{R}^n} u_0 \, dy G(t) \Big\|_p = 0.$$

For nonnegative solutions of (1.1)–(1.3), we need (1.6) only for $p = \infty$, because integrating (1.4) and (1.5) on $\mathbb{R}^n$ respectively, we observe that

$$\|u(t)\|_1 = \|u_0\|_1, \quad \|v(t)\|_1 = e^{-t} \|v_0\|_1 + (1 - e^{-t}) \|u_0\|_1.$$

Concerning the existence of bounded solutions, it is possible for a nonnegative solution of (1.1)–(1.3) in $\mathbb{R}^n (n \geq 2)$ to blow up in finite time(see [2]). For a simplified version of (1.1)–(1.2) replaced (1.2) by $0 = \Delta v - v + u$ in $\mathbb{R}^2$, it is shown that the nonnegative solution exists globally in time under the condition $\int_{\mathbb{R}^2} u_0 \, dx < 8\pi$(see [3]), and that the finite-time blowup of nonnegative solutions may occur under the condition $\int_{\mathbb{R}^2} u_0 \, dx > 8\pi$(see [3, 12]). It is also shown in [12] that the finite-time blowup of nonnegative solutions in $\mathbb{R}^n (n \geq 3)$ may occur even if $\int_{\mathbb{R}^n} u_0 \, dx$ is small. For blowup problems to (1.1), (1.2) in a bounded domain, we refer to [1, 4, 5, 7, 8, 9, 13, 15, 16, 18, 19] and the references therein.

The following theorem gives the existence of bounded solutions to (1.1)–(1.3) in $\mathbb{R}^n, n \geq 2$, when $\|u_0\|_1, \|\nabla v_0\|_1, \|\nabla v_0\|_\infty$ are small but $\|u_0\|_\infty$ is not necessarily small. The uniqueness of solutions is also given in the theorem. The nonnegativity of solutions is not assumed.

Theorem 1.2 (i) *The uniqueness of solutions to* (1.1)–(1.3) *holds.*

(ii) *Let* $n \geq 2$. *For any given* $K > 0$ *let* (u_0, v_0) *be an initial function satisfying*

$$\|u_0\|_\infty \leq K.$$

Then there exists a small $\delta > 0$, *depending on* K, *such that* (1.1)–(1.3) *admits a solution on* $\mathbb{R}^n \times [0, \infty)$ *satisfying* (1.6), *provided that*

$$\|u_0\|_1 \leq \delta, \ \|\nabla v_0\|_1 \leq \delta, \ \|\nabla v_0\|_\infty \leq \delta.$$

2 Estimates for the heat equation

We begin with mentioning $L^p - L^q$ estimates for $e^{t\Delta}$, which is proved by Young's inequality for convolution.

Lemma 2.1 *Let $1 \le q \le p \le \infty$ and $f \in L^q(\mathbb{R}^n)$. Then*

$$\|e^{t\Delta} f\|_p \le (4\pi t)^{-n(1/q-1/p)/2} \|f\|_q, \tag{2.1}$$

$$\|\nabla e^{t\Delta} f\|_p \le C t^{-n(1/q-1/p)/2-1/2} \|f\|_q, \tag{2.2}$$

where C is a positive constant depending only on p and q.

In this section, let $X = L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ or $X = \mathcal{B}(\mathbb{R}^n)$. We then see $e^{t\Delta} w_0 \in C([0, \infty) : X)$ for $w_0 \in X$.

Lemma 2.2 *Let $f \in C([0,T] : X)(0 < T < \infty)$ and define $F(t)$ by*

$$F(t) = \int_0^t e^{a(t-s)} e^{(t-s)\Delta} f(s)\, ds \ (0 < t \le T), \quad F(0) = 0,$$

where a is a constant. Then, $F, \partial_j F \in C([0,T] : X)$ $(1 \le j \le n)$.

Proof. We give the proof only for $\partial_j F$, because the proof for F is done in the same way.

By (2.2), we have

$$\|\partial_j e^{a(t-s)} e^{(t-s)\Delta} f(s)\|_p \le C e^{a(t-s)} (t-s)^{-1/2} \|f(s)\|_p,$$

which implies that

$$\partial_j F(t) = \int_0^t \partial_j e^{a(t-s)} e^{(t-s)\Delta} f(s)\, ds$$

belongs to $L^p(\mathbb{R}^n)$ if $X = L^p(\mathbb{R}^n)(1 \le p < \infty)$, and to $L^\infty(\mathbb{R}^n)$ if $X = \mathcal{B}(\mathbb{R}^n)$.

Let us show $\partial_j F(t) \in X$ when $X = \mathcal{B}(\mathbb{R}^n)$. For $x_1, x_2 \in \mathbb{R}^n$ we have

$$\begin{aligned}
&\partial_j F(x_1,t) - \partial_j F(x_2,t) \\
&= \int_0^t e^{a(t-s)}\, ds \int_{|x_1-y|\le 2|x_1-x_2|} \partial_j G(x_1-y, t-s) f(y,s)\, dy \\
&\quad - \int_0^t e^{a(t-s)}\, ds \int_{|x_1-y|\le 2|x_1-x_2|} \partial_j G(x_2-y, t-s) f(y,s)\, dy \\
&\quad + \int_0^t e^{a(t-s)}\, ds \int_{|x_1-y|\ge 2|x_1-x_2|} \big\{\partial_j G(x_1-y,t-s) - \partial_j G(x_2-y,t-s)\big\} f(y,s)\, dy \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Let $0 < \alpha < 1$. We observe that

$$\int_{|x_1 - y| \le 2|x_1 - x_2|} |\partial_j G(x_1 - y, t - s)|\, dy \le C|x_1 - x_2|^\alpha (t - s)^{-(1+\alpha)/2},$$

which implies that

$$|J_1| \le C|x_1 - x_2|^\alpha \int_0^t e^{a(t-s)}(t - s)^{-(1+\alpha)/2}\, ds \sup_{0<s<t} \|f(s)\|_\infty. \tag{2.3}$$

Since $\{y|\ |x_1 - y| \le 2|x_1 - x_2|\} \subset \{y|\ |x_2 - y| \le 3|x_1 - x_2|\}$, (2.3) holds for J_2. We next observe that for $|x_1 - y| \ge 2|x_1 - x_2|$,

$$|\partial_j G(x_1 - y, t - s) - \partial_j G(x_2 - y, t - s)| \le C|x_1 - x_2|^\alpha (t - s)^{-n/2-(1+\alpha)/2} e^{-c|x_1 - y|^2/(t-s)},$$

where c is a positive constant. Hence,

$$|J_3| \le C|x_1 - x_2|^\alpha \int_0^t e^{a(t-s)}(t - s)^{-(1+\alpha)/2}\, ds \sup_{0<s<t} \|f(s)\|_\infty.$$

Therefore, the function $x \mapsto \partial_j F(x, t)$ is Hölder continuous on $\mathbb{R}^n$. Hence, $\partial_j F(t) \in X$.

Let $0 \le t < t + h \le T$. $\partial_j F(t + h) - \partial_j F(t)$ is written as

$$\begin{aligned}
&\partial_j F(t + h) - \partial_j F(t) \\
&= \int_0^t e^{a(t-s)} \partial_j e^{(t-s)\Delta}(f(s + h) - f(s))\, ds + \int_{-h}^0 e^{a(t-s)} \partial_j e^{(t-s)\Delta} f(s + h)\, ds \\
&= I_1 + I_2.
\end{aligned}$$

For $1 \le p \le \infty$ we have

$$\begin{aligned}
\|I_1\|_p &\le C \int_0^t e^{a(t-s)}(t - s)^{-1/2} \|f(s + h) - f(s)\|_p\, ds \\
&\le C \sup_{0 \le s \le T} \|f(s + h) - f(s)\|_p,
\end{aligned}$$

and

$$\begin{aligned}
\|I_2\|_p &\le C \int_{-h}^0 e^{a(t-s)}(t - s)^{-1/2} \|f(s + h)\|_p\, ds \\
&\le C\sqrt{h} \sup_{0 \le s \le T} \|f(s)\|_p.
\end{aligned}$$

Hence,

$$\|\partial_j F(t + h) - \partial_j F(t)\|_p \le C \sup_{0 \le s \le T} \|f(s + h) - f(s)\|_p + C\sqrt{h} \sup_{0 \le s \le T} \|f(s)\|_p,$$

which implies $\partial_j F \in C([0, T] : X)$. The proof is complete.

Lemma 2.3 *Define v by*

$$v(t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s)\,ds \ (t>0), \quad v(0)=v_0.$$

Then the following holds.

(i) *If $v_0, \partial_j v_0 \in X$ and $u \in C([0,\infty):X)$, then $v, \partial_j v \in C([0,\infty):X)$.*

(ii) *Let $1 \le q \le p \le \infty$ and $1/q - 1/p < 1/n$. If $v_0, |\nabla v_0| \in L^p(\mathbb{R}^n)$ and $u \in C([0,\infty):L^q(\mathbb{R}^n))$, then*

$$\|\nabla v(t)\|_p \le e^{-t}\|\nabla v_0\|_p + C\Gamma(\beta)\sup_{0<s<t}\|u(s)\|_q, \tag{2.4}$$

$$\begin{aligned} \|\nabla v(t)\|_p \le & e^{-t}\|\nabla v_0\|_p + Ct^{\beta-1}e^{-t/2}\sup_{0<s<t/2}\|u(s)\|_q \\ & + C\Gamma(\beta)\sup_{t/2<s<t}\|u(s)\|_q \end{aligned} \tag{2.5}$$

where C is a positive constant depending only on p and q, $\Gamma(z)$ is the gamma function and $\beta = 1/2 - n(1/q-1/p)/2$.

Proof. The functions $t \mapsto e^{t\Delta}v_0$ and $t \mapsto \partial_j e^{t\Delta}v_0 = e^{t\Delta}\partial_j v_0$ belong to $C([0,\infty):X)$. Hence, $v, \partial_j v \in C([0,\infty):X)$ follows from Lemma 2.2.

To estimate ∇v, we use the form

$$\begin{aligned} \nabla v(t) &= e^{-t}e^{t\Delta}\nabla v_0 + \int_0^t e^{-(t-s)}\nabla e^{(t-s)\Delta}u(s)ds \\ &= e^{-t}e^{t\Delta}\nabla v_0 + V(t). \end{aligned}$$

By (2.1),

$$\|e^{-t}e^{t\Delta}\nabla v_0\|_p \le e^{-t}\|\nabla v_0\|_p.$$

By (2.2),

$$\begin{aligned} \|V(t)\|_p &\le C\int_0^t e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2}\|u(s)\|_q ds \\ &\le C\sup_{0<s<t}\|u(s)\|_q \int_0^\infty e^{-(t-s)}(t-s)^{\beta-1}ds \\ &= C\Gamma(\beta)\sup_{0<s<t}\|u(s)\|_q. \end{aligned}$$

Hence, we have (2.4).

Next, to get (2.5) we rewrite $V(t)$ in the form

$$\begin{aligned} V(t) &= \Big(\int_0^{t/2} + \int_{t/2}^t\Big)e^{-(t-s)}\nabla e^{(t-s)\Delta}u(s)ds \\ &= V_1(t) + V_2(t). \end{aligned}$$

By (2.2),

$$\begin{aligned}\|V_1(t)\|_p &\le C\int_0^{t/2} e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2}\|u(s)\|_q ds\\ &\le C\sup_{0<s<t/2}\|u(s)\|_q\int_0^{t/2} e^{-(t-s)}(t-s)^{\beta-1}ds\\ &\le Ce^{-t/2}t^{\beta-1}\sup_{0<s<t/2}\|u(s)\|_q,\end{aligned}$$

and

$$\begin{aligned}\|V_2(t)\|_p &\le C\int_{t/2}^{t} e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2}\|u(s)\|_q ds\\ &\le C\sup_{t/2<s<t}\|u(s)\|_q\int_{t/2}^{t} e^{-(t-s)}(t-s)^{\beta-1}ds\\ &\le C\Gamma(\beta)\sup_{t/2<s<t}\|u(s)\|_q.\end{aligned}$$

Hence, we have (2.5). The proof is complete.

3 Proof of Theorem 1.1

To prove the decay properties, we need the following lemma.

Lemma 3.1 ([6], Lemma 7.1.1) *Suppose $b \ge 0, \beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \le t < T \le \infty$, and suppose $f(t)$ is nonnegative and locally integrable on $0 \le t < T$ with*

$$f(t) \le a(t) + b\int_0^t (t-s)^{\beta-1}f(s)ds \quad (0 < t < T).$$

Then

$$f(t) \le a(t) + \theta\int_0^t E'_\beta(\theta(t-s))a(s)ds \quad (0 \le t < T),$$

where $\theta = (b\Gamma(\beta))^{1/\beta}$ and

$$E_\beta(z) = \sum_{n=0}^{\infty}\frac{z^{n\beta}}{\Gamma(n\beta+1)}, \quad E'_\beta(z) = \frac{d}{dz}E_\beta(z) \simeq C_\beta z^{\beta-1} \ (|z| \to 0).$$

By (1.6) and (2.4), we observe that

$$\sup_{t>0} \|\nabla v(t)\|_p < \infty \quad \text{for every } p \in [1,\infty]. \tag{3.1}$$

In the following lemma, we show that $\|u(t)\|_p$ decays to zero as $t \to \infty$ for p in a restricted range of $(1,\infty]$.

Lemma 3.2 *Let $1 < p < n/(n-2)$ if $n \geq 3$, and $1 < p < \infty$ if $n = 2$. Let $0 < \beta \leq 1/2$. Then*

$$\|u(t)\|_p \leq C(1+A)\|u_0\|_1(1+t)^{-n(1-1/p)/2+\beta} \quad (t > 0), \tag{3.2}$$

where C is a positive constant, depending on p, β, such that $C \to \infty$ as $\beta \to +0$, and

$$A = \sup_{t>0} \|\nabla v(t)\|_r, \quad r = \begin{cases} \dfrac{n}{1-2\beta} & \text{if } 0 < \beta < \dfrac{1}{2}, \\ \infty & \text{if } \beta = \dfrac{1}{2}. \end{cases}$$

Proof. For $\lambda > 0$ define (u_λ, v_λ) by

$$u_\lambda(x,t) = \lambda^n u(\lambda x, \lambda^2 t), \quad v_\lambda(x,t) = v(\lambda x, \lambda^2 t) \quad (x \in \mathbb{R}^n, t \geq 0).$$

The function (u_λ, v_λ) satisfies

$$\begin{cases} \dfrac{\partial u_\lambda}{\partial t} = \Delta u_\lambda - \nabla \cdot (u_\lambda \nabla v_\lambda), & x \in \mathbb{R}^n,\ t > 0, \\ \dfrac{\partial v_\lambda}{\partial t} = \Delta v_\lambda - \lambda^2 v_\lambda + \lambda^{-n+2} u_\lambda, & x \in \mathbb{R}^n,\ t > 0. \end{cases}$$

We write u_λ in the form

$$u_\lambda(t) = e^{t\Delta}u_{0,\lambda} - \int_0^t \nabla \cdot e^{(t-s)\Delta}(u_\lambda \nabla v_\lambda)(s)\, ds, \tag{3.3}$$

where $u_{0,\lambda}(x) = \lambda^n u_0(\lambda x)$.

We set

$$A_\lambda = \sup_{0<s<1} \|\nabla v_\lambda(s)\|_r.$$

Take $q \in [1,\infty)$ satisfying $1/q = 1/p + 1/r$. By using (2.2) and Hölder's inequality, it follows from (3.3) that for $0 < t \leq 1$,

$$\begin{aligned} \|u_\lambda(t)\|_p &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + C\int_0^t (t-s)^{-n(1/q-1/p)/2-1/2}\|u_\lambda(s)\nabla v_\lambda(s)\|_q ds \\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + C\int_0^t (t-s)^{-n(1/q-1/p)/2-1/2}\|u_\lambda(s)\|_p\|\nabla v_\lambda(s)\|_r ds \\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda\int_0^t (t-s)^{\beta-1}\|u_\lambda(s)\|_p ds. \end{aligned}$$

Here, we used that

$$-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)=-\frac{n}{2r}=\begin{cases}\beta-\frac{1}{2} & \text{if } 0<\beta<\frac{1}{2},\\ 0 & \text{if } \beta=\frac{1}{2}\end{cases}$$

by virtue of the definition of r for $\beta \in (0,1/2]$. Hence, applying Lemma 3.1 yields that for $0<t\leq 1$,

$$\begin{aligned}\|u_\lambda(t)\|_p &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda\Gamma(\beta)\int_0^t C_\beta(t-s)^{\beta-1}\|e^{s\Delta}u_{0,\lambda}\|_p ds\\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda\int_0^t (t-s)^{\beta-1}s^{-n(1-1/p)/2}\|u_0\|_1 ds\\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda\|u_0\|_1 t^{-n(1-1/p)/2+\beta},\end{aligned}$$

where C is a constant, depending on p,q,β, such that $C\to\infty$ as $\beta\to+0$. Note that

$$\begin{aligned}\|u_\lambda(t)\|_p &= \lambda^{n(1-1/p)}\|u(\lambda^2 t)\|_p,\\ \|\nabla v_\lambda(t)\|_r &= \lambda^{1-n/r}\|\nabla v(\lambda^2 t)\|_r = \lambda^{2\beta}\|\nabla v(\lambda^2 t)\|_r \leq A\lambda^{2\beta}.\end{aligned}$$

Then, for $0<t\leq 1$,

$$\begin{aligned}\|u(\lambda^2 t)\|_p \;\leq\; & \lambda^{-n(1-1/p)}(4\pi t)^{-n(1-1/p)/2}\|u_0\|_1\\ & + CA\|u_0\|_1 t^{-n(1-1/p)/2+\beta}\lambda^{-n(1-1/p)+2\beta}.\end{aligned}\tag{3.4}$$

Putting $t=1$ in (3.4) and taking $\lambda=\sqrt{t}$ for each $t>0$, we arrive at the decay estimate (3.2). The proof is complete.

We are now in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. By (1.4),

$$\begin{aligned}u(t)-e^{t\Delta}u_0 &= -\int_0^t \nabla\cdot e^{(t-s)\Delta}(u\nabla v)(s)\,ds\\ &= -I(t).\end{aligned}$$

We show that

$$\|I(t)\|_\infty \leq C(1+t)^{-n/2-\gamma} \quad \text{for some } \gamma>0. \tag{3.5}$$

Let $0<t\leq 2$. By (2.2) we have

$$\|I(t)\|_\infty \leq C\int_0^t (t-s)^{-1/2}\|u(s)\nabla v(s)\|_\infty\,ds.$$

Since $\|u(s)\nabla v(s)\|_\infty$ is bounded on $[0,\infty)$ by (1.6) and (3.1), we have

$$\|I(t)\|_\infty \le C \quad (0 < t \le 2).$$

Let $t \ge 2$. We divide $I(t)$ into two parts:

$$\begin{aligned} I(t) &= \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\nabla\cdot e^{(t-s)\Delta}(u\nabla v)(s)\,ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

We estimate $\|I_1(t)\|_\infty$. Let p be such that

$$\frac{n}{n-1} < p < \begin{cases} \infty & (n=2), \\ \dfrac{n}{n-2} & (n \ge 3). \end{cases}$$

By using (2.2), Hölder's inequality and (3.2), $I_1(t)$ is estimated as

$$\begin{aligned} \|I_1(t)\|_\infty &\le C\int_0^{t/2}(t-s)^{-n/2-1/2}\|u(s)\nabla v(s)\|_1 ds \\ &\le C\int_0^{t/2}(t-s)^{-n/2-1/2}\|u(s)\|_p\|\nabla v(s)\|_{p/(p-1)}ds \\ &\le C\int_0^{t/2}(t-s)^{-n/2-1/2}s^{-n(1-1/p)/2+\beta}ds\sup_{s>0}\|\nabla v(s)\|_{p/(p-1)} \\ &\le Ct^{-n/2-n(1-1/n-1/p)/2+\beta} \end{aligned}$$

for $0 < \beta < 1/2$. Given γ by

$$\gamma = \frac{n}{2}(\frac{n-1}{n} - \frac{1}{p}) - \beta \quad \text{for } 0 < \beta < \frac{n}{2}(\frac{n-1}{n} - \frac{1}{p}),$$

we then have

$$\|I_1(t)\|_\infty \le Ct^{-n/2-\gamma} \quad \text{for } 0 < \gamma < \frac{n}{2}(\frac{n-1}{n} - \frac{1}{p}). \tag{3.6}$$

We estimate $\|I_2(t)\|_\infty$. Let $n < q < \infty$. By (2.2), we have

$$\|I_2(t)\|_\infty \le \int_{t/2}^t (t-s)^{-n/(2q)-1/2}\|u(s)\nabla v(s)\|_q\,ds. \tag{3.7}$$

Let $1 < p < n/(n-2)$ if $n \ge 3$, and $p = q$ if $n = 2$. Using (3.2) and the boundedness of $\|u(s)\|_\infty$ on $[0,\infty)$, for $0 < \beta \le 1/2$ we have

$$\|u(s)\|_q \le \|u(s)\|_p^{p/q}\|u(s)\|_\infty^{1-p/q} \le C(1+s)^{-n(1-1/p)p/(2q)+\beta p/q},$$

which together with (2.5) in Lemma 2.3 yields that

$$\|\nabla v(s)\|_\infty \le C(1+s)^{-n(1-1/p)p/(2q)+\beta p/q}.$$

Hence,

$$\|u(s)\nabla v(s)\|_q \le \|u(s)\|_q \|\nabla v(s)\|_\infty \le C(1+s)^{-n(1-1/p)p/q+2\beta p/q}.$$

Substituting this estimate into (3.7) gives

$$\begin{aligned} \|I_2(t)\|_\infty &\le Ct^{-n/(2q)+1/2-n(1-1/p)p/q+2\beta p/q} \\ &= Ct^{-n/2-\ell(p,q)+2\beta p/q}, \end{aligned} \tag{3.8}$$

where

$$\ell(p,q) = \frac{np}{q} - \frac{n}{2} - \frac{n}{2q} - \frac{1}{2}.$$

Let $n = 2, 3$. In the case $n = 2$, from $p = q > 2$ we see that $\ell(p,q) = 1/2 - 1/q > 0$. In the case $n = 3$, take p satisfying $n/2 + 1 < p < n/(n-2)$. We then see that $\ell(p,n) = p - n/2 - 1 > 0$, which implies that we can take $q > n$ such that $\ell(p,q) > 0$. Hence, for such p and q we have

$$\ell(p,q) - 2\beta \frac{p}{q} > 0 \quad \text{for sufficiently small } \beta > 0.$$

Therefore,

$$\|I_2(t)\|_\infty \le Ct^{-n/2-\gamma} \quad \text{for some } \gamma > 0.$$

Thus we have (3.5).

Let $n \ge 4$. Let us put

$$\Pi = \{(p,q) |\ 1 < p < \frac{n}{n-2},\ n < q < \infty\ \}.$$

We observe that for $(p,q) \in \Pi$,

$$\ell(p,q) = \frac{n}{q}(p - \frac{1}{2}) - \frac{n+1}{2} < \ell(\frac{n}{n-2}, n) = -\frac{(n-1)^2 - 5}{2(n-2)} < 0.$$

By (2.1), (3.6) and (3.8), for $0 < \beta \le 1/2$ we have

$$\|u(t)\|_\infty \le \|e^{t\Delta}u_0\|_\infty + \|I(t)\|_\infty \le Ct^{-n/2-\ell(p,q)+2\beta p/q} \quad (t \ge 2).$$

Hence, for $s \ge 1$ we have

$$\begin{aligned} \|u(s)\|_q &\le \|u(s)\|_1^{1/q} \|u(s)\|_\infty^{1-1/q} \\ &\le Cs^{-n(1-1/q)/2-\ell(p,q)(1-1/q)+2\beta(1-1/q)p/q}, \end{aligned}$$

and applying (2.5) with $p = q = \infty$ gives

$$\|\nabla v(s)\|_\infty \le Cs^{-n/2-\ell(p,q)+2\beta p/q}.$$

Putting together these estimates gives

$$\|u(s)\nabla v(s)\|_q \le Cs^{-n(2-1/q)/2-\ell(p,q)(2-1/q)+2\beta(2-1/q)p/q},$$

and substituting this estimate into (3.7) yields that

$$\|I_2(t)\|_\infty \le Ct^{-n/2-(n-1)/2-\ell(p,q)(2-1/q)+2\beta(2-1/q)p/q}.$$

If $(n-1)/2 + \ell(p,q)(2-1/q) > 0$ for some $(p,q) \in \Pi$, then we have (3.5). Otherwise we have

$$\|u(t)\|_\infty \le Ct^{-n/2-(n-1)/2-\ell(p,q)(2-1/q)+2\beta(2-1/q)p/q}$$

and the argument above gives

$$\|u(s)\nabla v(s)\|_q \le Cs^{-n(2-1/q)/2-(n-1)(2-1/q)/2-\ell(p,q)(2-1/q)^2+2\beta(2-1/q)^2p/q},$$

and then

$$\|I_2(t)\|_\infty \le Ct^{-n/2-(n-1)/2-(n-1)(2-1/q)/2-\ell(p,q)(2-1/q)^2+2\beta(2-1/q)^2p/q}.$$

When we repeat this procedure m times, we get

$$\|I_2(t)\|_\infty \le Ct^{-n/2-\ell(p,q,m)+2\beta\alpha(q)^m p/q},$$

where

$$\ell(p,q,m) = \frac{n-1}{2}\sum_{k=0}^{m-1}\alpha(q)^k + \ell(p,q)\alpha(q)^m \quad (m \ge 1),$$
$$\ell(p,q,0) = \ell(p,q), \quad \alpha(q) = 2 - \frac{1}{q}.$$

We observe that

$$\ell(\frac{n}{n-2}, n, m) = \frac{2}{n-2}\{\alpha(n)^m - \frac{n(n-2)}{4}\}.$$

Taking a natural number m_0 such that

$$m_0 - 1 \le \frac{1}{\log\alpha(n)}\log\frac{n(n-2)}{4} < m_0,$$

we see that

$$\ell(\frac{n}{n-2}, n, m_0) > 0,$$

which implies that

$$\ell(p,q,m_0) > 0 \text{ for some } (p,q) \in \Pi.$$

Hence, there is a unique m_1 with $m_1 \le m_0$ satisfying

(i) $\ell(p,q,k) \le 0$ for all $(p,q) \in \Pi,\ k = 0,1,\dots,m_1-1$,

(ii) $\ell(p_1,q_1,m_1) > 0$ for some $(p_1,q_1) \in \Pi$.

Then,

$$\ell(p_1,q_1,m_1) - 2\beta\alpha(q_1)^{m_1}\frac{p_1}{q_1} > 0 \text{ for sufficiently small } \beta > 0.$$

Therefore,

$$\|I_2(t)\|_\infty \le Ct^{-n/2-\gamma} \quad \text{for some } \gamma > 0.$$

Thus we have (3.5).

It follows from (3.5) that

$$\|u(t)\|_\infty \le \|e^{t\Delta}u_0\|_\infty + \|I(t)\|_\infty \le C(1+t)^{-n/2}$$

and

$$\lim_{t\to\infty} t^{n/2}\|u(t) - e^{t\Delta}u_0\|_\infty = \lim_{t\to\infty} t^{n/2}\|I(t)\|_\infty = 0.$$

Noting that

$$\lim_{t\to\infty} t^{n/2}\Big\|e^{t\Delta}u_0 - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_\infty = 0,$$

we have

$$\lim_{t\to\infty} t^{n/2}\Big\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_\infty = 0.$$

For $1 < p < \infty$, observing

$$\begin{aligned}\Big\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_p &\le \Big\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_1^{1/p}\Big\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_\infty^{1-1/p} \\ &\le C\Big\|u(t) - \int_{\mathbb{R}^n} u_0 dy G(t)\Big\|_\infty^{1-1/p},\end{aligned}$$

we establish the conclusion for u.

To get the conclusion for v, we write v as follows.

$$\begin{aligned}v(t) &= e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-t+s}e^{(t-s)\Delta}u(s)ds \\ &= e^{t\Delta}u_0 + e^{-t}e^{t\Delta}(v_0 - u_0) + \int_0^t e^{-t+s}e^{(t-s)\Delta}(u(s) - e^{s\Delta}u_0)\,ds \\ &= e^{t\Delta}u_0 + e^{-t}e^{t\Delta}(v_0 - u_0) + J(t).\end{aligned}$$

By (2.1),

$$\|e^{-t}e^{t\Delta}(v_0 - u_0)\|_\infty \le e^{-t}\|v_0 - u_0\|_\infty.$$

Next,

$$J(t) = -\Big(\int_0^{t/2} + \int_{t/2}^{t}\Big) e^{-t+s} e^{(t-s)\Delta} I(s)\, ds$$
$$= J_1(t) + J_2(t).$$

By (2.1) and the boundedness of $\|I(s)\|_\infty$ on $(0,\infty)$, we have

$$\|J_1(t)\|_\infty \le \int_0^{t/2} e^{-t+s}\|I(s)\|_\infty ds \le Ce^{-t/2}.$$

Using (2.1) and the decay estimate (3.5) of $I(s)$ yields that

$$\|J_2(t)\|_\infty \le \int_{t/2}^{t} e^{-t+s}\|I(s)\|_\infty\, ds \le C\int_{t/2}^{t} e^{-t+s}(1+s)^{-n/2-\gamma} ds$$
$$\le C(1+t)^{-n/2-\gamma}.$$

for some $\gamma > 0$. Hence,

$$\|J(t)\|_\infty \le C(1+t)^{-n/2-\gamma}.$$

Therefore,

$$\|v(t) - e^{t\Delta}u_0\|_\infty \le C(1+t)^{-n/2-\gamma}.$$

The proof of Theorem 1.1 is complete.

4 Proof of Theorem 1.2

We first show the uniqueness of solutions to (1.1)–(1.3). Let (u_1, v_1) and (u_2, v_2) be two solutions of (1.1)–(1.3) on $\mathbb{R}^n \times [0,T]$ with the same initial function (u_0, v_0). Put $u = u_1 - u_2$ and $v = v_1 - v_2$. Then,

$$u(t) = -\int_0^t \nabla\cdot e^{(t-s)\Delta}(u\nabla v_1 + u_2\nabla v)(s) ds,$$
$$v(t) = \int_0^t e^{-t+s}e^{(t-s)\Delta}u(s) ds.$$

By (2.2) we have

$$\|u(t)\|_1 \le C\int_0^t (t-s)^{-1/2}\|(u\nabla v_1 + u_2\nabla v)(s)\|_1\, ds$$
$$\le C\int_0^t (t-s)^{-1/2}\big(\|u(s)\|_1\|\nabla v_1(s)\|_\infty + \|u_2(s)\|_\infty\|\nabla v(s)\|_1\big)\, ds,$$

and

$$\|\nabla v(t)\|_1 \le C \int_0^t (t-s)^{-1/2} \|u(s)\|_1 \, ds.$$

Combining the two inequalities gives that

$$\begin{aligned} &\|u(t)\|_1 + \|\nabla v(t)\|_1 \\ &\le K \int_0^t (t-s)^{-1/2} \big(\|u(s)\|_1 + \|\nabla v(s)\|_1 \big) \, ds, \end{aligned} \tag{4.1}$$

where

$$K = C\big(\sup_{0\le s\le T} \|\nabla v_1(s)\|_\infty + \sup_{0\le s\le T} \|u_2(s)\|_\infty + 1 \big).$$

Applying Lemma 3.1 to (4.1) yields that

$$\|u(t)\|_1 + \|\nabla v(t)\|_1 \equiv 0 \quad (0 \le t \le T),$$

which implies that $u \equiv 0$, $v \equiv 0$ on $\mathbb{R}^n \times [0, T]$. Therefore, we establish the uniqueness of solutions.

We next show the existence of bounded solutions to the integral equation (1.4), (1.5) by applying the contraction mapping principle. We proceed in several steps.

Step 1. Put

$$\mathcal{X}_0 = C([0,\infty) : L^1(\mathbb{R}^n)) \cap C([0,\infty) : \mathcal{B}(\mathbb{R}^n))$$

and consider the Banach space

$$\mathcal{X} = \{ u \in \mathcal{X}_0 \mid \sup_{t>0} \|u(t)\|_1 < \infty, \ \sup_{t>0} (1+t)^{n/2} \|u(t)\|_\infty < \infty \}$$

with the norm

$$\|u\|_{\mathcal{X}} = \sup_{t>0} \|u(t)\|_1 + \sup_{t>0} (1+t)^{n/2} \|u(t)\|_\infty.$$

For simplicity, we use the notation

$$\|u\|_{(p)} = \sup_{t>0} (1+t)^{n(1-1/p)/2} \|u(t)\|_p \quad \text{for} \quad 1 \le p \le \infty.$$

We note that $u \in \mathcal{X}$ implies that $\|u\|_{(p)} < +\infty$ for every $1 < p < \infty$ and

$$\|u\|_{(p)} \le \|u\|_{(1)}^{1/p} \|u\|_{(\infty)}^{1-1/p}.$$

For $K_1 > 0$ define the closed subset B_{K_1} of $\mathcal{X}$ by

$$B_{K_1} = \left\{ u \in \mathcal{X} \;\middle|\; \|u - w\|_{(1)} \le K_1,\ \|u - w\|_{(\infty)} \le 1 \right\},$$

where

$$w(t) = e^{t\Delta} u_0.$$

We see that $w \in \mathcal{X}_0$ and that by (2.1), w satisfies

$$\|w\|_{(1)} \le \|u_0\|_1, \quad \|w\|_{(\infty)} \le 2^{n/2} \max\{\|u_0\|_1, \|u_0\|_\infty\}, \tag{4.2}$$

which implies $w \in \mathcal{X}$.

Given $u \in B_{K_1}$, define v by

$$v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-t+s} e^{(t-s)\Delta} u(s) ds \tag{4.3}$$

and then $\Phi(u)$ by

$$[\Phi(u)](t) = w(t) - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds.$$

Lemma 2.3 gives $v, \partial_j v \in \mathcal{X}_0$ $(1 \le j \le n)$, and for every $1 \le p \le \infty$ we have

$$\|v(t)\|_p \le e^{-t} \|v_0\|_p + \sup_{s>0} \|u(s)\|_p \quad (t \ge 0).$$

Lemma 2.2 gives that the function $t \mapsto \int_0^t \nabla \cdot e^{(t-s)\Delta}(u\nabla v)(s)ds$ belongs to $\mathcal{X}_0$ because of $u\partial_j v \in \mathcal{X}_0$ $(1 \le j \le n)$, which implies $\Phi(u) \in \mathcal{X}_0$. If $u \in B_{K_1}$ is a fixed-point of Φ, then (u, v), where v is defined by (4.3), is a solution of (1.1)–(1.3) on $\mathbb{R}^n \times [0, \infty)$ satisfying (1.6).

In order to estimate $\Phi(u) - w$, we need the following estimates on ∇v, which are direct consequences of Lemma 2.3: for $1 \le q \le p \le \infty$ and $1/q - 1/p < 1/n$,

$$\|\nabla v(t)\|_p \le e^{-t} \|\nabla v_0\|_p + C \|u\|_{(q)}, \tag{4.4}$$

$$\|\nabla v(t)\|_p \le e^{-t} \|\nabla v_0\|_p + C(t^{\beta-1} + 1)(1+t)^{-n(1-1/q)/2} \|u\|_{(q)}, \tag{4.5}$$

where C is a positive constant depending only on p and q, and $\beta = 1/2 - n(1/q - 1/p)/2$.

Step 2. We estimate $\|\Phi(u) - w\|_{(\infty)}$. In what follows, for simplicity we put

$$I(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds.$$

We note that for $1 \leq p \leq \infty$,

$$\|u(t)\|_p \leq (1+t)^{-n(1-1/p)/2}\|u\|_{(p)} \quad (t>0). \tag{4.6}$$

Let $0 < t \leq 1$. For $n < p \leq \infty$ we use (2.2), (4.4) and $\|u(t)\|_p \leq \|u\|_{(p)}$ to get

$$\begin{aligned}
\|I(t)\|_\infty &\leq C\int_0^t (t-s)^{-n/(2p)-1/2}\|u(s)\nabla v(s)\|_p ds \\
&\leq C\int_0^t (t-s)^{-n/(2p)-1/2}\|u(s)\|_p\|\nabla v(s)\|_\infty ds \\
&\leq C\|u\|_{(p)}\Big(\|\nabla v_0\|_\infty + \|u\|_{(p)}\Big)\int_0^t (t-s)^{-n/(2p)-1/2}ds \\
&\leq C\|u\|_{(p)}\Big(\|\nabla v_0\|_\infty + \|u\|_{(p)}\Big).
\end{aligned} \tag{4.7}$$

Let $t \geq 1$. We divide $I(t)$ into two parts:

$$\begin{aligned}
I(t) &= \Big(\int_0^{t/2} + \int_{t/2}^t\Big)\nabla\cdot e^{(t-s)\Delta}(u\nabla v)(s)\,ds \\
&= I_1(t) + I_2(t).
\end{aligned}$$

For $n < p < \infty$ we use (2.2), (4.4) and (4.6) to get

$$\begin{aligned}
\|I_1(t)\|_\infty &\leq C\int_0^{t/2}(t-s)^{-n/2-1/2}\|u(s)\nabla v(s)\|_1 ds \\
&\leq C\int_0^{t/2}(t-s)^{-n/2-1/2}\|u(s)\|_p\|\nabla v(s)\|_{p/(p-1)}ds \\
&\leq C\|u\|_{(p)}\Big(\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1))}\Big)\int_0^{t/2}(t-s)^{-n/2-1/2}(1+s)^{-n(1-1/p)/2}ds \\
&\leq C\|u\|_{(p)}\Big(\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1))}\Big)(1+t)^{-n/2}.
\end{aligned}$$

For $2n < p < \infty$, using (4.5) gives

$$\begin{aligned}
\|I_2(t)\|_\infty &\le C\int_{t/2}^t (t-s)^{-n/p-1/2}\|u(s)\nabla v(s)\|_{p/2}ds \\
&\le C\int_{t/2}^t (t-s)^{-n/p-1/2}\|u(s)\|_p\|\nabla v(s)\|_p ds \\
&\le C\|u\|_{(p)}\int_{t/2}^t (t-s)^{-n/p-1/2}(1+s)^{-n(1-1/p)/2} \\
&\quad\times\Big\{e^{-s}\|\nabla v_0\|_p + (s^{-1/2}+1)(1+s)^{-n(1-1/p)/2}\|u\|_{(p)}\Big\}ds \\
&\le C\|u\|_{(p)}\Big(\|\nabla v_0\|_p + \|u\|_{(p)}\Big)\int_{t/2}^t (t-s)^{-n/p-1/2}(1+s)^{-n(1-1/p)}ds \\
&\le C\|u\|_{(p)}\Big(\|\nabla v_0\|_p + \|u\|_{(p)}\Big)(1+t)^{-n/2}.
\end{aligned}$$

Putting together the estimates of $\|I_1(t)\|_\infty$ and $\|I_2(t)\|_\infty$, we have

$$\begin{aligned}
&\|I(t)\|_\infty \\
&\quad\le C\|u\|_{(p)}\Big(\|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u\|_{(1)} + \|u\|_{(p)}\Big)(1+t)^{-n/2},
\end{aligned} \tag{4.8}$$

where $2n < p < \infty$. Here we used

$$\|\nabla v_0\|_q \le \frac{1}{q}\|\nabla v_0\|_1 + \frac{q-1}{q}\|\nabla v_0\|_\infty \quad \text{for } q \in (1,\infty). \tag{4.9}$$

Hence, (4.7) and (4.8) yield that

$$\begin{aligned}
&\|\Phi(u) - w\|_{(\infty)} = \|I\|_{(\infty)} \\
&\le C\Big\{\|u\|_{(p)}^2 + \|u\|_{(p)}(\|u\|_{(1)} + \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty)\Big\},
\end{aligned} \tag{4.10}$$

where $2n < p < \infty$.

Step 3. We estimate $\|\Phi(u) - w\|_{(1)}$. Let $n < p < \infty$. By (4.4) and (4.6), we have

$$\begin{aligned}
\|I(t)\|_1 &\le C\int_0^t (t-s)^{-1/2}\|u(s)\nabla v(s)\|_1 ds \\
&\le C\int_0^t (t-s)^{-1/2}\|u(s)\|_p\|\nabla v(s)\|_{p/(p-1)}ds \\
&\le C\|u\|_{(p)}(\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)})\int_0^t (t-s)^{-1/2}(1+s)^{-n(1-1/p)/2}\,ds \\
&\le C\|u\|_{(p)}(\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)}).
\end{aligned}$$

Here we used

$$\sup_{t>0} \int_0^t (t-s)^{-\alpha}(1+s)^{-\beta}\, ds < \infty \quad \text{for } \ 0<\alpha<1,\ \beta \geq 1-\alpha. \tag{4.11}$$

Hence, for $n < p < \infty$ we have

$$\|\Phi(u) - w\|_{(1)} \leq C\|u\|_{(p)}(\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)}). \tag{4.12}$$

Step 4. We estimate $\|\Phi(u_1) - \Phi(u_2)\|_{(\infty)}$ for $u_1, u_2 \in B_{K_1}$. To do so, we write

$$\begin{aligned}
&[\Phi(u_1) - \Phi(u_2)](t) \\
&= -\int_0^t \nabla\cdot e^{(t-s)\Delta}(u_1\nabla v_1)(s)ds + \int_0^t \nabla\cdot e^{(t-s)\Delta}(u_2\nabla v_2)(s)ds \\
&= \int_0^t \nabla\cdot e^{(t-s)\Delta}((u_2-u_1)\nabla v_1)(s)ds + \int_0^t \nabla\cdot e^{(t-s)\Delta}(u_2\nabla(v_2-v_1))(s)ds \\
&= J_1(t) + J_2(t)
\end{aligned}$$

Since the function $v_1 - v_2$ satisfies

$$(v_1 - v_2)(s) = \int_0^t e^{-t+s}e^{(t-s)\Delta}(u_1-u_2)(s)ds,$$

applying (4.4) and (4.5) with $v_0 = 0, v = v_1 - v_2, u = u_1 - u_2$ gives

$$\|\nabla(v_1-v_2)(t)\|_p \leq C\|u_1-u_2\|_{(q)}, \tag{4.13}$$

$$\|\nabla(v_1-v_2)(t)\|_p \leq C(t^{\beta-1}+1)(1+t)^{-n(1-1/q)/2}\|u_1-u_2\|_{(q)}, \tag{4.14}$$

where $1 \leq q \leq p \leq +\infty, 1/q - 1/p < 1/n$ and $\beta = 1/2 - n(1/q-1/p)/2$.

Let $0 < t \leq 1$ and $n < p < \infty$. We observe that

$$\begin{aligned}
\|J_1(t)\|_\infty &\leq C\int_0^t (t-s)^{-1/2}\|(u_1(s)-u_2(s))\nabla v_1(s)\|_\infty ds \\
&\leq C\int_0^t (t-s)^{-1/2}\|u_1(s)-u_2(s)\|_\infty\|\nabla v_1(s)\|_\infty ds \\
&\leq C\|u_1-u_2\|_{(\infty)}\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\int_0^t (t-s)^{-1/2}ds \\
&\leq C\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\|u_1-u_2\|_{(\infty)}.
\end{aligned}$$

Using (4.13), we observe that

$$\begin{aligned}\|J_2(t)\|_\infty &\le C\int_0^t (t-s)^{-n/(2p)-1/2}\|u_2(s)\nabla(v_1(s)-v_2(s))\|_p ds\\ &\le C\int_0^t (t-s)^{-n/(2p)-1/2}\|u_2(s)\|_p\|\nabla v_1(s)-\nabla v_2(s)\|_\infty ds\\ &\le C\|u_2\|_{(p)}\|u_1-u_2\|_{(\infty)}\int_0^t (t-s)^{-n/(2p)-1/2}\,ds\\ &\le C\|u_2\|_{(p)}\|u_1-u_2\|_{(\infty)}.\end{aligned}$$

Putting together the two inequalities just above gives

$$\begin{aligned}&\|\Phi(u_1)(t)-\Phi(u_2)(t)\|_\infty\\ &\le C(\|\nabla v_0\|_\infty+\|u_1\|_{(p)}+\|u_2\|_{(p)})\|u_1-u_2\|_{(\infty)},\end{aligned}\tag{4.15}$$

where $n < p < \infty$.

Let $t \ge 1$. We rewrite $J_1(t)$ and $J_2(t)$ as follows.

$$\begin{aligned}J_1(t) &= \Big(\int_0^{t/2}+\int_{t/2}^t\Big)\nabla\cdot e^{(t-s)\Delta}((u_2-u_1)\nabla v_1)(s)ds\\ &= J_{11}(t)+J_{12}(t),\\ J_2(t) &= \Big(\int_0^{t/2}+\int_{t/2}^t\Big)\nabla\cdot e^{(t-s)\Delta}(u_2\nabla(v_2-v_1))(s)ds\\ &= J_{21}(t)+J_{22}(t).\end{aligned}$$

Applying (4.6) with $u = u_1 - u_2$ and $p = \infty$ and (4.4), we have

$$\begin{aligned}\|J_{11}(t)\|_\infty &\le C\int_0^{t/2}(t-s)^{-n/2-1/2}\|u_1(s)-u_2(s)\|_\infty\|\nabla v_1(s)\|_1 ds\\ &\le C\|u_1-u_2\|_{(\infty)}\Big(\|\nabla v_0\|_1+\|u_1\|_{(1)}\Big)\int_0^{t/2}(t-s)^{-n/2-1/2}(1+s)^{-n/2}ds\\ &\le C\|u_1-u_2\|_{(\infty)}\Big(\|\nabla v_0\|_1+\|u_1\|_{(1)}\Big)(1+t)^{-n/2}.\end{aligned}$$

Let $n < p < \infty$. Applying (4.6) again with $u = u_1 - u_2$ and using (4.5), we have

$$\begin{aligned}
\|J_{12}(t)\|_\infty &\le C\int_{t/2}^t (t-s)^{-n/(2p)-1/2}\|u_1(s)-u_2(s)\|_\infty\|\nabla v_1(s)\|_p ds\\
&\le C\|u_1-u_2\|_{(\infty)}\int_{t/2}^t (t-s)^{-n/(2p)-1/2}(1+s)^{-n/2}\\
&\quad\times\{e^{-s}\|\nabla v_0\|_p+(s^{-1/2}+1)(1+s)^{-n(1-1/p)/2}\|u_1\|_{(p)}\}ds\\
&\le C\|u_1-u_2\|_{(\infty)}\Big(\|\nabla v_0\|_p+\|u_1\|_{(p)}\Big)\int_{t/2}^t (t-s)^{-n/(2p)-1/2}(1+s)^{-n+n/(2p)}ds\\
&\le C\|u_1-u_2\|_{(\infty)}\Big(\|\nabla v_0\|_p+\|u_1\|_{(p)}\Big)(1+t)^{-n/2}.
\end{aligned}$$

Similarly, by (4.14) we have

$$\begin{aligned}
\|J_{21}(t)\|_\infty &\le C\int_0^{t/2} (t-s)^{-n/2-1/2}\|u_2(s)\|_1\|\nabla(v_1(s)-v_2(s))\|_\infty ds\\
&\le C\|u_2\|_{(1)}\|u_1-u_2\|_{(\infty)}\int_0^{t/2}(t-s)^{-n/2-1/2}(s^{-1/2}+1)(1+s)^{-n/2}ds\\
&\le C\|u_2\|_{(1)}\|u_1-u_2\|_{(\infty)}(1+t)^{-n/2}
\end{aligned}$$

and

$$\begin{aligned}
\|J_{22}(t)\|_\infty &\le C\int_{t/2}^t (t-s)^{-n/(2p)-1/2}\|u_2(s)\|_p\|\nabla(v_1(s)-v_2(s))\|_\infty ds\\
&\le C\|u_2\|_{(p)}\|u_1-u_2\|_{(\infty)}\int_{t/2}^t (t-s)^{-n/(2p)-1/2}(1+s)^{-n+n/(2p)}ds\\
&\le C\|u_2\|_{(p)}\|u_1-u_2\|_{(\infty)}(1+t)^{-n/2}.
\end{aligned}$$

Putting together these inequalities and using (4.9) yields that

$$\begin{aligned}
&\|\Phi(u_1)(t)-\Phi(u_2)(t)\|_\infty \le \sum_{1\le i,j\le 2}\|J_{ij}(t)\|_\infty\\
&\le C\Big(\|\nabla v_0\|_1+\|\nabla v_0\|_\infty+\|u_1\|_{(1)}+\|u_1\|_{(p)}+\|u_2\|_{(1)}+\|u_2\|_{(p)}\Big)\\
&\quad\times\|u_1-u_2\|_{(\infty)}(1+t)^{-n/2},
\end{aligned}$$

from which together with (4.15) it follows that

$$\begin{aligned}
&\|\Phi(u_1)-\Phi(u_2)\|_{(\infty)}\\
&\le C\Big(\|\nabla v_0\|_1+\|\nabla v_0\|_\infty+\|u_1\|_{(1)}+\|u_1\|_{(p)}+\|u_2\|_{(1)}+\|u_2\|_{(p)}\Big)\\
&\quad\times\|u_1-u_2\|_{(\infty)},
\end{aligned}\tag{4.16}$$

where $n < p < \infty$.

Step 5. We estimate $\|\Phi(u_1) - \Phi(u_2)\|_{(1)}$ for $u_1, u_2 \in B_{K_1}$. To do so, we estimate $J_1(t)$ and $J_2(t)$ given in Step 4.

Let $n < p < \infty$. For $0 < t \leq 1$, by (4.4) we have

$$\begin{aligned}\|J_1(t)\|_1 &\leq C\int_0^t (t-s)^{-1/2}\|u_1(s) - u_2(s)\|_1\|\nabla v_1(s)\|_\infty ds \\ &\leq C\|u_1 - u_2\|_{(1)}\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\int_0^t (t-s)^{-1/2} ds \\ &\leq C\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\|u_1 - u_2\|_{(1)}.\end{aligned}$$

For $t \geq 1$, by (4.5) we have

$$\begin{aligned}\|J_1(t)\|_1 &\leq C\int_0^t (t-s)^{-1/2}\|u_1(s) - u_2(s)\|_1\|\nabla v_1(s)\|_\infty ds \\ &\leq C\|u_1 - u_2\|_{(1)} \\ &\quad \times \int_0^t (t-s)^{-1/2}\Big\{e^{-s}\|\nabla v_0\|_\infty + (s^{-1/2-n/(2p)} + 1)(1+s)^{-n(1-1/p)/2}\|u_1\|_{(p)}\Big\} ds \\ &\leq C\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\|u_1 - u_2\|_{(1)}.\end{aligned}$$

Hence,

$$\|J_1\|_{(1)} \leq C\Big(\|\nabla v_0\|_\infty + \|u_1\|_{(p)}\Big)\|u_1 - u_2\|_{(1)}. \tag{4.17}$$

Next, let $n < p < \infty$. Using (4.14) and applying (4.11), we observe that

$$\begin{aligned}\|J_2(t)\|_1 &\leq C\int_0^t (t-s)^{-1/2}\|u_2(s)\|_p\|\nabla(v_1(s) - v_2(s))\|_{p/(p-1)} ds \\ &\leq C\|u_2\|_{(p)}\|u_1 - u_2\|_{(1)}\int_0^t (t-s)^{-1/2}(1+s)^{-n(1-1/p)/2} ds \\ &\leq C\|u_2\|_{(p)}\|u_1 - u_2\|_{(1)},\end{aligned}$$

which implies that

$$\|J_2\|_{(1)} \leq C\|u_2\|_{(p)}\|u_1 - u_2\|_{(1)}. \tag{4.18}$$

Hence, it follows from (4.17) and (4.18) that

$$\|\Phi(u_1) - \Phi(u_2)\|_{(1)} \leq C(\|\nabla v_0\|_\infty + \|u_1\|_{(p)} + \|u_2\|_{(p)})\|u_1 - u_2\|_{(1)}, \tag{4.19}$$

where $n < p < \infty$.

Step 6. We show that Φ admits a unique fixed-point u in B_{K_1}. We may assume $\|u_0\|_1 \le 1$, because $\|u_0\|_1$ is taken small as will be mentioned below. By (4.2), we observe that for $u \in B_{K_1}$,

$$\|u\|_{(1)} \le \|u-w\|_{(1)} + \|w\|_{(1)} \le K_1 + \|u_0\|_1 \tag{4.20}$$

and

$$\|u\|_{(\infty)} \le \|u-w\|_{(\infty)} + \|w\|_{(\infty)} \le 1 + 2^{n/2}\max\{\|u_0\|_1, \|u_0\|_\infty\}.$$

Then, since $\|u_0\|_\infty \le K$, we have

$$\|u\|_{(\infty)} \le K_2, \quad K_2 = 1 + 2^{n/2}(1+K). \tag{4.21}$$

Using (4.20) and (4.21) implies

$$\|u\|_{(p)} \le \|u\|_{(1)}^{1/p}\|u\|_{(\infty)}^{1-1/p} \le K_2^{1-1/p}(K_1 + \|u_0\|_1)^{1/p} \quad (1 < p < \infty). \tag{4.22}$$

From (4.10) we then see that there is a small $\alpha > 0$ such that

$$\|\Phi(u) - w\|_{(\infty)} \le 1, \tag{4.23}$$

provided that

$$K_1 + \|u_0\|_1 \le \alpha.$$

Using (4.20) and (4.22) in the right-hand side of (4.12) and then Young's inequality, we have

$$\begin{aligned}
&\|\Phi(u) - w\|_{(1)} \\
&\le CK_2^{1-1/p}\|\nabla v_0\|_{p/(p-1)}(K_1 + \|u_0\|_1)^{1/p} + CK_2^{1-1/p}(K_1 + \|u_0\|_1)^{1+1/p} \\
&\le \frac{p-1}{p}CK_2\|\nabla v_0\|_{p/(p-1)} + \frac{1}{p}\|\nabla v_0\|_{p/(p-1)}^p(K_1 + \|u_0\|_1) \\
&\quad + CK_2^{1-1/p}2^{1+1/p}(K_1^{1+1/p} + \|u_0\|_1^{1+1/p}) \\
&= a + bK_1 + cK_1^\ell,
\end{aligned}$$

where $\ell = 1 + 1/p$ and

$$\begin{aligned}
a &= \frac{p-1}{p}CK_2\|\nabla v_0\|_{p/(p-1)} + \frac{1}{p}\|\nabla v_0\|_{p/(p-1)}^p\|u_0\|_1 + CK_2^{1-1/p}2^{1+1/p}\|u_0\|_1^{1+1/p}, \\
b &= \frac{1}{p}\|\nabla v_0\|_{p/(p-1)}^p, \quad c = CK_2^{1-1/p}2^{1+1/p}.
\end{aligned}$$

We observe that if a, b and c satisfy the condition

$$0 < b < 1, \quad 0 < a < (1-b)^{\ell/(\ell-1)}(c\ell)^{-1/(\ell-1)}(1 - 1/\ell),$$

then there exist β and γ, satisfying $0 < \beta < \gamma < ((1-b)/c)^{1/(\ell-1)}$, such that

$$c\xi^\ell + b\xi + a \le \xi \quad \text{for every } \xi \in [\beta, \gamma].$$

We also see that

$$\beta \to 0, \ \gamma \to \gamma_0 \quad \text{as} \quad a \to 0 \text{ and } b \to 0,$$

where $\gamma_0 = (1/c)^{1/(\ell-1)}$. Hence, we can take a small $\delta > 0$ with $\delta \le \alpha$ such that the condition

$$\|u_0\|_1 \le \delta, \ \|\nabla v_0\|_1 \le \delta, \ \|\nabla v_0\|_\infty \le \delta$$

ensures the existence of K_1 satisfying

$$K_1 + \|u_0\|_1 \le \alpha, \quad a + bK_1 + cK_1^\ell \le K_1,$$

which implies (4.23) and

$$\|\Phi(u) - w\|_{(1)} \le K_1 \quad \text{for every } u \in B_{K_1}.$$

We therefore obtain

$$\Phi(u) \in B_{K_1} \quad \text{for every } u \in B_{K_1}.$$

Next, it follows from (4.16) and (4.19) that

$$\|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{X}} \le L(u_0, v_0, u_1, u_2)\|u_1 - u_2\|_{\mathcal{X}},$$

where

$$\begin{aligned} &L(u_0, v_0, u_1, u_2) \\ &= C\Big\{\|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u_1\|_{(1)} + \|u_1\|_{(p)} + \|u_2\|_{(1)} + \|u_2\|_{(p)}\Big\}. \end{aligned}$$

We choose again α and δ so small that

$$L(u_0, v_0, u_1, u_2) \le \frac{1}{2} \quad \text{for every} \ \ u_1, u_2 \in B_{K_1},$$

which implies that Φ is a contraction mapping in the closed subset B_{K_1} of X. Therefore, Φ admits a unique fixed-point u in B_{K_1}. The proof of Theorem 1.2 is complete.

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