

# Decay Properties and Asymptotic Profiles of Bounded Solutions to a Parabolic System of Chemotaxis in $\mathbb{R}^n$

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## 1 Introduction

In this paper, we study the large time behavior of bounded solutions to the Cauchy problem for the following system of partial differential equations in  $\mathbb{R}^n$ ,  $n \geq 2$ :

$$(1.1) \quad \frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \nabla v), \quad x \in \mathbb{R}^n, t > 0,$$

$$(1.2) \quad \frac{\partial v}{\partial t} = \Delta v - v + u, \quad x \in \mathbb{R}^n, t > 0,$$

$$(1.3) \quad u(x, 0) = u_0, \quad v(x, 0) = v_0, \quad x \in \mathbb{R}^n.$$

The system (1.1), (1.2) is a mathematical model describing chemotaxis, that is, the directed movement of an organism in response to gradients of a chemical attractant (see [2, 10, 17]).  $u(x, t)$  corresponds to the population of the

organism at place  $x$  and time  $t$ , and  $v(x, t)$  to the concentration of the chemical.

Throughout this paper, it is always assumed that

$$u_0, v_0, \partial_j v_0 \in L^1(\mathbb{R}^n) \cap \mathcal{B}(\mathbb{R}^n) \quad (1 \leq j \leq n).$$

Here, we use the notation  $\partial_j = \partial/\partial x_j$  for simplicity, and  $\mathcal{B}(\mathbb{R}^n)$  is the Banach space of all bounded and uniformly continuous functions on  $\mathbb{R}^n$  with the usual supremum norm. We write (1.1)–(1.3) in the form of the integral equation:

$$(1.4) \quad u(t) = e^{t\Delta} u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds,$$

$$(1.5) \quad v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-t+s} e^{(t-s)\Delta} u(s) ds,$$

where

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^n} G(x-y, t) f(y) dy$$

and  $G(x, t)$  is the heat kernel

$$G(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right).$$

A function  $(u(x, t), v(x, t))$  on  $\mathbb{R}^n \times [0, T]$  ( $0 < T < \infty$ ) is said to be a solution of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, T]$  if

$$u, v, \partial_j v \in C([0, T] : L^1(\mathbb{R}^n)) \cap C([0, T] : \mathcal{B}(\mathbb{R}^n)) \quad (1 \leq j \leq n)$$

and  $(u, v)$  satisfies (1.4), (1.5) on  $[0, T]$ . It is also said that  $(u, v)$  is a solution of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, \infty)$  if  $(u, v)$  is a solution of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, T]$  for every  $0 < T < \infty$ . Using standard regularity arguments for the heat equation (see Chapter IV of [11]), we see that  $(u, v)$  is a classical solution of (1.1)–(1.3), which satisfies

$$u, v \in C((0, T) : W^{2,p}(\mathbb{R}^n)) \cap C^1((0, T) : L^p(\mathbb{R}^n)) \quad \text{for every } 1 < p < \infty.$$

It is shown in [14] that every bounded solution of (1.1)–(1.3) on  $\mathbb{R}^2 \times [0, \infty)$  decays to zero as  $t \rightarrow \infty$  and behaves like the heat kernel. We give the large time behavior for higher dimensional case in the following theorem. In what follows,  $\|\cdot\|_p$  represents the usual  $L^p$ -norm.

**Theorem 1.1** *Let  $(u, v)$  be a solution of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, \infty)$  and  $n \geq 2$ . Suppose that*

$$(1.6) \quad \sup_{t>0} (\|u(t)\|_p + \|v(t)\|_p) < \infty \quad \text{for } p = 1, \infty.$$

*Then, for every  $1 < p \leq \infty$ ,*

$$\begin{aligned} \sup_{t>0} (1+t)^{n(1-1/p)/2} (\|u(t)\|_p + \|v(t)\|_p) &< \infty, \\ \lim_{t \rightarrow \infty} t^{n(1-1/p)/2} \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_p &= 0, \\ \lim_{t \rightarrow \infty} t^{n(1-1/p)/2} \left\| v(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_p &= 0. \end{aligned}$$

For nonnegative solutions of (1.1)–(1.3), we need (1.6) only for  $p = \infty$ , because integrating (1.4) and (1.5) on  $\mathbb{R}^n$  respectively, we observe that

$$\|u(t)\|_1 = \|u_0\|_1, \quad \|v(t)\|_1 = e^{-t}\|v_0\|_1 + (1 - e^{-t})\|u_0\|_1.$$

Concerning the existence of bounded solutions, it is possible for a nonnegative solution of (1.1)–(1.3) in  $\mathbb{R}^n$  ( $n \geq 2$ ) to blow up in finite time (see [2]). For a simplified version of (1.1)–(1.2) replaced (1.2) by  $0 = \Delta v - v + u$  in  $\mathbb{R}^2$ , it is shown that the nonnegative solution exists globally in time under the condition  $\int_{\mathbb{R}^2} u_0 dx < 8\pi$  (see [3]), and that the finite-time blowup of nonnegative solutions may occur under the condition  $\int_{\mathbb{R}^2} u_0 dx > 8\pi$  (see [3, 12]). It is also shown in [12] that the finite-time blowup of nonnegative solutions in  $\mathbb{R}^n$  ( $n \geq 3$ ) may occur even if  $\int_{\mathbb{R}^n} u_0 dx$  is small. For blowup problems to (1.1), (1.2) in a bounded domain, we refer to [1, 4, 5, 7, 8, 9, 13, 15, 16, 18, 19] and the references therein.

The following theorem gives the existence of bounded solutions to (1.1)–(1.3) in  $\mathbb{R}^n$ ,  $n \geq 2$ , when  $\|u_0\|_1, \|\nabla v_0\|_1, \|\nabla v_0\|_\infty$  are small but  $\|u_0\|_\infty$  is not necessarily small. The uniqueness of solutions is also given in the theorem. The nonnegativity of solutions is not assumed.

**Theorem 1.2** (i) *The uniqueness of solutions to (1.1)–(1.3) holds.*

(ii) *Let  $n \geq 2$ . For any given  $K > 0$  let  $(u_0, v_0)$  be an initial function satisfying*

$$\|u_0\|_\infty \leq K.$$

*Then there exists a small  $\delta > 0$ , depending on  $K$ , such that (1.1)–(1.3) admits a solution on  $\mathbb{R}^n \times [0, \infty)$  satisfying (1.6), provided that*

$$\|u_0\|_1 \leq \delta, \quad \|\nabla v_0\|_1 \leq \delta, \quad \|\nabla v_0\|_\infty \leq \delta.$$

## 2 Estimates for the heat equation

We begin with mentioning  $L^p - L^q$  estimates for  $e^{t\Delta}$ , which is proved by Young's inequality for convolution.

**Lemma 2.1** *Let  $1 \leq q \leq p \leq \infty$  and  $f \in L^q(\mathbb{R}^n)$ . Then*

$$(2.1) \quad \|e^{t\Delta} f\|_p \leq (4\pi t)^{-n(1/q-1/p)/2} \|f\|_q,$$

$$(2.2) \quad \|\nabla e^{t\Delta} f\|_p \leq C t^{-n(1/q-1/p)/2-1/2} \|f\|_q,$$

where  $C$  is a positive constant depending only on  $p$  and  $q$ .

In this section, let  $X = L^p(\mathbb{R}^n)$  for  $1 \leq p < \infty$  or  $X = \mathcal{B}(\mathbb{R}^n)$ . We then see  $e^{t\Delta} w_0 \in C([0, \infty) : X)$  for  $w_0 \in X$ .

**Lemma 2.2** *Let  $f \in C([0, T] : X)$  ( $0 < T < \infty$ ) and define  $F(t)$  by*

$$F(t) = \int_0^t e^{a(t-s)} e^{(t-s)\Delta} f(s) ds \quad (0 < t \leq T), \quad F(0) = 0,$$

where  $a$  is a constant. Then,  $F, \partial_j F \in C([0, T] : X)$  ( $1 \leq j \leq n$ ).

*Proof.* We give the proof only for  $\partial_j F$ , because the proof for  $F$  is done in the same way.

By (2.2), we have

$$\|\partial_j e^{a(t-s)} e^{(t-s)\Delta} f(s)\|_p \leq C e^{a(t-s)} (t-s)^{-1/2} \|f(s)\|_p,$$

which implies that

$$\partial_j F(t) = \int_0^t \partial_j e^{a(t-s)} e^{(t-s)\Delta} f(s) ds$$

belongs to  $L^p(\mathbb{R}^n)$  if  $X = L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ), and to  $L^\infty(\mathbb{R}^n)$  if  $X = \mathcal{B}(\mathbb{R}^n)$ .

Let us show  $\partial_j F(t) \in X$  when  $X = \mathcal{B}(\mathbb{R}^n)$ . For  $x_1, x_2 \in \mathbb{R}^n$  we have

$$\begin{aligned} & \partial_j F(x_1, t) - \partial_j F(x_2, t) \\ &= \int_0^t e^{a(t-s)} ds \int_{|x_1-y| \leq 2|x_1-x_2|} \partial_j G(x_1-y, t-s) f(y, s) dy \\ & \quad - \int_0^t e^{a(t-s)} ds \int_{|x_1-y| \leq 2|x_1-x_2|} \partial_j G(x_2-y, t-s) f(y, s) dy \\ & \quad + \int_0^t e^{a(t-s)} ds \int_{|x_1-y| \geq 2|x_1-x_2|} \{\partial_j G(x_1-y, t-s) - \partial_j G(x_2-y, t-s)\} f(y, s) dy \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Let  $0 < \alpha < 1$ . We observe that

$$\int_{|x_1-y| \leq 2|x_1-x_2|} |\partial_j G(x_1-y, t-s)| dy \leq C|x_1-x_2|^\alpha (t-s)^{-(1+\alpha)/2},$$

which implies that

$$(2.3) \quad |J_1| \leq C|x_1-x_2|^\alpha \int_0^t e^{a(t-s)} (t-s)^{-(1+\alpha)/2} ds \sup_{0 < s < t} \|f(s)\|_\infty.$$

Since  $\{y \mid |x_1-y| \leq 2|x_1-x_2|\} \subset \{y \mid |x_2-y| \leq 3|x_1-x_2|\}$ , (2.3) holds for  $J_2$ . We next observe that for  $|x_1-y| \geq 2|x_1-x_2|$ ,

$$|\partial_j G(x_1-y, t-s) - \partial_j G(x_2-y, t-s)| \leq C|x_1-x_2|^\alpha (t-s)^{-n/2-(1+\alpha)/2} e^{-c|x_1-y|^2/(t-s)},$$

where  $c$  is a positive constant. Hence,

$$|J_3| \leq C|x_1-x_2|^\alpha \int_0^t e^{a(t-s)} (t-s)^{-(1+\alpha)/2} ds \sup_{0 < s < t} \|f(s)\|_\infty.$$

Therefore, the function  $x \mapsto \partial_j F(x, t)$  is Hölder continuous on  $\mathbb{R}^n$ . Hence,  $\partial_j F(t) \in X$ .

Let  $0 \leq t < t+h \leq T$ .  $\partial_j F(t+h) - \partial_j F(t)$  is written as

$$\begin{aligned} & \partial_j F(t+h) - \partial_j F(t) \\ &= \int_0^t e^{a(t-s)} \partial_j e^{(t-s)\Delta} (f(s+h) - f(s)) ds + \int_{-h}^0 e^{a(t-s)} \partial_j e^{(t-s)\Delta} f(s+h) ds \\ &= I_1 + I_2. \end{aligned}$$

For  $1 \leq p \leq \infty$  we have

$$\begin{aligned} \|I_1\|_p &\leq C \int_0^t e^{a(t-s)} (t-s)^{-1/2} \|f(s+h) - f(s)\|_p ds \\ &\leq C \sup_{0 \leq s \leq T} \|f(s+h) - f(s)\|_p, \end{aligned}$$

and

$$\begin{aligned} \|I_2\|_p &\leq C \int_{-h}^0 e^{a(t-s)} (t-s)^{-1/2} \|f(s+h)\|_p ds \\ &\leq C\sqrt{h} \sup_{0 \leq s \leq T} \|f(s)\|_p. \end{aligned}$$

Hence,

$$\|\partial_j F(t+h) - \partial_j F(t)\|_p \leq C \sup_{0 \leq s \leq T} \|f(s+h) - f(s)\|_p + C\sqrt{h} \sup_{0 \leq s \leq T} \|f(s)\|_p,$$

which implies  $\partial_j F \in C([0, T] : X)$ . The proof is complete.

**Lemma 2.3** Define  $v$  by

$$v(t) = e^{-t}e^{t\Delta}v_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}u(s) ds \quad (t > 0), \quad v(0) = v_0.$$

Then the following holds.

(i) If  $v_0, \partial_j v_0 \in X$  and  $u \in C([0, \infty) : X)$ , then  $v, \partial_j v \in C([0, \infty) : X)$ .

(ii) Let  $1 \leq q \leq p \leq \infty$  and  $1/q - 1/p < 1/n$ . If  $v_0, |\nabla v_0| \in L^p(\mathbb{R}^n)$  and  $u \in C([0, \infty) : L^q(\mathbb{R}^n))$ , then

$$(2.4) \quad \|\nabla v(t)\|_p \leq e^{-t}\|\nabla v_0\|_p + C\Gamma(\beta) \sup_{0 < s < t} \|u(s)\|_q,$$

$$(2.5) \quad \|\nabla v(t)\|_p \leq e^{-t}\|\nabla v_0\|_p + Ct^{\beta-1}e^{-t/2} \sup_{0 < s < t/2} \|u(s)\|_q \\ + C\Gamma(\beta) \sup_{t/2 < s < t} \|u(s)\|_q$$

where  $C$  is a positive constant depending only on  $p$  and  $q$ ,  $\Gamma(z)$  is the gamma function and  $\beta = 1/2 - n(1/q - 1/p)/2$ .

*Proof.* The functions  $t \mapsto e^{t\Delta}v_0$  and  $t \mapsto \partial_j e^{t\Delta}v_0 = e^{t\Delta}\partial_j v_0$  belong to  $C([0, \infty) : X)$ . Hence,  $v, \partial_j v \in C([0, \infty) : X)$  follows from Lemma 2.2.

To estimate  $\nabla v$ , we use the form

$$\begin{aligned} \nabla v(t) &= e^{-t}e^{t\Delta}\nabla v_0 + \int_0^t e^{-(t-s)}\nabla e^{(t-s)\Delta}u(s) ds \\ &= e^{-t}e^{t\Delta}\nabla v_0 + V(t). \end{aligned}$$

By (2.1),

$$\|e^{-t}e^{t\Delta}\nabla v_0\|_p \leq e^{-t}\|\nabla v_0\|_p.$$

By (2.2),

$$\begin{aligned} \|V(t)\|_p &\leq C \int_0^t e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2} \|u(s)\|_q ds \\ &\leq C \sup_{0 < s < t} \|u(s)\|_q \int_0^\infty e^{-(t-s)}(t-s)^{\beta-1} ds \\ &= C\Gamma(\beta) \sup_{0 < s < t} \|u(s)\|_q. \end{aligned}$$

Hence, we have (2.4).

Next, to get (2.5) we rewrite  $V(t)$  in the form

$$\begin{aligned} V(t) &= \left( \int_0^{t/2} + \int_{t/2}^t \right) e^{-(t-s)}\nabla e^{(t-s)\Delta}u(s) ds \\ &= V_1(t) + V_2(t). \end{aligned}$$

By (2.2),

$$\begin{aligned}
\|V_1(t)\|_p &\leq C \int_0^{t/2} e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2} \|u(s)\|_q ds \\
&\leq C \sup_{0 < s < t/2} \|u(s)\|_q \int_0^{t/2} e^{-(t-s)}(t-s)^{\beta-1} ds \\
&\leq C e^{-t/2} t^{\beta-1} \sup_{0 < s < t/2} \|u(s)\|_q,
\end{aligned}$$

and

$$\begin{aligned}
\|V_2(t)\|_p &\leq C \int_{t/2}^t e^{-(t-s)}(t-s)^{-n(1/q-1/p)/2-1/2} \|u(s)\|_q ds \\
&\leq C \sup_{t/2 < s < t} \|u(s)\|_q \int_{t/2}^t e^{-(t-s)}(t-s)^{\beta-1} ds \\
&\leq C \Gamma(\beta) \sup_{t/2 < s < t} \|u(s)\|_q.
\end{aligned}$$

Hence, we have (2.5). The proof is complete.

### 3 Proof of Theorem 1.1

To prove the decay properties, we need the following lemma.

**Lemma 3.1** ([6], Lemma 7.1.1) *Suppose  $b \geq 0, \beta > 0$  and  $a(t)$  is a non-negative function locally integrable on  $0 \leq t < T \leq \infty$ , and suppose  $f(t)$  is nonnegative and locally integrable on  $0 \leq t < T$  with*

$$f(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} f(s) ds \quad (0 < t < T).$$

Then

$$f(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s)) a(s) ds \quad (0 \leq t < T),$$

where  $\theta = (b\Gamma(\beta))^{1/\beta}$  and

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta + 1)}, \quad E'_\beta(z) = \frac{d}{dz} E_\beta(z) \simeq C_\beta z^{\beta-1} \quad (|z| \rightarrow 0).$$

By (1.6) and (2.4), we observe that

$$(3.1) \quad \sup_{t>0} \|\nabla v(t)\|_p < \infty \quad \text{for every } p \in [1, \infty].$$

In the following lemma, we show that  $\|u(t)\|_p$  decays to zero as  $t \rightarrow \infty$  for  $p$  in a restricted range of  $(1, \infty]$ .

**Lemma 3.2** *Let  $1 < p < n/(n-2)$  if  $n \geq 3$ , and  $1 < p < \infty$  if  $n = 2$ . Let  $0 < \beta \leq 1/2$ . Then*

$$(3.2) \quad \|u(t)\|_p \leq C(1+A)\|u_0\|_1(1+t)^{-n(1-1/p)/2+\beta} \quad (t > 0),$$

where  $C$  is a positive constant, depending on  $p, \beta$ , such that  $C \rightarrow \infty$  as  $\beta \rightarrow +0$ , and

$$A = \sup_{t>0} \|\nabla v(t)\|_r, \quad r = \begin{cases} \frac{n}{1-2\beta} & \text{if } 0 < \beta < \frac{1}{2}, \\ \infty & \text{if } \beta = \frac{1}{2}. \end{cases}$$

*Proof.* For  $\lambda > 0$  define  $(u_\lambda, v_\lambda)$  by

$$u_\lambda(x, t) = \lambda^n u(\lambda x, \lambda^2 t), \quad v_\lambda(x, t) = v(\lambda x, \lambda^2 t) \quad (x \in \mathbb{R}^n, t \geq 0).$$

The function  $(u_\lambda, v_\lambda)$  satisfies

$$\begin{cases} \frac{\partial u_\lambda}{\partial t} = \Delta u_\lambda - \nabla \cdot (u_\lambda \nabla v_\lambda), & x \in \mathbb{R}^n, t > 0, \\ \frac{\partial v_\lambda}{\partial t} = \Delta v_\lambda - \lambda^2 v_\lambda + \lambda^{-n+2} u_\lambda, & x \in \mathbb{R}^n, t > 0. \end{cases}$$

We write  $u_\lambda$  in the form

$$(3.3) \quad u_\lambda(t) = e^{t\Delta} u_{0,\lambda} - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u_\lambda \nabla v_\lambda)(s) ds,$$

where  $u_{0,\lambda}(x) = \lambda^n u_0(\lambda x)$ .

We set

$$A_\lambda = \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_r.$$

Take  $q \in [1, \infty)$  satisfying  $1/q = 1/p + 1/r$ . By using (2.2) and Hölder's inequality, it follows from (3.3) that for  $0 < t \leq 1$ ,

$$\begin{aligned} \|u_\lambda(t)\|_p &\leq \|e^{t\Delta} u_{0,\lambda}\|_p + C \int_0^t (t-s)^{-n(1/q-1/p)/2-1/2} \|u_\lambda(s) \nabla v_\lambda(s)\|_q ds \\ &\leq \|e^{t\Delta} u_{0,\lambda}\|_p + C \int_0^t (t-s)^{-n(1/q-1/p)/2-1/2} \|u_\lambda(s)\|_p \|\nabla v_\lambda(s)\|_r ds \\ &\leq \|e^{t\Delta} u_{0,\lambda}\|_p + C A_\lambda \int_0^t (t-s)^{\beta-1} \|u_\lambda(s)\|_p ds. \end{aligned}$$



Here, we used that

$$-\frac{n}{2}\left(\frac{1}{q} - \frac{1}{p}\right) = -\frac{n}{2r} = \begin{cases} \beta - \frac{1}{2} & \text{if } 0 < \beta < \frac{1}{2}, \\ 0 & \text{if } \beta = \frac{1}{2} \end{cases}$$

by virtue of the definition of  $r$  for  $\beta \in (0, 1/2]$ . Hence, applying Lemma 3.1 yields that for  $0 < t \leq 1$ ,

$$\begin{aligned} \|u_\lambda(t)\|_p &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda\Gamma(\beta) \int_0^t C_\beta(t-s)^{\beta-1} \|e^{s\Delta}u_{0,\lambda}\|_p ds \\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda \int_0^t (t-s)^{\beta-1} s^{-n(1-1/p)/2} \|u_0\|_1 ds \\ &\leq \|e^{t\Delta}u_{0,\lambda}\|_p + CA_\lambda \|u_0\|_1 t^{-n(1-1/p)/2+\beta}, \end{aligned}$$

where  $C$  is a constant, depending on  $p, q, \beta$ , such that  $C \rightarrow \infty$  as  $\beta \rightarrow +0$ . Note that

$$\begin{aligned} \|u_\lambda(t)\|_p &= \lambda^{n(1-1/p)} \|u(\lambda^2 t)\|_p, \\ \|\nabla v_\lambda(t)\|_r &= \lambda^{1-n/r} \|\nabla v(\lambda^2 t)\|_r = \lambda^{2\beta} \|\nabla v(\lambda^2 t)\|_r \leq A\lambda^{2\beta}. \end{aligned}$$

Then, for  $0 < t \leq 1$ ,

$$(3.4) \quad \|u(\lambda^2 t)\|_p \leq \lambda^{-n(1-1/p)} (4\pi t)^{-n(1-1/p)/2} \|u_0\|_1 + CA \|u_0\|_1 t^{-n(1-1/p)/2+\beta} \lambda^{-n(1-1/p)+2\beta}.$$

Putting  $t = 1$  in (3.4) and taking  $\lambda = \sqrt{t}$  for each  $t > 0$ , we arrive at the decay estimate (3.2). The proof is complete.

We are now in a position to give the proof of Theorem 1.1.

*Proof of Theorem 1.1.* By (1.4),

$$\begin{aligned} u(t) - e^{t\Delta}u_0 &= - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds \\ &= -I(t). \end{aligned}$$

We show that

$$(3.5) \quad \|I(t)\|_\infty \leq C(1+t)^{-n/2-\gamma} \quad \text{for some } \gamma > 0.$$

Let  $0 < t \leq 2$ . By (2.2) we have

$$\|I(t)\|_\infty \leq C \int_0^t (t-s)^{-1/2} \|u(s) \nabla v(s)\|_\infty ds.$$

Since  $\|u(s)\nabla v(s)\|_\infty$  is bounded on  $[0, \infty)$  by (1.6) and (3.1), we have

$$\|I(t)\|_\infty \leq C \quad (0 < t \leq 2).$$

Let  $t \geq 2$ . We divide  $I(t)$  into two parts:

$$\begin{aligned} I(t) &= \left( \int_0^{t/2} + \int_{t/2}^t \right) \nabla \cdot e^{(t-s)\Delta} (u\nabla v)(s) ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

We estimate  $\|I_1(t)\|_\infty$ . Let  $p$  be such that

$$\frac{n}{n-1} < p < \begin{cases} \infty & (n=2), \\ \frac{n}{n-2} & (n \geq 3). \end{cases}$$

By using (2.2), Hölder's inequality and (3.2),  $I_1(t)$  is estimated as

$$\begin{aligned} \|I_1(t)\|_\infty &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u(s)\nabla v(s)\|_1 ds \\ &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u(s)\|_p \|\nabla v(s)\|_{p/(p-1)} ds \\ &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} s^{-n(1-1/p)/2+\beta} ds \sup_{s>0} \|\nabla v(s)\|_{p/(p-1)} \\ &\leq C t^{-n/2-n(1-1/n-1/p)/2+\beta} \end{aligned}$$

for  $0 < \beta < 1/2$ . Given  $\gamma$  by

$$\gamma = \frac{n}{2} \left( \frac{n-1}{n} - \frac{1}{p} \right) - \beta \quad \text{for } 0 < \beta < \frac{n}{2} \left( \frac{n-1}{n} - \frac{1}{p} \right),$$

we then have

$$(3.6) \quad \|I_1(t)\|_\infty \leq C t^{-n/2-\gamma} \quad \text{for } 0 < \gamma < \frac{n}{2} \left( \frac{n-1}{n} - \frac{1}{p} \right).$$

We estimate  $\|I_2(t)\|_\infty$ . Let  $n < q < \infty$ . By (2.2), we have

$$(3.7) \quad \|I_2(t)\|_\infty \leq \int_{t/2}^t (t-s)^{-n/(2q)-1/2} \|u(s)\nabla v(s)\|_q ds.$$

Let  $1 < p < n/(n-2)$  if  $n \geq 3$ , and  $p = q$  if  $n = 2$ . Using (3.2) and the boundedness of  $\|u(s)\|_\infty$  on  $[0, \infty)$ , for  $0 < \beta \leq 1/2$  we have

$$\|u(s)\|_q \leq \|u(s)\|_p^{p/q} \|u(s)\|_\infty^{1-p/q} \leq C(1+s)^{-n(1-1/p)p/(2q)+\beta p/q},$$

which together with (2.5) in Lemma 2.3 yields that

$$\|\nabla v(s)\|_\infty \leq C(1+s)^{-n(1-1/p)p/(2q)+\beta p/q}.$$

Hence,

$$\|u(s)\nabla v(s)\|_q \leq \|u(s)\|_q \|\nabla v(s)\|_\infty \leq C(1+s)^{-n(1-1/p)p/q+2\beta p/q}.$$

Substituting this estimate into (3.7) gives

$$(3.8) \quad \begin{aligned} \|I_2(t)\|_\infty &\leq Ct^{-n/(2q)+1/2-n(1-1/p)p/q+2\beta p/q} \\ &= Ct^{-n/2-\ell(p,q)+2\beta p/q}, \end{aligned}$$

where

$$\ell(p, q) = \frac{np}{q} - \frac{n}{2} - \frac{n}{2q} - \frac{1}{2}.$$

Let  $n = 2, 3$ . In the case  $n = 2$ , from  $p = q > 2$  we see that  $\ell(p, q) = 1/2 - 1/q > 0$ . In the case  $n = 3$ , take  $p$  satisfying  $n/2 + 1 < p < n/(n-2)$ . We then see that  $\ell(p, n) = p - n/2 - 1 > 0$ , which implies that we can take  $q > n$  such that  $\ell(p, q) > 0$ . Hence, for such  $p$  and  $q$  we have

$$\ell(p, q) - 2\beta \frac{p}{q} > 0 \quad \text{for sufficiently small } \beta > 0.$$

Therefore,

$$\|I_2(t)\|_\infty \leq Ct^{-n/2-\gamma} \quad \text{for some } \gamma > 0.$$

Thus we have (3.5).

Let  $n \geq 4$ . Let us put

$$\Pi = \{(p, q) \mid 1 < p < \frac{n}{n-2}, n < q < \infty\}.$$

We observe that for  $(p, q) \in \Pi$ ,

$$\ell(p, q) = \frac{n}{q}(p - \frac{1}{2}) - \frac{n+1}{2} < \ell(\frac{n}{n-2}, n) = -\frac{(n-1)^2 - 5}{2(n-2)} < 0.$$

By (2.1), (3.6) and (3.8), for  $0 < \beta \leq 1/2$  we have

$$\|u(t)\|_\infty \leq \|e^{t\Delta}u_0\|_\infty + \|I(t)\|_\infty \leq Ct^{-n/2-\ell(p,q)+2\beta p/q} \quad (t \geq 2).$$

Hence, for  $s \geq 1$  we have

$$\begin{aligned} \|u(s)\|_q &\leq \|u(s)\|_1^{1/q} \|u(s)\|_\infty^{1-1/q} \\ &\leq Cs^{-n(1-1/q)/2-\ell(p,q)(1-1/q)+2\beta(1-1/q)p/q}, \end{aligned}$$

and applying (2.5) with  $p = q = \infty$  gives

$$\|\nabla v(s)\|_\infty \leq C s^{-n/2 - \ell(p,q) + 2\beta p/q}.$$

Putting together these estimates gives

$$\|u(s)\nabla v(s)\|_q \leq C s^{-n(2-1/q)/2 - \ell(p,q)(2-1/q) + 2\beta(2-1/q)p/q},$$

and substituting this estimate into (3.7) yields that

$$\|I_2(t)\|_\infty \leq C t^{-n/2 - (n-1)/2 - \ell(p,q)(2-1/q) + 2\beta(2-1/q)p/q}.$$

If  $(n-1)/2 + \ell(p,q)(2-1/q) > 0$  for some  $(p,q) \in \Pi$ , then we have (3.5). Otherwise we have

$$\|u(t)\|_\infty \leq C t^{-n/2 - (n-1)/2 - \ell(p,q)(2-1/q) + 2\beta(2-1/q)p/q}$$

and the argument above gives

$$\|u(s)\nabla v(s)\|_q \leq C s^{-n(2-1/q)/2 - (n-1)(2-1/q)/2 - \ell(p,q)(2-1/q)^2 + 2\beta(2-1/q)^2 p/q},$$

and then

$$\|I_2(t)\|_\infty \leq C t^{-n/2 - (n-1)/2 - (n-1)(2-1/q)/2 - \ell(p,q)(2-1/q)^2 + 2\beta(2-1/q)^2 p/q}.$$

When we repeat this procedure  $m$  times, we get

$$\|I_2(t)\|_\infty \leq C t^{-n/2 - \ell(p,q,m) + 2\beta\alpha(q)^m p/q},$$

where

$$\ell(p,q,m) = \frac{n-1}{2} \sum_{k=0}^{m-1} \alpha(q)^k + \ell(p,q)\alpha(q)^m \quad (m \geq 1),$$

$$\ell(p,q,0) = \ell(p,q), \quad \alpha(q) = 2 - \frac{1}{q}.$$

We observe that

$$\ell\left(\frac{n}{n-2}, n, m\right) = \frac{2}{n-2} \left\{ \alpha(n)^m - \frac{n(n-2)}{4} \right\}.$$

Taking a natural number  $m_0$  such that

$$m_0 - 1 \leq \frac{1}{\log \alpha(n)} \log \frac{n(n-2)}{4} < m_0,$$

we see that

$$\ell\left(\frac{n}{n-2}, n, m_0\right) > 0,$$

which implies that

$$\ell(p,q,m_0) > 0 \text{ for some } (p,q) \in \Pi.$$

Hence, there is a unique  $m_1$  with  $m_1 \leq m_0$  satisfying

(i)  $\ell(p, q, k) \leq 0$  for all  $(p, q) \in \Pi$ ,  $k = 0, 1, \dots, m_1 - 1$ ,

(ii)  $\ell(p_1, q_1, m_1) > 0$  for some  $(p_1, q_1) \in \Pi$ .

Then,

$$\ell(p_1, q_1, m_1) - 2\beta\alpha(q_1)^{m_1} \frac{p_1}{q_1} > 0 \text{ for sufficiently small } \beta > 0.$$

Therefore,

$$\|I_2(t)\|_\infty \leq Ct^{-n/2-\gamma} \quad \text{for some } \gamma > 0.$$

Thus we have (3.5).

It follows from (3.5) that

$$\|u(t)\|_\infty \leq \|e^{t\Delta}u_0\|_\infty + \|I(t)\|_\infty \leq C(1+t)^{-n/2}$$

and

$$\lim_{t \rightarrow \infty} t^{n/2} \|u(t) - e^{t\Delta}u_0\|_\infty = \lim_{t \rightarrow \infty} t^{n/2} \|I(t)\|_\infty = 0.$$

Noting that

$$\lim_{t \rightarrow \infty} t^{n/2} \left\| e^{t\Delta}u_0 - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_\infty = 0,$$

we have

$$\lim_{t \rightarrow \infty} t^{n/2} \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_\infty = 0.$$

For  $1 < p < \infty$ , observing

$$\begin{aligned} \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_p &\leq \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_1^{1/p} \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_\infty^{1-1/p} \\ &\leq C \left\| u(t) - \int_{\mathbb{R}^n} u_0 dy G(t) \right\|_\infty^{1-1/p}, \end{aligned}$$

we establish the conclusion for  $u$ .

To get the conclusion for  $v$ , we write  $v$  as follows.

$$\begin{aligned} v(t) &= e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-t+s} e^{(t-s)\Delta} u(s) ds \\ &= e^{t\Delta} u_0 + e^{-t} e^{t\Delta} (v_0 - u_0) + \int_0^t e^{-t+s} e^{(t-s)\Delta} (u(s) - e^{s\Delta} u_0) ds \\ &= e^{t\Delta} u_0 + e^{-t} e^{t\Delta} (v_0 - u_0) + J(t). \end{aligned}$$

By (2.1),

$$\|e^{-t} e^{t\Delta} (v_0 - u_0)\|_\infty \leq e^{-t} \|v_0 - u_0\|_\infty.$$

Next,

$$\begin{aligned} J(t) &= -\left(\int_0^{t/2} + \int_{t/2}^t\right) e^{-t+s} e^{(t-s)\Delta} I(s) ds \\ &= J_1(t) + J_2(t). \end{aligned}$$

By (2.1) and the boundedness of  $\|I(s)\|_\infty$  on  $(0, \infty)$ , we have

$$\|J_1(t)\|_\infty \leq \int_0^{t/2} e^{-t+s} \|I(s)\|_\infty ds \leq C e^{-t/2}.$$

Using (2.1) and the decay estimate (3.5) of  $I(s)$  yields that

$$\begin{aligned} \|J_2(t)\|_\infty &\leq \int_{t/2}^t e^{-t+s} \|I(s)\|_\infty ds \leq C \int_{t/2}^t e^{-t+s} (1+s)^{-n/2-\gamma} ds \\ &\leq C(1+t)^{-n/2-\gamma}. \end{aligned}$$

for some  $\gamma > 0$ . Hence,

$$\|J(t)\|_\infty \leq C(1+t)^{-n/2-\gamma}.$$

Therefore,

$$\|v(t) - e^{t\Delta} u_0\|_\infty \leq C(1+t)^{-n/2-\gamma}.$$

The proof of Theorem 1.1 is complete.

## 4 Proof of Theorem 1.2

We first show the uniqueness of solutions to (1.1)–(1.3). Let  $(u_1, v_1)$  and  $(u_2, v_2)$  be two solutions of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, T]$  with the same initial function  $(u_0, v_0)$ . Put  $u = u_1 - u_2$  and  $v = v_1 - v_2$ . Then,

$$\begin{aligned} u(t) &= -\int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v_1 + u_2 \nabla v)(s) ds, \\ v(t) &= \int_0^t e^{-t+s} e^{(t-s)\Delta} u(s) ds. \end{aligned}$$

By (2.2) we have

$$\begin{aligned} \|u(t)\|_1 &\leq C \int_0^t (t-s)^{-1/2} \|(u \nabla v_1 + u_2 \nabla v)(s)\|_1 ds \\ &\leq C \int_0^t (t-s)^{-1/2} (\|u(s)\|_1 \|\nabla v_1(s)\|_\infty + \|u_2(s)\|_\infty \|\nabla v(s)\|_1) ds, \end{aligned}$$

and

$$\|\nabla v(t)\|_1 \leq C \int_0^t (t-s)^{-1/2} \|u(s)\|_1 ds.$$

Combining the two inequalities gives that

$$(4.1) \quad \begin{aligned} & \|u(t)\|_1 + \|\nabla v(t)\|_1 \\ & \leq K \int_0^t (t-s)^{-1/2} (\|u(s)\|_1 + \|\nabla v(s)\|_1) ds, \end{aligned}$$

where

$$K = C \left( \sup_{0 \leq s \leq T} \|\nabla v_1(s)\|_\infty + \sup_{0 \leq s \leq T} \|u_2(s)\|_\infty + 1 \right).$$

Applying Lemma 3.1 to (4.1) yields that

$$\|u(t)\|_1 + \|\nabla v(t)\|_1 \equiv 0 \quad (0 \leq t \leq T),$$

which implies that  $u \equiv 0$ ,  $v \equiv 0$  on  $\mathbb{R}^n \times [0, T]$ . Therefore, we establish the uniqueness of solutions.

We next show the existence of bounded solutions to the integral equation (1.4), (1.5) by applying the contraction mapping principle. We proceed in several steps.

*Step 1.* Put

$$\mathcal{X}_0 = C([0, \infty) : L^1(\mathbb{R}^n)) \cap C([0, \infty) : \mathcal{B}(\mathbb{R}^n))$$

and consider the Banach space

$$\mathcal{X} = \{u \in \mathcal{X}_0 \mid \sup_{t>0} \|u(t)\|_1 < \infty, \sup_{t>0} (1+t)^{n/2} \|u(t)\|_\infty < \infty\}$$

with the norm

$$\|u\|_{\mathcal{X}} = \sup_{t>0} \|u(t)\|_1 + \sup_{t>0} (1+t)^{n/2} \|u(t)\|_\infty.$$

For simplicity, we use the notation

$$\|u\|_{(p)} = \sup_{t>0} (1+t)^{n(1-1/p)/2} \|u(t)\|_p \quad \text{for } 1 \leq p \leq \infty.$$

We note that  $u \in \mathcal{X}$  implies that  $\|u\|_{(p)} < +\infty$  for every  $1 < p < \infty$  and

$$\|u\|_{(p)} \leq \|u\|_{(1)}^{1/p} \|u\|_{(\infty)}^{1-1/p}.$$

For  $K_1 > 0$  define the closed subset  $B_{K_1}$  of  $\mathcal{X}$  by

$$B_{K_1} = \left\{ u \in \mathcal{X} \mid \|u - w\|_{(1)} \leq K_1, \|u - w\|_{(\infty)} \leq 1 \right\},$$

where

$$w(t) = e^{t\Delta} u_0.$$

We see that  $w \in \mathcal{X}_0$  and that by (2.1),  $w$  satisfies

$$(4.2) \quad \|w\|_{(1)} \leq \|u_0\|_1, \quad \|w\|_{(\infty)} \leq 2^{n/2} \max\{\|u_0\|_1, \|u_0\|_\infty\},$$

which implies  $w \in \mathcal{X}$ .

Given  $u \in B_{K_1}$ , define  $v$  by

$$(4.3) \quad v(t) = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-t+s} e^{(t-s)\Delta} u(s) ds$$

and then  $\Phi(u)$  by

$$[\Phi(u)](t) = w(t) - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds.$$

Lemma 2.3 gives  $v, \partial_j v \in \mathcal{X}_0$  ( $1 \leq j \leq n$ ), and for every  $1 \leq p \leq \infty$  we have

$$\|v(t)\|_p \leq e^{-t} \|v_0\|_p + \sup_{s>0} \|u(s)\|_p \quad (t \geq 0).$$

Lemma 2.2 gives that the function  $t \mapsto \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds$  belongs to  $\mathcal{X}_0$  because of  $u \partial_j v \in \mathcal{X}_0$  ( $1 \leq j \leq n$ ), which implies  $\Phi(u) \in \mathcal{X}_0$ . If  $u \in B_{K_1}$  is a fixed-point of  $\Phi$ , then  $(u, v)$ , where  $v$  is defined by (4.3), is a solution of (1.1)–(1.3) on  $\mathbb{R}^n \times [0, \infty)$  satisfying (1.6).

In order to estimate  $\Phi(u) - w$ , we need the following estimates on  $\nabla v$ , which are direct consequences of Lemma 2.3: for  $1 \leq q \leq p \leq \infty$  and  $1/q - 1/p < 1/n$ ,

$$(4.4) \quad \|\nabla v(t)\|_p \leq e^{-t} \|\nabla v_0\|_p + C \|u\|_{(q)},$$

$$(4.5) \quad \|\nabla v(t)\|_p \leq e^{-t} \|\nabla v_0\|_p + C(t^{\beta-1} + 1)(1+t)^{-n(1-1/q)/2} \|u\|_{(q)},$$

where  $C$  is a positive constant depending only on  $p$  and  $q$ , and  $\beta = 1/2 - n(1/q - 1/p)/2$ .

*Step 2.* We estimate  $\|\Phi(u) - w\|_{(\infty)}$ . In what follows, for simplicity we put

$$I(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (u \nabla v)(s) ds.$$



We note that for  $1 \leq p \leq \infty$ ,

$$(4.6) \quad \|u(t)\|_p \leq (1+t)^{-n(1-1/p)/2} \|u\|_{(p)} \quad (t > 0).$$

Let  $0 < t \leq 1$ . For  $n < p \leq \infty$  we use (2.2), (4.4) and  $\|u(t)\|_p \leq \|u\|_{(p)}$  to get

$$(4.7) \quad \begin{aligned} \|I(t)\|_\infty &\leq C \int_0^t (t-s)^{-n/(2p)-1/2} \|u(s)\nabla v(s)\|_p ds \\ &\leq C \int_0^t (t-s)^{-n/(2p)-1/2} \|u(s)\|_p \|\nabla v(s)\|_\infty ds \\ &\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_\infty + \|u\|_{(p)} \right) \int_0^t (t-s)^{-n/(2p)-1/2} ds \\ &\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_\infty + \|u\|_{(p)} \right). \end{aligned}$$

Let  $t \geq 1$ . We divide  $I(t)$  into two parts:

$$\begin{aligned} I(t) &= \left( \int_0^{t/2} + \int_{t/2}^t \right) \nabla \cdot e^{(t-s)\Delta} (u\nabla v)(s) ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

For  $n < p < \infty$  we use (2.2), (4.4) and (4.6) to get

$$\begin{aligned} \|I_1(t)\|_\infty &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u(s)\nabla v(s)\|_1 ds \\ &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u(s)\|_p \|\nabla v(s)\|_{p/(p-1)} ds \\ &\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)} \right) \int_0^{t/2} (t-s)^{-n/2-1/2} (1+s)^{-n(1-1/p)/2} ds \\ &\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)} \right) (1+t)^{-n/2}. \end{aligned}$$

For  $2n < p < \infty$ , using (4.5) gives

$$\begin{aligned}
\|I_2(t)\|_\infty &\leq C \int_{t/2}^t (t-s)^{-n/p-1/2} \|u(s)\nabla v(s)\|_{p/2} ds \\
&\leq C \int_{t/2}^t (t-s)^{-n/p-1/2} \|u(s)\|_p \|\nabla v(s)\|_p ds \\
&\leq C \|u\|_{(p)} \int_{t/2}^t (t-s)^{-n/p-1/2} (1+s)^{-n(1-1/p)/2} \\
&\quad \times \left\{ e^{-s} \|\nabla v_0\|_p + (s^{-1/2} + 1)(1+s)^{-n(1-1/p)/2} \|u\|_{(p)} \right\} ds \\
&\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_p + \|u\|_{(p)} \right) \int_{t/2}^t (t-s)^{-n/p-1/2} (1+s)^{-n(1-1/p)/2} ds \\
&\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_p + \|u\|_{(p)} \right) (1+t)^{-n/2}.
\end{aligned}$$

Putting together the estimates of  $\|I_1(t)\|_\infty$  and  $\|I_2(t)\|_\infty$ , we have

$$\begin{aligned}
(4.8) \quad &\|I(t)\|_\infty \\
&\leq C \|u\|_{(p)} \left( \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u\|_{(1)} + \|u\|_{(p)} \right) (1+t)^{-n/2},
\end{aligned}$$

where  $2n < p < \infty$ . Here we used

$$(4.9) \quad \|\nabla v_0\|_q \leq \frac{1}{q} \|\nabla v_0\|_1 + \frac{q-1}{q} \|\nabla v_0\|_\infty \quad \text{for } q \in (1, \infty).$$

Hence, (4.7) and (4.8) yield that

$$\begin{aligned}
(4.10) \quad &\|\Phi(u) - w\|_{(\infty)} = \|I\|_{(\infty)} \\
&\leq C \left\{ \|u\|_{(p)}^2 + \|u\|_{(p)} (\|u\|_{(1)} + \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty) \right\},
\end{aligned}$$

where  $2n < p < \infty$ .

*Step 3.* We estimate  $\|\Phi(u) - w\|_{(1)}$ . Let  $n < p < \infty$ . By (4.4) and (4.6), we have

$$\begin{aligned}
\|I(t)\|_1 &\leq C \int_0^t (t-s)^{-1/2} \|u(s)\nabla v(s)\|_1 ds \\
&\leq C \int_0^t (t-s)^{-1/2} \|u(s)\|_p \|\nabla v(s)\|_{p/(p-1)} ds \\
&\leq C \|u\|_{(p)} (\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)}) \int_0^t (t-s)^{-1/2} (1+s)^{-n(1-1/p)/2} ds \\
&\leq C \|u\|_{(p)} (\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)}).
\end{aligned}$$

Here we used

$$(4.11) \quad \sup_{t>0} \int_0^t (t-s)^{-\alpha} (1+s)^{-\beta} ds < \infty \quad \text{for } 0 < \alpha < 1, \beta \geq 1 - \alpha.$$

Hence, for  $n < p < \infty$  we have

$$(4.12) \quad \|\Phi(u) - w\|_{(1)} \leq C \|u\|_{(p)} (\|\nabla v_0\|_{p/(p-1)} + \|u\|_{(1)}).$$

*Step 4.* We estimate  $\|\Phi(u_1) - \Phi(u_2)\|_{(\infty)}$  for  $u_1, u_2 \in B_{K_1}$ . To do so, we write

$$\begin{aligned} & [\Phi(u_1) - \Phi(u_2)](t) \\ &= - \int_0^t \nabla \cdot e^{(t-s)\Delta} (u_1 \nabla v_1)(s) ds + \int_0^t \nabla \cdot e^{(t-s)\Delta} (u_2 \nabla v_2)(s) ds \\ &= \int_0^t \nabla \cdot e^{(t-s)\Delta} ((u_2 - u_1) \nabla v_1)(s) ds + \int_0^t \nabla \cdot e^{(t-s)\Delta} (u_2 \nabla (v_2 - v_1))(s) ds \\ &= J_1(t) + J_2(t) \end{aligned}$$

Since the function  $v_1 - v_2$  satisfies

$$(v_1 - v_2)(s) = \int_0^t e^{-t+s} e^{(t-s)\Delta} (u_1 - u_2)(s) ds,$$

applying (4.4) and (4.5) with  $v_0 = 0, v = v_1 - v_2, u = u_1 - u_2$  gives

$$(4.13) \quad \|\nabla(v_1 - v_2)(t)\|_p \leq C \|u_1 - u_2\|_{(q)},$$

$$(4.14) \quad \|\nabla(v_1 - v_2)(t)\|_p \leq C(t^{\beta-1} + 1)(1+t)^{-n(1-1/q)/2} \|u_1 - u_2\|_{(q)},$$

where  $1 \leq q \leq p \leq +\infty, 1/q - 1/p < 1/n$  and  $\beta = 1/2 - n(1/q - 1/p)/2$ .

Let  $0 < t \leq 1$  and  $n < p < \infty$ . We observe that

$$\begin{aligned} \|J_1(t)\|_{\infty} &\leq C \int_0^t (t-s)^{-1/2} \|(u_1(s) - u_2(s)) \nabla v_1(s)\|_{\infty} ds \\ &\leq C \int_0^t (t-s)^{-1/2} \|u_1(s) - u_2(s)\|_{\infty} \|\nabla v_1(s)\|_{\infty} ds \\ &\leq C \|u_1 - u_2\|_{(\infty)} \left( \|\nabla v_0\|_{\infty} + \|u_1\|_{(p)} \right) \int_0^t (t-s)^{-1/2} ds \\ &\leq C \left( \|\nabla v_0\|_{\infty} + \|u_1\|_{(p)} \right) \|u_1 - u_2\|_{(\infty)}. \end{aligned}$$

Using (4.13), we observe that

$$\begin{aligned}
\|J_2(t)\|_\infty &\leq C \int_0^t (t-s)^{-n/(2p)-1/2} \|u_2(s) \nabla(v_1(s) - v_2(s))\|_p ds \\
&\leq C \int_0^t (t-s)^{-n/(2p)-1/2} \|u_2(s)\|_p \|\nabla v_1(s) - \nabla v_2(s)\|_\infty ds \\
&\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(\infty)} \int_0^t (t-s)^{-n/(2p)-1/2} ds \\
&\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(\infty)}.
\end{aligned}$$

Putting together the two inequalities just above gives

$$\begin{aligned}
(4.15) \quad &\|\Phi(u_1)(t) - \Phi(u_2)(t)\|_\infty \\
&\leq C (\|\nabla v_0\|_\infty + \|u_1\|_{(p)} + \|u_2\|_{(p)}) \|u_1 - u_2\|_{(\infty)},
\end{aligned}$$

where  $n < p < \infty$ .

Let  $t \geq 1$ . We rewrite  $J_1(t)$  and  $J_2(t)$  as follows.

$$\begin{aligned}
J_1(t) &= \left( \int_0^{t/2} + \int_{t/2}^t \right) \nabla \cdot e^{(t-s)\Delta} ((u_2 - u_1) \nabla v_1)(s) ds \\
&= J_{11}(t) + J_{12}(t), \\
J_2(t) &= \left( \int_0^{t/2} + \int_{t/2}^t \right) \nabla \cdot e^{(t-s)\Delta} (u_2 \nabla(v_2 - v_1))(s) ds \\
&= J_{21}(t) + J_{22}(t).
\end{aligned}$$

Applying (4.6) with  $u = u_1 - u_2$  and  $p = \infty$  and (4.4), we have

$$\begin{aligned}
\|J_{11}(t)\|_\infty &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u_1(s) - u_2(s)\|_\infty \|\nabla v_1(s)\|_1 ds \\
&\leq C \|u_1 - u_2\|_{(\infty)} \left( \|\nabla v_0\|_1 + \|u_1\|_{(1)} \right) \int_0^{t/2} (t-s)^{-n/2-1/2} (1+s)^{-n/2} ds \\
&\leq C \|u_1 - u_2\|_{(\infty)} \left( \|\nabla v_0\|_1 + \|u_1\|_{(1)} \right) (1+t)^{-n/2}.
\end{aligned}$$

Let  $n < p < \infty$ . Applying (4.6) again with  $u = u_1 - u_2$  and using (4.5), we

have

$$\begin{aligned}
\|J_{12}(t)\|_\infty &\leq C \int_{t/2}^t (t-s)^{-n/(2p)-1/2} \|u_1(s) - u_2(s)\|_\infty \|\nabla v_1(s)\|_p ds \\
&\leq C \|u_1 - u_2\|_{(\infty)} \int_{t/2}^t (t-s)^{-n/(2p)-1/2} (1+s)^{-n/2} \\
&\quad \times \{e^{-s} \|\nabla v_0\|_p + (s^{-1/2} + 1)(1+s)^{-n(1-1/p)/2} \|u_1\|_{(p)}\} ds \\
&\leq C \|u_1 - u_2\|_{(\infty)} \left( \|\nabla v_0\|_p + \|u_1\|_{(p)} \right) \int_{t/2}^t (t-s)^{-n/(2p)-1/2} (1+s)^{-n+n/(2p)} ds \\
&\leq C \|u_1 - u_2\|_{(\infty)} \left( \|\nabla v_0\|_p + \|u_1\|_{(p)} \right) (1+t)^{-n/2}.
\end{aligned}$$

Similarly, by (4.14) we have

$$\begin{aligned}
\|J_{21}(t)\|_\infty &\leq C \int_0^{t/2} (t-s)^{-n/2-1/2} \|u_2(s)\|_1 \|\nabla(v_1(s) - v_2(s))\|_\infty ds \\
&\leq C \|u_2\|_{(1)} \|u_1 - u_2\|_{(\infty)} \int_0^{t/2} (t-s)^{-n/2-1/2} (s^{-1/2} + 1)(1+s)^{-n/2} ds \\
&\leq C \|u_2\|_{(1)} \|u_1 - u_2\|_{(\infty)} (1+t)^{-n/2}
\end{aligned}$$

and

$$\begin{aligned}
\|J_{22}(t)\|_\infty &\leq C \int_{t/2}^t (t-s)^{-n/(2p)-1/2} \|u_2(s)\|_p \|\nabla(v_1(s) - v_2(s))\|_\infty ds \\
&\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(\infty)} \int_{t/2}^t (t-s)^{-n/(2p)-1/2} (1+s)^{-n+n/(2p)} ds \\
&\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(\infty)} (1+t)^{-n/2}.
\end{aligned}$$

Putting together these inequalities and using (4.9) yields that

$$\begin{aligned}
\|\Phi(u_1)(t) - \Phi(u_2)(t)\|_\infty &\leq \sum_{1 \leq i, j \leq 2} \|J_{ij}(t)\|_\infty \\
&\leq C \left( \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u_1\|_{(1)} + \|u_1\|_{(p)} + \|u_2\|_{(1)} + \|u_2\|_{(p)} \right) \\
&\quad \times \|u_1 - u_2\|_{(\infty)} (1+t)^{-n/2},
\end{aligned}$$

from which together with (4.15) it follows that

$$\begin{aligned}
(4.16) \quad &\|\Phi(u_1) - \Phi(u_2)\|_{(\infty)} \\
&\leq C \left( \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u_1\|_{(1)} + \|u_1\|_{(p)} + \|u_2\|_{(1)} + \|u_2\|_{(p)} \right) \\
&\quad \times \|u_1 - u_2\|_{(\infty)},
\end{aligned}$$

where  $n < p < \infty$ .

*Step 5.* We estimate  $\|\Phi(u_1) - \Phi(u_2)\|_{(1)}$  for  $u_1, u_2 \in B_{K_1}$ . To do so, we estimate  $J_1(t)$  and  $J_2(t)$  given in Step 4.

Let  $n < p < \infty$ . For  $0 < t \leq 1$ , by (4.4) we have

$$\begin{aligned} \|J_1(t)\|_1 &\leq C \int_0^t (t-s)^{-1/2} \|u_1(s) - u_2(s)\|_1 \|\nabla v_1(s)\|_\infty ds \\ &\leq C \|u_1 - u_2\|_{(1)} \left( \|\nabla v_0\|_\infty + \|u_1\|_{(p)} \right) \int_0^t (t-s)^{-1/2} ds \\ &\leq C \left( \|\nabla v_0\|_\infty + \|u_1\|_{(p)} \right) \|u_1 - u_2\|_{(1)}. \end{aligned}$$

For  $t \geq 1$ , by (4.5) we have

$$\begin{aligned} \|J_1(t)\|_1 &\leq C \int_0^t (t-s)^{-1/2} \|u_1(s) - u_2(s)\|_1 \|\nabla v_1(s)\|_\infty ds \\ &\leq C \|u_1 - u_2\|_{(1)} \\ &\quad \times \int_0^t (t-s)^{-1/2} \left\{ e^{-s} \|\nabla v_0\|_\infty + (s^{-1/2-n/(2p)} + 1)(1+s)^{-n(1-1/p)/2} \|u_1\|_{(p)} \right\} ds \\ &\leq C \left( \|\nabla v_0\|_\infty + \|u_1\|_{(p)} \right) \|u_1 - u_2\|_{(1)}. \end{aligned}$$

Hence,

$$(4.17) \quad \|J_1\|_{(1)} \leq C \left( \|\nabla v_0\|_\infty + \|u_1\|_{(p)} \right) \|u_1 - u_2\|_{(1)}.$$

Next, let  $n < p < \infty$ . Using (4.14) and applying (4.11), we observe that

$$\begin{aligned} \|J_2(t)\|_1 &\leq C \int_0^t (t-s)^{-1/2} \|u_2(s)\|_p \|\nabla(v_1(s) - v_2(s))\|_{p/(p-1)} ds \\ &\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(1)} \int_0^t (t-s)^{-1/2} (1+s)^{-n(1-1/p)/2} ds \\ &\leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(1)}, \end{aligned}$$

which implies that

$$(4.18) \quad \|J_2\|_{(1)} \leq C \|u_2\|_{(p)} \|u_1 - u_2\|_{(1)}.$$

Hence, it follows from (4.17) and (4.18) that

$$(4.19) \quad \|\Phi(u_1) - \Phi(u_2)\|_{(1)} \leq C \left( \|\nabla v_0\|_\infty + \|u_1\|_{(p)} + \|u_2\|_{(p)} \right) \|u_1 - u_2\|_{(1)},$$

where  $n < p < \infty$ .

*Step 6.* We show that  $\Phi$  admits a unique fixed-point  $u$  in  $B_{K_1}$ . We may assume  $\|u_0\|_1 \leq 1$ , because  $\|u_0\|_1$  is taken small as will be mentioned below. By (4.2), we observe that for  $u \in B_{K_1}$ ,

$$(4.20) \quad \|u\|_{(1)} \leq \|u - w\|_{(1)} + \|w\|_{(1)} \leq K_1 + \|u_0\|_1$$

and

$$\|u\|_{(\infty)} \leq \|u - w\|_{(\infty)} + \|w\|_{(\infty)} \leq 1 + 2^{n/2} \max\{\|u_0\|_1, \|u_0\|_{\infty}\}.$$

Then, since  $\|u_0\|_{\infty} \leq K$ , we have

$$(4.21) \quad \|u\|_{(\infty)} \leq K_2, \quad K_2 = 1 + 2^{n/2}(1 + K).$$

Using (4.20) and (4.21) implies

$$(4.22) \quad \|u\|_{(p)} \leq \|u\|_{(1)}^{1/p} \|u\|_{(\infty)}^{1-1/p} \leq K_2^{1-1/p} (K_1 + \|u_0\|_1)^{1/p} \quad (1 < p < \infty).$$

From (4.10) we then see that there is a small  $\alpha > 0$  such that

$$(4.23) \quad \|\Phi(u) - w\|_{(\infty)} \leq 1,$$

provided that

$$K_1 + \|u_0\|_1 \leq \alpha.$$

Using (4.20) and (4.22) in the right-hand side of (4.12) and then Young's inequality, we have

$$\begin{aligned} & \|\Phi(u) - w\|_{(1)} \\ & \leq CK_2^{1-1/p} \|\nabla v_0\|_{p/(p-1)} (K_1 + \|u_0\|_1)^{1/p} + CK_2^{1-1/p} (K_1 + \|u_0\|_1)^{1+1/p} \\ & \leq \frac{p-1}{p} CK_2 \|\nabla v_0\|_{p/(p-1)} + \frac{1}{p} \|\nabla v_0\|_{p/(p-1)}^p (K_1 + \|u_0\|_1) \\ & \quad + CK_2^{1-1/p} 2^{1+1/p} (K_1^{1+1/p} + \|u_0\|_1^{1+1/p}) \\ & = a + bK_1 + cK_1^\ell, \end{aligned}$$

where  $\ell = 1 + 1/p$  and

$$\begin{aligned} a &= \frac{p-1}{p} CK_2 \|\nabla v_0\|_{p/(p-1)} + \frac{1}{p} \|\nabla v_0\|_{p/(p-1)}^p \|u_0\|_1 + CK_2^{1-1/p} 2^{1+1/p} \|u_0\|_1^{1+1/p}, \\ b &= \frac{1}{p} \|\nabla v_0\|_{p/(p-1)}^p, \quad c = CK_2^{1-1/p} 2^{1+1/p}. \end{aligned}$$

We observe that if  $a, b$  and  $c$  satisfy the condition

$$0 < b < 1, \quad 0 < a < (1-b)^{\ell/(\ell-1)} (c\ell)^{-1/(\ell-1)} (1-1/\ell),$$

then there exist  $\beta$  and  $\gamma$ , satisfying  $0 < \beta < \gamma < ((1-b)/c)^{1/(\ell-1)}$ , such that

$$c\xi^\ell + b\xi + a \leq \xi \quad \text{for every } \xi \in [\beta, \gamma].$$

We also see that

$$\beta \rightarrow 0, \quad \gamma \rightarrow \gamma_0 \quad \text{as } a \rightarrow 0 \text{ and } b \rightarrow 0,$$

where  $\gamma_0 = (1/c)^{1/(\ell-1)}$ . Hence, we can take a small  $\delta > 0$  with  $\delta \leq \alpha$  such that the condition

$$\|u_0\|_1 \leq \delta, \quad \|\nabla v_0\|_1 \leq \delta, \quad \|\nabla v_0\|_\infty \leq \delta$$

ensures the existence of  $K_1$  satisfying

$$K_1 + \|u_0\|_1 \leq \alpha, \quad a + bK_1 + cK_1^\ell \leq K_1,$$

which implies (4.23) and

$$\|\Phi(u) - w\|_{(1)} \leq K_1 \quad \text{for every } u \in B_{K_1}.$$

We therefore obtain

$$\Phi(u) \in B_{K_1} \quad \text{for every } u \in B_{K_1}.$$

Next, it follows from (4.16) and (4.19) that

$$\|\Phi(u_1) - \Phi(u_2)\|_{\mathcal{X}} \leq L(u_0, v_0, u_1, u_2) \|u_1 - u_2\|_{\mathcal{X}},$$

where

$$\begin{aligned} & L(u_0, v_0, u_1, u_2) \\ &= C \left\{ \|\nabla v_0\|_1 + \|\nabla v_0\|_\infty + \|u_1\|_{(1)} + \|u_1\|_{(p)} + \|u_2\|_{(1)} + \|u_2\|_{(p)} \right\}. \end{aligned}$$

We choose again  $\alpha$  and  $\delta$  so small that

$$L(u_0, v_0, u_1, u_2) \leq \frac{1}{2} \quad \text{for every } u_1, u_2 \in B_{K_1},$$

which implies that  $\Phi$  is a contraction mapping in the closed subset  $B_{K_1}$  of  $X$ . Therefore,  $\Phi$  admits a unique fixed-point  $u$  in  $B_{K_1}$ . The proof of Theorem 1.2 is complete.



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