

Asymptotic behavior of a Mindlin-Timoshenko plate with viscoelastic dissipation on the boundary ^{*}

by

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Abstract

A Mindlin-Timoshenko plate system with memory dissipation on the boundary is considered and the asymptotic behavior of the energy when the time goes to infinity is showed, more precisely, we prove that the energy decays with the same rate as the relaxation function, that is, if the relaxation function decay exponentially (respectively polynomially) then the energy decay exponentially (respectively polynomially).

1 Introduction

In this paper we deal with the Mindlin-Timoshenko plate system. We consider a plate of small thickness h occupying a region $\Omega \subset \mathbb{R}^2$. Denoting by $\psi(x, y, t)$ and $\varphi(x, y, t)$ the angles of rotation of a filament and by $\omega(x, y, t)$ the vertical displacement of the middle surface, the equations (see [3]) that model this system are given by

$$(1.1) \quad \frac{\rho h^3}{12} \psi_{tt} - D \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{1-\mu}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{1+\mu}{2} \frac{\partial^2 \varphi}{\partial x \partial y} \right) + K \left(\psi + \frac{\partial \omega}{\partial x} \right) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$(1.2) \quad \frac{\rho h^3}{12} \varphi_{tt} - D \left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{1-\mu}{2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{1+\mu}{2} \frac{\partial^2 \psi}{\partial x \partial y} \right) + K \left(\varphi + \frac{\partial \omega}{\partial y} \right) = 0 \quad \text{in } \Omega \times \mathbb{R}^+,$$

$$(1.3) \quad \rho h \omega_{tt} - K \left[\frac{\partial}{\partial x} \left(\psi + \frac{\partial \omega}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varphi + \frac{\partial \omega}{\partial y} \right) \right] = 0 \quad \text{in } \Omega \times \mathbb{R}^+.$$

Here, the positive constants ρ , D , μ and K denote

- ρ : the mass density per unit volume,
- D : the modulus of flexural rigidity,
- μ : the Poisson's ratio ($0 < \mu < 1/2$),
- K : the shear modulus.

The boundary Γ of Ω is a smooth surface composed by two components Γ_0, Γ_1 ($\Gamma = \Gamma_0 \cup \Gamma_1$) such that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$.

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We assume that the plate is clamped along Γ_0 , that is

$$(1.4) \quad \psi = \varphi = \omega = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+,$$

and the other part of its boundary Γ_1 is in connection with a viscoelastic element, which produces the following boundary condition

$$(1.5) \quad \begin{aligned} \psi + \int_0^t g_1(t-s) \mathfrak{B}_1(\psi(s), \varphi(s)) ds &= 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ \varphi + \int_0^t g_2(t-s) \mathfrak{B}_2(\psi(s), \varphi(s)) ds &= 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ \omega + \int_0^t g_3(t-s) \mathfrak{B}_3(\psi(s), \varphi(s), \omega(s)) ds &= 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \end{aligned}$$

where the boundary operators \mathfrak{B}_1 , \mathfrak{B}_2 and \mathfrak{B}_3 are given by

$$\begin{aligned} \mathfrak{B}_1(\psi, \varphi) &= D \left[\nu_1 \frac{\partial \psi}{\partial x} + \mu \nu_1 \frac{\partial \varphi}{\partial y} + \frac{1-\mu}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \nu_2 \right], \\ \mathfrak{B}_2(\psi, \varphi) &= D \left[\nu_2 \frac{\partial \varphi}{\partial y} + \mu \nu_2 \frac{\partial \psi}{\partial x} + \frac{1-\mu}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \nu_1 \right], \\ \mathfrak{B}_3(\psi, \varphi, \omega) &= K \left(\frac{\partial \omega}{\partial \nu} + \nu_1 \psi + \nu_2 \varphi \right), \end{aligned}$$

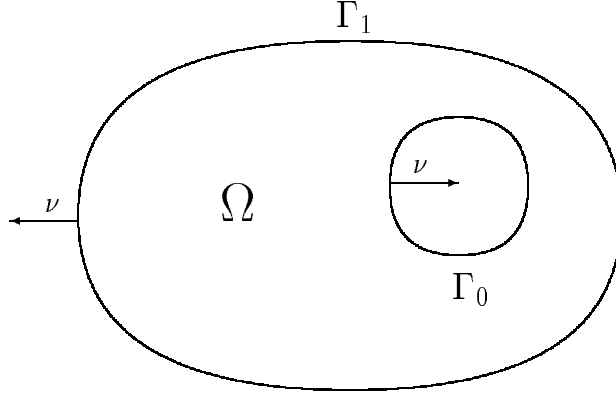
and g_i , $i = 1, 2, 3$ are non negative decreasing functions. The initial position for ψ , φ and ω are prescribed by

$$(1.6) \quad \begin{aligned} \psi(x, 0) &= \psi^0(x), & \varphi(x, 0) &= \varphi^0(x), & \omega(x, 0) &= \omega^0(x) \\ \psi_t(x, 0) &= \psi^1(x), & \varphi_t(x, 0) &= \varphi^1(x), & \omega_t(x, 0) &= \omega^1(x) \end{aligned} \quad \text{in} \quad \Omega.$$

where ψ^0 , ψ^1 , φ^0 , φ^1 , ω^0 and ω^1 are known functions. A typical example of Ω is given in the next figure

Concerning memory condition on the boundary we can cite only a few works. In [1] Ciarletta established theorems of existence, uniqueness and asymptotic stability for a linear model of heat conduction. In this case the memory condition describes a boundary that can absorb heat and due to the hereditary term, can retain part of it. In [2] Fabrizio and Morro consider a linear electromagnetic model and proved the existence, uniqueness and asymptotic stability of the solutions. In [6] Muñoz Rivera and Andrade showed exponential stability for a non homogeneous anisotropic system when the resolvent kernel of the memory is exponential type. Polynomial resolvent kernel was not considered in that work.

The nonlinear one-dimensional wave equation with memory condition on the boundary was studied by Qin [7], he showed existence, uniqueness and stability of global solutions provided the initial data is small in $H^3 \times H^2$, this result was improved by Muñoz Rivera and Andrade [5], by taking small initial data in $H^2 \times H^1$.



The set Ω

The uniform stabilization of system (1.1)–(1.4) for frictional boundary conditions instead of condition (1.5) was studied by Lagnese [3], who proved, under some geometrical conditions that the energy associated to the Mindlin-Timoshenko's plate decays exponentially as time goes to infinity.

The aim of this paper is to study the asymptotic behavior of solutions of system (1.1)–(1.6). We show that the solution decays exponentially to zero provided the relaxation functions g_i decays exponentially to zero. Moreover, if g_i decays polynomially, then we show that the corresponding solution also decays polynomially to zero with the same rate of decay.

The remainder part of this paper is organized as follows. In the next section we establish the existence, uniqueness and regularity of solutions. In section 3 we show the exponential decay of the first order energy and finally, in section 4 we prove the polynomial decay.

2 Existence of solutions

In this section the existence and uniqueness of strong solutions will be established. From (1.5) the following boundary conditions is obtained

$$\begin{aligned}
 \mathfrak{B}_1(\psi, \varphi) &= -\frac{1}{g_1(0)} \{ \psi_t + k_1(0)\psi + k'_1 * \psi - k_1(t)\psi^0 \} \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\
 \mathfrak{B}_2(\psi, \varphi) &= -\frac{1}{g_2(0)} \{ \varphi_t + k_2(0)\varphi + k'_2 * \varphi - k_2(t)\varphi^0 \} \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\
 \mathfrak{B}_3(\psi, \varphi, \omega) &= -\frac{1}{g_3(0)} \{ \omega_t + k_3(0)\omega + k'_3 * \omega - k_3(t)\omega^0 \} \quad \text{on } \Gamma_1 \times \mathbb{R}^+,
 \end{aligned}
 \tag{2.1}$$

where $*$ is the convolution product given by $(g * h)(t) := \int_0^t g(t-s)h(s) ds$ and $k_i, i = 1, 2, 3$ are the resolvent kernels of the functions $-g'_i/g_i(0)$, that is, k_i satisfies

$$k_i + \frac{1}{g_i(0)}g'_i * k_i = -\frac{1}{g_i(0)}g'_i.$$

Indeed, if we differentiate condition (1.5)₁ for example, we arrive at the following Volterra equation

$$(2.2) \quad \mathfrak{B}_1(\psi, \varphi) + \frac{1}{g_1(0)}g'_1 * \mathfrak{B}_1(\psi, \varphi) = -\frac{1}{g_1(0)}\psi_t,$$

Now, we Apply the Volterra inverse operator to get

$$\begin{aligned} \mathfrak{B}_1(\psi, \varphi) &= -\frac{1}{g_1(0)} \{ \psi_t + k_1 * \psi_t \} \\ &= -\frac{1}{g_1(0)} \{ \psi_t + k_1(0)\psi + k'_1 * \psi - k_1(t)\psi^0 \}. \end{aligned}$$

The other terms are similar. Note that we can retrieve condition (1.5)₁ when the initial data ψ^0 vanishes on Γ_1 , to see this, we integrate (2.2) to obtain

$$\psi + \int_0^t g_1(t-s)\mathfrak{B}_1(\psi(s), \varphi(s)) ds = \psi^0 = 0 \quad \text{on } \Gamma_1.$$

Therefore, we can work with the condition (2.1) instead of (1.5). In the next Lemma we establish some asymptotic properties of the solution of the Volterra's equation

$$(2.3) \quad k(t) - k * h(t) = h(t).$$

Lemma 2.1 *Let us suppose that k and h satisfies equation (2.3). If h is a positive continuous function, then k also is a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that*

$$h(t) \leq c_0 e^{-\gamma t},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0} e^{-\epsilon t},$$

for all $0 < \epsilon < \gamma - c_0$.

2. *Given $p > 1$ let us denote by $c_p := \sup_{t \in \mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} ds$. If there exists a positive constant c_0 with $c_0 c_p < 1$ such that*

$$h(t) \leq c_0 (1+t)^{-p},$$

then, the function k satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_p} (1+t)^{-p}.$$

Proof: Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0]$. If $t_0 \in \mathbb{R}^+$, from equation (2.3) we get that $-\int_0^{t_0} k(t_0 - s)h(s) ds = h(t_0)$ but this is contradictory. Therefore $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix ϵ , such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (2.3) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$, hence

$$\sup_{s \in [0, t]} k_\epsilon(s) \leq \sup_{s \in [0, t]} h_\epsilon(s) + \left(\int_0^\infty c_0 e^{(\epsilon - \gamma)s} ds \right) \sup_{s \in [0, t]} k_\epsilon(s) \leq c_0 + \frac{c_0}{(\gamma - \epsilon)} \sup_{s \in [0, t]} k_\epsilon(s).$$

Therefore

$$k_\epsilon(t) \leq \frac{c_0(\gamma - \epsilon)}{\gamma - \epsilon - c_0},$$

which implies our first assertion. To show the second part let us consider the following notations

$$k_p(t) := (1 + t)^p k(t), \quad h_p(t) := (1 + t)^p h(t).$$

Multiplying equation (2.3) by $(1 + t)^p$ we get $k_p(t) = h_p(t) + \int_0^t k_p(t - s)(1 + t - s)^{-p}(1 + t)^p h(s) ds$, hence

$$\sup_{s \in [0, t]} k_p(s) \leq \sup_{s \in [0, t]} h_p(s) + c_0 c_p \sup_{s \in [0, t]} k_p(s) \leq c_0 + c_0 c_p \sup_{s \in [0, t]} k_p(s).$$

Therefore

$$k_p(t) \leq \frac{c_0}{1 - c_0 c_p},$$

which proves our second assertion. \square

The existence of solutions to the system (1.1)-(1.6) can be established by using the standard Galerkin method. This result is given by the following theorem:

Theorem 2.2 *We assume that k_i , $-k'_i$ and k''_i , $i = 1, 2, 3$, are positive functions. Let $(\psi^0, \varphi^0, \omega^0)$ be in $[H^1(\Omega)]^3$ and $(\psi^1, \varphi^1, \omega^1)$ in $[L^2(\Omega)]^3$ satisfying*

$$(2.4) \quad \psi^0 = \varphi^0 = \omega^0 = 0 \quad \text{on} \quad \Gamma_0.$$

Then there exists only one weak solution of (1.1)-(1.6)

$$(\psi, \varphi, \omega) \in \bigcap_{\mu=0}^1 W^{\mu, \infty}(\mathbb{R}_0^+; [H^{1-\mu}(\Omega)]^3).$$

Moreover, if $(\psi^0, \varphi^0, \omega^0) \in [H^2(\Omega)]^3$, $(\psi^1, \varphi^1, \omega^1) \in [H^1(\Omega)]^3$ and satisfy

$$(2.5) \quad \begin{aligned} \psi^1 = \varphi^1 = \omega^1 = 0 \quad \text{on} \quad \Gamma_0, \quad \mathfrak{B}_1(\psi^0, \varphi^0) &= -\frac{\psi^1}{g_1(0)} \quad \text{on} \quad \Gamma_1, \\ \mathfrak{B}_2(\psi^0, \varphi^0) &= -\frac{\varphi^1}{g_2(0)} \quad \text{and} \quad \mathfrak{B}_3(\psi^0, \varphi^0, \omega^0) &= -\frac{\omega^1}{g_3(0)} \quad \text{on} \quad \Gamma_1, \end{aligned}$$

the solution has the following regularity property

$$(\psi, \varphi, \omega) \in \bigcap_{\mu=0}^2 W^{\mu, \infty}(\mathbb{R}_0^+; [H^{2-\mu}(\Omega)]^3).$$

which is called strong solution.

Finally, let us denote by Λ_1 , Λ_2 and Λ_3 the differential operators

$$\begin{aligned} \Lambda_1(\psi, \varphi, \omega) &= D \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{1-\mu}{2} \frac{\partial^2 \psi}{\partial y^2} + \frac{1+\mu}{2} \frac{\partial^2 \varphi}{\partial x \partial y} \right) - K \left(\psi + \frac{\partial \omega}{\partial x} \right), \\ \Lambda_2(\psi, \varphi, \omega) &= D \left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{1-\mu}{2} \frac{\partial^2 \varphi}{\partial x^2} + \frac{1+\mu}{2} \frac{\partial^2 \psi}{\partial x \partial y} \right) - K \left(\varphi + \frac{\partial \omega}{\partial y} \right), \\ \Lambda_3(\psi, \varphi, \omega) &= K \left[\frac{\partial}{\partial x} \left(\psi + \frac{\partial \omega}{\partial x} \right) + \frac{\partial}{\partial y} \left(\varphi + \frac{\partial \omega}{\partial y} \right) \right], \end{aligned}$$

and by \mathbf{a}_0 and \mathbf{a}_1 the following linear forms

$$\begin{aligned} \mathbf{a}_0(\psi, \varphi; \hat{\psi}, \hat{\varphi}) &= \\ & D \int_{\Omega} \left[\frac{\partial \psi}{\partial x} \frac{\partial \hat{\psi}}{\partial x} + \frac{\partial \varphi}{\partial y} \frac{\partial \hat{\varphi}}{\partial y} + \mu \frac{\partial \psi}{\partial x} \frac{\partial \hat{\varphi}}{\partial y} + \mu \frac{\partial \varphi}{\partial y} \frac{\partial \hat{\psi}}{\partial x} + \frac{1-\mu}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \left(\frac{\partial \hat{\psi}}{\partial y} + \frac{\partial \hat{\varphi}}{\partial x} \right) \right] dx, \\ \mathbf{a}_1(\psi, \varphi, \omega; \hat{\psi}, \hat{\varphi}, \hat{\omega}) &= \int_{\Omega} \left[\left(\psi + \frac{\partial \omega}{\partial x} \right) \left(\hat{\psi} + \frac{\partial \hat{\omega}}{\partial x} \right) + \left(\varphi + \frac{\partial \omega}{\partial y} \right) \left(\hat{\varphi} + \frac{\partial \hat{\omega}}{\partial y} \right) \right] dx, \end{aligned}$$

then for any $(\psi, \varphi, \omega) \in [H^2(\Omega)]^3$ and $(\hat{\psi}, \hat{\varphi}, \hat{\omega}) \in [H^1(\Omega)]^3$ we have

$$\begin{aligned} & \int_{\Omega} \{ \Lambda_1(\psi, \varphi, \omega) \hat{\psi} + \Lambda_2(\psi, \varphi, \omega) \hat{\varphi} + \Lambda_3(\psi, \varphi, \omega) \hat{\omega} \} dx \\ &= \int_{\Gamma} \{ \mathfrak{B}_1(\psi, \varphi) \hat{\psi} + \mathfrak{B}_2(\psi, \varphi) \hat{\varphi} + \mathfrak{B}_3(\psi, \varphi, \omega) \hat{\omega} \} d\Gamma + \mathbf{a}(\psi, \varphi, \omega; \hat{\psi}, \hat{\varphi}, \hat{\omega}) \end{aligned}$$

where $\mathbf{a}(\psi, \varphi, \omega; \hat{\psi}, \hat{\varphi}, \hat{\omega}) = \mathbf{a}_0(\psi, \varphi; \hat{\psi}, \hat{\varphi}) + K \mathbf{a}_1(\psi, \varphi, \omega; \hat{\psi}, \hat{\varphi}, \hat{\omega})$. This result can be found in [3]. The above formula will be used frequently in that follows.

3 Exponential decay

In this section we shall prove that the solutions of system (1.1)-(1.6) decay exponentially to zero provided the resolvent kernel also decay exponentially to zero. We shall assume that there exists $x_0 \in \mathbb{R}^n$ such that

$$(3.1) \quad r(x)\nu(x) \leq 0 \quad \text{on } \Gamma_0, \quad r(x)\nu(x) \geq \delta_0 > 0 \quad \text{on } \Gamma_1,$$

where $r(x) := x - x_0$, that is to say, Ω is star-shaped respect to x_0 . The first order energy associated to the system (1.1)-(1.5) is given by

$$\begin{aligned} E(t) = E(t, \psi, \varphi, \omega) &:= \frac{\rho h}{2} \int_{\Omega} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} dx + \frac{1}{2} \mathbf{a}(\psi, \varphi, \omega) \\ &+ \frac{1}{2g_1(0)} \int_{\Gamma_1} \{-k'_1 \square \psi + k_1(t) |\psi|^2\} d\Gamma \\ &+ \frac{1}{2g_2(0)} \int_{\Gamma_1} \{-k'_2 \square \varphi + k_2(t) |\varphi|^2\} d\Gamma \\ &+ \frac{1}{2g_3(0)} \int_{\Gamma_1} \{-k'_3 \square \omega + k_3(t) |\omega|^2\} d\Gamma. \end{aligned}$$

Here we are denoting by $\mathbf{a}(\psi, \varphi, \omega) = \mathbf{a}(\psi, \varphi, \omega; \psi, \varphi, \omega)$ and the operator \square is given by $(k \square h)(t) := \int_0^t k(t-s) |h(t) - h(s)|^2 ds$. The exponential decay of the energy is given by the following theorem

Theorem 3.1 *Let us take $(\psi^0, \varphi^0, \omega^0) \in [H^1(\Omega)]^3$ satisfying the compatibility conditions (2.4) and $(\psi^1, \varphi^1, \omega^1) \in [L^2(\Omega)]^3$. Let us assume that the resolvent kernel k is a non-negative function such that*

$$(3.2) \quad k_i(0) > 0, \quad k'_i(t) \leq -c_1 k_i(t), \quad k''_i(t) \geq c_2 [-k'_i(t)], \quad i = 1, 2, 3,$$

where c_1, c_2 are positive constants. Then there exist positive constants C and γ such that

$$E(t) \leq CE(0)e^{-\gamma t}.$$

We prove this theorem for initial data $(\psi^0, \varphi^0, \omega^0) \in [H^2(\Omega)]^3$ and $(\psi^1, \varphi^1, \omega^1) \in [H^1(\Omega)]^3$ satisfying the compatibility condition (2.4)–(2.5), hence, our conclusion follows by density argument. Before to prove the above theorem, let us establish the following Lemmas.

Lemma 3.2 *Let \mathcal{E} be a real positive function of class C^1 . If there exists positive constants γ_0, γ_1 and c_0 such that*

$$\mathcal{E}'(t) \leq -\gamma_0 \mathcal{E}(t) + c_0 e^{-\gamma_1 t},$$

then there exist positive constants γ and c such that

$$\mathcal{E}(t) \leq (\mathcal{E}(0) + c)e^{-\gamma t}.$$

Proof: First, let us suppose that $\gamma_0 < \gamma_1$. Define $\mathcal{I}(t)$ by

$$\mathcal{I}(t) := \mathcal{E}(t) + \frac{c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t}.$$

Then

$$\mathcal{I}'(t) = \mathcal{E}'(t) - \frac{\gamma_1 c_0}{\gamma_1 - \gamma_0} e^{-\gamma_1 t} \leq -\gamma_0 \mathcal{I}(t).$$

Integrating from 0 to t we arrive at

$$\mathcal{I}(t) \leq \mathcal{I}(0)e^{-\gamma_0 t} \quad \Rightarrow \quad \mathcal{E}(t) \leq \left(\mathcal{E}(0) + \frac{c_0}{\gamma_1 - \gamma_0} \right) e^{-\gamma_0 t}.$$

Now, we shall assume that $\gamma_0 \geq \gamma_1$. In this conditions we get

$$\mathcal{E}'(t) \leq -\gamma_1 \mathcal{E}(t) + c_0 e^{-\gamma_1 t} \quad \Rightarrow \quad [e^{\gamma_1 t} \mathcal{E}(t)]' \leq c_0.$$

Integrating from 0 to t we get

$$\mathcal{E}(t) \leq (\mathcal{E}(0) + c_0 t) e^{-\gamma_1 t}.$$

Since $t \leq (\gamma_1 - \epsilon) e^{(\gamma_1 - \epsilon)t}$ for any $0 < \epsilon < \gamma_1$ we conclude that

$$\mathcal{E}(t) \leq [\mathcal{E}(0) + c_0(\gamma_1 - \epsilon)] e^{-\epsilon t}.$$

This completes the proof. □

Let us denote by \diamond the operator given by $(g \diamond h)(t) := \int_0^t g(t-s)(h(t) - h(s)) ds$. Some relations between the operators $*$, \square and \diamond are given by the following lemma.

Lemma 3.3 *For any functions $g, h \in C^1(\mathbb{R})$ and $\theta \in [0, 1]$, the following inequalities hold*

$$\begin{aligned} 2[g * h] h' &= g' \square h - g(t) |h|^2 - \frac{d}{dt} \left\{ g \square h - \left(\int_0^t g ds \right) |h|^2 \right\}, \\ |(g \diamond h)(t)|^2 &\leq \left[\int_0^t |g(s)|^{2(1-\theta)} ds \right] |g|^{2\theta} \square h. \end{aligned}$$

Proof: Differentiating the expression

$$g \square h - \left(\int_0^t g ds \right) |h|^2,$$

with respect to the time, the first part of our conclusion follows. The second part is a consequence of Hölder's inequality. □

Lemma 3.4 *The energy of the Mindlin-Timoshenko plate system satisfies*

$$\begin{aligned} \frac{d}{dt} E(t) &\leq -\frac{1}{2g_1(0)} \int_{\Gamma_1} \{ |\psi_t|^2 + k_1'' \square \psi - k_1'(t) |\psi|^2 - |k_1(t) \psi^0|^2 \} d\Gamma \\ &\quad - \frac{1}{2g_2(0)} \int_{\Gamma_1} \{ |\varphi_t|^2 + k_2'' \square \varphi - k_2'(t) |\varphi|^2 - |k_2(t) \varphi^0|^2 \} d\Gamma \\ &\quad - \frac{1}{2g_3(0)} \int_{\Gamma_1} \{ |\omega_t|^2 + k_3'' \square \omega - k_3'(t) |\omega|^2 - |k_3(t) \omega^0|^2 \} d\Gamma. \end{aligned}$$

Proof: Multiplying equation (1.1) by ψ_t , equation (1.2) by φ_t , equation (1.3) by ω_t , integrating by parts and using Lemma 3.3 we obtain

$$\begin{aligned}\frac{d}{dt}E(t) &= -\frac{1}{2g_1(0)} \int_{\Gamma_1} \{2|\psi_t|^2 + k_1'' \square \psi - k_1'(t)|\psi|^2 - 2k_1(t)\psi^0\psi_t\} d\Gamma \\ &\quad -\frac{1}{2g_2(0)} \int_{\Gamma_1} \{2|\varphi_t|^2 + k_2'' \square \varphi - k_2'(t)|\varphi|^2 - 2k_2(t)\varphi^0\varphi_t\} d\Gamma \\ &\quad -\frac{1}{2g_3(0)} \int_{\Gamma_1} \{2|\omega_t|^2 + k_3'' \square \omega - k_3'(t)|\omega|^2 - 2k_3(t)\omega^0\omega_t\} d\Gamma.\end{aligned}$$

Applying Young's inequality our Lemma follows. \square

Let us introduce the following functionals

$$\begin{aligned}R(t) &:= \rho h \int_{\Omega} \left\{ \frac{h^2}{12} (\psi_t(\mathbf{r} \cdot \nabla \psi) + \varphi_t(\mathbf{r} \cdot \nabla \varphi)) + \omega_t(\mathbf{r} \cdot \nabla \omega) \right\} dx \\ &\quad + (1 - \epsilon_0) \int_{\Omega} \frac{\rho h^3}{12} (\psi_t \psi + \varphi_t \varphi) dx + \epsilon_0 \int_{\Omega} \rho h \omega_t \omega dx, \\ \mathcal{N}(t) &:= \frac{\rho h}{2} \int_{\Omega} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} dx + \frac{1}{2} \mathbf{a}(\psi, \varphi, \omega),\end{aligned}$$

where ϵ_0 is a small positive constant to be fixed later and $r = x - x_0$.

Lemma 3.5 *There exists a positive constant C_0 such that*

$$\begin{aligned}\frac{d}{dt}R(t) &\leq -\epsilon_0 \mathcal{N}(t) + C_0 \int_{\Gamma_1} (|\psi_t|^2 + |k_1(t)\psi|^2 + |k_1'(t)\psi|^2 + |k_1(t)\psi^0|^2) d\Gamma \\ &\quad + C_0 \int_{\Gamma_1} (|\varphi_t|^2 + |k_2(t)\varphi|^2 + |k_2'(t)\varphi|^2 + |k_2(t)\varphi^0|^2) d\Gamma \\ &\quad + C_0 \int_{\Gamma_1} (|\omega_t|^2 + |k_3(t)\omega|^2 + |k_3'(t)\omega|^2 + |k_3(t)\omega^0|^2) d\Gamma.\end{aligned}$$

Proof: Let us denote by J the following functional

$$J(t) := \rho h \int_{\Omega} \left\{ \frac{h^2}{12} (\psi_t(\mathbf{r} \cdot \nabla \psi) + \varphi_t(\mathbf{r} \cdot \nabla \varphi)) + \omega_t(\mathbf{r} \cdot \nabla \omega) \right\} dx.$$

Multiplying equation (1.1) by $\mathbf{r} \cdot \nabla \psi$, equation (1.2) by $\mathbf{r} \cdot \nabla \varphi$, equation (1.3) by $\mathbf{r} \cdot \nabla \omega$ and performing an integration by parts we obtain the following identity

$$\begin{aligned}(3.3) \quad \frac{d}{dt}J(t) &= \frac{\rho h}{2} \int_{\Gamma_1} \mathbf{r} \cdot \nu \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} d\Gamma \\ &\quad - \rho h \int_{\Omega} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} dx \\ &\quad + \int_{\Gamma_1} \{ \mathfrak{B}_1(\psi, \varphi) \mathbf{r} \cdot \nabla \psi + \mathfrak{B}_2(\psi, \varphi) \mathbf{r} \cdot \nabla \varphi + \mathfrak{B}_3(\psi, \varphi, \omega) \mathbf{r} \cdot \nabla \omega \} d\Gamma \\ &\quad - \frac{1}{2} \int_{\Gamma_1} \mathbf{r} \cdot \nu \mathfrak{D}(\psi, \varphi, \omega) d\Gamma + \frac{1}{2} \int_{\Gamma_0} \mathbf{r} \cdot \nu \mathfrak{D}(\psi, \varphi, \omega) d\Gamma \\ &\quad + K \int_{\Omega} [(\psi + \omega_x)\psi + (\varphi + \omega_x)\varphi] dx,\end{aligned}$$

where the operator \mathfrak{D} is given by

$$\mathfrak{D}(\psi, \varphi, \omega) := D \left[|\psi_x|^2 + |\varphi_y|^2 + 2\mu\psi_x\varphi_y + \frac{1-\mu}{2}|\psi_y + \varphi_x|^2 \right] + K [|\psi + \omega_x|^2 + |\varphi + \omega_y|^2].$$

On the other hand, multiplying equation (1.1) by ψ , equation (1.2) by φ and integrating by parts, we arrive at

$$(3.4) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \frac{\rho h^3}{12} (\psi_t \psi + \varphi_t \varphi) dx &= \frac{\rho h^3}{12} \int_{\Omega} (|\psi_t|^2 + |\varphi_t|^2) dx \\ &+ \int_{\Gamma_1} \{ \mathfrak{B}_1(\psi, \varphi) \psi + \mathfrak{B}_2(\psi, \varphi) \varphi \} d\Gamma \\ &- \mathfrak{a}_0(\psi, \varphi) - K \int_{\Omega} [(\psi + \omega_x) \psi + (\varphi + \omega_y) \varphi] dx. \end{aligned}$$

Finally, multiplying equation (1.3) by ω and integrating by parts we get

$$(3.5) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} \rho h \omega_t \omega dx &= \int_{\Omega} \rho h |\omega_t|^2 dx + \int_{\Gamma_1} \mathfrak{B}_3(\psi, \varphi, \omega) \omega d\Gamma \\ &- K \int_{\Omega} [(\psi + \omega_x) \omega_x + (\varphi + \omega_y) \omega_y] dx. \end{aligned}$$

Now, for $\epsilon > 0$ let us define the functional R_ϵ given by

$$R_\epsilon(t) := J_0(t) + (1 - \epsilon) \int_{\Omega} \frac{\rho h^3}{12} (\psi_t \psi + \varphi_t \varphi) dx + \epsilon \int_{\Omega} \rho h \omega_t \omega dx.$$

From the identities (3.3), (3.4) and (3.5) we conclude that R_ϵ satisfies

$$\begin{aligned} \frac{d}{dt} R_\epsilon(t) &= \frac{\rho h}{2} \int_{\Gamma_1} \mathbf{r} \cdot \boldsymbol{\nu} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} d\Gamma - \int_{\Omega} \frac{\epsilon \rho h^3}{12} (|\psi_t|^2 + |\varphi_t|^2) dx \\ &- \int_{\Omega} (1 - \epsilon) \rho h |\omega_t|^2 dx - \frac{1}{2} \int_{\Gamma_1} \mathbf{r} \cdot \boldsymbol{\nu} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma + \frac{1}{2} \int_{\Gamma_0} \mathbf{r} \cdot \boldsymbol{\nu} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma \\ &+ \int_{\Gamma_1} \{ \mathfrak{B}_1(\psi, \varphi) \mathbf{r} \cdot \nabla \psi + \mathfrak{B}_2(\psi, \varphi) \mathbf{r} \cdot \nabla \varphi + \mathfrak{B}_3(\psi, \varphi, \omega) \mathbf{r} \cdot \nabla \omega \} d\Gamma \\ &+ \int_{\Gamma_1} \{ (1 - \epsilon) [\mathfrak{B}_1(\psi, \varphi) \psi + \mathfrak{B}_2(\psi, \varphi) \varphi] + \epsilon \mathfrak{B}_3(\psi, \varphi, \omega) \omega \} d\Gamma \\ &+ 2\epsilon K \int_{\Omega} [(\psi + \omega_x) \psi + (\varphi + \omega_x) \varphi] dx - (1 - \epsilon) \mathfrak{a}_0(\psi, \varphi) - \epsilon K \mathfrak{a}_1(\psi, \varphi, \omega). \end{aligned}$$

Taking into account the inequalities in (3.1) we get

$$\begin{aligned}
(3.6) \quad \frac{d}{dt}R_\epsilon(t) &\leq \frac{\rho h}{2} \int_{\Gamma_1} \mathbf{r} \cdot \boldsymbol{\nu} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} d\Gamma \\
&\quad - \int_{\Omega} \frac{\epsilon \rho h^3}{12} (|\psi_t|^2 + |\varphi_t|^2) dx - \int_{\Omega} (1 - \epsilon) \rho h |\omega_t|^2 dx - \frac{\delta_0}{2} \int_{\Gamma_1} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma \\
&\quad + \underbrace{\int_{\Gamma_1} \{ \mathfrak{B}_1(\psi, \varphi) \mathbf{r} \cdot \nabla \psi + \mathfrak{B}_2(\psi, \varphi) \mathbf{r} \cdot \nabla \varphi + \mathfrak{B}_3(\psi, \varphi, \omega) \mathbf{r} \cdot \nabla \omega \} d\Gamma}_{:=I_1} \\
&\quad + \underbrace{\int_{\Gamma_1} \{ (1 - \epsilon) [\mathfrak{B}_1(\psi, \varphi) \psi + \mathfrak{B}_2(\psi, \varphi) \varphi] + \epsilon \mathfrak{B}_3(\psi, \varphi, \omega) \omega \} d\Gamma}_{:=I_2} \\
&\quad + 2\epsilon K \underbrace{\int_{\Omega} [(\psi + \omega_x) \psi + (\varphi + \omega_x) \varphi] dx}_{:=I_3} - (1 - \epsilon) \mathfrak{a}_0(\psi, \varphi) - \epsilon K \mathfrak{a}_1(\psi, \varphi, \omega).
\end{aligned}$$

From Korn's inequality and Trace Theorem there exist positive constants $C_1, C_2 > 1$ such that

$$\begin{aligned}
\int_{\Gamma_1} (|\nabla \psi|^2 + |\nabla \varphi|^2 + |\nabla \omega|^2) d\Gamma &\leq C_1 \left\{ \int_{\Gamma_1} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma + \int_{\Gamma_1} (|\psi|^2 + |\varphi|^2) d\Gamma \right\}, \\
\int_{\Gamma_1} (|\psi|^2 + |\varphi|^2 + |\omega|^2) d\Gamma &\leq C_2 \mathfrak{a}(\psi, \varphi, \omega),
\end{aligned}$$

hence it follows that

$$\|(\psi, \varphi, \omega)\|_{[H^1(\Gamma_1)]^3}^2 \leq 2C_1 C_2 \left\{ \int_{\Gamma_1} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma + \mathfrak{a}(\psi, \varphi, \omega) \right\}.$$

From Young's inequality and the above estimate we conclude that for $\epsilon > 0$ there exists C_ϵ such that

$$\begin{aligned}
(3.7) \quad I_1 + I_2 &\leq \frac{\epsilon}{4} \left\{ \int_{\Gamma_1} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma + \mathfrak{a}(\psi, \varphi, \omega) \right\} \\
&\quad + C_\epsilon \int_{\Omega} (|\mathfrak{B}_1(\psi, \varphi)|^2 + |\mathfrak{B}_2(\psi, \varphi)|^2 + |\mathfrak{B}_3(\psi, \varphi, \omega)|^2) dx.
\end{aligned}$$

The Trace's theorem implies that

$$\int_{\Gamma_1} (|\psi|^2 + |\varphi|^2) d\Gamma \leq C_3 \mathfrak{a}_0(\psi, \varphi),$$

for some $C_3 > 0$. Using the above inequality we conclude that

$$(3.8) \quad I_3 \leq \frac{\epsilon K}{4} \mathfrak{a}_1(\psi, \varphi, \omega) + 4K C_3 \epsilon \mathfrak{a}_0(\psi, \varphi).$$

Substitution of (3.7)-(3.8) into (3.6) yields

$$\begin{aligned}
(3.9) \quad \frac{d}{dt}R_\epsilon(t) &\leq \frac{\rho h}{2} \int_{\Gamma_1} \mathbf{r} \cdot \boldsymbol{\nu} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} d\Gamma - \int_{\Omega} \frac{\epsilon \rho h^3}{12} (|\psi_t|^2 + |\varphi_t|^2) dx \\
&\quad - \int_{\Omega} (1 - \epsilon) \rho h |\omega_t|^2 dx - \frac{(2\delta_0 - \epsilon)}{4} \int_{\Gamma_1} \mathfrak{D}(\psi, \varphi, \omega) d\Gamma \\
&\quad - [1 - \epsilon(1 + 4KC_3 + 1/4)] \mathfrak{a}_0(\psi, \varphi) - \frac{\epsilon}{2} K \mathfrak{a}_1(\psi, \varphi, \omega) \\
&\quad + C_\epsilon \int_{\Omega} \left(|\mathfrak{B}_1(\psi, \varphi)|^2 + |\mathfrak{B}_2(\psi, \varphi)|^2 + |\mathfrak{B}_3(\psi, \varphi, \omega)|^2 \right) dx.
\end{aligned}$$

Let us take $\epsilon > 0$ small enough such that

$$\min \{2\delta_0 - \epsilon, 1 - (1 + 4KC_3 + 1/4)\epsilon\} \geq \frac{\epsilon}{2}$$

then we can rewrite the inequality (3.9) as

$$\begin{aligned}
\frac{d}{dt}R_\epsilon(t) &\leq \frac{\rho h}{2} \int_{\Gamma_1} \mathbf{r} \cdot \boldsymbol{\nu} \left\{ \frac{h^2}{12} (|\psi_t|^2 + |\varphi_t|^2) + |\omega_t|^2 \right\} d\Gamma - \epsilon \mathcal{N}(t) \\
&\quad + C_\epsilon \int_{\Omega} \left(|\mathfrak{B}_1(\psi, \varphi)|^2 + |\mathfrak{B}_2(\psi, \varphi)|^2 + |\mathfrak{B}_3(\psi, \varphi, \omega)|^2 \right) dx.
\end{aligned}$$

Since \mathfrak{B}_1 , \mathfrak{B}_2 and \mathfrak{B}_3 can be written as

$$\begin{aligned}
\mathfrak{B}_1(\psi, \varphi) &= -\frac{1}{g_1(0)} \{ \psi_t + k_1(t)\psi - k'_1 \diamond \psi - k_1(t)\psi^0 \}, \\
\mathfrak{B}_2(\psi, \varphi) &= -\frac{1}{g_2(0)} \{ \varphi_t + k_2(t)\varphi - k'_2 \diamond \varphi - k_2(t)\varphi^0 \}, \\
\mathfrak{B}_3(\psi, \varphi, \omega) &= -\frac{1}{g_3(0)} \{ \omega_t + k_3(t)\omega - k'_3 \diamond \omega - k_3(t)\omega^0 \},
\end{aligned}$$

Our conclusion follows. \square

Proof of Theorem 3.1: From Lemma 3.4 and hypothesis (3.2) we get

$$\begin{aligned}
\frac{d}{dt}E(t) &\leq -\frac{1}{2g_1(0)} \int_{\Gamma_1} \{ |\psi_t|^2 - c_2 k'_1 \square \psi + c_1 k_1(t) |\psi|^2 - |k_1(t)\psi^0|^2 \} d\Gamma \\
&\quad - \frac{1}{2g_2(0)} \int_{\Gamma_1} \{ |\varphi_t|^2 - c_2 k'_2 \square \varphi + c_1 k_2(t) |\varphi|^2 - |k_2(t)\varphi^0|^2 \} d\Gamma \\
&\quad - \frac{1}{2g_3(0)} \int_{\Gamma_1} \{ |\omega_t|^2 - c_2 k'_3 \square \omega + c_1 k_3(t) |\omega|^2 - |k_3(t)\omega^0|^2 \} d\Gamma.
\end{aligned}$$

Let us denote by for $(\phi_1, \phi_2, \phi_3) = (\psi, \varphi, \omega)$. Since k_i is bounded the second inequality of Lemma 3.3 implies that $|k'_i \diamond \phi_i|^2 \leq C[-k'_i] \square \phi_i$ and $|k_i(t)\phi_i|^2 \leq Ck_i(t)|\phi_i|^2$. From Lemma 3.5 we get

$$\begin{aligned}
\frac{d}{dt}R(t) &\leq -\epsilon_0 \mathcal{N}(t) + C \int_{\Gamma_1} (|\psi_t|^2 + k_1(t)|\psi|^2 - k'_1 \square \psi + |k_1(t)\psi^0|^2) d\Gamma \\
&\quad + C \int_{\Gamma_1} (|\varphi_t|^2 + k_2(t)|\varphi|^2 - k'_2 \square \varphi + |k_2(t)\varphi^0|^2) d\Gamma \\
&\quad + C \int_{\Gamma_1} (|\omega_t|^2 + k_3(t)|\omega|^2 - k'_3 \square \omega + |k_3(t)\omega^0|^2) d\Gamma.
\end{aligned}$$

Let us denote by \mathcal{F} the following functional

$$\mathcal{F}(t) := E(t) + \delta R(t).$$

Taking δ small enough the above inequalities imply that

$$(3.10) \quad \frac{d}{dt}\mathcal{F}(t) \leq -\epsilon_0 E(t) + M_0 \int_{\Gamma_1} (|k_1(t)\psi^0|^2 + |k_2(t)\varphi^0|^2 + |k_3(t)\omega^0|^2) d\Gamma,$$

where $M_0 := 1/\min\{g_i(0) : i = 1, 2, 3\}$. It is easy to verify that for δ small enough we also have

$$(3.11) \quad \frac{1}{2}E(t) \leq \mathcal{F}(t) \leq 2E(t).$$

Combining inequalities (3.10) and (3.11) we arrive at

$$\frac{d}{dt}\mathcal{F}(t) \leq -\frac{\epsilon_0}{2}\mathcal{F}(t) + M_0 \int_{\Gamma_1} (|k_1(t)\psi^0|^2 + |k_2(t)\varphi^0|^2 + |k_3(t)\omega^0|^2) d\Gamma.$$

Since $k_i(t) \leq k_i(0)e^{-c_1 t}$, Lemma 3.2 implies that \mathcal{F} decays exponentially. In view of (3.11) we conclude that the energy also decays exponentially. \square

4 Polynomial decay

Here our attention will be focused on the uniform rate of decay when the resolvent kernel k decays polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decay polynomially with the same rate.

Theorem 4.1 *Let us take $(\psi^0, \varphi^0, \omega^0) \in [H^1(\Omega)]^3$ satisfying the compatibility conditions (2.4) and $(\psi^1, \varphi^1, \omega^1) \in [L^2(\Omega)]^3$. Let us assume that the resolvent kernel k is a non-negative function such that*

$$(4.1) \quad k_i(0) > 0, \quad k_i'(t) \leq -c_1 k_i(t)^{1+\frac{1}{p}}, \quad k_i''(t) \geq c_2 [-k_i'(t)]^{1+\frac{1}{p}}, \quad i = 1, 2, 3,$$

for some $p > 2$ and c_1, c_2 positive constants. Then there exists positive constants $C = C(E(0))$ such that

$$E(t) \leq \frac{C}{(1+t)^p}.$$

To show this theorem we shall use the same ideas of the previous section, that is, we will construct a Lyapunov functional satisfying the nonlinear differential inequality of the following Lemma

Lemma 4.2 *Let us suppose that \mathcal{E} is a nonnegative C^1 function satisfying*

$$\mathcal{E}'(t) \leq -k_0[\mathcal{E}(t)]^{1+\frac{1}{p}} + \frac{k_1}{(1+t)^{p+1}}.$$

For some $p \geq 1$ and positive constant k_0 and k_1 . In these conditions, there exists a positive constant c such that

$$\mathcal{E}(t) \leq \left([\mathcal{E}(0) + 2k_1/p]^{-\frac{1}{p}} + ct \right)^{-p}.$$

Proof: Let us denote by $\mathcal{I}(t)$ the function defined by

$$\mathcal{I}(t) := \mathcal{E}(t) + \frac{2\kappa_1}{p(1+t)^p}.$$

Differentiating the above identity we have

$$\begin{aligned} \mathcal{I}'(t) &= \mathcal{E}'(t) - \frac{2\kappa_1}{(1+t)^{p+1}} \\ &\leq -\kappa_0[\mathcal{E}(t)]^{1+\frac{1}{p}} - \frac{\kappa_1}{(1+t)^{p+1}} \\ &\leq -a_0 \left\{ [\mathcal{E}(t)]^{1+\frac{1}{p}} + \left[\frac{2\kappa_1}{p(1+t)^p} \right]^{1+\frac{1}{p}} \right\} \\ &\leq -\frac{a_0}{2^{1/p}} [\mathcal{I}(t)]^{1+\frac{1}{p}}, \end{aligned}$$

where $a_0 = \min \left\{ k_0, \left(\frac{p}{2}\right)^{1+\frac{1}{p}} k_1^{-\frac{1}{p}} \right\}$. From where it follows that

$$\mathcal{I}(t) \leq \left(\mathcal{I}(0)^{-\frac{1}{p}} + \frac{a_0}{2^{1/p}p} t \right)^{-p}.$$

which gives the required inequality. \square

Lemma 4.3 Suppose that $g \in C([0, \infty])$, $h \in L^1_{loc}(0, \infty)$ and $0 \leq \theta \leq 1$, then we have that:

$$\int_0^t |g(\tau)h(\tau)| d\tau \leq \left\{ \int_0^t |g(\tau)|^{1-\theta} |h(\tau)| d\tau \right\}^{\frac{1}{q+1}} \left\{ \int_0^t |g(\tau)|^{1+\frac{\theta}{q}} |h(\tau)| d\tau \right\}^{\frac{q}{q+1}}.$$

Proof: For any fixed t we have:

$$\int_0^t |g(\tau)h(\tau)| d\tau = \int_0^t \underbrace{|g(\tau)|^{\frac{1-\theta}{q+1}} |h(\tau)|^{\frac{1}{q+1}}}_{:=w_1} \underbrace{|g(\tau)|^{1-\frac{1-\theta}{q+1}} |h(\tau)|^{\frac{q}{q+1}}}_{:=w_2} d\tau.$$

Note that $w_1 \in L^p_{loc}(0, \infty)$, $w_2 \in L^{p'}_{loc}(0, \infty)$, where $p = q+1$ and $p' = \frac{q+1}{q}$. Using Hölder's inequality, we get

$$\int_0^t |g(\tau)h(\tau)| d\tau \leq \left\{ \int_0^t |g(\tau)|^{1-\theta} |h(\tau)| d\tau \right\}^{\frac{1}{q+1}} \left\{ \int_0^t |g(\tau)|^{1+\frac{\theta}{q}} |h(\tau)| d\tau \right\}^{\frac{q}{q+1}}.$$

This completes the proof. \square

Lemma 4.4 *Let us suppose that $v \in L^\infty(0, T; L^2(\Gamma_1))$ and g is a continuous function. Then*

$$\int_{\Gamma_1} g \square v \, d\Gamma \leq \sqrt{2} \left\{ \int_0^t \|v(\tau)\|_{L^2(\Gamma_1)}^2 \, d\tau + t \|v(t)\|_{L^2(\Gamma_1)}^2 \right\}^{\frac{1}{p+1}} \left\{ \int_{\Gamma_1} g^{1+\frac{1}{p}} \square v \, d\Gamma \right\}^{\frac{p}{p+1}}.$$

Moreover, If there exists $0 < \theta < 1$ such that $\int_0^\infty g^{1-\theta}(s) \, ds < \infty$, then we have

$$\int_{\Gamma_1} g \square v \, d\Gamma \leq 4 \left\{ \left(\int_0^\infty g^{1-\theta} \, d\tau \right) \|v\|_{L^\infty(0, T; L^2(\Gamma_1))}^2 \right\}^{\frac{1}{\theta p+1}} \left\{ \int_{\Gamma_1} g^{1+\frac{1}{p}} \square v \, d\Gamma \right\}^{\frac{\theta p}{\theta p+1}},$$

Proof: From Lemma 4.3 we get:

$$\begin{aligned} (4.2) \quad \int_{\Gamma_1} g \square v \, d\Gamma &= \int_{\Gamma_1} \int_0^t g(t-\tau) \underbrace{(v(t)-v(\tau))(v(t)-v(\tau))}_{=h(\tau)} \, d\tau \, d\Gamma \\ &\leq \left\{ \int_{\Gamma_1} \int_0^t g^{1-\theta}(t-\tau) h(\tau) \, d\tau \, d\Gamma \right\}^{\frac{1}{\theta p+1}} \left\{ \int_{\Gamma_1} \int_0^t g^{1+\frac{1}{p}}(t-\tau) h(\tau) \, d\tau \, d\Gamma \right\}^{\frac{\theta p}{\theta p+1}} \\ &\leq \left\{ \int_{\Gamma_1} g^{1-\theta} \square v \, d\Gamma \right\}^{\frac{1}{\theta p+1}} \left\{ \int_{\Gamma_1} g^{1+\frac{1}{p}} \square v \, d\Gamma \right\}^{\frac{\theta p}{\theta p+1}}. \end{aligned}$$

Now, for $0 < \theta < 1$ we have

$$\begin{aligned} \int_{\Gamma_1} g^{1-\theta} \square v \, d\Gamma &= \int_0^t g^{1-\theta}(t-\tau) \int_{\Gamma_1} (v(t)-v(\tau))(v(t)-v(\tau)) \, d\Gamma \, d\tau \\ &\leq 4 \left(\int_0^t g^{1-\theta}(\tau) \, d\tau \right) \|v\|_{L^\infty(0, T; L^2(\Gamma_1))}^2. \end{aligned}$$

From where the second inequality of Lemma 4.4 follows. When $\theta = 1$ we get

$$\begin{aligned} \int_{\Gamma_1} 1 \square v \, d\Gamma &= \int_0^t \int_{\Gamma_1} (v(t)-v(\tau))(v(t)-v(\tau)) \, d\Gamma \, d\tau \\ &\leq 2t \int_{\Gamma_1} |v(t)|^2 \, d\Gamma + 2 \int_0^t \int_{\Gamma_1} |v(\tau)|^2 \, d\Gamma \, d\tau. \end{aligned}$$

Substitution of the above inequality into (4.2) yields the first inequality. The proof is now complete. \square

Proof of Theorem 4.1: We use some estimates of the previous section which are independent of the behavior of the resolvent kernel k . Using hypothesis (4.1) in Lemma 3.4 yields

$$\begin{aligned} (4.3) \quad \frac{d}{dt} E(t) &\leq -\frac{1}{2g_1(0)} \int_{\Gamma_1} \left\{ |\psi_t|^2 + c_2[-k'_1]^{1+\frac{1}{p}} \square \psi + c_1 k_1(t)^{1+\frac{1}{p}} |\psi|^2 - |k_1(t) \psi^0|^2 \right\} \, d\Gamma \\ &\quad -\frac{1}{2g_2(0)} \int_{\Gamma_1} \left\{ |\varphi_t|^2 + c_2[-k'_2]^{1+\frac{1}{p}} \square \varphi + c_1 k_2(t)^{1+\frac{1}{p}} |\varphi|^2 - |k_2(t) \varphi^0|^2 \right\} \, d\Gamma \\ &\quad -\frac{1}{2g_3(0)} \int_{\Gamma_1} \left\{ |\omega_t|^2 + c_2[-k'_3]^{1+\frac{1}{p}} \square \omega + c_1 k_3(t)^{1+\frac{1}{p}} |\omega|^2 - |k_3(t) \omega^0|^2 \right\} \, d\Gamma. \end{aligned}$$

Let us denote by $(\phi_1, \phi_2, \phi_3) = (\psi, \varphi, \omega)$. Applying the second inequality of Lemma 3.3 to k'_i and ϕ_i with $\theta = \frac{1}{2} \left(1 + \frac{1}{p}\right)$ we get

$$|k'_i \diamond \phi_i|^2 \leq \left[\int_0^t [-k'_i(s)]^{1-\frac{1}{p}} ds \right] [-k'_i]^{1+\frac{1}{p}} \square \phi_i.$$

Hypothesis (4.1) implies that $-k'_i(t) \leq C(1+t)^{-p}$. Since $p > 2$ we conclude that

$$\int_0^\infty [-k'_i(s)]^{1-\frac{1}{p}} ds \leq C \int_0^\infty \frac{1}{(1+t)^{p-1}} dt < \infty.$$

Therefore $|k'_i \diamond \phi_i|^2 \leq C[-k'_i]^{1+\frac{1}{p}} \square \phi_i$. Additionally, since k_i is bounded we have that $|k_i(t)\phi_i|^2 \leq Ck_i(t)^{1+1/p}|\phi_i|^2$. Substituting these inequalities in Lemma 3.5 we get

$$(4.4) \quad \begin{aligned} \frac{d}{dt}R(t) &\leq -\epsilon_0\mathcal{N}(t) + C \int_{\Gamma_1} (|\psi_t|^2 + k_1(t)^{1+\frac{1}{p}}|\psi|^2 + [-k'_1]^{1+\frac{1}{p}}\square\psi + |k_1(t)\psi^0|^2) d\Gamma \\ &+ C \int_{\Gamma_1} (|\varphi_t|^2 + k_2(t)^{1+\frac{1}{p}}|\varphi|^2 + [-k'_2]^{1+\frac{1}{p}}\square\varphi + |k_2(t)\varphi^0|^2) d\Gamma \\ &+ C \int_{\Gamma_1} (|\omega_t|^2 + k_3(t)^{1+\frac{1}{p}}|\omega|^2 + [-k'_3]^{1+\frac{1}{p}}\square\omega + |k_3(t)\omega^0|^2) d\Gamma, \end{aligned}$$

Let us denote by \mathcal{F} the Lyapunov functional

$$\mathcal{F}(t) := E(t) + \delta R(t).$$

From inequalities (4.3)-(4.4) we get

$$(4.5) \quad \begin{aligned} \frac{d}{dt}\mathcal{F}(t) &\leq -\epsilon_0\mathcal{N}(t) + M_0 \int_{\Gamma_1} (|k_1(t)\psi^0|^2 + |k_2(t)\varphi^0|^2 + |k_3(t)\omega^0|^2) d\Gamma \\ &- M_1 \int_{\Gamma_1} \left([-k'_1]^{1+\frac{1}{p}}\square\psi + k_1(t)^{1+\frac{1}{p}}|\psi|^2 \right) d\Gamma \\ &- M_1 \int_{\Gamma_1} \left([-k'_2]^{1+\frac{1}{p}}\square\varphi + k_2(t)^{1+\frac{1}{p}}|\varphi|^2 \right) d\Gamma \\ &- M_1 \int_{\Gamma_1} \left([-k'_3]^{1+\frac{1}{p}}\square\omega + k_3(t)^{1+\frac{1}{p}}|\omega|^2 \right) d\Gamma, \end{aligned}$$

provided δ is small enough, where $M_0 := 1/\min\{g_i(0) : i = 1, 2, 3\}$ and $M_1 := \min\{c_1, c_2\}/\max\{4g_i(0) : i = 1, 2, 3\}$. Fix $\theta = 1/2$. Hypothesis (4.1) and $p > 2$ imply that $\int_0^\infty [-k'_i(t)]^{1-\theta} dt < \infty$. From the second part of Lemma 4.4 we get

$$(4.6) \quad \int_{\Gamma_1} [-k'_i]^{1+\frac{1}{p}} \square \phi_i d\Gamma \geq \frac{1}{C} \left[\int_{\Gamma_1} [-k'_i] \square \phi_i d\Gamma \right]^{\frac{\theta p + 1}{\theta p}}.$$

Since k_i is bounded, there exists a positive constant $C > 0$ such that

$$(4.7) \quad \int_{\Gamma_1} k_i(t)^{1+\frac{1}{p}} |\phi_i|^2 d\Gamma \geq \frac{1}{C} \left[\int_{\Gamma_1} k_i(t) |\phi_i|^2 d\Gamma \right]^{\frac{\theta p + 1}{\theta p}}.$$

The energy E is also bounded therefore there exists a positive constant $C > 0$ such that

$$(4.8) \quad \mathcal{N}(t) \geq \frac{1}{C} \mathcal{N}(t)^{\frac{\theta p + 1}{\theta p}}.$$

Using (4.6)-(4.8) in (4.6) we conclude that there exist a positive constant $C > 0$ such that

$$\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{C} E(t)^{\frac{\theta p + 1}{\theta p}} + M_0 \int_{\Gamma_1} (|k_1(t)\psi^0|^2 + |k_2(t)\varphi^0|^2 + |k_3(t)\omega^0|^2) d\Gamma.$$

The estimative (3.11) implies that

$$\frac{d}{dt} \mathcal{F}(t) \leq -\frac{1}{C} \mathcal{F}(t)^{\frac{\theta p + 1}{\theta p}} + M_0 \int_{\Gamma_1} (|k_1(t)\psi^0|^2 + |k_2(t)\varphi^0|^2 + |k_3(t)\omega^0|^2) d\Gamma$$

From (4.1) it follows that $|k_i(t)|^2 \leq C(1+t)^{-2p} \leq C(1+t)^{-(\theta p + 1)}$. Applying Lemma 4.2 we conclude that

$$\mathcal{F}(t) \leq \frac{C}{(1+t)^{\theta p}} \quad \text{consequently} \quad E(t) \leq \frac{C}{(1+t)^{\theta p}}.$$

Since $p > 2$ and $\theta = 1/2$, this last inequality implies that

$$\int_0^\infty \|v(\tau)\|_{L^2(\Gamma_1)}^2 d\tau + t \|v(\tau)\|_{L^2(\Gamma_1)}^2 \leq C \left\{ \int_0^\infty E(\tau) d\tau + t E(t) \right\} < \infty$$

for $v = \psi, \varphi, \omega$. Repeating the same reasoning as above and using the first part of Lemma 4.4 we arrive at

$$E(t) \leq \frac{C}{(1+t)^p}.$$

This completes the proof. □

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