# Extension of a Geometric Stability Switch Criterion

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## 1 Introduction

In this paper we study the occurrence of any possible stability switch from the increase of the value of the time delay $\tau$ for general delay equation

$$x^{'}(t) = F(x(t),\ x(t-\tau)) \tag{1.1}$$

where $x \in \Re^n$, $\tau \in \Re_{+0} = [0, +\infty)$ is a fixed delay and $F : C\left([-\tau, 0], \Re^n\right) \to \Re^n$ is of class $C^1$ with respect both $x(t)$ and $x(t-\tau)$.

We assume that any equilibrium $x^*$ of (1.1) is delay dependent, i.e. $F(x_t) = 0$ gives a constant solution

$$x^* = x^*(\tau) \tag{1.2}$$

which is continuous and differentiable in $\tau$.

The variation equation around $x^*$ (set $u(t) = x(t) - x^*$)

$$\dot{u}(t) = \left(\frac{\partial F}{\partial x(t)}\right)_{x^*(\tau)} u(t) + \left(\frac{\partial F}{\partial x(t-\tau)}\right)_{x^*(\tau)} u(t-\tau) \tag{1.3}$$

gives the characteristic equation

$$\det\left\{\left(\frac{\partial F}{\partial x(t)}\right)_{x^*(\tau)} + e^{-\lambda\tau}\left(\frac{\partial F}{\partial x(t-\tau)}\right)_{x^*(\tau)} - \lambda I\right\} = 0 \tag{1.4}$$

which in general has delay dependent coefficients, where det denotes the determinant of a matrix, $I$ is an identity matrix and $\lambda$ are the corresponding characteristic roots.

We give the following definition:

**Definition 1.1** A stability switch occurs at $\tau^* \in \Re_{+0}$ if crossing $\tau^*$ for increasing $\tau$ the stability of $x^*(\tau)$ changes from asymptotic stability to instability or vice versa.

The most general structure of characteristic equation (1.4) results to be

$$D(\lambda, \tau) = 0 \tag{1.5}$$

where

$$\begin{cases} D(\lambda, \tau) = P(\lambda, \tau) + \sum_{k=1}^{m} Q^{(k)}(\lambda, \tau) e^{-k\lambda\tau}, \\ P(\lambda, \tau) = \sum_{j=0}^{n} p_j(\tau)\lambda^j, \\ Q^{(k)}(\lambda, \tau) = \sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j, k = 1, \dots, m, \\ m, n, m_k \in N_0, \end{cases} \tag{1.6}$$

and $p_j(\tau)$, $q_j^{(k)}(\tau)$: $\Re_{+0} \to \Re$ are continuous and differentiable functions of $\tau \in \Re_{+0}$. Generally in (1.6) is $m \leq n$ but herefollowing we remove this condition.

We assume that $\lambda = 0$ cannot be a characteristic root, i.e.

$$D(0, \tau) \neq 0 \qquad \forall \tau \in \Re_{+0}. \tag{1.7}$$

In the study of the occurrence of stability switches the following is an essential result. Assume that we rewrite (1.5) as

$$D(\lambda, \tau) = \lambda^n + g(\lambda, \tau). \tag{1.8}$$

Then, the following theorem holds (see Freedman and Kuang [5]):

**Theorem 1.1** *Assume that $g(\lambda, \tau)$ in (1.8) is an analytic function in $\lambda$ and continuous in $\tau$ such that*

$$\alpha = \limsup_{\Re e\lambda \geq 0, |\lambda| \to \infty} \left| \lambda^{-n} g(\lambda, \tau) \right| < 1. \tag{1.9}$$

*Then, as $\tau$ varies in $\Re_{+0}$, the sum of multiplicities of roots of $D(\lambda, \tau) = 0$ in the open right half-plane can change only if a root appears on or crosses the imaginary axis.*

It is to be noticed that if in (1.6)

$$m_k < n, k = 1, \dots, n, \tag{1.10}$$

i.e. the degree of polynomial $Q^{(k)}$ in $\lambda$ is lower than the degree $n$ polynomial $P$ in $\lambda$, then assumption (1.9) holds true and Theorem 1.1 applies to the characteristic equation (1.5) and (1.6).

Rewrite $D(\lambda, \tau)$ in (1.6) like

$$D(\lambda, \tau) = p_n(\tau)\lambda^n + \left[ \sum_{j=0}^{n-1} p_j(\tau)\lambda^j + \sum_{k=1}^{m} \left( \sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j \right) e^{-k\lambda\tau} \right]$$

without loss of generality we assume $p_n(\tau) \equiv 1$ and define

$$g(\lambda, \tau) = \sum_{j=0}^{n-1} p_j(\tau)\lambda^j + \sum_{k=1}^{m} \left( \sum_{j=0}^{m_k} q_j^{(k)}(\tau)\lambda^j \right) e^{-k\lambda\tau}. \tag{1.11}$$

Assume $\lambda$ such that $Re(\lambda) \geq 0$. Then, for any $\tau \geq 0$ :

$$\begin{aligned} \left|\lambda^{-n} g(\lambda,\tau)\right| &\leq \sum_{j=0}^{n-1} \frac{|p_j(\tau)|}{|\lambda|^{n-j}} + \sum_{k=1}^{m} \left|e^{-k\lambda\tau}\right| \left( \sum_{j=0}^{m_k} \frac{\left|q_j^{(k)}(\tau)\right|}{|\lambda|^{n-j}} \right) \\ &\leq \sum_{j=0}^{n-1} \frac{|p_j(\tau)|}{|\lambda|^{n-j}} + \sum_{k=1}^{m} \left( \sum_{j=0}^{m_k} \frac{\left|q_j^{(k)}(\tau)\right|}{|\lambda|^{n-j}} \right) \to 0, |\lambda| \to \infty, \end{aligned} \tag{1.12}$$

since on the right hand side of (1.12) $n-j > n-m_k > 0$. Therefore, the assumption (1.9) of the Theorem 1.1 holds true. Hence, characteristic equations (1.5) with structure (1.6) which satisfy (1.7) and (1.10), i.e.

$$(A.1) \qquad \begin{cases} p_0(\tau) + \sum_{j=0}^{m_k} q_0^{(j)}(\tau) \neq 0 & \forall \tau \geq 0 \\ n > m_k, & k = 1, 2, \ldots, m \end{cases}$$

may have a stability switch for some $\tau \geq 0$, say $\tau^*$, only if $\lambda = \pm i\omega(\tau^*)$, $\omega(\tau^*) \in \Re_+$ are characteristic roots.

In the following of the paper we still study the occurrence of stability switches for the subclass of characteristic equations (1.5) where

$$D(\lambda,\tau) = P(\lambda,\tau) + Q^{(1)}(\lambda,\tau)e^{-\lambda\tau} + Q^{(2)}(\lambda,\tau)e^{-2\lambda\tau} \tag{1.13}$$

which is already sufficiently general to include as a particular case the characteristic equation

$$D(\lambda,\tau) = 0, \quad D(\lambda,\tau) = P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} \tag{1.14}$$

recently studied by Beretta and Kuang [3]. An application is also shown in a paper by Beretta, Carletti and Solimano [2]. Therefore, we extend the geometric stability switch criterion developed by Beretta and Kuang [3] for (1.14) to the more general case (1.13). This will be done in the next section.

However, we don't feel that the method we are presenting in the next section could be applied to general characteristic equations (1.6) with $m > 2$.

In Section 3 we present an application of the geometric stability switch criterion.

We conclude the paper with Section 4 showing that many of the characteristic equations known in literature are included in the case with structure (1.13), and that related stability switch results are obtained as particular cases of the geometric stability switch criterion presented in Section 2.

# 2 Stability switch geometric criterion

In this section we study the occurrence of stability switches for the characteristic equation

$$D(\lambda,\tau) = 0, \tag{2.1}$$

$$D(\lambda,\tau) = P(\lambda,\tau) + Q^{(1)}(\lambda,\tau)e^{-\lambda\tau} + Q^{(2)}(\lambda,\tau)e^{-2\lambda\tau}, \tag{2.2}$$

where $P$, $Q^{(i)}$, $i = 1, 2$ are polynomials defined in (1.6), satisfying the assumption (A.1), namely

$$(A.1) \qquad \begin{cases} p_0(\tau) + q_0^{(1)}(\tau) + q_0^{(2)}(\tau) \neq 0 & \forall \tau \geq 0 \\ n > m_k, & k = 1, 2. \end{cases}$$

According to (A.1) a stability switch may only occur with a pair of simple pure imaginary roots $\lambda = \pm i\omega$, $\omega \in \Re_+$ of the characteristic equation (2.1). Since $P$, $Q^{(k)}$, $k = 1, 2$ are polynomials with real coefficients

$$\overline{P(-i\omega,\tau)} = P(i\omega,\tau), \quad \overline{Q^{(k)}(-i\omega,\tau)} = Q^{(k)}(i\omega,\tau), \quad k = 1,2$$

where "—" denotes complex and conjugate, thus implying that if $\lambda = i\omega$, $\omega > 0$ is a root of (2.1), then even $\lambda = -i\omega$, $\omega > 0$, is a root of (2.1).

Finally, we assume that if $\lambda = i\omega$, $\omega > 0$ is a root of $D(\lambda,\tau) = 0$, then

$$(A.2) \qquad \begin{cases} either \\ \begin{cases} P_R(i\omega,\tau) + Q_R^{(2)}(i\omega,\tau) \neq 0 \\ P_R(i\omega,\tau) + Q_R^{(1)}(i\omega,\tau) + Q_R^{(2)}(i\omega,\tau) \neq 0 \qquad \forall \tau \geq 0 \end{cases} \\ or \\ \begin{cases} P_I(i\omega,\tau) + Q_I^{(2)}(i\omega,\tau) \neq 0 \\ P_I(i\omega,\tau) + Q_I^{(1)}(i\omega,\tau) + Q_I^{(2)}(i\omega,\tau) \neq 0 \qquad \forall \tau \geq 0 \end{cases} \end{cases}$$

i.e. $P$, $Q^{(1)}$, $Q^{(2)}$ have no common imaginary roots. The meaning of this assumption will be clear herefollowing.

Assume that $\lambda = i\omega$, $\omega > 0$ is a root of (2.1), (2.2). Denote by $P_R$, $Q_R^{(k)}$ $(k = 1, 2)$ and by $P_I$, $Q_I^{(k)}$ $(k = 1, 2)$ respectively real and imaginary parts of the polynomials $P(i\omega,\tau)$, $Q^{(k)}(i\omega,\tau)$, $k = 1, 2$. From the characteristic equation (2.1), (2.2), separating real and imaginary parts, we get:

$$\begin{cases} (P_R(i\omega,\tau) + Q_R^{(2)}(i\omega,\tau))\cos\omega\tau - (P_I(i\omega,\tau) - Q_I^{(2)}(i\omega,\tau))\sin\omega\tau = -Q_R^{(1)}(i\omega,\tau), \\ (P_I(i\omega,\tau) + Q_I^{(2)}(i\omega,\tau))\cos\omega\tau + (P_R(i\omega,\tau) - Q_R^{(2)}(i\omega,\tau))\sin\omega\tau = -Q_I^{(1)}(i\omega,\tau). \end{cases} \tag{2.3}$$

Hence, from (2.3), $\omega = \omega(\tau) > 0$ must satisfy the equations

$$\begin{cases} \cos\omega\tau = \frac{Q_R^{(1)}(Q_R^{(2)} - P_R) + Q_I^{(1)}(Q_I^{(2)} - P_I)}{|P(i\omega,\tau)|^2 - \left|Q^{(2)}(i\omega,\tau)\right|^2}, \\ \sin\omega\tau = \frac{Q_R^{(1)}(P_I + Q_I^{(2)}) - Q_I^{(1)}(P_R + Q_R^{(2)})}{|P(i\omega,\tau)|^2 - \left|Q^{(2)}(i\omega,\tau)\right|^2}. \end{cases} \tag{2.4}$$

A necessary condition in order that (2.4) holds true is that $\omega = \omega(\tau) > 0$ is a root of

$$F(\omega,\tau) = 0 \tag{2.5}$$

where

$$\begin{aligned} F(\omega,\tau) =: & \left[|P(i\omega,\tau)|^2 - \left|Q^{(2)}(i\omega,\tau)\right|^2\right]^2 \\ & - \left[Q_R^{(1)}(i\omega,\tau)(Q_R^{(2)}(i\omega,\tau) - P_R(i\omega,\tau)) + Q_I^{(1)}(i\omega,\tau)(Q_I^{(2)}(i\omega,\tau) - P_I(i\omega,\tau))\right]^2 \\ & - \left[Q_R^{(1)}(i\omega,\tau)(P_I(i\omega,\tau) + Q_I^{(2)}(i\omega,\tau)) - Q_I^{(1)}(i\omega,\tau)(P_R(i\omega,\tau) + Q_R^{(2)}(i\omega,\tau))\right]^2. \end{aligned} \tag{2.6}$$

Assume that $\omega = \omega(\tau)$ is a positive root of (2.5) for $\tau \in I \subseteq \Re_{+0}$ and that for $\tau \notin I$ such a root is not defined. We assume that each positive root $\omega = \omega(\tau), \tau \in I$ of (2.5) is a continuous and differentiable function of $\tau$. Since $\omega = \omega(\tau), \tau \in I$, if we substitute $\omega(\tau)$

into the right hand side of (2.4) we can define the angle $\theta(\tau) \in [0, 2\pi]$ as solution of (for the sake of simplicity we omit the arguments on the right hand side):

$$\begin{cases} \cos\theta(\tau) = \frac{Q_R^{(1)}(Q_R^{(2)}-P_R)+Q_I^{(1)}(Q_I^{(2)}-P_I)}{|P|^2-\left|Q^{(2)}\right|^2} =: \frac{T(\omega(\tau),\tau)}{R(\omega(\tau),\tau)}, \\ \sin\theta(\tau) = \frac{Q_R^{(1)}(P_I+Q_I^{(2)})-Q_I^{(1)}(P_R+Q_R^{(2)})}{|P|^2-\left|Q^{(2)}\right|^2} =: \frac{S(\omega(\tau),\tau)}{R(\omega(\tau),\tau)}. \end{cases} \tag{2.7}$$

where, of course

$$R(\omega(\tau), \tau) = |P(i\omega, \tau)|^2 - \left|Q^{(2)}(i\omega, \tau)\right|^2. \tag{2.8}$$

In order that $\lambda = \pm i\omega(\tau)$, $\omega(\tau) > 0$ solution of (2.5) for $\tau \in I$, are characteristic roots of (2.1) the necessary and sufficient condition is that the arguments "$\omega(\tau)\tau$" in (2.4) and "$\theta(\tau)$" in (2.7) are in the relationship:

$$\omega(\tau)\tau = \theta(\tau) + n2\pi, \quad n \in N_0 =: N \cup \{0\}. \tag{2.9}$$

Hence, we can define the maps $\tau_n : I \to \Re_{+0}$

$$\tau_n(\tau) =: \frac{\theta(\tau) + n2\pi}{\omega(\tau)}, \qquad n \in N_0, \tau \in I \tag{2.10}$$

where $\omega(\tau)$ is a positive solution of (2.5).

Let us introduce the functions $S_n : I \to \Re :$

$$S_n(\tau) = \tau - \tau_n(\tau), \qquad n \in N_0, \tau \in I. \tag{2.11}$$

Of course $\lambda = \pm i\omega(\tau)$, $\omega(\tau) > 0$ are characteristic roots of (2.1) at the $\tau$ values, say $\tau^* \in I$, which are zeros of $S_n(\tau)$ for some $n \in N_0$. We can prove:

**Lemma 2.1** *Assume that $\omega(\tau)$ is a positive solution of $F(\omega, \tau) = 0$ defined for $\tau \in I$, which is continuous and differentiable. Assume further that (A.2) holds true. Then the functions $S_n(\tau)$, $n \in N_0$, are continuous and differentiable on $I$.*

**Proof** Remark that

$$S_n(\tau) = \tau - \frac{\theta(\tau) + n2\pi}{\omega(\tau)}, \qquad n \in N_0, \tau \in I$$

where for all $\tau \in I$, $\omega(\tau)$ is positive, continuous and differentiable. Hence it is enough to prove that $\theta(\tau) \in [0, 2\pi]$ is continuous and differentiable for all $\tau \in I$. First we prove that $\theta(\tau) \in (0, 2\pi)$, i.e. $\theta(\tau) \neq 0, 2\pi$ for all $\tau \in I$, thus excluding $2\pi$ jump discontinuity as $\theta(\tau) = 0$ or $\theta(\tau) = 2\pi$. By the assumption (A.2) at least either

$$\begin{cases} P_R(i\omega, \tau) + Q_R^{(2)}(i\omega, \tau) \neq 0 \\ P_R(i\omega, \tau) + Q_R^{(1)}(i\omega, \tau) + Q_R^{(2)}(i\omega, \tau) \neq 0 \qquad \forall \tau \geq 0 \end{cases} \tag{2.12a}$$

or

$$\begin{cases} P_I(i\omega, \tau) + Q_I^{(2)}(i\omega, \tau) \neq 0 \\ P_I(i\omega, \tau) + Q_I^{(1)}(i\omega, \tau) + Q_I^{(2)}(i\omega, \tau) \neq 0 \qquad \forall \tau \geq 0. \end{cases} \tag{2.12b}$$

Assume first that (2.12a) holds true. From (2.7) if $\theta(\tau) = 0, 2\pi$ then $(\sin\theta(\tau) = 0)$

$$Q_I^{(1)} = Q_R^{(1)} \frac{P_I + Q_I^{(2)}}{P_R + Q_R^{(2)}}, \tag{2.13}$$

and substituting in $\cos\theta(\tau)$ one gets:

$$\begin{aligned} \cos\theta(\tau) &= \frac{1}{|P|^2 - |Q^{(2)}|^2} \left[ Q_R^{(1)}(Q_R^{(2)} - P_R) + Q_R^{(1)} \frac{P_I + Q_I^{(2)}}{P_R + Q_R^{(2)}}(Q_I^{(2)} - P_I) \right] \\ &= \frac{-Q_R^{(1)}(|P|^2 - \left|Q^{(2)}\right|^2)}{(P_R + Q_R^{(2)})(|P|^2 - |Q^{(2)}|^2)} = \frac{-Q_R^{(1)}}{P_R + Q_R^{(2)}}. \end{aligned} \tag{2.14}$$

Thus, if $\theta(\tau) = 0, 2\pi$ from (2.14) we obtain $P_R(i\omega,\tau) + Q_R^{(1)}(i\omega,\tau) + Q_R^{(2)}(i\omega,\tau) = 0$ in contradiction to the second of (2.12a). Similarly we prove $\theta(\tau) \neq 0, 2\pi$ for the case (2.12b). Hence $\theta(\tau) \in (0, 2\pi)$, $\forall\ \tau \in I$. According to (2.7) we can define $\theta(\tau)$ as

$$\theta(\tau) = \begin{cases} \arctan\left(\frac{S(\tau)}{T(\tau)}\right) & if \quad \sin\theta(\tau) > 0, \cos\theta(\tau) > 0, \\ \frac{\pi}{2} & if \quad \sin\theta(\tau) = 1, \cos\theta(\tau) = 0, \\ \pi + \arctan\left(\frac{S(\tau)}{T(\tau)}\right) & if \quad \cos\theta(\tau) < 0, \\ \frac{3\pi}{2} & if \quad \sin\theta(\tau) = -1, \cos\theta(\tau) = 0, \\ 2\pi + \arctan\left(\frac{S(\tau)}{T(\tau)}\right) & if \quad \sin\theta(\tau) < 0, \cos\theta(\tau) > 0, \end{cases} \tag{2.15}$$

where $S(\tau) =: S(\omega(\tau),\tau)$, $T(\tau) =: T(\omega(\tau),\tau)$ are continuous and differentiable functions of $\tau \in I$. It is easy to check that $\theta(\tau)$ is continuous on $I$. Furthermore $\theta'(\tau)$ is well defined for $\theta(\tau) \in (0, 2\pi)$ and is indeed given by

$$\theta'(\tau) = \frac{-S(\tau)T'(\tau) + S'(\tau)T(\tau)}{T^2(\tau) + S^2(\tau)}. \tag{2.16}$$

Observe that if $\theta(\tau) \neq \frac{\pi}{2}, \frac{3\pi}{2}$, then $T(\tau) \neq 0$ and (2.16) simply follows from (2.15). When $\theta(\tau) = \frac{\pi}{2}, \frac{3\pi}{2}$, we have $T(\tau) = 0$ and we compute $\theta'(\tau)$ directly from $\cos\theta(\tau) = T(\tau)/R(\tau)$, where $R(\tau) =: R(\omega(\tau),\tau)$ and $R^2(\tau) = T^2(\tau) + S^2(\tau)$. We obtain

$$-\sin\theta(\tau)\theta'(\tau) = \left(T(\tau)/R(\tau)\right)'. \tag{2.17}$$

It is easy to see that in the limit $T(\tau) \to 0$ (2.17) implies (2.16) as well.

Therefore, if $\theta(\tau) \in (0, 2\pi)$, $\tau \in I$, then $\theta(\tau)$ is a continuous and differentiable function on $I$. Since $\omega(\tau)$ is positive, continuous and differentiable on $I$, the functions $S_n(\tau)$, $n \in N_0$ are all continuous and differentiable on $I$. This completes the proof of Lemma.

We can now give the main result of the paper:

**Theorem 2.1** *Let $\omega(\tau)$ be a positive root of (2.5) for $\tau \in I \subseteq \Re_{+0}$. Assume that at some $\tau^* \in I$*

$$S_n(\tau^*) = 0 \tag{2.18}$$

*for some $n \in N_0$. Then a pair of simple conjugate pure imaginary roots $\lambda_+(\tau^*) = i\omega(\tau^*)$ and $\lambda_-(\tau^*) = -i\omega(\tau^*)$ of (2.1) exists at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where*

$$\begin{aligned} \delta(\tau^*) &= sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda=i\omega(\tau^*)}\right\} \\ &= sign\left\{\left[R(\omega(\tau),\tau)F'_\omega(\omega(\tau),\tau)\right]\Big|_{\tau=\tau^*}\right\}\cdot sign\left\{S'_n(\tau^*)\right\}. \end{aligned} \tag{2.19}$$

**Proof** If $\omega(\tau)$ is a positive root of (2.5) for $\tau \in I$, the relationship (2.9) implies that (2.18) is sufficient in order that $\lambda_+(\tau^*) = i\omega(\tau^*)$ and $\lambda_-(\tau^*) = -i\omega(\tau^*)$ is a simple pair of pure imaginary and conjugate roots of the characteristic equations (2.1), (2.2) which occurs at $\tau^* \in I$. Since $\delta(\tau^*) = sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda=i\omega(\tau^*)}\right\}$, $\lambda_+(\tau^*)$ and $\lambda_-(\tau^*)$ are crossing the imaginary axis for increasing $\tau$ according to the sign of $\delta(\tau^*)$. To prove the geometric stability switch criterion it is therefore necessary to prove equality (2.19). Let us first derive an expression for $S'_n(\tau^*)$ where at $\tau^*$, $S_n(\tau^*) = 0$. Since $S_n(\tau)$ is given by (2.10) and (2.11), then

$$S'_n(\tau^*) = \frac{\omega^2(\tau^*) - \theta'(\tau^*)\omega(\tau^*) + \tau^*\omega'(\tau^*)\omega(\tau^*)}{\omega^2(\tau^*)} \tag{2.20}$$

where $\theta'(\tau^*)$ is given by (2.16).

Taking into account that $S(\tau) = S(\omega(\tau),\tau)$, $T(\tau) = T(\omega(\tau),\tau)$ and that $S'(\tau)$, $T'(\tau)$ are total derivatives in "$\tau$", then in (2.16)

$$\begin{cases} T'(\tau) = T'_\omega(\omega(\tau),\tau)\omega'(\tau) + T'_\tau(\omega(\tau),\tau) \\ S'(\tau) = S'_\omega(\omega(\tau),\tau)\omega'(\tau) + S'_\tau(\omega(\tau),\tau) \end{cases} \tag{2.21}$$

where $T'_\omega, S'_\omega$ and $T'_\tau$ , $S'_\tau$ are respectively the partial derivatives of $T$ and $S$ first with respect to "$\omega$" and then with respect to "$\tau$". Hence, by substitution of (2.21) into (2.16) and using (2.16) in (2.20) we obtain

$$\begin{aligned} S'_n(\tau^*) &= \frac{1}{\omega^2(\tau^*)R^2(\tau^*)}\Big[(\omega^2(\tau^*) + \tau^*\omega(\tau^*)\omega'(\tau^*))R^2(\tau^*) \\ &\quad -\omega(\tau^*)(S'_\omega(\tau^*)T(\tau^*) - S(\tau^*)T'_\omega(\tau^*))\omega'(\tau^*) \\ &\quad -\omega(\tau^*)(S'_\tau(\tau^*)T(\tau^*) - S(\tau^*)T'_\tau(\tau^*))\Big] \end{aligned} \tag{2.22}$$

where of course, for the sake of simplicity, we have also set $R(\tau) = R(\omega(\tau),\tau)$.

In the following we will use this nomenclature for derivative. The total derivative, say of $P(\lambda,\tau)$ with respect to $\tau$ will be denoted by

$$D_\tau P(\lambda,\tau) := P'_\lambda(\lambda,\tau)\frac{d\lambda}{d\tau} + P'_\tau(\lambda,\tau) \tag{2.23}$$

where $P'_\lambda(\lambda,\tau) := \partial_\lambda P(\lambda,\tau)$, $P'_\tau(\lambda,\tau) := \partial_\tau P(\lambda,\tau)$ are the partial derivatives with respect to $\lambda$ and respectively with respect to $\tau$. Same nomenclature applies for derivatives of $Q^{(k)}$, $k = 1,2$. Return to the characteristic equations (2.1), (2.2), i.e.

$$P(\lambda,\tau) + Q^{(1)}(\lambda,\tau)e^{-\lambda\tau} + Q^{(2)}(\lambda,\tau)e^{-2\lambda\tau} = 0. \tag{2.24}$$

Differentiating (2.24) with respect to $\tau$ we obtain

$$\frac{d\lambda}{d\tau} = -\frac{\lambda(P - Q^{(2)}e^{-2\lambda\tau}) + P'_\tau + Q^{(1)'}_\tau e^{-\lambda\tau} + Q^{(2)'}_\tau e^{-2\lambda\tau}}{\tau(P - Q^{(2)}e^{-2\lambda\tau}) + P'_\lambda + Q^{(1)'}_\lambda e^{-\lambda\tau} + Q^{(2)'}_\lambda e^{-2\lambda\tau}}. \quad (2.25)$$

Let $\lambda = i\omega$, $\omega > 0$. Since $iP'_\lambda = P'_\omega$, then $P'_\lambda = -iP'_\omega$. Similarly we have $Q^{(1)'}_\lambda = -iQ^{(1)'}_\omega$, $Q^{(2)'}_\lambda = -iQ^{(2)'}_\omega$. Hence

$$\begin{aligned} \frac{d\lambda}{d\tau}\,|_{\lambda=i\omega} &= \frac{\omega(Pe^{i\omega\tau} - Q^{(2)}e^{-i\omega\tau}) - i\left[P'_\tau e^{i\omega\tau} + Q^{(1)'}_\tau + Q^{(2)'}_\tau e^{-i\omega\tau}\right]}{i\tau(Pe^{i\omega\tau} - Q^{(2)}e^{-i\omega\tau}) + P'_\omega e^{i\omega\tau} + Q^{(1)'}_\omega + Q^{(2)'}_\omega e^{-i\omega\tau}} \\ &= \frac{\omega A - iB}{i\tau A + C}. \end{aligned} \quad (2.26)$$

Since

$$sign\left\{\frac{dRe\lambda}{d\tau}\,|_{\lambda=i\omega}\right\} = sign\left\{Re(\frac{d\lambda}{d\tau}\,|_{\lambda=i\omega})\right\},$$

we compute:

$$Re(\frac{d\lambda}{d\tau}\,|_{\lambda=i\omega}) = \frac{(\omega A_R + B_I)(C_R - \tau A_I) + (\omega A_I - B_R)(C_I + \tau A_R)}{(C_R - \tau A_I)^2 + (C_I + \tau A_R)^2},$$

from which

$$\begin{aligned} & sign\left\{Re(\frac{d\lambda}{d\tau}\,|_{\lambda=i\omega})\right\} \\ &= sign\left\{\omega(A_R C_R + A_I C_I) - \tau(A_R B_R + A_I B_I) + B_I C_R - B_R C_I\right\} \\ &= sign\left\{Re\left[\bar{A}(\omega C - \tau B) + i\bar{B}C\right]\right\} \end{aligned} \quad (2.27)$$

where "–" means complex and conjugate. According to (2.26), in (2.27) we have:

$$\begin{aligned} A &= (Pe^{i\omega\tau} - Q^{(2)}e^{-i\omega\tau})R \\ &= P(T + iS) - Q^{(2)}(T - iS) \\ &= (P - Q^{(2)})T + i(P + Q^{(2)})S, \quad (2.28) \\ B &= (P'_\tau e^{i\omega\tau} + Q^{(1)'}_\tau + Q^{(2)'}_\tau e^{-i\omega\tau})R \\ &= P'_\tau(T + iS) + Q^{(1)'}_\tau R + Q^{(2)'}_\tau(T - iS) \\ &= (P'_\tau + Q^{(2)'}_\tau)T + Q^{(1)'}_\tau R + i(P'_\tau - Q^{(2)'}_\tau)S, \quad (2.29) \\ C &= (P'_\omega e^{i\omega\tau} + Q^{(1)'}_\omega + Q^{(2)'}_\omega e^{-i\omega\tau})R \\ &= P'_\omega(T + iS) + Q^{(1)'}_\omega R + Q^{(2)'}_\omega(T - iS) \\ &= (P'_\omega + Q^{(2)'}_\omega)T + Q^{(1)'}_\omega R + i(P'_\omega - Q^{(2)'}_\omega)S. \quad (2.30) \end{aligned}$$

Furthermore, from (2.7)

$$\begin{aligned} S &= Q^{(1)}_R(P_I + Q^{(2)}_I) - Q^{(1)}_I(P_R + Q^{(2)}_R) = Re\left[iQ^{(1)}(\bar{P} + \bar{Q}^{(2)})\right], \quad (2.31) \\ T &= Q^{(1)}_R(Q^{(2)}_R - P_R) + Q^{(1)}_I(Q^{(2)}_I - P_I) = Re\left[Q^{(1)}(\bar{Q}^{(2)} - \bar{P})\right]. \quad (2.32) \end{aligned}$$

Therefore, from (2.8), (2.31), (2.32) we get:

$$
\begin{aligned}
R'_\tau &= (|P|^2 - \left|Q^{(2)}\right|^2)'_\tau = \bar{P}P'_\tau + P\bar{P}'_\tau - \bar{Q}^{(2)}Q^{(2)\prime}_\tau - Q^{(2)}\bar{Q}^{(2)\prime}_\tau, && (2.33)\\
R'_\omega &= (|P|^2 - \left|Q^{(2)}\right|^2)'_\omega = \bar{P}P'_\omega + P\bar{P}'_\omega - \bar{Q}^{(2)}Q^{(2)\prime}_\omega - Q^{(2)}\bar{Q}^{(2)\prime}_\omega, && (2.34)\\
S'_\omega &= Re\left[iQ^{(1)\prime}_\omega(\bar{P} + \bar{Q}^{(2)}) + iQ^{(1)}(\bar{P}'_\omega + \bar{Q}^{(2)\prime}_\omega)\right], && (2.35)\\
T'_\omega &= Re\left[-Q^{(1)\prime}_\omega(\bar{P} - \bar{Q}^{(2)}) - Q^{(1)}(\bar{P}'_\omega - \bar{Q}^{(2)\prime}_\omega)\right], && (2.36)\\
S'_\tau &= Re\left[iQ^{(1)\prime}_\tau(\bar{P} + \bar{Q}^{(2)}) + iQ^{(1)}(\bar{P}'_\tau + \bar{Q}^{(2)\prime}_\tau)\right], && (2.37)\\
T'_\tau &= Re\left[-Q^{(1)\prime}_\tau(\bar{P} - \bar{Q}^{(2)}) - Q^{(1)}(\bar{P}'_\tau - \bar{Q}^{(2)\prime}_\tau)\right]. && (2.38)
\end{aligned}
$$

Finally remark that since

$$F(\omega, \tau) =: R^2 - T^2 - S^2 = 0$$

for all $\tau \in I$, where $\omega = \omega(\tau) > 0$ is a root of $F = 0$ for $\tau \in I$, then we have

$$F'_\omega(\omega, \tau) = 2(RR'_\omega - TT'_\omega - SS'_\omega), \quad F'_\tau(\omega, \tau) = 2(RR'_\tau - TT'_\tau - SS'_\tau)$$

and

$$\omega'(\tau) = -\frac{F'_\tau(\omega, \tau)}{F'_\omega(\omega, \tau)} = -\frac{RR'_\tau - TT'_\tau - SS'_\tau}{RR'_\omega - TT'_\omega - SS'_\omega} \qquad (2.39)$$

for all $\tau \in I$.

Now, we are in position to consider the terms appearing in (2.27). Let us start with $Re(\tau\bar{A}B)$. Here and in the following we present the main results leaving the detailed computations to the reader. We have

$$Re(\bar{A}B) = R(RR'_\tau - TT'_\tau - SS'_\tau) + Re\{\delta_0\} \qquad (2.40)$$

where

$$
\begin{aligned}
\delta_0 = \; & -R^2(P\bar{P}'_\tau - Q^{(2)}\bar{Q}^{(2)\prime}_\tau) + RTQ^{(1)}(\bar{Q}^{(2)\prime}_\tau - \bar{P}'_\tau) + RSQ^{(1)}i(\bar{Q}^{(2)\prime}_\tau + \bar{P}'_\tau)\\
& +(T^2 - S^2)(P\bar{Q}^{(2)\prime}_\tau - Q^{(2)}\bar{P}'_\tau) + 2iTS(P\bar{Q}^{(2)\prime}_\tau + Q^{(2)}\bar{P}'_\tau). \qquad (2.41)
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
\delta_0 &= \bar{Q}^{(2)\prime}_\tau\left[R^2Q^{(2)} + R(T + iS)Q^{(1)} + (T + iS)^2P\right]\\
&\quad -\bar{P}'_\tau\left[R^2P + R(T - iS)Q^{(1)} + (T - iS)^2Q^{(2)}\right]\\
&= R^2e^{2i\omega\tau}\bar{Q}^{(2)\prime}_\tau(P + Q^{(1)}e^{-i\omega\tau} + Q^{(2)}e^{-2i\omega\tau})\\
&\quad -R^2\bar{P}'_\tau(P + Q^{(1)}e^{-i\omega\tau} + Q^{(2)}e^{-2i\omega\tau}) = 0. \qquad (2.42)
\end{aligned}
$$

Therefore

$$Re(\bar{A}B) = R(RR'_\tau - TT'_\tau - SS'_\tau) = \frac{1}{2}RF'_\tau(\omega, \tau), \qquad (2.43)$$

and from (2.39)

$$Re(-\tau\bar{A}B) = -\frac{1}{2}\tau RF'_\tau(\omega, \tau) = \frac{1}{2}\tau R\omega'(\tau)F'_\omega(\omega, \tau). \qquad (2.44)$$

Now we consider the term $Re(\omega\bar{A}C)$ in equation (2.27) we obtain

$$Re(\omega\bar{A}C) = R(RR_\omega^{'} - TT_\omega^{'} - SS_\omega^{'}) + Re\left\{\delta_1\right\} \tag{2.45}$$

where

$$\begin{aligned}\delta_1 \quad = \quad & -R^2(P\bar{P}_\omega^{'} - Q^{(2)}\bar{Q}_\omega^{(2)'}) + (T^2 - S^2)(P\bar{Q}_\omega^{(2)'} - Q^{(2)}\bar{P}_\omega^{'}) + RTQ^{(1)}(\bar{Q}_\omega^{(2)'} - \bar{P}_\omega^{'}) \\ & +RSQ^{(1)}i(\bar{Q}_\omega^{(2)'} + \bar{P}_\omega^{'}) + 2iTS(P\bar{Q}_\omega^{(2)'} + Q^{(2)}\bar{P}_\omega^{'}).\end{aligned} \tag{2.46}$$

It is easy to check that

$$\begin{aligned}\delta_1 \quad &= \quad \bar{Q}_\omega^{(2)'}\left[R^2Q^{(2)} + R(T+iS)Q^{(1)} + (T+iS)^2P\right] \\ & \quad\quad -\bar{P}_\omega^{'}\left[R^2P + R(T-iS)Q^{(1)} + (T-iS)^2Q^{(2)}\right] \\ &= \quad R^2e^{2i\omega\tau}\bar{Q}_\omega^{(2)'}(P + Q^{(1)}e^{-i\omega\tau} + Q^{(2)}e^{-2i\omega\tau}) \\ & \quad\quad -R^2\bar{P}_\omega^{'}(P + Q^{(1)}e^{-i\omega\tau} + Q^{(2)}e^{-2i\omega\tau}) = 0.\end{aligned} \tag{2.47}$$

Therefore

$$Re(\bar{A}C) = R(RR_\omega^{'} - TT_\omega^{'} - SS_\omega^{'}), \tag{2.48}$$

i.e.

$$Re(\omega\bar{A}C) = \frac{1}{2}\omega RF_\omega^{'}(\omega,\tau). \tag{2.49}$$

Finally, we consider the term $Re(i\bar{B}C)$ in equation (2.27) we obtain:

$$Re(i\bar{B}C) = T(S_\omega^{'}R_\tau^{'} - S_\tau^{'}R_\omega^{'}) + R(S_\tau^{'}T_\omega^{'} - S_\omega^{'}T_\tau^{'}) + S(R_\omega^{'}T_\tau^{'} - R_\tau^{'}T_\omega^{'}) + \delta_2, \tag{2.50}$$

where

$$\begin{aligned}\delta_2 \quad = \quad & S_\omega^{'}Re\left\{\delta_{21}\right\} + T_\omega^{'}Re\left\{\delta_{22}\right\} + S_\tau^{'}Re\left\{\delta_{23}\right\} + T_\tau^{'}Re\left\{\delta_{24}\right\} \\ & +Re\left[iQ^{(1)}(\bar{P}_\omega^{'} + \bar{Q}_\omega^{(2)'})\right]Re\left\{\delta_{25}\right\} + Re\left[Q^{(1)}(\bar{P}_\omega^{'} - \bar{Q}_\omega^{(2)'})\right]Re\left\{\delta_{26}\right\} \\ & +\delta_{27} + \delta_{28} + \delta_{29},\end{aligned} \tag{2.51}$$

and

$$\delta_{21} = -T(\bar{P} + \bar{Q}^{(2)})(P_\tau^{'} - Q_\tau^{(2)'}) - RQ^{(1)}(\bar{P}_\tau^{'} - \bar{Q}_\tau^{(2)'}) + iS(\bar{P} - \bar{Q}^{(2)})(P_\tau^{'} - Q_\tau^{(2)'}), \tag{2.52}$$

$$\delta_{22} = iT(\bar{P} + \bar{Q}^{(2)})(P_\tau^{'} + Q_\tau^{(2)'}) - iRQ^{(1)}(\bar{P}_\tau^{'} + \bar{Q}_\tau^{(2)'}) + S(\bar{P} - \bar{Q}^{(2)})(P_\tau^{'} - Q_\tau^{(2)'}), \tag{2.53}$$

$$\delta_{23} = T(\bar{P} + \bar{Q}^{(2)})(P_\omega^{'} - Q_\omega^{(2)'}) + RQ^{(1)}(\bar{P}_\omega^{'} - \bar{Q}_\omega^{(2)'}) - iS(\bar{P} - \bar{Q}^{(2)})(P_\omega^{'} - Q_\omega^{(2)'}), \tag{2.54}$$

$$\delta_{24} = -iT(\bar{P} + \bar{Q}^{(2)})(P_\omega^{'} + Q_\omega^{(2)'}) + iRQ^{(1)}(\bar{P}_\omega^{'} + \bar{Q}_\omega^{(2)'}) - S(\bar{P} - \bar{Q}^{(2)})(P_\omega^{'} + Q_\omega^{(2)'}), \tag{2.55}$$

$$\delta_{25} = T(\bar{P} + \bar{Q}^{(2)})(P_\tau^{'} - Q_\tau^{(2)'}) + RQ^{(1)}(\bar{P}_\tau^{'} - \bar{Q}_\tau^{(2)'}) - iS(\bar{P} - \bar{Q}^{(2)})(P_\tau^{'} - Q_\tau^{(2)'}), \tag{2.56}$$

$$\delta_{26} = iT(\bar{P} + \bar{Q}^{(2)})(P_\tau^{'} + Q_\tau^{(2)'}) - iRQ^{(1)}(\bar{P}_\tau^{'} + \bar{Q}_\tau^{(2)'}) + S(\bar{P} - \bar{Q}^{(2)})(P_\tau^{'} + Q_\tau^{(2)'}), \tag{2.57}$$

$$\begin{aligned}
\delta_{27} &= -TRe\left[Q^{(1)}(\bar{P}'_\tau - \bar{Q}^{(2)'}_\tau)\right] Re\left[i(\bar{P}+\bar{Q}^{(2)})(P'_\omega + Q^{(2)'}_\omega)\right] \\
&\quad -SRe\left[Q^{(1)}(\bar{P}'_\tau - \bar{Q}^{(2)'}_\tau)\right] Re\left[(\bar{P}-\bar{Q}^{(2)})(P'_\omega + Q^{(2)'}_\omega)\right] \\
&\quad -TRe\left[iQ^{(1)}(\bar{P}'_\tau + \bar{Q}^{(2)'}_\tau)\right] Re\left[(\bar{P}+\bar{Q}^{(2)})(P'_\omega - Q^{(2)'}_\omega)\right] \\
&\quad +SRe\left[iQ^{(1)}(\bar{P}'_\tau + \bar{Q}^{(2)'}_\tau)\right] Re\left[i(\bar{P}-\bar{Q}^{(2)})(P'_\omega - Q^{(2)'}_\omega)\right], \quad (2.58) \\
\delta_{28} &= -TR'_\tau Re\left[iQ^{(1)}(\bar{P}'_\omega + \bar{Q}^{(2)'}_\omega)\right] + TR'_\omega Re\left[iQ^{(1)}(\bar{P}'_\tau + \bar{Q}^{(2)'}_\tau)\right] \\
&\quad -SR'_\tau Re\left[Q^{(1)}(\bar{P}'_\omega - \bar{Q}^{(2)'}_\omega)\right] + SR'_\omega Re\left[Q^{(1)}(\bar{P}'_\tau - \bar{Q}^{(2)'}_\tau)\right], \quad (2.59) \\
\delta_{29} &= R^2 Re\left[i(\bar{P}'_\tau P'_\omega + \bar{Q}^{(2)'}_\tau Q^{(2)'}_\omega)\right] \\
&\quad +Re\left[i(T-iS)^2 \bar{P}'_\tau Q^{(2)'}_\omega + i(T+iS)^2 P'_\omega \bar{Q}^{(2)'}_\tau)\right]. \quad (2.60)
\end{aligned}$$

Let us prove that $\delta_2 = 0$. First we consider $Re\{\delta_{21}\}$. From (2.52):

$$\begin{aligned}
Re\{\delta_{21}\} &= Re\Big\{-T(\bar{P}+\bar{Q}^{(2)})(P'_\tau - Q^{(2)'}_\tau) - R\bar{Q}^{(1)}(P'_\tau - Q^{(2)'}_\tau) \\
&\quad +iS(\bar{P}-\bar{Q}^{(2)})(P'_\tau - Q^{(2)'}_\tau)\Big\} \\
&= Re\Big\{-(P'_\tau - Q^{(2)'}_\tau)\left[R\bar{Q}^{(1)} + (T-iS)\bar{P} + (T+iS)\bar{Q}^{(2)}\right]\Big\}. \quad (2.61)
\end{aligned}$$

Notice that $\lambda = i\omega$ is a characteristic root of (2.24), i.e.

$$P(i\omega,\tau) + Q^{(1)}(i\omega,\tau)e^{-i\omega\tau} + Q^{(2)}(i\omega,\tau)e^{-2i\omega\tau} = 0. \quad (2.62)$$

Since

$$Re^{i\omega\tau} = T + iS, \quad Re^{-i\omega\tau} = T - iS, \quad (2.63)$$

(2.62) implies that:

$$\begin{aligned}
RQ^{(1)} + (T+iS)P + (T-iS)Q^{(2)} &= 0, \\
R\bar{Q}^{(1)} + (T-iS)\bar{P} + (T+iS)\bar{Q}^{(2)} &= 0.
\end{aligned} \quad (2.64)$$

Therefore, from the second of (2.64), we have $Re\{\delta_{21}\} = 0$. Similarly, we can obtain $Re\{\delta_{2i}\} = 0$, $i = 2,3,4,5,6$. We now consider the terms $\delta_{2i}$, $i = 7,8,9$.

Complicate and tedious computations lead to

$$\begin{aligned}
\delta_2 &= \delta_{27} + \delta_{28} + \delta_{29} \\
&= Re\left\{\bar{P}'_\tau P'_\omega \delta_{31}\right\} + Re\left\{Q^{(2)'}_\tau \bar{Q}^{(2)'}_\omega \delta_{32}\right\} \\
&\quad +Re\left\{P'_\omega \bar{Q}^{(2)'}_\tau \delta_{33}\right\} + Re\left\{\bar{P}'_\tau Q^{(2)'}_\omega \delta_{34}\right\},
\end{aligned} \quad (2.65)$$

where

$$\begin{aligned}
\delta_{31} &= iRQ^{(1)}\bar{Q}^{(1)} + Q^{(1)}(iT+S)\bar{P} - \bar{Q}^{(1)}(-iT+S)P + iR^2, \quad (2.66) \\
\delta_{32} &= iRQ^{(1)}\bar{Q}^{(1)} + Q^{(1)}(iT-S)\bar{Q}^{(2)} - \bar{Q}^{(1)}(-iT-S)Q^{(2)} - iR^2, \quad (2.67) \\
\delta_{33} &= Q^{(1)}(iT-S)\bar{P} + \bar{Q}^{(1)}(-iT+S)Q^{(2)} + i(T+iS)^2, \quad (2.68) \\
\delta_{34} &= -Q^{(1)}(iT+S)\bar{Q}^{(2)} - \bar{Q}^{(1)}(-iT-S)P + i(T-iS)^2. \quad (2.69)
\end{aligned}$$

Let us start considering $\delta_{31}$. From (2.66)

$$\delta_{31} = iRQ^{(1)}\bar{Q}^{(1)} + Q^{(1)}(iT+S)\bar{P} - \bar{Q}^{(1)}(-iT+S)P + i(T^2+S^2). \quad (2.70)$$

Now remark that from the second of (2.64) we obtain:

$$iRQ^{(1)}\bar{Q}^{(1)} + iQ^{(1)}(T - iS)\bar{P} + iQ^{(1)}(T + iS)\bar{Q}^{(2)} = 0,$$

i.e.

$$iRQ^{(1)}\bar{Q}^{(1)} + iQ^{(1)}(T - iS)\bar{P} = -iQ^{(1)}(T + iS)\bar{Q}^{(2)}. \tag{2.71}$$

By substituting (2.71) in (2.70)

$$\begin{aligned} \delta_{31} &= -iQ^{(1)}(T + iS)\bar{Q}^{(2)} - \bar{Q}^{(1)}(-iT + S)P + i(T^2 + S^2) \\ &= (S - iT)(Q^{(1)}\bar{Q}^{(2)} - \bar{Q}^{(1)}P - T + iS). \end{aligned} \tag{2.72}$$

Notice that from (2.64) we obtain

$$\begin{aligned} &RQ^{(1)}\bar{Q}^{(2)} = -(T + iS)P\bar{Q}^{(2)} - (T - iS)Q^{(2)}\bar{Q}^{(2)}, \\ &R\bar{Q}^{(1)}P = -(T - iS)\bar{P}P - (T + iS)\bar{Q}^{(2)}P, \end{aligned} \tag{2.73}$$

from which

$$R(Q^{(1)}\bar{Q}^{(2)} - \bar{Q}^{(1)}P) = (T - iS)R,$$

i.e.

$$R(Q^{(1)}\bar{Q}^{(2)} - \bar{Q}^{(1)}P - T + iS) = 0. \tag{2.74}$$

Since $R \neq 0$, then

$$Q^{(1)}\bar{Q}^{(2)} - \bar{Q}^{(1)}P - T + iS = 0 \tag{2.75}$$

and this implies that

$$\delta_{31} = 0.$$

Similarly, we get $\delta_{3i} = 0$, $i = 2, 3, 4$. Therefore

$$\delta_2 = 0, \tag{2.76}$$

i.e.

$$Re(i\bar{B}C) = T(S'_\omega R'_\tau - S'_\tau R'_\omega) + R(S'_\tau T'_\omega - S'_\omega T'_\tau) + S(R'_\omega T'_\tau - R'_\tau T'_\omega). \tag{2.77}$$

Now, by standard computations we obtain :

$$Re(i\bar{B}C) = \frac{F'_\omega(\omega,\tau)}{2\omega R}\left\{-(TS'_\omega - ST'_\omega)\omega\omega' - (TS'_\tau - ST'_\tau)\omega\right\}. \tag{2.78}$$

Combining (2.44), (2.49), (2.78), we obtain

$$\begin{aligned} &sign\left\{Re(\frac{d\lambda}{d\tau}|_{\lambda=i\omega})\right\} = sign\left\{Re\left[\bar{A}(\omega C - \tau B) + i\bar{B}C\right]\right\} \\ &= sign\left\{\frac{1}{2}\tau R\omega'(\tau)F'_\omega(\omega,\tau) + \frac{1}{2}\omega RF'_\omega(\omega,\tau)\right. \\ &\left.+\frac{F'_\omega(\omega,\tau)}{2\omega R}\left\{-(TS'_\omega - ST'_\omega)\omega\omega' - (TS'_\tau - ST'_\tau)\omega\right\}\right\} \\ &= sign\left\{\frac{F'_\omega(\omega,\tau)}{2\omega R}\left[(\omega^2 + \tau\omega\omega')R^2 - (TS'_\omega - ST'_\omega)\omega\omega' - (TS'_\tau - ST'_\tau)\omega\right]\right\}. \end{aligned}$$

Since $\omega(\tau) > 0$ for all $\tau \in I$, then

$$sign\left\{Re(\frac{d\lambda}{d\tau}|_{\lambda=i\omega})\right\} = sign\left\{R(\omega,\tau)F'_\omega(\omega,\tau)\right\}\cdot$$
$$sign\left\{(\omega^2+\tau\omega\omega')R^2-(TS'_\omega-ST'_\omega)\omega\omega'-(TS'_\tau-ST'_\tau)\omega\right\}. \quad (2.79)$$

Furthermore if $\tau^* \in I$ is a delay value at which $S_n(\tau^*) = 0$ then (2.22) holds true and (2.79) gives

$$sign\left\{Re(\frac{d\lambda}{d\tau}|_{\lambda=i\omega(\tau^*)})\right\} = sign\left\{R(\omega(\tau^*),\tau)F'_\omega(\omega(\tau^*),\tau^*)\right\}\cdot sign\left\{S'_n(\tau^*)\right\}, \quad (2.80)$$

and this proves the Theorem.

# 3 An application

To show the applicability of Theorem 2.1 we notice that for a second order characteristic equation (i.e. $P(\lambda,\tau)$ is a second order polynomial in $\lambda$ and $Q^{(i)}(\lambda,\tau), i = 1,2$ are at most first order polynomials in $\lambda$ ) the algebraic equation $F(\omega,\tau) = 0$ in (2.6) gives a polynomial equation in $\omega$ which is of degree 8. Thus, unless for a specific model and to avoid length algebraic computations, here we show how to apply Theorem 2.1 considering the first order characteristic equation:

$$D(\lambda,\tau) = \lambda + a(\tau) + b_1(\tau)e^{-\lambda\tau} + b_2(\tau)e^{-2\lambda\tau} = 0 \quad (3.1)$$

which belongs to the general class (2.1) with

$$P(\lambda,\tau) = \lambda + a(\tau), \quad Q^{(1)}(\lambda,\tau) = b_1(\tau), \quad Q^{(2)}(\lambda,\tau) = b_2(\tau), \quad (3.2)$$

where $a(\tau), b_1(\tau)$ and $b_2(\tau)$ are real smooth functions of delay $\tau$, assumed to have continuous derivatives in $\tau$. Then

$$P(i\omega,\tau) = i\omega + a(\tau), \quad Q^{(1)}(i\omega,\tau) = b_1(\tau), \quad Q^{(2)}(i\omega,\tau) = b_2(\tau)$$

thus implying that

$$\begin{array}{l} P_R(i\omega,\tau) = a(\tau), \quad P_I(i\omega,\tau) = \omega, \quad Q_R^{(1)}(i\omega,\tau) = b_1(\tau), \\ Q_I^{(1)}(i\omega,\tau) = 0, \quad Q_R^{(2)}(i\omega,\tau) = b_2(\tau), \quad Q_I^{(2)}(i\omega,\tau) = 0. \end{array} \quad (3.3)$$

Substituting (3.3) in (2.6) we find that $\omega(\tau)$ must be a positive root of

$$\begin{aligned} F(\omega,\tau) &= \left[\omega^2 + a^2(\tau) - b_2^2(\tau)\right]^2 - \left[b_1(\tau)(b_2(\tau) - a(\tau))\right]^2 - b_1^2(\tau)\omega^2 \\ &= \left[\omega^2 + a^2(\tau) - b_2^2(\tau)\right]^2 - b_1^2(\tau)\left[\omega^2 + (b_2(\tau) - a(\tau))^2\right] = 0. \end{aligned} \quad (3.4)$$

Hence, the equation $F(\omega,\tau) = 0$ gives

$$\omega^4 + \omega^2(2a^2(\tau) - 2b_2^2(\tau) - b_1^2(\tau)) + (a^2(\tau) - b_2^2(\tau))^2 - b_1^2(\tau)(b_2(\tau) - a(\tau))^2 = 0.$$

Its zeros are

$$\begin{cases} \omega_{\pm}^2(\tau) = \frac{1}{2}\left[b_1^2(\tau) - 2a^2(\tau) + 2b_2^2(\tau) \pm \sqrt{\Delta(\tau)}\right], \\ \Delta(\tau) = b_1^2(\tau)\left[b_1^2(\tau) + 8b_2^2(\tau) - 8a(\tau)b_2(\tau)\right]. \end{cases} \tag{3.5}$$

Of course $b_1(\tau) \neq 0 \ \forall \tau \geq 0$ and we assume that

$$b_1^2(\tau) + 8b_2^2(\tau) - 8a(\tau)b_2(\tau) \neq 0 \quad \forall \tau \geq 0. \tag{3.6}$$

This assumption guarantees that $\omega_{\pm}(\tau)$ (if exist) are the simple roots of $F(\omega(\tau), \tau) = 0$. Assume that $\omega_+(\tau)$ (or $\omega_-(\tau)$) exists for $\tau \in I$, then (3.4) shows that

$$R(\omega(\tau), \tau) = \omega_{\pm}^2(\tau) + a^2(\tau) - b_2^2(\tau) \neq 0 \tag{3.7}$$

for all $\tau \in I$. Hence, according to (2.7) we can define the angles $\theta_{\pm}(\tau) \in (0, 2\pi)$ for $\tau \in I$ as solution of

$$\sin\theta_{\pm}(\tau) = \frac{b_1(\tau)\omega_{\pm}(\tau)}{\omega_{\pm}^2(\tau) + a^2(\tau) - b_2^2(\tau)}, \cos\theta_{\pm}(\tau) = \frac{b_1(\tau)(b_2(\tau) - a(\tau))}{\omega_{\pm}^2(\tau) + a^2(\tau) - b_2^2(\tau)}. \tag{3.8}$$

According to (2.10), (2.11) we define the functions $S_n^{\pm}(\tau) : I \to \Re$ :

$$S_n^{\pm}(\tau) = \tau - \tau_n^{\pm}(\tau) = \tau - \frac{\theta_{\pm}(\tau) + n2\pi}{\omega_{\pm}(\tau)}, \qquad n \in N_0, \tag{3.9}$$

and the study of stability switches becomes the study of the zeros of the functions $S_n^{\pm}(\tau)$ in (3.9). Using the popular software such as MATLAB, we can get the values $\tau^*$ such that $S_n^{\pm}(\tau^*) = 0$, and determine the sign of $\frac{d}{d\tau}S_n^{\pm}(\tau^*)$ . The sign of the stability switch is determined by $\delta(\tau^*) = sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda = i\omega(\tau^*)}\right\}$ which according to (2.19) is given by

$$\delta(\tau^*) = sign\left\{R(\omega(\tau^*), \tau)F_{\omega}'(\omega(\tau^*), \tau)\right\} \cdot sign\left\{S_n'(\tau^*)\right\}.$$

Then, notice that

$$F_{\omega}'(\omega, \tau) = 2\omega\left[2\omega^2 + 2a^2(\tau) - 2b_2^2(\tau) - b_1^2(\tau)\right]. \tag{3.10}$$

If $\omega_{\pm}(\tau)$ is a positive root of $F(\omega, \tau) = 0$, then according to (3.5),

$$2\omega_{\pm}^2 + 2a^2(\tau) - 2b_2^2(\tau) - b_1^2(\tau) = \pm\sqrt{\Delta(\tau)}.$$

Hence

$$F_{\omega}'(\omega_{\pm}(\tau^*), \tau^*) = 2\omega_{\pm}(\tau^*)\left[\pm\sqrt{\Delta(\tau^*)}\right]. \tag{3.11}$$

Finally, if $\omega_{\pm}(\tau)$ is a positive root of $F(\omega, \tau) = 0$, then

$$R(\omega_{\pm}(\tau^*), \tau^*) = \omega_{\pm}^2(\tau^*) + a^2(\tau^*) - b_2^2(\tau^*) = \frac{1}{2}(b_1^2(\tau^*) \pm \sqrt{\Delta(\tau^*)}). \tag{3.12}$$

In conclusion, from (3.11), (3.12) we get

$$\begin{aligned} \delta(\tau^*) &= sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda = i\omega_{\pm}(\tau^*)}\right\} \\ &= sign\left\{\pm\sqrt{\Delta(\tau^*)}(b_1^2(\tau^*) \pm \sqrt{\Delta(\tau^*)})\right\} \cdot sign\left\{\frac{d}{d\tau}S_n^{\pm}(\tau^*)\right\}. \end{aligned} \tag{3.13}$$

Notice that, in correspondence of the characteristic roots $\lambda = \pm i\omega_+(\tau^*)$, (3.13) becomes

$$\delta(\tau^*) = sign\left\{\frac{d}{d\tau}S_n^+(\tau^*)\right\}, \tag{3.14a}$$

whereas, if $\lambda = \pm i\omega_-(\tau^*)$, (3.13) gives

$$\delta(\tau^*) = sign\left\{-(b_1^2(\tau^*) - \sqrt{\Delta(\tau^*)})\right\} \cdot sign\left\{\frac{d}{d\tau}S_n^-(\tau^*)\right\}. \tag{3.14b}$$

**Remark 3.1.** *If $Q^{(2)}(\lambda,\tau) = b_2(\tau) \equiv 0$ in equation (3.1), i.e.*

$$\lambda + a(\tau) + b_1(\tau)e^{-\lambda\tau} = 0,$$

*then $\omega_+^2(\tau) = b_1^2(\tau) - a^2(\tau)$ whereas $\omega_-(\tau)$ is never feasible. Accordingly (3.14a) holds true, in agreement with Theorem 3.1 in Beretta and Kuang [3].*

As an example, we consider a model proposed by Bélair and Campbell [1] suitably modified to give a characteristic equation belonging to the class (3.1). The model equation proposed by Bélair and Campbell [1] is

$$\dot{x}(t) = f_1(x(t - T_1)) + f_2(x(t - T_2)) \tag{3.15}$$

where $f_i(x) = -A_i \tanh(x), i = 1,2$ and where $A_i, i = 1,2$ were assumed to be positive constants.

We modify (3.15) in the following way

$$\dot{x}(t) = -A_1(\tau)\tanh(x(t-\tau)) - A_2(\tau)\tanh(x(t-2\tau)) \tag{3.16}$$

where for $A_i(\tau)$ we assume the simple structure:

$$A_i(\tau) = A_i e^{-\mu_i\tau}, \quad A_i, \mu_i \in \Re_+, \quad i = 1,2. \tag{3.17}$$

Of course (3.16), (3.17) have no biological (physical, etc.) meaning but it is a simple mathematical test for Theorem 2.1.

The characteristic equation at the equilibrium solution $x = 0$ gives

$$\lambda + A_1(\tau)e^{-\lambda\tau} + A_2(\tau)e^{-2\lambda\tau} = 0 \tag{3.18}$$

which belongs to the class of first order characteristic equation (3.1) with

$$a(\tau) \equiv 0, b_1(\tau) = A_1(\tau), b_2(\tau) = A_2(\tau). \tag{3.19}$$

From (3.5) we see that

$$\omega_\pm^2(\tau) = \frac{1}{2}\left[A_1^2(\tau) + 2A_2^2(\tau) \pm \sqrt{\Delta(\tau)}\right] \tag{3.20}$$

where $\omega_+$ ($\omega_-$) is the root obtained choosing the sign + (respectively −) in the right hand side of (3.20) and

$$\Delta(\tau) = A_1^2(\tau)\left[A_1^2(\tau) + 8A_2^2(\tau)\right]. \tag{3.21}$$

It is easy to see that $\omega_+(\tau)$ exists for all $\tau \geq 0$, i.e. for all $\tau \in I_+ \equiv \Re_{+0}$. Furthermore, $\omega_-(\tau)$ exists for all $\tau$ such that $A_2(\tau) > A_1(\tau)$. For example, if in (3.17) we choose $A_2 > A_1, \mu_2 > \mu_1$ then $\omega_-(\tau) > 0$ exists for all $\tau \in I_- =: [0, \tau_1)$ where

$$\tau_1 = \frac{1}{\mu_2 - \mu_1} \log \frac{A_2}{A_1}. \tag{3.22}$$

According to (3.8) we can define angles $\theta_\pm(\tau)$ such that

$$\sin\theta_+(\tau) = \frac{A_1(\tau)\omega_+(\tau)}{\omega_+^2(\tau) - A_2^2(\tau)}, \quad \cos\theta_+(\tau) = \frac{A_1(\tau)A_2(\tau)}{\omega_+^2(\tau) - A_2^2(\tau)}, \tau \in I_+, \tag{3.23a}$$

and

$$\sin\theta_-(\tau) = \frac{A_1(\tau)\omega_-(\tau)}{\omega_-^2(\tau) - A_2^2(\tau)}, \quad \cos\theta_-(\tau) = \frac{A_1(\tau)A_2(\tau)}{\omega_-^2(\tau) - A_2^2(\tau)}, \tau \in I_-, \tag{3.23b}$$

and, according to (3.9) the two sequences of functions are:

$$S_n^+(\tau) = \tau - \frac{\theta_+(\tau) + n2\pi}{\omega_+(\tau)}, \qquad n \in N_0, \tau \in I_+, \tag{3.24a}$$

$$S_n^-(\tau) = \tau - \frac{\theta_-(\tau) + n2\pi}{\omega_-(\tau)}, \qquad n \in N_0, \tau \in I_-. \tag{3.24b}$$

Stability switch may occur at the delay values, say $\tau^*$, which are zeros of the functions $S_n^+(\tau)$ in (3.24a) and $S_n^-(\tau)$ in (3.24b) and the direction of the stability switches is respectively given by (3.14a) and by (3.14b).

We may notice that in (3.14b)

$$b_1^2(\tau^*) - \sqrt{\Delta(\tau^*)} = A_1^2(\tau^*) - \sqrt{A_1^2(\tau^*)(A_1^2(\tau^*) + 8A_2^2(\tau^*))} < 0$$

for any $\tau^* \geq 0$ and accordingly (3.14b) becomes:

$$sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda = i\omega_-(\tau^*)}\right\} = sign\left\{\frac{d}{d\tau}S_n^-(\tau^*)\right\}. \tag{3.25}$$

For the following set of parameters

$$A_1 = 1, \quad A_2 = 8, \quad \mu_1 = 0.3, \quad \mu_2 = 0.4, \tag{3.26}$$

the graphs of functions $S_n^\pm(\tau)$ versus $\tau$ are depicted in Fig.3.1, where $I_+ = [0, +\infty)$, $I_+ = [0, \tau_1)$, $\tau_1 = 20.7944$.

**Figure 3.1** Distribution of zeros of $S_n^\pm(\tau)$ for the set of parameters (3.26). The equilibrium $x = 0$ of (3.16) is unstable for all $\tau \in (0.12, 12.34)$, asymptotically stable outside this interval. We have two stability switches: the first at $\tau = 0.12$ toward instability, the second at $\tau = 12.34$ toward stability.

Notice that, from the structure of functions $S_n^+(\tau)$, $S_n^-(\tau)$ (see (3.24a,b)) if for some $n$, say $n^* \in N$ it is $S_{n^*}^+(\tau) < 0$ for all $\tau \in I_+$ (or $S_{n^*}^-(\tau) < 0$ for all $\tau \in I_-$), then , for every $n > n^*$ , $S_n^+(\tau) < 0$ for all $\tau \in I_+$ (respectively, $S_n^-(\tau) < 0$ for all $\tau \in I_-$, $n > n^*$). Hence, it is sufficient to depict the graphs versus $\tau$ only of the first nonnegative functions in the sequences $S_n^+(\tau)$, $S_n^-(\tau)$, since those negative cannot give rise to stability switches.

In Fig 3.1, we see that only the functions $S_0^+(\tau)$, $S_1^+(\tau)$ are nonnegative in the sequence $S_n^+(\tau)$, and the function $S_0^-(\tau)$ in the sequence $S_n^-(\tau)$. All the functions have same shape and therefore each nonnegative function has two zeros. Starting from $\tau = 0$ (notice that at $\tau = 0$ the characteristic equation (3.18) gives $\lambda = -(A_1 + A_2) < 0$, i.e. asymptotic stability) for increasing $\tau$ we have the sequence of zeros:

$$0 < \tau_{01}^+ < \tau_{01}^- < \tau_{11}^+ < \tau_{12}^+ < \tau_{02}^- < \tau_{02}^+$$

with

$$\begin{aligned} S_0^+ &: \tau_{01}^+ = 0.12; \quad \tau_{02}^+ = 12.34, \\ S_0^- &: \tau_{01}^- = 1.12; \quad \tau_{02}^- = 5.56, \\ S_1^+ &: \tau_{11}^+ = 1.72; \quad \tau_{12}^+ = 3.92. \end{aligned}$$

Furthermore, let $M(\tau)$ the total multiplicity of characteristic roots of (3.18) on the right hand side of complex plane. Hence, Fig 3.1 says us that from $\tau = 0$ up to $\tau_{01}^+$ (excluded) $M(\tau) = 0$ and the equilibrium is still asymptotically stable. At $\tau = \tau_{01}^+$ a couple of characteristic roots $\lambda = \pm i\omega_+(\tau_{01}^+)$ exist which are entering in the right half complex plane according to (3.14a) (since $\frac{d}{d\tau}S_0^+(\tau_{01}^+) > 0$) and hence $M(\tau) = 2$ for $\tau \in (\tau_{01}^+, \tau_{01}^-)$ and the equilibrium is now unstable. According to our Definition 1.1, $\tau_{01}^+ = 0.12$ is a delay value at which a stability switch occurs toward instability. Now at $\tau = \tau_{01}^-$ a couple of characteristic roots $\lambda = \pm i\omega_-(\tau_{01}^-)$ exist which are entering in the right half complex plane according to (3.25) (since $\frac{d}{d\tau}S_0^-(\tau_{01}^-) > 0$). Therefore crossing $\tau_{01}^-$ the total multiplicity changes from $M(\tau) = 2$ to $M(\tau) = 4$ but equilibrium remains unstable. Hence, according to Definition 1.1, $\tau_{01}^-$ is not a stability switch delay value. Now, we can easily extend this analysis of $M(\tau)$ to the whole sequence of zeros. We obtain easily

$$\begin{aligned} M(\tau) &= 0, \quad \tau \in \left[0, \tau_{01}^+\right); \qquad M(\tau) = 2, \quad \tau \in (\tau_{01}^+, \tau_{01}^-); \\ M(\tau) &= 4, \quad \tau \in (\tau_{01}^-, \tau_{11}^+); \qquad M(\tau) = 6, \quad \tau \in (\tau_{11}^+, \tau_{12}^+); \\ M(\tau) &= 4, \quad \tau \in (\tau_{12}^+, \tau_{02}^-); \qquad M(\tau) = 2, \quad \tau \in (\tau_{02}^-, \tau_{02}^+); \\ M(\tau) &= 0, \quad \tau > \tau_{02}^+. \end{aligned}$$

In conclusion $M(\tau) = 0$, $\tau \in \left[0, \tau_{01}^+\right)$, $M(\tau) \geq 2$ for $\tau \in (\tau_{01}^+, \tau_{02}^+)$ and $M(\tau) = 0$, $\tau > \tau_{02}^+$. Hence the equilibrium $x = 0$ of (3.16) is unstable in the interval $(\tau_{01}^+ = 0.12, \tau_{02}^+ = 12.34)$ and asymptotically stable for $0 \leq \tau < \tau_{01}^+ = 0.12$ and $\tau > \tau_{02}^+ = 12.34$. We have two stability switch delay values. The first toward instability at $\tau = \tau_{01}^+ = 0.12$ and the second toward stability at $\tau = \tau_{02}^+ = 12.34$. As already noticed in the paper by Beretta and Kuang [3], in delay differential systems with delay dependent coefficient large delays seem to have a stability effect.

# 4 Discussion

In this paper we have generalized the geometric stability switch criterion presented by Beretta and Kuang [3] for the class of characteristic equations (1.18) with delay dependent coefficients
$$D(\lambda,\tau) = P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0$$
to the wider class of characteristic equation (1.17)
$$D(\lambda,\tau) = P(\lambda,\tau) + Q^{(1)}(\lambda,\tau)e^{-\lambda\tau} + Q^{(2)}(\lambda,\tau)e^{-2\lambda\tau} = 0.$$
Why "geometric" stability switch criterion? This term is justified shortly recalling the theory developed in Section 2.

Assume that we know the stability of the steady state when $\tau = 0$. The assumptions made ensure that the multiplicity of characteristic roots in the right half complex plane can only change if a couple of simple pure imaginary roots $\lambda = \pm i\omega(\tau)$, $\omega(\tau) > 0$ cross the imaginary axis at some delay value, say $\tau^* > 0$. Then $\omega(\tau) > 0$ must be an isolated root of $F(\omega,\tau) = 0$ given by (2.6). This defines the function $\omega = \omega(\tau)$ for $\tau \in I \subseteq \Re_{+0}$. With $\omega = \omega(\tau)$, $\tau \in I$ we define the function $\theta(\tau) \in (0, 2\pi)$, $\tau \in I$ as a solution of (2.7), function which is continuous and continuously differentiable for all $\tau \in I$ . The couple of simple pure imaginary roots of (1.17) occur at the $\tau$ values, say $\tau^*$, which are zeros of the function $S_n(\tau)$, $\tau \in I$ in (2.11) and which are continuous and continuously differentiable functions for all $\tau \in I$ . The direction of this couple of simple pure imaginary roots $\lambda = \pm i\omega(\tau^*)$ (i.e. if they are entering in the left or right complex plane for increasing $\tau$) is given by (2.19) in Theorem 2.1, i.e. by the sign of $S_n'(\tau^*)$.

Hence, the knowledge of the geometric shape of the functions $S_n = S_n(\tau)$, $\tau \in I$, i.e. location of their zeros $\tau^*$ and the sign of $S_n'(\tau^*)$ give us through Theorem 2.1 the way to determine at which delay values the stability switches occur. This justifies the term "geometric" stability switches. Therefore, it is sufficient to get by some simple mathematical software the graphs of the functions $S_n(\tau)$ versus $\tau$ for $n \in N_0$.

**Remark 4.1.** It is interesting to notice that the results of this paper are in agreement with the results presented by Beretta and Kuang [3] for the class of characteristic equations (1.18). The characteristic equations (1.18) is obtained from (1.17) by setting
$$Q^{(2)}(\lambda,\tau) \equiv 0, Q^{(1)}(\lambda,\tau) = Q(\lambda,\tau). \tag{4.1}$$
For example, if we consider (2.6) in this paper with conditions (4.1) we get
$$\begin{aligned} F(\omega,\tau) &= |P(i\omega,\tau)|^4 - [-Q_R(i\omega,\tau)P_R(i\omega,\tau) - Q_I(i\omega,\tau)P_I(i\omega,\tau)]^2 \\ &\quad - [Q_R(i\omega,\tau)P_I(i\omega,\tau) - Q_I(i\omega,\tau)P_R(i\omega,\tau)]^2 \\ &= |P(i\omega,\tau)|^2 \left(|P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2\right) = 0. \end{aligned} \tag{4.2}$$
Since $|P(i\omega,\tau)| \neq 0$, $\omega$ must be a positive root of
$$|P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2 = 0, \tag{4.3}$$
in agreement with Beretta and Kuang [3].

Furthermore, from (2.7) with conditions (4.1) we get that the angle $\theta(\tau) \in (0, 2\pi)$, $\tau \in I$ will be a solution of

$$\begin{cases} \sin\theta(\tau) = \frac{-P_R(i\omega,\tau)Q_I(i\omega,\tau)+Q_R(i\omega,\tau)P_I(i\omega,\tau)}{|P(i\omega,\tau)|^2}, \\ \cos\theta(\tau) = -\frac{Q_R(i\omega,\tau)P_R(i\omega,\tau)+Q_I(i\omega,\tau)P_I(i\omega,\tau)}{|P(i\omega,\tau)|^2}, \end{cases} \tag{4.4}$$

still in agreement with Beretta and Kuang [3]. Therefore (4.3), (4.4) define the functions

$$S_n(\tau) = \tau - \frac{\theta(\tau) + n2\pi}{\omega(\tau)}, \qquad n \in N_0,$$

for which our Theorem 2.1, once observed that under conditions (4.1)

$$R(\omega(\tau), \tau) = |P(i\omega, \tau)|^2 - \left|Q^{(2)}(i\omega, \tau)\right|^2 = |P(i\omega, \tau)|^2, \tag{4.5}$$

gives

$$\begin{aligned} \delta(\tau^*) &= sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda=i\omega(\tau^*)}\right\} \\ &= sign\left\{F_\omega^{'}(\omega(\tau^*), \tau^*)\right\} \cdot sign\left\{\frac{d}{d\tau}S_n(\tau^*)\right\}. \end{aligned} \tag{4.6}$$

in complete agreement with Theorem 2.1 in Beretta and Kuang [3].

**Remark 4.2.** We wish conclude observing that the geometric stability switch criterion even applies to the particular case in which the characteristic equations have delay independent coefficients, i.e.

$$D(\lambda, \tau) = P(\lambda) + Q^{(1)}(\lambda)e^{-\lambda\tau} + Q^{(2)}(\lambda)e^{-2\lambda\tau} = 0 \tag{4.7}$$

with

$$P(\lambda) = \sum_{j=0}^{n} p_j\lambda^j, \quad Q^{(i)}(\lambda) = \sum_{j=0}^{m_i} q_j^{(i)}\lambda^j, \quad i = 1, 2 \tag{4.8}$$

where $n > m_i$, $i = 1, 2$ and $p_j$, $q_j^{(i)}$ are given real numbers.

Since $P, Q^{(i)}$ are independent of $\tau$, any solution $\omega > 0$ of (2.6) will be independent of $\tau$, so as the angle $\theta \in (0, 2\pi)$ solution of (2.7). Denote by $\omega^*$ the solution of (2.6) and $\theta^*$ the corresponding solution of (2.7).

Therefore the functions $S_n$ given by (2.10), (2.11) will be

$$S_n(\tau) = \tau - \frac{\theta^* + n2\pi}{\omega^*}, \qquad n \in N_0 \tag{4.9}$$

where the second term in (4.9) is independent of $\tau$. Hence $S_n(\tau)$ is a family of straight-lines versus $\tau$ which zeros are:

$$\tau_n^* = \frac{\theta^* + n2\pi}{\omega^*}, \qquad n \in N_0 \tag{4.10}$$

at which $S_n^{'}(\tau^*) = 1$. Hence, from Theorem 2.1 of this paper, the direction of stability switches is given by

$$\delta(\tau^*) = sign\left\{\frac{dRe\lambda}{d\tau}\Big|_{\lambda=i\omega^*}\right\} = sign\left\{R(\omega^*)F_\omega^{'}(\omega^*)\right\} \tag{4.11}$$

where

$$R(\omega^*) = |P(\omega^*)|^2 - \left|Q^{(2)}(\omega^*)\right|^2.$$

Of course, these results even apply to the characteristic equations

$$D(\lambda,\tau) = P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0 \tag{4.12}$$

where $P(\lambda)$, $Q(\lambda)$ are polynomials in $\lambda$ with $n > m$ and with time delay independent coefficients.

Let $\omega^*$ be the isolated positive zeros of

$$F(\omega) = |P(i\omega)|^2 - |Q(i\omega)|^2 \tag{4.13}$$

and $\theta^*$ be the corresponding angle solution of (4.4) with $P, Q$ independent of $\tau$. Then the stability switches may occur at the $\tau_n^*$ values given by (4.10) and the direction of the switches will be given by:

$$\delta(\tau^*) = sign\left\{\frac{dRe\lambda}{d\tau}|_{\lambda=i\omega^*}\right\} = sign\left\{F'_\omega(\omega^*)\right\}. \tag{4.14}$$

Assume for example that (4.12) is a second order characteristic equation,i.e.

$$P(\lambda) = \lambda^2 + a\lambda + c, \quad Q(\lambda) = b\lambda + d \tag{4.15}$$

where $a, b, c, d$ are given real numbers with $c + d \neq 0$. Then $P(i\omega) = -\omega^2 + ia\omega + c$, $Q(i\omega) = ib\omega + d$ and (4.13) becomes

$$\omega^4 - \omega^2(b^2 + 2c - a^2) + (c^2 - d^2) = 0. \tag{4.16}$$

We may have two positive roots, say $\omega_+, \omega_-$, satisfying

$$\omega_\pm^2 = \frac{1}{2}\left[(b^2 + 2c - a^2) \pm \sqrt{\Delta}\right], \tag{4.17}$$

where

$$\Delta = (b^2 + 2c - a^2)^2 - 4(c^2 - d^2). \tag{4.18}$$

Furthermore, it is easy to check that

$$F'_\omega(\omega_\pm) = 2\omega_\pm\left[\pm\sqrt{\Delta}\right]. \tag{4.19}$$

From (4.4) we get the angles $\theta_+(\tau), \theta_-(\tau) \in (0, 2\pi)$, respectively corresponding to $\omega_+$and to $\omega_-$, as solution of

$$\sin\theta_\pm = \frac{-(c - \omega_\pm^2)\omega_\pm b + \omega_\pm ad}{\omega_\pm^2 b^2 + d^2}, \quad \cos\theta_\pm = -\frac{(c - \omega_\pm^2)d + \omega_\pm^2 ab}{\omega_\pm^2 b^2 + d^2}. \tag{4.20}$$

If both the positive roots $\omega_+$and $\omega_-$ of (4.16) exist then (4.10) gives two families of delays, say

$$\tau_n^+ = \frac{\theta_+ + n2\pi}{\omega_+}, \qquad \tau_n^- = \frac{\theta_- + n2\pi}{\omega_-}, \qquad n \in N_0. \tag{4.21}$$

At each delay $\tau_n^+$ we have a couple of pure imaginary roots $\lambda = \pm i\omega_+$ and at each delay $\tau_n^-$ we have a couple of pure imaginary roots $\lambda = \pm i\omega_-$ and their direction for increasing $\tau$ (i.e. if they are entering in the right or left half complex plane) is given by

$$\delta(\tau_n^+) = sign\left\{+\sqrt{\Delta}\right\} = 1, \quad (4.22)$$

and respectively by

$$\delta(\tau_n^-) = sign\left\{-\sqrt{\Delta}\right\} = -1. \quad (4.23)$$

Hence for increasing $\tau$, crossing the delay value $\tau_n^+$, the total multiplicity $M(\tau)$ of the roots in the right half complex plane increases of two, whereas crossing $\tau_n^-$, $M(\tau)$ decreases of two units. These results are in agreement with those obtained in Section 4 of the paper by Freedman and Kuang [5].

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