

# Commuting differential operators of type $B_2$

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## 1 Introduction

**1.1.** Several integral systems are accidentally related to root systems. Olshanetsky-Perelomov ([OP1], [OP2]) considered integrable  $n$ -particle models in dimension one arising from root systems. The systems of differential operators satisfied by zonal spherical functions give such integrable systems and these were generalized by Sekiguchi and Heckman-Opdam ([Sj], [HO]).

In [OOS] we announce a classification of integrable systems invariant under simple classical Weyl groups. The precise discussion has already been given by [OS] and [O] except for the case of type  $B_2$ . As is shown in [OS], the classification problem for type  $B_2$  is reduced to a functional differential equation (1.4).

In §2 we give a complete list of solutions of this functional equation. Some solutions have already been obtained, after [OP2], by Inozemtsev [IM], [I] (See also [P]). The main result of §2 is Theorem 2.9, which is stated in §1.3 in a different form.

In §3 we examine the reducibility of the system obtained in §2. We note that if the system coincides with the system satisfied by zonal spherical functions of a semisimple Lie group, the reducibility is related to degenerate series representations.

The final draft of this paper was completed when the authors were visiting University of Leiden in the fall of 1994. The authors express their sincere gratitude to Prof. Dr. van Dijk for his hospitality during their stay there.

**1.2.** Now we give a quick review of the results in [OS, §6] concerning with type  $B_2$ . Let  $W(B_2)$  be the Weyl group of type  $B_2$ , which is identified with the group of coordinate transformations of  $(x_1, x_2)$  generated by

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$(x_1, x_2) \mapsto (x_2, x_1)$  and  $(x_1, x_2) \mapsto (x_1, -x_2)$ . Consider  $W(B_2)$ -invariant differential operators

$$(1.1) \quad \begin{cases} P_1 = \partial_1^2 + \partial_2^2 + R(x), \\ P_2 = \partial_1^2 \partial_2^2 + \text{lower order terms} \end{cases}$$

which satisfies  $[P_1, P_2] = 0$  and  ${}^t P_2 = P_2$ . Here we denote  $\partial_1 = \frac{\partial}{\partial x_1}$  and  $\partial_2 = \frac{\partial}{\partial x_2}$  for simplicity and the map  ${}^t$  is the anti-automorphism of the algebra of differential operators such that  ${}^t a(x) = a(x)$  for functions  $a(x)$  and  ${}^t \partial_i = -\partial_i$  for  $i = 1$  and  $2$ . We assume that the coefficients of differential operators are extended to holomorphic functions on a Zariski open subset of an open connected neighborhood of the origin of the complexification  $\mathbb{C}^2$  of  $\mathbb{R}^2$ .

The operators are proved to be expressed by even functions  $u$  and  $v$  of one variable as follows ([OS, Proposition 6.3]):

$$(1.2) \quad \begin{cases} P_1 = \partial_1^2 + \partial_2^2 + u(x_1 + x_2) + u(x_1 - x_2) + v(x_1) + v(x_2), \\ P_2 = \left( \partial_1 \partial_2 + \frac{u(x_1 + x_2) - u(x_1 - x_2)}{2} \right)^2 + v(x_2) \partial_1^2 + v(x_1) \partial_2^2 \\ \quad + v(x_1)v(x_2) + T(x_1, x_2), \end{cases}$$

where  $T$  is determined by the following equations up to a constant.

$$(1.3) \quad \begin{cases} \partial_2 T = \frac{1}{2} v'(x_1) (u(x_1 + x_2) - u(x_1 - x_2)) + v(x_1) (u'(x_1 + x_2) - u'(x_1 - x_2)), \\ \partial_1 T = \frac{1}{2} v'(x_2) (u(x_1 + x_2) - u(x_1 - x_2)) + v(x_2) (u'(x_1 + x_2) + u'(x_1 - x_2)). \end{cases}$$

As the compatibility condition for the existence of the solution  $T$  of the equation (1.3), we have an equation

$$(1.4) \quad \begin{aligned} & \partial_2 \left( v'(x_2) (u(x_1 + x_2) - u(x_1 - x_2)) + 2v(x_2) (u'(x_1 + x_2) + u'(x_1 - x_2)) \right) \\ &= \partial_1 \left( v'(x_1) (u(x_1 + x_2) - u(x_1 - x_2)) + 2v(x_1) (u'(x_1 + x_2) - u'(x_1 - x_2)) \right), \end{aligned}$$

which have been posed in [OS, Proposition 6.3] (cf. [P, §2.2.C]).

Conversely for any solution  $(u, v)$  of (1.4) and the pair  $(P_1, P_2)$  of the operators which are given by (1.2) with

$$(1.5) \quad T = \frac{1}{2} \left( \partial_1^2 - \partial_2^2 \right) \left( V(x_1) (U(x_1 + x_2) + U(x_1 - x_2)) - G(x_1) \right)$$

under the notation in Remark 2.1 and Lemma 2.2, we have  $[P_1, P_2] = 0$ .

**1.3.** We give a complete list of solutions of the functional equation (1.4). Remind that the Schrödinger operator  $P_1$  is explicitly expressed as in (1.2) using  $u$  and  $v$ .

1) (Trivial case)  $u = \text{constant}$ ,  $v = \text{an arbitrary even function}$ ,

1<sup>d</sup>)  $u = \text{an arbitrary even function}$ ,  $v = \text{constant}$ .

Let  $\omega_1$  and  $\omega_2$  denote the primitive half periods of the Weierstrass elliptic function  $\wp(t)$  and put  $\omega_3 = -\omega_1 - \omega_2$  and  $\omega_4 = 0$ .

2) (Elliptic case) For  $\omega_1, \omega_2 < \infty$

$$\begin{cases} u(t) = C_6\wp(t) + C_7, \\ v(t) = \sum_{i=1}^4 C_i\wp(t + \omega_i) + C_5, \end{cases}$$

2<sup>d</sup>)

$$\begin{cases} u(t) = \sum_{i=1}^4 C_i\wp(t + \omega_i) + C_5, \\ v(t) = C_6\wp(2t) + C_7. \end{cases}$$

2)' (Trigonometric case)

$$\begin{cases} u(t) = C_6 \sinh^{-2} \lambda t + C_7, \\ v(t) = C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3 \sinh^2 \lambda t + C_4 \sinh^2 2\lambda t + C_5, \end{cases}$$

2<sup>d</sup>)'

$$\begin{cases} u(t) = C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3 \sinh^2 \lambda t + C_4 \sinh^2 2\lambda t + C_5 \\ v(t) = C_6 \sinh^{-2} 2\lambda t + C_7. \end{cases}$$

2)'' (Rational case)

$$\begin{cases} u(t) = C_6 t^{-2} + C_7, \\ v(t) = C_1 t^{-2} + C_2 + C_3 t^2 + C_4 t^4 + C_5 t^6, \end{cases}$$

2<sup>d</sup>)''

$$\begin{cases} u(t) = C_1 t^{-2} + C_2 + C_3 t^2 + C_4 t^4 + C_5 t^6, \\ v(t) = C_6 t^{-2} + C_7. \end{cases}$$

3) (Elliptic case) For  $\omega_1, \omega_2 < \infty$

$$\begin{cases} u(t) = C_1 \left( \wp\left(\frac{t}{2} + \omega_1\right) + \wp\left(\frac{t}{2} + \omega_2\right) \right) + C_2 \wp(t) + C_3, \\ v(t) = C_4 \wp(t) + C_5 \wp(t + \omega_3) + C_6. \end{cases}$$

3)' (Trigonometric case)

$$\begin{cases} u(t) &= C_1 \sinh^{-2} \frac{\lambda}{2} t + C_2 \sinh^{-2} \lambda t + C_3, \\ v(t) &= C_4 \sinh^{-2} \lambda t + C_5 \sinh^2 \lambda t + C_6, \end{cases}$$

3<sup>d</sup>)'

$$\begin{cases} u(t) &= C_4 \sinh^{-2} \lambda t + C_5 \sinh^2 \lambda t + C_6, \\ v(t) &= C_1 \sinh^{-2} \lambda t + C_2 \sinh^{-2} 2\lambda t + C_3. \end{cases}$$

3)'' (Rational case)

$$\begin{cases} u(t) &= C_1 t^{-2} + C_2 + C_3 t^2, \\ v(t) &= C_4 t^{-2} + C_5 + C_6 t^2. \end{cases}$$

**1.4.** Although we deal with the commuting differential operators of type  $B_2$  with the Weyl group symmetry in the main body of this paper, we will give a brief summary of the related works.

The commuting differential operators of type  $A$  have been studied very well. The commuting differential operators of type  $A$  with the Weyl group invariant condition are classified in [OS]. This work is generalized to the commuting differential operators of type  $A_2$  without Weyl group invariant condition

$$\begin{cases} \Delta_1 = \partial_1 + \partial_2 + \partial_3, \\ \Delta_2 = \partial_1 \partial_2 + \partial_2 \partial_3 + \partial_3 \partial_1 + R(x), \\ \Delta_3 = \partial_1 \partial_2 \partial_3 + \text{lower order terms.} \end{cases}$$

To classify the potential function  $R(x)$ , we may assume that  ${}^t \Delta_3 = -\Delta_3$ . Then there exist one-variable functions  $u_1 = u_1(x_2 - x_3)$ ,  $u_2 = u_2(x_3 - x_1)$  and  $u_3 = u_3(x_1 - x_2)$  such that  $R(x) = -u_1 - u_2 - u_3$ , and

$$(1.6) \quad \begin{vmatrix} 1 & 1 & 1 \\ u_1(x) & u_2(y) & u_3(z) \\ u_1'(x) & u_2'(y) & u_3'(z) \end{vmatrix} = 0 \quad \text{for } x + y + z = 0.$$

For the Weyl group invariant case, we have  $u_1(z) = u_2(z) = u_3(z)$  and the proof of this fact is given in Proposition 4.2 (with  $m = 3$ ) of [OS], which is valid for the general case with no change. For the Weyl group invariant case, the functional differential equation (1.6) is solved in [WW] and the solution is a Weierstrass elliptic function  $\wp$ . The corresponding potential  $R(x)$  is of Calogero-Moser type. For the general case, the equation (1.6) is solved in [BP] and [BB]. Besides the  $\wp$  solutions, we also have solutions expressed by exponential functions. The corresponding potential is known as of type

periodic/non-periodic Toda, which can be regarded as a degenerating limit of a Weyl group invariant potential [vD].

For type  $B_2$ , the classification of the commuting differential operators (1.1) without the Weyl group symmetry has not been done yet. It is known that the similar functional differential equation (see (2.3)) is related to such operators. The following results are obtained in [Oc]:

(i) We have the expression of the (non Weyl group invariant) operators  $P_1$  and  $P_2$  by using four functions  $u_1 = u_1(x_1 + x_2)$ ,  $u_2 = u_2(x_1 - x_2)$ ,  $v_1 = v_1(x_1)$  and  $v_2 = v_2(x_2)$  with one-variable. Actually, if we replace  $u(x_1 + x_2)$  by  $u_1(x_1 + x_2)$ ,  $u(x_1 - x_2)$  by  $u_2(x_1 - x_2)$ , and so on, the formula (1.2) is also valid for non-invariant operators. These functions satisfy the functional differential equation like (1.4).

(ii) Suppose  $P_1$  be non-trivial (cf. Lemma 2.4 i)). If  $P_1$  is holomorphic at some point, then  $P_1$  and  $P_2$  can be meromorphically continued to whole plane  $\mathbb{C}^2$ . The orders of poles of  $P_1$  are at most two.

(iii) Suppose, moreover, that  $v_2(z)$  has poles at three points  $z = z_1, z_2, z_3$  such that  $z_1 - z_2$  and  $z_2 - z_3$  are linearly independent over  $\mathbb{Q}$ . Then the function  $v_2$  can be expressed as

$$v_2(z) = \sum_{i=1}^4 C_i \wp(z + \omega_i) + C_5,$$

with an elliptic function  $\wp$  and constants  $C_1, \dots, C_5$ .

## 2 Functional differential equation for type $B_2$

### 2.1 Functional differential equations

In this section we solve (1.4).

*Remark 2.1.* For even holomorphic functions  $u$  and  $v$  on  $0 < |t| \ll 1$ , there exist unique odd holomorphic functions  $U$  and  $V$  with  $U' = u$  and  $V' = v$  on  $0 < |t| \ll 1$ . Then the equation (1.4) is equivalent to

$$(2.1) \quad \partial_1 \partial_2 (\partial_1^2 - \partial_2^2) \left( V(x_1) (U(x_1 + x_2) + U(x_1 - x_2)) \right. \\ \left. + V(x_2) (U(x_1 + x_2) - U(x_1 - x_2)) \right) = 0.$$

**Lemma 2.2.** *Odd holomorphic functions  $U$  and  $V$  on a small punctured disk satisfy the equation (2.1) if and only if there exist even holomorphic functions*

$F$  and  $G$  on a small punctured disk such that

$$(2.2) \quad \begin{aligned} & V(x_1)(U(x_1 + x_2) + U(x_1 - x_2)) + V(x_2)(U(x_1 + x_2) - U(x_1 - x_2)) \\ & = F(x_1 + x_2) + F(x_1 - x_2) + G(x_1) + G(x_2). \end{aligned}$$

*Proof.* The “if” part is clear. Now we assume (2.1) and set the left hand side of (2.2) to be

$$W(x_1, x_2) \in \mathcal{O}(\{(x_1, x_2) \in \mathbb{C}^2 \mid 0 < |x_1| < \varepsilon/2, 0 < |x_2| < \varepsilon/2, x_1 \neq \pm x_2\}).$$

Then the function  $\partial_2(\partial_1^2 - \partial_2^2)W \in \mathcal{O}(\{(x_1, x_2) \in \mathbb{C}^2 \mid 0 < |x_1| < \varepsilon/2, 0 < |x_2| < \varepsilon/2, x_1 \neq \pm x_2\})$  is locally constant with respect to  $x_1$  and consequently it is constant with respect to  $x_1$ . Then this is an element of  $\mathcal{O}(\{x_2 \in \mathbb{C} \mid 0 < |x_2| < \varepsilon/2\})$ . Moreover, the residue  $\text{Res}_{x_2=0} \partial_2(\partial_1^2 - \partial_2^2)W =$

$\int_{\gamma} \partial_2(\partial_1^2 - \partial_2^2)W(x_1, x_2)dx_2 = 0$ . Hence we have a holomorphic function  $g_2(x_2) \in \mathcal{O}(\{x_2 \in \mathbb{C} \mid 0 < |x_2| < \varepsilon/2\})$  such that  $\partial_2(\partial_1^2 - \partial_2^2)W(x_1, x_2) = \partial_2 g_2$ . Then the difference  $(\partial_1^2 - \partial_2^2)W - g_2$  is locally constant with respect to  $x_2$ . The same argument tells us that there exists a holomorphic function  $g_1 \in \mathcal{O}(\{x_1 \in \mathbb{C} \mid 0 < |x_1| < \varepsilon/2\})$  such that  $(\partial_1^2 - \partial_2^2)W = g_1 + g_2$ .

Next we change the coordinates  $\xi_1 = (x_1 + x_2)/2$ ,  $\xi_2 = (x_1 - x_2)/2$  and write  $\partial'_1 = \frac{\partial}{\partial \xi_1}$ ,  $\partial'_2 = \frac{\partial}{\partial \xi_2}$  for short. Then  $\partial'_1 \partial'_2 W = g_1(\xi_1 + \xi_2) + g_2(\xi_1 - \xi_2)$ . The residue  $\text{Res}_{\xi_1 = -\xi_2} g_1(\xi_1 + \xi_2) = \int_{\gamma} \partial'_1 \partial'_2 W d\xi_1 - \int_{\gamma} g_2(\xi_1 - \xi_2) d\xi_1 = 0$ . Then we have an integral  $g_3(t) \in \mathcal{O}(\{t \in \mathbb{C} \mid 0 < |t| < \varepsilon/2\})$  such that  $g'_3 = g_1$ . Similarly we have  $g_4$  with  $g'_4 = g_2$ , and  $\partial'_1(\partial'_2 W - g_3 - g_4) = 0$ . Then  $g_5 := \partial'_2 W - g_3 - g_4$  is locally constant with respect to  $\xi_1$ , that is,  $g_5$  is constant with respect to  $\xi_1$ . As before  $g_3, g_4$  and  $g_5$  have integrals  $G_3, G_4$  and  $G_5$ , and the difference  $G_6 := W - G_3 - G_4 - G_5$  depends only on  $\xi_1$ .

Taking the averages of  $G_3, G_4, G_5$  and  $G_6$  under the action of the Weyl group  $W(B_2)$ , we get functions  $F$  and  $G$  with required property.  $\square$

This lemma can be generalized to the case when the Weyl group invariance is not imposed. In fact, the functional equation mentioned in Section 1.4 (ii) can be expressed as

$$(2.3) \quad \begin{aligned} & \partial_1 \partial_2 (\partial_1^2 - \partial_2^2) \left( V_1(x_1)(U_1(x_1 + x_2) + U_2(x_1 - x_2)) \right. \\ & \quad \left. + V_2(x_2)(U_1(x_1 + x_2) - U_2(x_1 - x_2)) \right) = 0. \end{aligned}$$

This can be integrated as

$$(2.4) \quad \begin{aligned} & V_1(x_1)(U_1(x_1 + x_2) + U_2(x_1 - x_2)) + V_2(x_2)(U_1(x_1 + x_2) - U_2(x_1 - x_2)) \\ & = F_1(x_1 + x_2) + F_2(x_1 - x_2) + G_1(x_1) + G_2(x_2). \end{aligned}$$

For detail, see [Oc, Proposition 2.4].

*Remark 2.3.* The same argument holds for type  $A_2$ . The equation (1.6) with  $u_1 = u_2 = u_3$  is equivalent to the equation

$$(2.5) \quad \partial_x \partial_y (\partial_x - \partial_y) \left( (U(x) + U(y) + U(-x - y))^2 \right) = 0,$$

where  $U$  is the odd primitive function of  $u$ . By the same argument as in the proof of the previous lemma this is also equivalent to

$$(2.6) \quad ((U(x) + U(y) + U(z))^2 = F(x) + F(y) + F(z) \quad \text{for } x + y + z = 0$$

with some even function  $F$ . Remark that  $u = \wp$  satisfies (1.6) and that  $U = -\zeta$  and  $F = \wp$  satisfy (2.6).

**Lemma 2.4.** i) *If  $u$  or  $v$  is constant, then  $(u, v)$  is a solution of (1.4). A solution of this form is called a trivial solution.*

ii) *If there are functions  $F_1$  and  $G_1$  such that*

$$(2.7) \quad (U(x_1 + x_2) + V(-x_1) + V(-x_2))^2 = F_1(x_1 + x_2) + G_1(x_1) + G_1(x_2),$$

*then  $(U, V)$  is a solution of (2.2).*

iii) *If  $u = v = \wp$ , then  $(u, v)$  is a solution of (1.4).*

*Proof.* For ii),  $(U, V)$  satisfy (2.2) with  $F(t) = \frac{1}{2}(U(t)^2 - F_1(t))$  and  $G(t) = V(t)^2 - G_1(t)$ . iii) follows from ii) and Remark 2.3.  $\square$

We summarize several elementary properties of the equation (1.4).

**Lemma 2.5.** i) *The equation (1.4) is bilinear with respect to  $(u, v)$ .*

ii) *For a solution  $(u_0(t), v_0(t))$  of (1.4) and a non-zero constant  $C$ ,  $(u(t), v(t)) = (u_0(Ct), v_0(Ct))$  is also a solution.*

iii) *For a solution  $(u_0(t), v_0(t))$  of (1.4),  $(u(t), v(t)) = (v_0(t), u_0(2t))$  is also a solution.*

iv) *For a solution  $(u_0(t), v_0(t))$  of (1.4) with  $u_0(t + 2\omega) = u_0(t)$  satisfying some constant  $\omega$ ,  $(u(t), v(t)) = (u_0(t), v_0(t + \omega))$  is also a solution.*

*Proof.* All but iv) are shown in [OS, Proposition 6.3 iv)]. iv) follows from  $u(x_1 - x_2) = u((x_1 + \omega) - (x_2 + \omega))$  and  $u(x_1 + x_2) = u((x_1 + \omega) + (x_2 + \omega))$ .  $\square$

*Remark 2.6.* The equations (2.1) and (2.5) above are written in a uniform manner. Let the root system  $(E, \Sigma)$  be  $(\mathbb{R}^2, \Sigma(A_2))$  or  $(\mathbb{R}^2, \Sigma(B_2))$  with the Weyl group  $W$ . Consider an element  $\mathbb{V}$  of the space of  $W$ -invariants  $(\mathcal{O}(E) \otimes E^*)^W$  in  $\mathcal{O}(E) \otimes E^*$ . Extend the natural invariant inner bilinear form

$\langle \cdot, \cdot \rangle$  on  $E^*$  to a  $\mathcal{O}(E)$ -linear form on this space of  $W$ -invariants. Consider the differential equations

$$(2.8) \quad \left\{ \begin{array}{l} (\prod_{\alpha \in \Sigma^+} \partial_\alpha) \mathbb{V} = 0, \\ (\prod_{\alpha \in \Sigma^+} \partial_\alpha) \langle \mathbb{V}, \mathbb{V} \rangle = 0. \end{array} \right.$$

Here differential operators act on the first factor of  $\mathcal{O}(E) \otimes E^*$ .

This is equivalent to the equations (2.1) or (2.5). In fact, if we set

$$(2.9) \quad \mathbb{V} = \sum_{\alpha \in \Sigma^+} V_\alpha(\langle \alpha, \cdot \rangle) \otimes \alpha = \frac{1}{2} \sum_{\alpha \in \Sigma} V_\alpha(\langle \alpha, \cdot \rangle) \otimes \alpha$$

with  $V_\alpha$  corresponding to the solutions (2.1) or (2.5), it satisfies the equation (2.8). On the other hand, any solution of the former equation of (2.8) is written in the form (2.9) with odd functions  $V_\alpha$ , and the  $W$ -invariance and the latter equation of (2.8) are sufficient for the equations (2.1) or (2.5).

## 2.2 Elliptic functions

We summarize several well-known properties of the elliptic functions  $\wp$  and  $\zeta$  of Weierstrass type for latter convenience (cf. [WW]).

They are given by

$$(2.10) \quad \wp(z) = \wp(z|2\omega_1, 2\omega_2) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left( \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right),$$

$$(2.11) \quad \zeta(z) = \zeta(z|2\omega_1, 2\omega_2) = \frac{1}{z} + \sum_{\omega \neq 0} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right),$$

where the sum ranges over all non-zero periods  $2m_1\omega_1 + 2m_2\omega_2$  of  $\wp$ . They satisfy

$$\begin{aligned} \zeta'(z) &= -\wp(z), \\ \wp(z + 2m_1\omega_1 + 2m_2\omega_2|2\omega_1, 2\omega_2) &= \wp(z|2\omega_1, 2\omega_2), \\ \zeta(z + 2m_1\omega_1 + 2m_2\omega_2|2\omega_1, 2\omega_2) &= \zeta(z|2\omega_1, 2\omega_2) + 2m_1\eta_1 + 2m_2\eta_2 \\ &\quad \text{for } m_1, m_2 \in \mathbb{Z}, \\ (\wp')^2 &= 4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3). \end{aligned}$$



Here the constants have the relations

$$\begin{aligned}
g_2 &= 60 \sum_{\omega \neq 0} \omega^{-4}, & g_3 &= 140 \sum_{\omega \neq 0} \omega^{-6}, \\
\omega_3 &= -\omega_1 - \omega_2, & e_j &= \wp(\omega_j), & \eta_j &= \zeta(\omega_j), \\
e_1 + e_2 + e_3 &= 0, & g_2 &= -4(e_1e_2 + e_2e_3 + e_3e_1), & g_3 &= 4e_1e_2e_3, \\
\eta_1 + \eta_2 + \eta_3 &= 0, & \eta_2\omega_1 - \eta_1\omega_2 &= \pm \frac{\pi\sqrt{-1}}{2}.
\end{aligned}$$

The following are variants of addition formulas.

$$(2.12) \quad (\zeta(x) + \zeta(y) + \zeta(z))^2 = \wp(x) + \wp(y) + \wp(z) \quad \text{when } x + y + z = 0,$$

$$(2.13) \quad \zeta(x + y) - \zeta(x) - \zeta(y) = \frac{1}{2} \frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)}.$$

The Laurent expansion at the origin is

$$(2.14) \quad \wp(z|2\omega_1, 2\omega_2) = z^{-2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \dots$$

The complex numbers  $\omega_1$  and  $\omega_2$  are assumed to be linearly independent over  $\mathbb{R}$  but we allow the period to be infinity. In other words, the numbers  $g_2$  and  $g_3$  are any complex numbers. For example we have

$$(2.15) \quad \begin{aligned} \wp(z|\sqrt{-1}\pi, \infty) &= \sinh^{-2} z + \frac{1}{3} & \text{when } g_2 &= \frac{4}{3} \text{ and } g_3 = -\frac{8}{27}, \\ \wp(z|\infty, \infty) &= z^{-2} & \text{when } g_2 &= g_3 = 0. \end{aligned}$$

If  $\omega_1$  and  $\omega_2$  are finite, we have a formula

$$(2.16) \quad \wp(z + \omega_\nu|2\omega_1, 2\omega_2) = e_\lambda + \frac{(e_\nu - e_\lambda)(e_\nu - e_\mu)}{\wp(z|2\omega_1, 2\omega_2) - e_\nu}$$

with  $\{\nu, \mu, \lambda\} = \{1, 2, 3\}$

and every function of the form  $\wp'^{-2} \times$  (a polynomial of  $\wp$  of degree at most 4) is written by a linear combination of 1,  $\wp$ ,  $(\wp - e_1)^{-1}$ ,  $(\wp - e_2)^{-1}$  and  $(\wp - e_3)^{-1}$ , equivalently by a linear combination of 1,  $\wp(z)$ ,  $\wp(z + \omega_1)$ ,  $\wp(z + \omega_2)$  and  $\wp(z + \omega_3)$ .

Lastly we quote the Landen transformation

$$(2.17) \quad \wp(z|\omega_1, 2\omega_2) = \wp(z|2\omega_1, 2\omega_2) + \wp(z + \omega_1|2\omega_1, 2\omega_2) - e_1 \quad \text{if } \omega_1 \text{ is finite.}$$

## 2.3 Solutions of the functional equation

**Theorem 2.7.** *The functions*

(2.18)

$$\begin{aligned} u(t) &= c_6 \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} + c_7 \wp(t) + c_8, \\ v(t) &= \frac{(\wp(t) - e_1)(\wp(t) - e_2)(c_1 \wp(t)^2 + c_2 \wp(t) + c_3) + c_4 \wp(t) + c_5}{\wp'(t)^2} \end{aligned}$$

satisfy the equation (1.4) if  $c_4 c_6 = c_5 c_6 = 0$ .

*Proof.* Since the equation is bilinear, we may check for each *monomial* in  $u$  or  $v$ . Here we will give a proof for  $\omega_1, \omega_2 < \infty$ , which implies the theorem by the analytic continuation.

i) Case  $c_6 = c_7 = 0$ : It follows from Lemma 2.4 i).

ii) Case  $c_6 = c_8 = 0, c_7 = 1$ : We may assume that  $v = \wp(t + a)$  with  $a = 0, \omega_1, \omega_2$  or  $\omega_3$ . Moreover we may assume  $a = 0$  by Lemma 2.5 iv), that is,  $u = v = \wp$ . Then (2.2) follows from Lemma 2.4 ii) and Remark 2.3. This simplifies the proof of [OS, Proposition 7.3 ii)].

iii) Case  $c_7 = c_8 = 0, c_6 = 1$ : By §2.2 the function

$$\begin{aligned} v(t) &= \wp'(t)^{-2} (\wp(t) - e_1)(\wp(t) - e_2)(c_1 \wp(t)^2 + c_2 \wp(t) + c_3) \\ &= \frac{c_1 \wp(t)^2 + c_2 \wp(t) + c_3}{4(\wp(t) - e_3)} \end{aligned}$$

is a linear combination of 1,  $\wp(t)$  and  $\wp(t + \omega_3)$ . Since

$$\begin{aligned} \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} &= \frac{1}{4} \left( \frac{e_1 - e_3}{e_1 - e_2} \frac{1}{\wp(\frac{t}{2}) - e_1} + \frac{e_2 - e_3}{e_2 - e_1} \frac{1}{\wp(\frac{t}{2}) - e_2} \right) \\ (2.19) \quad &= \frac{1}{4(e_1 - e_2)^2} \left( \wp(\frac{t}{2} + \omega_1) - e_1 + \wp(\frac{t}{2} + \omega_2) - e_2 \right) \\ &= \frac{1}{4(e_1 - e_2)^2} \left( \wp(\frac{t}{2} + \omega_1 | 2\omega_1, \omega_3) + 2e_3 \right) \\ &= \frac{1}{(e_1 - e_2)^2} \left( \wp(t + 2\omega_1 | 4\omega_1, 2\omega_3) + \frac{e_3}{2} \right) \end{aligned}$$

has a period  $2\omega_3$ , we may assume  $v(t) = \wp(t)$  by Lemma 2.5 iv). By Lemma 2.5 iii) we can reduce to the case  $u(t) = \wp(t)$  and  $v(t) = \frac{1}{4(e_1 - e_2)^2} (\wp(t + \omega_1) + \wp(t + \omega_2) - e_1 - e_2)$ , which has already treated in ii).  $\square$

*Remark 2.8.* i) The solutions in §1.3 corresponds to (2.18) with the following conditions:

$$\begin{array}{ll}
2) & c_6 = 0, \quad e_1 \neq e_2 \neq e_3 \neq e_1, \\
2') & c_6 = 0, \quad e_1 = -\frac{2}{3}\lambda^2 \neq 0, \quad e_2 = e_3 = \frac{1}{3}\lambda^2, \\
2'') & c_6 = 0, \quad e_1 = e_2 = e_3 = 0, \\
3) & c_4 = c_5 = 0, \quad e_1 \neq e_2 \neq e_3 \neq e_1, \\
3') & c_4 = c_5 = 0, \quad e_1 = -\frac{2}{3}\lambda^2 \neq 0, \quad e_2 = e_3 = \frac{1}{3}\lambda^2, \\
3')^d & c_4 = c_5 = 0, \quad e_1 = e_2 = \frac{1}{3}\lambda^2 \neq 0, \quad e_3 = -\frac{2}{3}\lambda^2, \\
3'') & c_4 = c_5 = 0, \quad e_1 = e_2 = e_3 = 0.
\end{array}$$

ii) The family of solutions with  $c_4 = c_5 = 0$  are written in a more symmetric form under the symmetry in Lemma 2.5 iii). By the proof of Theorem 2.7 iii) we can write

$$\begin{aligned}
(2.20) \quad u(t) &= a_1\wp(t|4\omega_1, 2\omega_3) + a_2\wp(t|2\omega_1, 2\omega_3) + a_3, \\
v(t) &= b_1\wp(t|2\omega_1, 2\omega_3) + b_2\wp(t|2\omega_1, \omega_3) + b_3.
\end{aligned}$$

Then the solution  $(\bar{u}(t), \bar{v}(t)) = (v(t), u(2t))$  can be expressed in the same form as (2.20) by replacing  $2\bar{\omega}_1 = \omega_3$ ,  $2\bar{\omega}_3 = 2\omega_1$ ,  $\bar{a}_1 = b_1$ ,  $\bar{a}_2 = b_2$ ,  $\bar{a}_3 = b_3$ ,  $\bar{b}_1 = a_1/4$ ,  $\bar{b}_2 = a_2/4$  and  $\bar{b}_3 = a_3$ .

## 2.4 The main theorem

In this subsection we shall solve the functional differential equation (1.4) by the aid of a computer with the algebraic programming system REDUCE Ver3.4. The following is the main result in §2, which is proved at the end of § 2.5.4:

**Theorem 2.9.** *Any solution  $(u(t), v(t))$  of the equation (1.4) such that  $u(t)$  and  $v(t)$  are real analytic on  $\{t \in \mathbb{R} \mid 0 < |t| \ll 1\}$  is one of the following form.*

- i) *Functions  $(u(t), v(t))$  is of the form in Theorem 2.7 with  $c_4c_6 = c_5c_6 = 0$ .*
- ii) *Functions  $(v(t), u(2t))$  is of the form in Theorem 2.7 with  $c_4c_6 = c_5c_6 = 0$ .*
- iii) *Either  $u$  or  $v$  is constant.*
- iv)  *$u' = 0$  and  $v''$  is constant.*
- v)  *$v' = 0$  and  $u''$  is constant.*

*Here we note that if  $u(t)$  and  $v(t)$  are even or they are holomorphic on  $\{t \in \mathbb{C} \mid 0 < |t| \ll 1\}$ , then iv) and v) are reduced to iii).*

**2.4.1** The following lemma is a generalization of [OS, Lemma 7.1 i)].

**Lemma 2.10.** *Let  $u(t)$  and  $v(t)$  be real analytic functions on  $\{t \in \mathbb{R} \mid 0 < |t| \ll 1\}$  which satisfy (1.4). Suppose  $u' \neq 0$  and  $v' \neq 0$ . Then  $u(t)$  and  $v(t)$  can be extended to even meromorphic functions on  $\{t \in \mathbb{C} \mid 0 < |t| \ll 1\}$  with poles of order at most 2 at the origin.*

*Proof.* We may assume  $v'|_{t>0} \neq 0$  by replacing the following  $x$  by  $-x$  if necessary. Fix  $x$  with  $0 < x \ll 1$  and consider the Laurent expansion for  $0 < |y| \ll x$

$$(2.21) \quad u(x+y) - u(x-y) = 2 \left( \frac{u^{(1)}(x)}{1!} y + \frac{u^{(3)}(x)}{3!} y^3 + \cdots \right).$$

Then we have

$$(2.22) \quad \begin{aligned} & \frac{\partial}{\partial x} \left( v'(x) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+1)!} y^{2k+1} + 2v(x) \sum_{k=0}^{\infty} \frac{u^{(2k+2)}(x)}{(2k+1)!} y^{2k+1} \right) \\ & - \frac{\partial}{\partial y} \left( v'(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+1)!} y^{2k+1} + 2v(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k)!} y^{2k} \right) = 0 \end{aligned}$$

and for  $0 < |y| \ll x$

$$(2.23) \quad f(x, y) = y(u'(x) + yc_2(x, y))v''(y) + 3(u'(x) + yc_1(x, y))v'(y) + c_0(x, y)v(y)$$

with a suitable holomorphic functions  $f(x, y)$ ,  $c_0(x, y)$ ,  $c_1(x, y)$  and  $c_2(x, y)$  of  $y$  defined on a neighborhood of the origin. Since this equation for  $v(y)$  has regular singularities at the origin with the characteristic exponents 0 and  $-2$ ,

$$(2.24) \quad v(t) = a_{-1}t^{-2} + v_0(t) + v_1(t) \log t \quad \text{for } 0 < t \ll 1.$$

Here  $v_0(t)$  and  $v_1(t)$  are holomorphic function defined in a neighborhood of the origin and moreover  $v_1(0) = 0$  means  $v_1 = 0$ .

By the analytic continuation of (2.22) for the variable  $y$  around the origin we have

$$(2.25) \quad \frac{\partial}{\partial y} \left( v_1'(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+1)!} y^{2k+1} + 2v_1(y) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k)!} y^{2k} \right) = 0.$$

The coefficients of the term  $y^1$  in this equation mean

$$(2.26) \quad 2v_1(0)u^{(3)}(x) + 4v_1''(0)u'(x) = 0.$$

Suppose  $v_1 \neq 0$ . Let  $\lambda$  be a complex number with  $\lambda^2 = -2v_1''(0)/v_1(0)$ .

$$(2.27) \quad u^{(3)}(x) = \lambda^2 u'(x).$$

Then (2.25) is

$$(2.28) \quad \frac{\partial}{\partial y} \left( v_1'(y) u'(x) \frac{\sinh \lambda y}{\lambda} + 2v_1(y) u'(x) \cosh \lambda y \right) = 0.$$

Since  $u'(x_0) \neq 0$ , we have

$$\begin{aligned} \frac{\partial}{\partial y} \left( v_1(y) \left( \frac{\sinh \lambda y}{\lambda} \right)^2 \right) &= 0, \\ v_1(y) \left( \frac{\sinh \lambda y}{\lambda} \right)^2 &= v_1(y) \left( \frac{\sinh \lambda y}{\lambda} \right)^2 \Big|_{y=0} = 0, \end{aligned}$$

then  $v_1 = 0$ , which contradicts to the assumption  $v_1 \neq 0$ .

Thus we have proved that  $v_1 = 0$ . By (2.24) we can put

$$v(t) = a_{-1} t^{-2} + \sum_{j=0}^{\infty} (a_j t^{2j} + c_j t^{2j+1})$$

with suitable  $a_j, c_j \in \mathbb{C}$  on  $0 < t \ll 1$ . Suppose there exist  $c_k$  satisfying  $c_k \neq 0$  and  $c_j = 0$  for  $j = 0, \dots, k-1$ . Then the coefficients of  $y^{2k}$  in (2.22) shows

$$-((2k+1)^2 + 2(2k+1))c_k u^{(1)}(x) = 0,$$

which contradicts to the assumption  $c_k \neq 0$  and hence  $v(t) = a_{-1} t^{-2} + \sum_{j=0}^{\infty} a_j t^{2j}$  on  $0 < t \ll 1$ . Here we note that  $v'' \neq 0$  and that  $u'' \neq 0$  by the symmetry of  $u$  and  $v$ .

Substituting  $(x_1, x_2)$  in (1.4) by  $(x, y)$  and  $(x, -y)$ , respectively, and summing up the resulting equations, we have

$$\begin{aligned} \frac{\partial}{\partial y} \left( (v'(y) + v'(-y))(u(x+y) - u(x-y)) \right. \\ \left. + 2(v(y) - v(-y))(u'(x+y) + u'(x-y)) \right) = 0 \end{aligned}$$

and hence

$$\frac{\partial^2}{\partial y^2} \left( (v(y) - v(-y))(u(x+y) - u(x-y))^2 \right) = 0.$$

Thus we have  $v(-y) = v(y)$  because  $u'' \neq 0$ .

By the symmetry of  $u(t)$  and  $v(t)$  we have the lemma.  $\square$

First suppose that  $u(t)$  and  $v(t)$  are real analytic functions on  $\{t \in \mathbb{R} \mid 0 < |t| \ll 1\}$ . It is clear that  $(u, v)$  given by iii) or iv) or v) in Theorem 2.9 satisfies (1.4). Assume  $u' = 0$ . Then there exist  $C_1, C_2 \in \mathbb{C}$  such that  $u(t) = C_1$  and  $u(-t) = C_2$  for  $0 < t \ll 1$ . Suppose  $(u, v)$  satisfies (1.4) and suppose  $C_1 \neq C_2$  and let  $0 < x < y \ll 1$ . Substituting  $(x_1, x_2)$  in (1.4) by  $(x, y)$ ,  $(-x, -y)$  and  $(-x, y)$ , we have  $v''(y) = v''(x)$ ,  $v''(-y) = v''(-x)$  and  $v''(y) = v''(-x)$ , respectively, and therefore  $v''$  is constant. In the same way, if  $v' = 0$  and  $(u, v)$  satisfies (1.4), then  $v$  is constant or  $u''$  is constant.

Then owing to Lemma 2.10 we assume  $u(t)$  and  $v(t)$  are holomorphic on  $\{t \in \mathbb{C} \mid 0 < |t| \ll 1\}$  and satisfy (1.4) to the end of this section. By Lemma 2.10, the Laurent expansion at the origin can be assumed as follows.

$$(2.29) \quad u(t) = a_{-1}t^{-2} + \sum_{j=1}^{\infty} a_j t^{2j}, \quad v(t) = b_{-1}t^{-2} + \sum_{j=1}^{\infty} b_j t^{2j}.$$

Suppose  $0 < |y| \ll |x| \ll 1$ . It follows from (2.21) that

$$(2.30) \quad \begin{aligned} \frac{\partial^2}{\partial x \partial y} \left( v'(x) \sum_{k=0}^{\infty} \frac{u^{(2k+1)}(x)}{(2k+2)!} y^{2k+2} + 2v(x) \sum_{k=0}^{\infty} \frac{u^{(2k+2)}(x)}{(2k+2)!} y^{2k+2} \right. \\ \left. - \left( \sum_{j=-1}^{\infty} 2j b_j y^{2j-1} \right) \sum_{k=0}^{\infty} \frac{u^{(2k)}(x)}{(2k+1)!} y^{2k+1} \right. \\ \left. - \left( \sum_{j=-1}^{\infty} 2b_j y^{2j} \right) \sum_{k=0}^{\infty} \frac{u^{(2k)}(x)}{(2k)!} y^{2k} \right) = 0. \end{aligned}$$

Since the coefficient of the term  $b_j u^{(2m-2j)}(x) y^{2m}$  inside the above  $\left( \quad \right)$  equals

$$-\frac{2j}{(2m-2j+1)!} - \frac{2}{(2m-2j)!} = -2 \frac{2m-j+1}{(2m-2j+1)!},$$

for any positive integer  $m$ , we obtain

$$(2.31) \quad u^{(2m-1)}(x)v'(x) + 2u^{(2m)}(x)v(x) - \sum_{j=-1}^m \frac{2(2m)!(2m-j+1)}{(2m-2j+1)!} b_j u^{(2m-2j)}(x) = C_m$$

with suitable constant numbers  $C_m$ .

Let  $X(m, k)$  denote the the coefficients of  $x^{2k}$  in the left hand side of (2.22). Then the condition  $X(m, k) = 0$  for all  $m \geq 1$  and  $k \geq 1$  is equivalent to (2.22), and so is to (1.4).

For example, we have the following, all of which will be used in the proof of Theorem 2.9.

$$\begin{aligned}
X(1, 1) &= 0, \\
X(1, 2) &= 4(3a_1b_2 + 6a_2b_1 - 32a_4b_{-1} - a_{-1}b_4), \\
X(1, 3) &= 8(2a_1b_3 + 5a_2b_2 + 8a_3b_1 - 64a_5b_{-1} - a_{-1}b_5), \\
X(1, 4) &= 4(5a_1b_4 + 12a_2b_3 + 21a_3b_2 + 30a_4b_1 - 336a_6b_{-1} - 3a_{-1}b_6), \\
X(1, 5) &= \frac{8}{5}(15a_1b_5 + 35a_2b_4 + 60a_3b_3 + 90a_4b_2 + 120a_5b_1 \\
&\quad - 1792a_7b_{-1} - 10a_{-1}b_7), \\
X(1, 6) &= 4(7a_1b_6 + 16a_2b_5 + 27a_3b_4 + 40a_4b_3 + 55a_5b_2 + 70a_6b_1 \\
&\quad - 1344a_8b_{-1} - 5a_{-1}b_8), \\
X(1, 7) &= 8(4a_1b_7 + 9a_2b_6 + 15a_3b_5 + 22a_4b_4 + 30a_5b_3 + 39a_6b_2 + 48a_7b_1 \\
&\quad - 1152a_9b_{-1} - 3a_{-1}b_9), \\
X(1, 8) &= 4(9a_1b_8 + 20a_2b_7 + 33a_3b_6 + 48a_4b_5 + 65a_5b_4 + 84a_6b_3 + 105a_7b_2 \\
&\quad + 126a_8b_1 - 3696a_{10}b_{-1} - 7a_{-1}b_{10}), \\
X(1, 9) &= 8(5a_1b_9 + 11a_2b_8 + 18a_3b_7 + 26a_4b_6 + 35a_5b_5 + 45a_6b_4 + 56a_7b_3, \\
&\quad + 68a_8b_2 + 80a_9b_1 - 2816a_{11}b_{-1} - a_{-1}4b_{11}), \\
X(2, 1) &= 48(-3a_1b_2 - 6a_2b_1 + 32a_4b_{-1} + a_{-1}b_4), \\
X(2, 2) &= 0, \\
X(2, 3) &= 16(12a_2b_3 + 66a_3b_2 + 140a_4b_1 - 1056a_6b_{-1} - 3a_{-1}b_6), \\
X(2, 4) &= 48(5a_2b_4 + 30a_3b_3 + 95a_4b_2 + 180a_5b_1 - 1664a_7b_{-1} - 2a_{-1}b_7), \\
X(2, 5) &= 48(6a_2b_5 + 35a_3b_4 + 112a_4b_3 + 267a_5b_2 + 462a_6b_1 \\
&\quad - 5184a_8b_{-1} - 3a_{-1}b_8), \\
X(2, 6) &= 16(21a_2b_6 + 120a_3b_5 + 378a_4b_4 + 900a_5b_3 + 1806a_6b_2 + 2912a_7b_1 \\
&\quad - 39168a_9b_{-1} - 12a_{-1}b_9), \\
X(2, 7) &= \frac{48}{7}(56a_2b_7 + 315a_3b_6 + 980a_4b_5 + 2310a_5b_4 + 4620a_6b_3 \\
&\quad + 8260a_7b_2 + 12600a_8b_1 - 200640a_{10}b_{-1} - 35a_{-1}b_{10}), \\
X(2, 8) &= 48(+9a_2b_8 + 50a_3b_7 + 154a_4b_6 + 360a_5b_5 + 715a_6b_4 + 1274a_7b_3 \\
&\quad + 2097a_8b_2 + 3060a_9b_1 - 57024a_{11}b_{-1} - 6a_{-1}b_{11}), \\
X(3, 1) &= 2880(-2a_1b_3 - 5a_2b_2 - 8a_3b_1 + 64a_5b_{-1} + a_{-1}b_5), \\
X(3, 2) &= 480(-12a_2b_3 - 66a_3b_2 - 140a_4b_1 + 1056a_6b_{-1} + 3a_{-1}b_6), \\
X(3, 3) &= 0, \\
X(3, 4) &= 1440(5a_3b_4 + 52a_4b_3 + 219a_5b_2 + 462a_6b_1 - 4160a_8b_{-1} - a_{-1}b_8), \\
X(3, 5) &= 192(45a_3b_5 + 490a_4b_4 + 2490a_5b_3 + 8085a_6b_2 + 16016a_7b_1 \\
&\quad - 163200a_9b_{-1} - 15a_{-1}b_9),
\end{aligned}$$

$$\begin{aligned}
X(3, 6) &= 480(21a_3b_6 + 224a_4b_5 + 1134a_5b_4 + 3948a_6b_3 + 10556a_7b_2 \\
&\quad + 19656a_8b_1 - 227392a_{10}b_{-1} - 9a_{-1}b_{10}), \\
X(3, 7) &= 5760(2a_3b_7 + 21a_4b_6 + 105a_5b_5 + 363a_6b_4 + 1000a_7b_3 \\
&\quad + 2316a_8b_2 + 4080a_9b_1 - 53504a_{11}b_{-1} - a_{-1}b_{11}), \\
X(4, 1) &= 80640(-5a_1b_4 - 12a_2b_3 - 21a_3b_2 - 30a_4b_1 + 336a_6b_{-1} + 3a_{-1}b_6), \\
X(4, 2) &= 80640(-5a_2b_4 - 30a_3b_3 - 95a_4b_2 - 180a_5b_1 + 1664a_7b_{-1} + a_{-1}2b_7), \\
X(4, 3) &= 80640(-5a_3b_4 - 52a_4b_3 - 219a_5b_2 - 462a_6b_1 \\
&\quad + 4160a_8b_{-1} + a_{-1}b_8), \\
X(4, 4) &= 0, \\
X(4, 5) &= 16128(30a_4b_5 + 500a_5b_4 + 3300a_6b_3 + 12298a_7b_2 + 25740a_8b_1 \\
&\quad - 258400a_{10}b_{-1} - 5a_{-1}b_{10}), \\
X(4, 6) &= 80640(7a_4b_6 + 120a_5b_5 + 886a_6b_4 + 4108a_7b_3 + 13182a_8b_2 \\
&\quad + 26520a_9b_1 - 289408a_{11}b_{-1} - 2a_{-1}b_{11}),
\end{aligned}$$

We borrow the following notation from REDUCE. For a polynomial function  $p$ , we denote by  $\text{coeffn}(p, x, k)$  the coefficient of the term  $x^k$  of  $p$  with respect to one specific variable  $x$ . For example,  $\text{coeffn}(x^2 + 2xy + 3x + y^2, x, 1) = 2y + 3$ .

**2.4.2.** Now we shall prove Theorem 2.9 dividing into the cases classified by the order of zeros of  $(u(t), v(t))$ . Owing to the symmetry between  $u$  and  $v$ , [OS, Lemma 7.1 ii)] shows that we may assume the pair of orders of the zeros equal  $(-2, 6)$ ,  $(-2, 4)$ ,  $(2, 2)$ ,  $(-2, 2)$  or  $(-2, -2)$ .

### 2.4.3 Type $(-2, 6)$

We may assume  $a_{-1} = b_3 = 1$  and  $b_{-1} = b_1 = b_2 = 0$ . For  $k \geq 5$  we have

$$\begin{aligned}
&\begin{pmatrix} \text{coeffn}(X(1, k-2), a_{k-4}, 1) & \text{coeffn}(X(1, k-2), b_k, 1) \\ \text{coeffn}(X(2, k-3), a_{k-4}, 1) & \text{coeffn}(X(2, k-3), b_k, 1) \end{pmatrix} \\
&= \begin{pmatrix} (2k-8)(2k-6) & (-2)(2k-6) \\ (2k-8)(2k-9)(2k-8)(2k-10) & (-2)(-3)(-4)(2k-10) \end{pmatrix}.
\end{aligned}$$

The determinant of this matrix equals

$$4(2k-5)(2k-6)(2k-8)(2k-10)(2k-12).$$

Hence if  $k \geq 7$ , the equations  $X(1, k-2) = X(2, k-3) = 0$  assure that  $a_{k-4}$  and  $b_k$  are expressed suitable linear combinations of  $a_{k-j-1}b_j$  with  $j =$



$4, \dots, k-2$ , which proves that  $a_{k-4}$  and  $b_k$  with  $k \geq 7$  are expressed by polynomial functions of  $(a_1, a_2, b_1, b_4, b_5, b_6)$  by the induction on  $k$ .

Now we note that  $X(1, 2) = 0$  implies  $b_4 = 0$ . Moreover it follows from  $X(1, 3) = X(1, 4) = 0$  that  $b_5$  and  $b_6$  are expressed by polynomial functions of  $(a_1, a_2, b_4)$ . Hence we have proved that all the coefficients  $a_j$  and  $b_j$  are uniquely expressed by polynomial functions of  $(a_1, a_2)$ . In particular for any given  $(a_1, a_2) \in \mathbb{C}^2$  the solution is unique if it exists.

On the other hand we have the solution

$$\begin{aligned} u(t) &= \wp(t) = t^{-2} + \frac{g_2}{20}t^2 + \frac{g_3}{28}t^4 + \dots, \\ v(t) &= \frac{4}{\wp'(t)^2} = t^6 + \frac{g_2}{10}t^{10} + \dots. \end{aligned}$$

Hence the coefficients  $a_j$  and  $b_j$  which are uniquely determined by  $(a_1, a_2)$  equal the coefficients of the above  $u(t)$  and  $v(t)$  with  $g_2 = 20a_1$  and  $g_3 = 28a_2$ .

#### 2.4.4 Type (-2,4)

We may assume  $a_{-1} = b_2 = 1$  and  $b_{-1} = b_1 = 0$ . Then for  $k \geq 4$

$$\begin{aligned} &\begin{pmatrix} \text{coeffn}(X(1, k-2), a_{k-3}, 1) & \text{coeffn}(X(1, k-2), b_k, 1) \\ \text{coeffn}(X(2, k-3), a_{k-3}, 1) & \text{coeffn}(X(2, k-3), b_k, 1) \end{pmatrix} \\ &= \begin{pmatrix} 2(2k-6)(2k-5) & (-2)(2k-6) \\ 2(2k-4)(2k-10)(4k^2-28k+57) & (-2)(-3)(-4)(2k-10) \end{pmatrix}. \end{aligned}$$

Since the determinant of this matrix is

$$4(2k-3)(2k-6)(2k-7)(2k-8)(2k-10),$$

$a_{k-3}$  and  $b_k$  for  $k \geq 6$  are uniquely determined by  $(a_1, a_2, b_3, b_4, b_5)$ . Moreover  $X(1, 2) = X(1, 3) = 0$  imply that  $b_4$  and  $b_5$  are uniquely determined by  $(a_1, a_2, b_3)$ . On the other hand, we have the solution

$$\begin{aligned} u(t) &= \wp(t) = t^{-2} + \frac{g_2}{20}t^2 + \frac{g_3}{28}t^4 + \dots, \\ v(t) &= \frac{4(\wp(t) + C_5)}{\wp'(t)^2} = t^4 + C_5t^6 + \dots \end{aligned}$$

with parameters  $g_2, g_3$  and  $C_5$ . Thus the coefficients  $a_k$  and  $b_k$  uniquely determined by  $(a_1, a_2, b_3)$  corresponds to this solution with  $g_2 = 20a_1, g_3 = 28a_2$  and  $C_5 = b_3$ .

### 2.4.5 Type (2,2)

We may assume  $a_1 = b_1 = 1$  and  $a_{-1} = b_{-1} = 0$ . For  $k \geq 4$

$$\begin{pmatrix} \text{coeffn}(X(1, k-2), a_{k-2}, 1) & \text{coeffn}(X(1, k-2), b_{k-2}, 1) \\ \text{coeffn}(X(2, k-3), a_{k-2}, 1) & \text{coeffn}(X(2, k-3), b_{k-2}, 1) \end{pmatrix} \\ = \begin{pmatrix} 2(2k-2)(2k-6) & 2(2k-2) \\ 2(2k-2)(2k-4)(2k-5)(2k-10) & 0 \end{pmatrix}$$

and the determinant of this matrix equals

$$-4(2k-2)^2(2k-4)(2k-5)(2k-10).$$

Hence  $a_{k-2}$  and  $b_{k-2}$  for  $k \geq 6$  are uniquely determined by  $(a_2, a_3, b_2, b_3)$ .

Moreover since  $X(1, 2) = X(1, 3) = 0$ , for any given  $(a_2, a_3)$  the solution is unique if it exists and therefore it corresponds to the solution

$$u(t) = \frac{16(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} = t^2 - \frac{e_3}{2}t^4 + \frac{1}{16}\left(\frac{g_2}{5} + e_3^2\right)t^6 + \dots, \\ v(t) = \frac{1}{\wp(t) - e_3} = t^2 + \dots.$$

with  $e_3 = -2a_2$  and  $g_2 = 5(16a_3 - e_3^2)$ .

### 2.4.6 Type (-2,2)

We may assume  $a_{-1} = b_1 = 1$  and  $b_{-1} = 0$ . For  $k \geq 4$ ,

$$\begin{pmatrix} \text{coeffn}(X(1, k-2), a_{k-2}, 1) & \text{coeffn}(X(1, k-2), b_k, 1) \\ \text{coeffn}(X(2, k-3), a_{k-2}, 1) & \text{coeffn}(X(2, k-3), b_k, 1) \end{pmatrix} \\ = \begin{pmatrix} 2(2k-2)(2k-6) & (-2)(2k-6) \\ 2(2k-2)(2k-4)(2k-5)(2k-10) & (-2)(-3)(-4)(2k-10) \end{pmatrix}$$

and the determinant of this matrix equals

$$4(2k-1)(2k-2)(2k-6)(2k-8)(2k-10).$$

Hence  $a_{k-2}$  and  $b_k$  for  $k \geq 6$  are uniquely determined by  $(a_1, a_2, a_3, a_4, a_5, b_2, b_3)$ .

Owing to this with  $X(1, 2) = X(1, 3) = 0$ , for any given  $(a_1, a_2, a_3, b_2, b_3)$  the solution is unique if it exists.

Now putting

$$a_3 = c_3 + \frac{1}{3}a_1^2,$$

we write  $X(3, 4)$  and  $X(3, 5)$  by the variables  $(a_1, a_2, c_3, b_2, b_3)$ :

$$\begin{aligned} X(3, 4) &= 12096c_3(-7a_1b_2 - 14a_2 - b_2^3 + 3b_2b_3), \\ X(3, 5) &= 12096c_3(-16a_1^2 + 29a_1b_2^2 - 12a_1b_3 + 28a_2b_2 + 3b_2^4 - 11b_2^2b_3 \\ &\quad + 4b_3^2 - 48c_3). \end{aligned}$$

First suppose  $c_3 = 0$ . Then the solution is uniquely determined by  $(a_1, a_2, b_2, b_3)$ , which corresponds to the solution

$$\begin{aligned} u(t) &= \varphi(t) = t^{-2} + \frac{g_2}{20}t^2 + \frac{g_3}{28}t^4 + \dots, \\ v(t) &= \frac{1}{\varphi(t) - e_3} + \frac{4(C_4\varphi(t) + C_5)}{\varphi'(t)^2} \\ &= t^2 + (e_3 + C_4)t^4 + (C_5 + e_3^2 - \frac{g_2}{20})t^6 + \dots \end{aligned}$$

with  $g_2 = 20a_1$ ,  $g_3 = 28a_2$ ,  $C_4 = b_2 - e_3$  and  $C_5 = b_3 - e_3^2 + \frac{g_2}{20}$ .

Next suppose  $c_3 \neq 0$ . Then it follows from  $X(3, 4) = X(3, 5) = 0$  that  $(a_2, c_3)$  is uniquely determined by  $(a_1, b_2, b_3)$ . Hence the solution is uniquely determined by  $(a_1, b_2, b_3)$ , which corresponds to the solution

$$\begin{aligned} u(t) &= \varphi(t) + 16C_6 \frac{(\varphi(\frac{t}{2}) - e_3)^2}{\varphi'(\frac{t}{2})^2} = t^{-2} + (C_6 + \frac{g_2}{20})t^2 + \dots, \\ v(t) &= \frac{1}{\varphi(t) - e_3} = t^2 + e_3t^4 + (e_3^2 - \frac{g_2}{20})t^6 + \dots \end{aligned}$$

with  $e_3 = b_2$ ,  $g_2 = 20(e_3^2 - b_3)$  and  $C_6 = a_1 - \frac{g_2}{20}$ .

## 2.5 Type $(-2, -2)$

We shall do a similar but more complicated calculation for the type  $(-2, -2)$ . In § 2.5.3 and 2.5.6 we also use REDUCE.

**2.5.1.** We may assume  $a_{-1} = b_{-1} = 1$ . For  $k \geq 4$

$$\begin{pmatrix} \text{coeffn}(X(1, k-2), a_k, 1) & \text{coeffn}(X(1, k-2), b_k, 1) \\ \text{coeffn}(X(2, k-3), a_k, 1) & \text{coeffn}(X(2, k-3), b_k, 1) \end{pmatrix} = \begin{pmatrix} -\frac{4}{15}k(2k-2)(2k-6)(2k+2) & (-2)(2k-6) \\ \frac{4}{35}k(2k-1)(2k-2)(2k-4)(2k+2)(2k-10) & (-2)(-3)(-4)(2k-10) \end{pmatrix}$$

The determinant of this matrix equals

$$-\frac{4}{35}2k(2k+2)(2k+3)(2k-2)(2k-6)(2k-8)(2k-10).$$

Hence  $a_k$  and  $b_k$  with  $k \geq 6$  are uniquely determined by  $(a_1, \dots, a_5, b_1, \dots, b_5)$ . Moreover by the equations  $X(1, 2) = X(1, 3) = 0$ ,  $b_4$  and  $b_5$  are expressed by polynomial functions of  $(a_1, \dots, a_5, b_1, b_2, b_3)$ .

Thus  $a_i$  and  $b_j$  with  $i \geq 6$  and  $j \geq 4$  are determined by polynomial functions of

$$(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3).$$

Here we have used all of  $X(1, 1), X(1, 2), \dots$  and  $X(2, 1), X(2, 2), \dots$

We put

$$(2.32) \quad \begin{aligned} a_3 &= c_3 + \frac{1}{3}a_1^2, \\ b_3 &= d_3 + \frac{1}{3}b_1^2, \\ a_4 &= c_4 + \frac{3}{11}a_1a_2, \\ a_5 &= c_5 + \frac{2}{39}a_1^3 + \frac{1}{13}a_2^2. \end{aligned}$$

Then all coefficients are suitable polynomial functions of

$$(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3).$$

Similarly by denoting

$$\begin{aligned} b_4 &= d_4 + \frac{3}{11}b_1b_2, \\ b_5 &= d_5 + \frac{2}{39}b_1^3 + \frac{1}{13}b_2^2, \end{aligned}$$

we have

$$\begin{aligned} d_4 &= \frac{3}{11}(-32a_1a_2 + 11a_1b_2 + 22a_2b_1 - b_1b_2) - 32c_4, \\ d_5 &= \frac{1}{39}(-128a_1^3 + 104a_1^2b_1 + 26a_1b_1^2 + 78a_1d_3 - 192a_2^2 + 195a_2b_2 \\ &\quad - 2b_1^3 + 312b_1c_3 - 3b_2^2) - 64c_5. \end{aligned}$$

Here we remark that  $c_3 = c_4 = c_5 = 0$  (resp.  $d_3 = d_4 = d_5 = 0$ ) if  $u$  (resp.  $v$ ) is the Weierstrass function  $\wp$ .

**2.5.2.** Before going into the detail we prepare several notations. We are considering the set of solutions

$$\{(u, v) \mid \text{with } a_{-1} = b_{-1} = 1 \text{ satisfies (1.4)}\}.$$

By picking up the first several coefficients of these solutions, we define the set

$$V := \{(a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3) \in \mathbb{C}^8 \mid \text{a solution } (u, v) \\ \text{of the form (2.29) with } a_{-1} = b_{-1} = 1 \text{ satisfies (1.4)}\}.$$

It is easy to see that  $V$  is a closed algebraic subvariety. Since the map from such  $(u, v)$  to  $V$  defined by (2.29) and (2.32) is injective, we may identify the set of such solutions  $(u, v)$ 's with  $V$ .

**Lemma 2.11.** i) *The solutions*

$$\begin{aligned}
(2.33) \quad u(t) &= \wp(t) + 16C \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} \\
&= t^{-2} + (\frac{g_2}{20} + C)t^2 + (\frac{g_3}{28} - \frac{e_3 C}{2})t^4 + (\frac{g_2^2}{1200} + \frac{1}{16}(e_3^2 + \frac{g_2}{5})C)t^6 \\
&\quad + (\frac{3g_2 g_3}{6160} + \frac{3}{64}(\frac{g_3}{14} - \frac{g_2 e_3}{10})C)t^8 + \dots, \\
v(t) &= \wp(t) + \frac{C'}{\wp(t) - e_3} \\
&= t^{-2} + (\frac{g_2}{20} + C')t^2 + (\frac{g_3}{28} + e_3 C')t^4 + (\frac{g_2^2}{1200} + (e_3^2 - \frac{g_2}{20})C')t^6 \\
&\quad + (\frac{3g_2 g_3}{6160} + \frac{3}{64}(e_3^3 - \frac{g_3}{28} - \frac{g_2 e_3}{10})C')t^8 + \dots
\end{aligned}$$

belong to  $V$ . Therefore we can define a map

$$\Psi_3 : (e_1 + e_2, e_1 e_2, C, C') \in \mathbb{C}^4 \rightarrow V.$$

ii) *We can define a  $\mathbb{C}^\times$ -action by*

$$\begin{aligned}
(2.34) \quad \lambda.(A, B, C, C') &= (\lambda A, \lambda^2 B, \lambda^2 C, \lambda^2 C'), \\
\lambda.(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3) &= (\lambda^2 a_1, \lambda^3 a_2, \lambda^4 c_3, \lambda^5 c_4, \lambda^6 c_5, \lambda^2 b_1, \lambda^3 b_2, \lambda^4 d_3)
\end{aligned}$$

so that  $\Psi_3$  is  $\mathbb{C}^\times$ -equivariant. Moreover  $\Psi_3^{-1}(0) = 0$ .

iii) *The solutions*

$$\begin{aligned}
(2.35) \quad u(t) &= \wp(t), \\
v(t) &= \frac{4\wp(t)^4 + C\wp(t)^2 + C'\wp(t) + C''}{\wp'(t)^2}
\end{aligned}$$

belong to  $V$ . Then we have a map

$$\Psi_1 : (g_2, g_3, C, C', C'') \in \mathbb{C}^5 \rightarrow V.$$

Similarly the solutions

$$\begin{aligned}
(2.36) \quad u(t) &= \frac{\wp(\frac{t}{2})^4 + C\wp(\frac{t}{2})^2 + C'\wp(\frac{t}{2}) + C''}{\wp'(\frac{t}{2})^2}, \\
v(t) &= \wp(t)
\end{aligned}$$

belong to  $V$ , which define a map

$$\Psi_2 : (g_2, g_3, C, C', C'') \in \mathbb{C}^5 \rightarrow V.$$

We have a  $\mathbb{C}^\times$ -action

$$\lambda.(g_2, g_3, C, C', C'') = (\lambda^2 g_2, \lambda^3 g_3, \lambda^2 C, \lambda^3 C', \lambda^4 C'')$$

so that  $\Psi_1$  and  $\Psi_2$  are  $\mathbb{C}^\times$ -equivariant.

iv) *Cf. [OS, Proposition 7.3 ii)]*

$$\begin{aligned}
\text{Im } \Psi_1 &= V \cap \{c_3 = c_4 = c_5 = 0\}, \\
\text{Im } \Psi_2 &= V \cap \{d_3 = d_4 = d_5 = 0\}.
\end{aligned}$$

v) *The maps  $\Psi_1$  and  $\Psi_2$  are injective.*

*Proof.* i) follows from Theorem 2.7.

ii) The  $\mathbb{C}^\times$ -equivariance is easy. To prove  $\Psi_3^{-1}(0) = 0$ , suppose  $u(t) = v(t) = t^{-2}$  of the form (2.33). If one of  $e_i$  is not zero,  $u(t)$  and  $v(t)$  have a finite period. Since  $t^{-2}$  has no period,  $e_1 = e_2 = e_3 = 0$ . This means  $\wp(t) = t^{-2}$ . Then one should have  $C = C' = 0$ .

iii) is similarly proved as in the case of i) and ii).

iv) The Laurent expansion (2.14) of  $\wp(t)$  implies that the left hand side is contained in the right hand side. Conversely, for any  $(a_1, a_2)$  we can take  $g_2 = 20a_1$  and  $g_3 = 28a_2$  and moreover for any  $(b_1, b_2, b_3)$ , we can take  $(C, C', C'')$  so that the expansion of  $v(t)$  has desired coefficients.

v) For  $\Psi_1$ , the Taylor expansion (2.14) of  $u(t)$  determines  $g_2$  and  $g_3$ . The other coefficients  $C, C', C''$  are determined by the Taylor expansion of  $v(t)$ .  $\square$

**2.5.3.** By direct calculations we obtain that the vanishing of  $X(3, 4)$ ,  $X(3, 5)$ ,  $X(3, 6)$ ,  $X(4, 5)$ ,  $X(3, 7)$  and  $X(4, 6)$  are equivalent to

(2.37)

$$f_1 := 96a_1a_2c_3 - 33a_1b_2c_3 - 66a_2b_1c_3 + 3b_1b_2c_3 + 352c_3c_4 - 22c_4d_3 = 0,$$

(2.38)

$$\begin{aligned} f_2 := & 128a_1^3c_3 - 104a_1^2b_1c_3 + 1056a_1a_2c_4 - 26a_1b_1^2c_3 - 363a_1b_2c_4 - 54a_1c_3d_3 \\ & + 192a_2^2c_3 - 726a_2b_1c_4 - 195a_2b_2c_3 + 2b_1^3c_3 + 33b_1b_2c_4 - 312b_1c_3^2 \\ & + 6b_1c_3d_3 + 3b_2^2c_3 + 2496c_3c_5 + 3872c_4^2 - 156c_5d_3 = 0, \end{aligned}$$

(2.39)

$$\begin{aligned} f_3 := & 394240a_1^3c_4 + 446976a_1^2a_2c_3 - 320320a_1^2b_1c_4 - 153648a_1^2b_2c_3 \\ & - 272736a_1a_2b_1c_3 + 1946880a_1a_2c_5 - 80080a_1b_1^2c_4 + 2088a_1b_1b_2c_3 \\ & - 669240a_1b_2c_5 + 1638912a_1c_3c_4 - 268752a_1c_4d_3 + 591360a_2^2c_4 \\ & - 23760a_2b_1^2c_3 - 1338480a_2b_1c_5 - 600600a_2b_2c_4 - 116640a_2c_3d_3 \\ & + 6160b_1^3c_4 + 1080b_1^2b_2c_3 + 60840b_1b_2c_5 - 834240b_1c_3c_4 \\ & - 16170b_1c_4d_3 + 9240b_2^2c_4 - 10935b_2c_3d_3 + 14826240c_4c_5 = 0, \end{aligned}$$

(2.40)

$$\begin{aligned} f_4 := & 305536a_1^3c_4 + 257760a_1^2a_2c_3 - 248248a_1^2b_1c_4 - 88605a_1^2b_2c_3 \\ & - 165978a_1a_2b_1c_3 + 1812096a_1a_2c_5 - 62062a_1b_1^2c_4 + 4194a_1b_1b_2c_3 \\ & - 622908a_1b_2c_5 + 945120a_1c_3c_4 - 225390a_1c_4d_3 + 458304a_2^2c_4 \\ & - 7722a_2b_1^2c_3 - 1245816a_2b_1c_5 - 465465a_2b_2c_4 - 68526a_2c_3d_3 \\ & + 4774b_1^3c_4 + 351b_1^2b_2c_3 + 56628b_1b_2c_5 - 703560b_1c_3c_4 \\ & - 20328b_1c_4d_3 + 7161b_2^2c_4 - 13122b_2c_3d_3 + 12602304c_4c_5 = 0, \end{aligned}$$

(2.41)

$$\begin{aligned} f_5 := & 122880a_1^4c_3 - 89600a_1^3b_1c_3 + 1198080a_1^3c_5 + 4308480a_1^2a_2c_4 \\ & - 33280a_1^2b_1^2c_3 - 973440a_1^2b_1c_5 - 1481040a_1^2b_2c_4 - 120960a_1^2c_3d_3 \\ & + 1331712a_1a_2^2c_3 - 2085600a_1a_2b_1c_4 - 303696a_1a_2b_2c_3 - 160a_1b_1^3c_3 \\ & - 243360a_1b_1^2c_5 - 166650a_1b_1b_2c_4 - 299520a_1b_1c_3^2 + 7920a_1b_1c_3d_3 \\ & - 92655a_1b_2^2c_3 + 2396160a_1c_3c_5 + 15797760a_1c_4^2 - 767520a_1c_5d_3 \\ & - 773472a_2^2b_1c_3 + 1797120a_2^2c_5 - 602580a_2b_1^2c_4 - 170814a_2b_1b_2c_3 \\ & - 1825200a_2b_2c_5 + 4207104a_2c_3c_4 - 690624a_2c_4d_3 + 160b_1^4c_3 \\ & + 18720b_1^3c_5 + 27390b_1^2b_2c_4 - 24960b_1^2c_3^2 - 1140b_1^2c_3d_3 \\ & + 8925b_1b_2^2c_3 - 2720640b_1c_3c_5 + 3213760b_1c_4^2 + 1560b_1c_5d_3 \\ & + 28080b_2^2c_5 + 1019040b_2c_3c_4 - 10230b_2c_4d_3 - 155520c_3^2d_3 \\ & - 4860c_3d_3^2 + 23362560c_5^2 = 0, \end{aligned}$$

(2.42)

$$\begin{aligned} f_6 := & 2826240a_1^4c_3 - 2245120a_1^3b_1c_3 + 49121280a_1^3c_5 + 164482560a_1^2a_2c_4 \\ & - 615680a_1^2b_1^2c_3 - 39911040a_1^2b_1c_5 - 56540880a_1^2b_2c_4 - 2782080a_1^2c_3d_3 \\ & + 33864192a_1a_2^2c_3 - 74865120a_1a_2b_1c_4 - 917136a_1a_2b_2c_3 + 33760a_1b_1^3c_3 \\ & - 9977760a_1b_1^2c_5 - 7996890a_1b_1b_2c_4 - 6888960a_1b_1c_3^2 + 532080a_1b_1c_3d_3 \\ & - 4599135a_1b_2^2c_3 + 55111680a_1c_3c_5 + 603102720a_1c_4^2 - 32816160a_1c_5d_3 \\ & - 20290272a_2^2b_1c_3 + 73681920a_2^2c_5 - 26273940a_2b_1^2c_4 - 8482974a_2b_1b_2c_3 \\ & - 74833200a_2b_2c_5 + 108624384a_2c_3c_4 - 24323904a_2c_4d_3 + 800b_1^4c_3 \\ & + 767520b_1^3c_5 + 1194270b_1^2b_2c_4 - 124800b_1^2c_3^2 - 102900b_1^2c_3d_3 \\ & + 425325b_1b_2^2c_3 - 118734720b_1c_3c_5 + 140127680b_1c_4^2 - 497640b_1c_5d_3 \\ & + 1151280b_2^2c_5 + 49764000b_2c_3c_4 - 918390b_2c_4d_3 - 4510080c_3^2d_3 \\ & - 315900c_3d_3^2 + 957864960c_5^2 = 0, \end{aligned}$$

respectively.

Note that  $f_1 = 0$  is equivalent to

$$(2.43) \quad c_3d_4 + 2d_3c_4 = 0.$$

**Lemma 2.12.** i)  $V \cap \{c_3 = d_3 = 0\} \subset \text{Im } \Psi_1 \cup \text{Im } \Psi_2$ .

ii)  $V \cap \{c_3 = 0, d_3 \neq 0\} \subset \text{Im } \Psi_1$ .

iii)  $V \cap \{c_3 \neq 0, d_3 = 0\} \subset \text{Im } \Psi_2$ .

*Proof.* We examine the left hand sides.

i) First note that  $f_2 = -121c_4d_4$  when  $c_3 = d_3 = 0$ . Hence we may assume  $c_3 = c_4 = d_3 = 0$  by the symmetry of  $u$  and  $v$ . In this case  $f_3 =$

$-223080c_5d_4$  and  $f_5 = -365040c_5d_5$ . Hence we have  $c_5 = 0$  or  $d_4 = d_5 = 0$ . By Lemma 2.11 iv) the result holds.

ii) Since  $c_4 = 0$  by (2.43), we have  $f_2 = -156c_5d_3$ . Then we have  $c_3 = c_4 = c_5 = 0$ .

iii) By the symmetry between  $u$  and  $v$ , it is reduced to ii).  $\square$

**2.5.4.** The remaining case is  $c_3d_3 \neq 0$ . Since  $V \cap \{c_3d_3 \neq 0\} \cap (\text{Im } \Psi_1 \cup \text{Im } \Psi_2) = \emptyset$ , we have to prove

$$(2.44) \quad V \cap \{c_3d_3 \neq 0\} \subset \text{Im } \Psi_3,$$

which will be proved at the end of this subsection.

**Proposition 2.13.** *Recall the map  $\Psi_3 : \mathbb{C}^4 \rightarrow \mathbb{C}^8$  in Lemma 2.11. Let  $Y$  be a  $d$ -dimensional subspace of  $\mathbb{C}^4$ ,  $L$  a subspace of  $\mathbb{C}^8$  and  $\Omega$  a Zariski open subset of  $L$  such that*

- a)  $\Psi_3(Y) \subset L$ .
  - b)  $\Omega \cap \Psi_3(Y) \neq \emptyset$  and  $\Psi_3|_{\Psi_3^{-1}(\Omega) \cap Y}$  is locally injective at a certain point.
  - c)  $\Omega \cap V$  is contained in an irreducible  $d$ -dimensional subvariety of  $V \cap L$ .
- Then  $\Omega \cap V \subset \text{Im } \Psi_3$ .

*Proof.* By Lemma 2.11 ii),

$$\bar{\Psi}_3 : (Y - \{0\})/\mathbb{C}^\times \rightarrow (L - \{0\})/\mathbb{C}^\times$$

is well defined. Then the image of  $\bar{\Psi}_3$  is compact,  $\Psi_3(Y - \{0\})$  is closed in  $L - \{0\}$ ,  $\Psi_3(Y)$  is closed in  $L$  and then  $\Omega \cap \Psi_3(Y)$  is closed in  $\Omega \cap Y$ . On the other hand, by the assumption c),

$$\Psi_3(\Psi_3^{-1}(\Omega) \cap Y) \subset \Omega \cap \Psi_3(Y) \subset \Omega \cap V \subset (\text{a } d\text{-dimensional irreducible variety}).$$

By the assumption b), the first term is dense in the last term and then  $\Omega \cap \Psi_3(Y)$  is dense in  $\Omega \cap V$ . Hence  $\Omega \cap \Psi_3(Y) = \Omega \cap V$ .  $\square$

**Proposition 2.14.** *The following  $Y, L$  and  $\Omega$  satisfy the assumptions a), b) and c) in Proposition 2.13. Here  $(A, B, C, C')$  and  $(a_1, a_2, c_3, c_4, c_5, b_1, b_2, d_3)$  are the coordinates of  $\mathbb{C}^4$  and  $\mathbb{C}^8$ , respectively.*

- i)  $Y = \mathbb{C}^4$ ,  $L = \mathbb{C}^8$  and  $\Omega = \{c_3d_3c_4d_4 \neq 0\}$ .
- ii)  $Y = \{A = 0\}$ ,  $L = \{a_2 = b_2 = c_4 = 0\}$  and  $\Omega = \{c_3d_3(16c_3 - d_3) \neq 0\} \cap L$ .
- iii)  $Y = \{A = 4C - C' = 0\}$ ,  $L = \{a_2 = b_2 = c_4 = 16c_3 - d_3 = 0\}$  and  $\Omega = \{c_3d_3 \neq 0, 4a_1 + b_1 \neq 0\} \cap L$ .
- iv)  $Y = \{A = B - 4C - C' = 0\}$ ,  $L = \{a_2 = b_2 = c_4 = 16c_3 - d_3 = 4a_1 + b_1 = 0\}$  and  $\Omega = \{c_3d_3 \neq 0\} \cap L$ .



*Proof.* The explicit expression of  $\Psi_3$  shows

$$\begin{aligned}
(2.45) \quad a_1 &= \frac{g_2}{20} + C &= \frac{1}{5}(A^2 - B + 5C), \\
a_2 &= \frac{g_3}{28} - \frac{e_3 C}{2} &= \frac{1}{14}A(-2B + 7C), \\
c_3 &= -\frac{1}{3}C \left( C + \frac{1}{16}(g_2 - 3e_3^2) \right) &= \frac{1}{48}C(-A^2 + 4B - 16C), \\
c_4 &= \frac{3}{22}e_3 C \left( C + \frac{1}{16}(e_1 - e_2)^2 \right) &= \frac{3}{352}AC(-A^2 + 4B - 16C), \\
b_1 &= \frac{g_2}{20} + C' &= \frac{1}{5}(A^2 - B + 5C'), \\
b_2 &= \frac{g_3}{28} + e_3 C' &= -\frac{1}{7}A(B + 7C'), \\
d_3 &= -\frac{1}{3}C'(C' + \frac{1}{4}(g_2 - 12e_3^2)) &= \frac{1}{3}C'(2A^2 + B - C').
\end{aligned}$$

Hence if  $A = 0$ , we have

$$\begin{aligned}
(2.46) \quad a_1 &= C - \frac{1}{5}B, \\
c_3 &= -\frac{1}{3}C^2 + \frac{1}{12}BC, \\
b_1 &= C' - \frac{1}{5}B, \\
d_3 &= -\frac{1}{3}C'^2 + \frac{1}{3}BC', \\
a_2 &= c_4 = b_2 = 0, \\
4a_1 + b_1 &= C' + 4C - B, \\
d_3 - 16c_3 &= \frac{1}{3}(4C - C')(4a_1 + b_1)
\end{aligned}$$

which proves a) for ii), iii) and iv). The assumption b) is also clear from (2.45) and (2.46).

The assumption c) will be proved in Lemma 2.16, 2.18, 2.19 and 2.20, respectively.  $\square$

*Proof of Theorem 2.9.* As we have already remarked, we have to prove (2.44). By Proposition 2.12 with the help of Proposition 2.14, it is enough to show

$$(2.47) \quad V \cap \{c_3 d_3 \neq 0, c_4 d_4 = 0\} \subset V \cap \{c_3 d_3 \neq 0, a_2 = b_2 = c_4 = 0\}.$$

This is proved as follows: Take an element in  $V$  such that  $c_3 d_3 \neq 0$ , and  $c_4 d_4 = 0$ . First note that we have  $c_4 = d_4 = 0$  by (2.43). Putting  $c_4 = 0$ , we have

$$\begin{aligned}
(2.48) \quad &\text{coeffn}(f_1, b_2, 1)f_3 - \text{coeffn}(f_3, b_2, 1)f_1 \\
&= c_3(-11a_1 + b_1)f_3 - 3(-17072a_1^2c_3 + 232a_1b_1c_3 - 74360a_1c_5 \\
&\quad + 120b_1^2c_3 + 6760b_1c_5 - 1215c_3d_3)f_1 \\
&= 51030a_2c_3^2d_3(32a_1 - 7b_1).
\end{aligned}$$

Suppose  $a_2 \neq 0$ . Then we have  $b_1 = \frac{32}{7}a_1$  and therefore  $f_1 = -\frac{45}{7}a_1c_3(32a_2 + 3b_2)$ . Since the assumption  $b_2 = -\frac{32}{3}a_2$  implies  $f_4 = 71442a_2c_3d_3 \neq 0$ , we have  $a_1 = b_1 = 0$  and hence  $f_4 = 1458c_3d_3(-47a_2 - 9b_2)$ . Applying  $a_1 = b_1 = c_4 = 0$  and  $b_2 = -\frac{47}{9}a_2$  to  $f_3$ , we obtain  $f_3 = -59535a_2c_3d_3 \neq 0$ , which means a contradiction.

Thus we can conclude  $a_2 = 0$  and by the symmetry between  $u$  and  $v$ , we have  $a_2 = b_2 = c_4 = 0$ .  $\square$

**Proposition 2.15.** *The subset  $\text{Im } \Psi_1$ ,  $\text{Im } \Psi_2$  and  $\text{Im } \Psi_3$  are closed subvarieties.*

*Proof.* Lemma 2.11 iv) shows that  $\text{Im } \Psi_1$  and  $\text{Im } \Psi_2$  are closed. Proposition 2.14 and the proof of Proposition 2.13 imply that  $\text{Image } \Psi_3$  is closed.  $\square$

**2.5.5.** We shall examine the assumption c).

**Lemma 2.16.** *The restriction of the projection*

$$(2.49) \quad V \cap \{c_3 d_3 c_4 d_4 \neq 0\} \ni (a_1, \dots, d_3) \mapsto (a_1, a_2, c_3, c_4, b_1) \in \mathbb{C}^5$$

*is injective. Its image is contained in  $\{h_1 = 0\}$  with an irreducible polynomial  $h_1(a_1, a_2, c_3, c_4)$  in (2.57).*

*Proof.* If  $16c_3 \neq d_3$ , then we have

$$(2.50) \quad c_5 = \frac{1}{156(16c_3 - d_3)} \left( -128a_1^3 c_3 + 104a_1^2 b_1 c_3 - 1056a_1 a_2 c_4 + 26a_1 b_1^2 c_3 \right. \\ \left. + 363a_1 b_2 c_4 + 54a_1 c_3 d_3 - 192a_2^2 c_3 + 726a_2 b_1 c_4 + 195a_2 b_2 c_3 - 2b_1^3 c_3 \right. \\ \left. - 33b_1 b_2 c_4 + 312b_1 c_3^2 - 6b_1 c_3 d_3 - 3b_2^2 c_3 - 3872c_4^2 \right)$$

from  $f_2 = 0$ . If  $d_3 = 16c_3$ , then we have

$$(2.51) \quad c_5 = \frac{1}{1853280} c_4^{-1} \left( -49280a_1^3 c_4 - 55872a_1^2 a_2 c_3 + 40040a_1^2 b_1 c_4 + 19206a_1^2 b_2 c_3 \right. \\ \left. + 34092a_1 a_2 b_1 c_3 + 10010a_1 b_1^2 c_4 - 261a_1 b_1 b_2 c_3 + 332640a_1 c_3 c_4 - 73920a_2^2 c_4 \right. \\ \left. + 2970a_2 b_1^2 c_3 + 75075a_2 b_2 c_4 + 233280a_2 c_3^2 - 770b_1^3 c_4 - 135b_1^2 b_2 c_3 \right. \\ \left. + 136620b_1 c_3 c_4 - 1155b_2^2 c_4 + 21870b_2 c_3^2 \right)$$

from the relation  $c_3 f_3 - 20280c_5 f_1 = 0$ . In either case, the relation (2.50) or (2.51) shows that  $c_5$  is uniquely determined by  $(a_1, a_2, c_3, c_4, b_1, b_2, d_3)$ .

Next we will do eliminations of variables in  $f_1 = \dots = f_4 = 0$ . Put

$$r_1 = \text{coeffn}(f_1, d_3, 1)f_2 - \text{coeffn}(f_2, d_3, 1)f_1, \\ r_2 = 17f_3 - 20f_4.$$

Then

$$r_1 = (-22c_4)f_2 - 6(-9a_1 c_3 + b_1 c_3 - 26c_5)f_1.$$

Moreover motivated by  $\text{coeffn}(r_1, c_5, 1) / \text{coeffn}(r_2, c_5, 1) = -c_3/210$ , we put

$$\begin{aligned} r_3 &= -c_3 r_2 - 210 r_1, \\ r_4 &= \text{coeffn}(f_1, d_3, 1) r_3 - \text{coeffn}(r_3, d_3, 1) f_1, \\ &= -22c_4 r_3 - 63c_3(968a_1 c_4 + 9720a_2 c_3 - 2090b_1 c_4 - 1215b_2 c_3) f_1. \end{aligned}$$

Then  $r_4$  is a polynomial function of  $(a_1, a_2, c_3, c_4, b_1, b_2)$  and it is factored into

$$(2.52) \quad r_4 = -105d_4(7128a_1 c_3^2 c_4 - 5832a_2 c_3^3 + 1782b_1 c_3^2 c_4 + 729b_2 c_3^3 - 10648c_4^3).$$

Then we have

$$(2.53) \quad 7128a_1 c_3^2 c_4 - 5832a_2 c_3^3 + 1782b_1 c_3^2 c_4 + 729b_2 c_3^3 - 10648c_4^3 = 0$$

and hence

$$(2.54) \quad b_2 = \frac{2}{729} c_3^{-3} (-3564a_1 c_3^2 c_4 + 2916a_2 c_3^3 - 891b_1 c_3^2 c_4 + 5324c_4^3).$$

Finally from  $f_1 = 0$  we have

$$(2.55) \quad d_3 = \frac{1}{22} c_3 c_4^{-1} (96a_1 a_2 - 33a_1 b_2 - 66a_2 b_1 + 3b_1 b_2 + 352c_4).$$

Since  $c_5, b_2$  and  $d_3$  is given by (2.50) or (2.51), (2.54) and (2.55), all coefficients are uniquely determined by  $(a_1, a_2, c_3, c_4, b_1)$ . This proves the injectivity.

By substituting (2.54) and (2.55) we have

$$\begin{aligned} & \text{coeffn}(f_2, c_5, 1) f_3 - \text{coeffn}(f_3, c_5, 1) f_2 \\ (2.56) \quad &= 156(16c_3 - d_3) f_3 \\ & \quad - 4680(416a_1 a_2 - 143a_1 b_2 - 286a_2 b_1 + 13b_1 b_2 + 3168c_4) f_2 \\ &= \frac{1040}{24057} c_3^{-5} c_4^{-2} d_4 h_1 (108a_1 c_3^2 + 27b_1 c_3^2 - 484c_4^2) \end{aligned}$$

with

$$(2.57) \quad \begin{aligned} h_1 &= -39694050a_1^2 c_3^4 c_4^2 - 22733865a_1 a_2 c_3^5 c_4 + 59296050a_1 c_3^2 c_4^4 \\ & \quad + 26040609a_2^2 c_3^6 + 47544651a_2 c_3^3 c_4^3 - 85739148c_3^5 c_4^2 - 14172488c_4^6. \end{aligned}$$

Suppose  $h_1 \neq 0$ . Then from (2.56) we have

$$(2.58) \quad b_1 = \frac{4}{27} c_3^{-2} (-27a_1 c_3^2 + 121c_4^2).$$

When  $16c_3 \neq d_3$ , it follows from (2.50), (2.54), (2.55) and (2.58) that

$$f_5 = \frac{80}{29403} c_3^{-3} c_4^{-2} d_3 h_1 \neq 0,$$

which contradicts to the fact that  $f_5 = 0$ .

When  $16c_3 = d_3$ , substituting (2.51), (2.54), (2.58) and  $d_3 = 16c_3$  to  $f_1$  we have

$$f_1 = \frac{968}{6561}c_3^{-4}c_4^2(8910a_1c_3^2c_4 - 5103a_2c_3^3 - 10648c_4^3)$$

and therefore

$$a_2 = \frac{22}{5103}c_3^{-3}c_4(405a_1c_3^2 - 484c_4^2).$$

Combining this with (2.51), (2.54), (2.58) and  $d_3 = 16c_3$ , we have

$$f_5 = -3732480c_3^3 \neq 0,$$

which also leads a contradiction.

Hence we obtain

$$(2.59) \quad h_1(a_1, a_2, c_3, c_4) = 0. \quad \square$$

**Lemma 2.17.** *On  $V \cap \{c_3d_3 \neq 0, a_2 = b_2 = c_4 = d_4 = 0\}$ ,  $(a_1, c_3, b_1, d_3)$  satisfy an equation  $h_2(a_1, c_3, b_1, d_3) = 0$ , which is given in (2.61).*

*Proof.* By  $f_2 = 0$ , we have

$$(2.60) \quad 156c_5(16c_3 - d_3) = -c_3(128a_1^3 - 104a_1^2b_1 - 26a_1b_1^2 - 54a_1d_3 + 2b_1^3 - 312b_1c_3 + 6b_1d_3).$$

Now applying  $a_2 = b_2 = c_4 = 0$  and (2.60) to  $(16c_3 - d_3)^2f_5$  and  $(16c_3 - d_3)^2f_6$ , we obtain

$$\begin{aligned} (16c_3 - d_3)^2f_5 &= 60c_3d_3h_2r_5, \\ (16c_3 - d_3)^2f_6 &= 60c_3d_3h_2r_6 \end{aligned}$$

with

$$\begin{aligned} h_2 &= 256a_1^4 - 144a_1^3b_1 - 104a_1^2b_1^2 + 1536a_1^2c_3 - 204a_1^2d_3 - 9a_1b_1^3 - 432a_1b_1c_3 \\ &\quad - 27a_1b_1d_3 + b_1^4 - 204b_1^2c_3 + 6b_1^2d_3 + 2304c_3^2 - 288c_3d_3 + 9d_3^2, \\ r_5 &= 64a_1^2 - 68a_1b_1 + 4b_1^2 - 288c_3 - 9d_3, \\ r_6 &= 2624a_1^2 - 2788a_1b_1 + 164b_1^2 - 8352c_3 - 585d_3, \end{aligned}$$

respectively. We have  $h_2r_5 = h_2r_6 = 0$ . Moreover because of the identity

$$\begin{aligned} 5184h_2 &= 1681r_5^2 - 82r_5r_6 + r_6^2 \\ &\quad - (144576a_1^2 - 1512a_1b_1 - 9414b_1^2)r_5 + (4032a_1^2 + 216a_1b_1 - 198b_1^2)r_6 \end{aligned}$$

we can conclude

$$(2.61) \quad h_2(a_1, c_3, b_1, d_3) = 0. \quad \square$$

**Lemma 2.18.** *The map  $V \cap \{c_3 d_3(16c_3 - d_3) \neq 0, a_2 = b_2 = c_4 = d_4 = 0\} \ni (a_1, \dots, d_3) \mapsto (a_1, c_3, b_1, d_3) \in \mathbb{C}^4$  is injective. Its image is contained in  $\{h_2 = 0\}$ .*

*Proof.* Since  $d_3 \neq 16d_3$ ,  $c_5$  is uniquely determined by (2.60) and the lemma is clear from Lemma 2.17.  $\square$

**Lemma 2.19.** *The map*

$$V \cap \{c_3(4a_1 + b_1) \neq 0, a_2 = b_2 = c_4 = d_4 = 16c_3 - d_3 = 0\} \\ \ni (a_1, \dots, d_3) \mapsto (a_1, b_1)$$

*is injective.*

*Proof.* For an element of  $V$  such that  $a_2 = b_2 = c_4 = d_4 = 16c_3 - d_3 = 0$ , we have

$$h_2 = (4a_1 + b_1)^2(16a_1^2 - 17a_1b_1 + b_1^2 - 108c_3).$$

Moreover assume  $c_3(4a_1 + b_1) \neq 0$ , then we have

$$c_3 = \frac{1}{180}(16a_1^2 - 17a_1b_1 + b_1^2).$$

Then

$$f_5 = \frac{20}{27}h_4r_7, \\ f_6 = \frac{20}{27}h_4r_8$$

with

$$h_4 = 128a_1^3 - 152a_1^2b_1 + 25a_1b_1^2 - b_1^3 - 2808c_5, \\ r_7 = -384a_1^3 + 520a_1^2b_1 - 143a_1b_1^2 + 7b_1^3 - 11232c_5, \\ r_8 = -11136a_1^3 + 17576a_1^2b_1 - 6799a_1b_1^2 + 359b_1^3 - 460512c_5.$$

Now by the equality

$$r_8 - 41r_7 = 7776c_3(4a_1 + b_1) \neq 0,$$

we can conclude  $h_4 = 0$ . Then  $c_5$  is determined by  $(a_1, b_1)$ .  $\square$

**Lemma 2.20.** *The map  $V \cap \{c_3 \neq 0, a_2 = b_2 = c_4 = d_4 = 16c_3 - d_3 = 4a_1 + b_1 = 0\} \ni (a_1, \dots, d_3) \mapsto (a_1, c_3, c_5) \in \mathbb{C}^3$  is injective. Its image is contained in  $\{f_5 = 0\}$ .*

*Proof.* In this case,

$$f_5 = 69120(338c_5^2 + 13a_1c_3c_5 - 28a_1^2c_3^2 - 54c_3^3)$$

is an irreducible polynomial.  $\square$

### 3 Reducible systems of type $B_2$

**3.1.** For our commuting differential operators  $P_1$  and  $P_2$  we can consider the simultaneous eigenvalue problem

$$(3.1) \quad P_j u(x) = \lambda_j u(x) \quad \text{for } j = 1 \text{ and } 2$$

with  $\lambda_j \in \mathbb{C}$ . If the potential function of  $P_1$  is generic, the study of this problem seems to be difficult. For the first step to analyze (3.1) we examine the case when the system (3.1) is reducible. To be precise we study the operators  $P$  and  $Q$  in the following lemma such that  $P = P_1$  and  $P_2 = {}^t Q Q$ .

**Lemma 3.1.** *Let  $P$  be a self-adjoint differential operator and let  $Q$  be a differential operator satisfying*

$$(3.2) \quad [P, Q] = BQ$$

*with a self-adjoint operator  $B$ . Then*

$$[P, {}^t Q Q] = 0.$$

*Proof.* The assumption of the lemma implies  $[P, {}^t Q Q] = [P, {}^t Q]Q + {}^t Q[P, Q] = -{}^t[{}^t P, Q]Q + {}^t Q[P, Q] = -{}^t Q B Q + {}^t Q B Q = 0$ .  $\square$

**Theorem 3.2.** *Let  $\varepsilon$  be the one dimensional representation  $\varepsilon : W(B_2) \rightarrow \{\pm 1\}$  such that  $g(x_1 x_2) = \varepsilon(g) x_1 x_2$  for  $g \in W(B_2)$ . Let  $P$  and  $Q$  be holomorphic differential operators of the form*

$$(3.3) \quad \begin{cases} P = \partial_1^2 + \partial_2^2 + R(x_1, x_2), \\ Q = \partial_1 \partial_2 + a_1(x_1, x_2) \partial_1 + a_2(x_1, x_2) \partial_2 + a_0(x_1, x_2) \end{cases}$$

*defined on a Zariski open subset of a connected open neighborhood of the origin of  $\mathbb{C}^2$ . Suppose*

$$(3.4) \quad g(P) = P, \quad g(Q) = \varepsilon(g)Q \quad \text{for } g \in W(B_2)$$

*and*

$$(3.5) \quad [P, Q] = b(x_1, x_2)Q$$

*with a function  $b(x_1, x_2)$ . Then*

$$(3.6) \quad \begin{cases} R(x_1, x_2) &= u(x_1 + x_2) + u(x_1 - x_2) + w(x_1) + w(x_2), \\ w(t) &= V'(t) - V^2(t), \\ a_0(x_1, x_2) &= V(x_1)V(x_2) + \frac{1}{2}(u(x_1 + x_2) - u(x_1 - x_2)), \\ a_1(x_1, x_2) &= V(x_2), \\ a_2(x_1, x_2) &= V(x_1), \\ b(x_1, x_2) &= 2V'(x_1) + 2V'(x_2), \end{cases}$$

where

$$(3.7) \quad \begin{cases} u(t) &= c_4 \frac{(\wp(\frac{t}{2}) - e_3)^2}{\wp'(\frac{t}{2})^2} + c_5 \wp(t) + c_6, \\ V(t) &= \frac{c_1(\wp(t) - e_1)(\wp(t) - e_2) + c_2 \wp(t) + c_3}{\wp'(t)} \end{cases}$$

(3.8)

with suitable complex numbers  $c_1, \dots, c_6$  satisfying

$$(3.9) \quad c_2 c_4 = c_3 c_4 = 0$$

or

$$(3.10) \quad u(t) = c, \quad V(t) \text{ is any odd function with } c \in \mathbb{C}$$

or

$$(3.11) \quad u(t) \text{ is any even function, } V(t) = 0.$$

On the other hand the operators  $P$  and  $Q$  given by (3.6) satisfy the relation (3.5) by putting (3.7) for any complex numbers  $c_1, \dots, c_6$  with (3.9) and any periods of  $\wp(t)$  or by putting (3.10) or by putting (3.11).

The following Remark 3.3 and Remark 3.4 are easily obtained by direct calculations.

*Remark 3.3.* Under the notation of Theorem 3.2

$$\begin{aligned} {}^tQQ &= \left( \partial_1 \partial_2 + \frac{(u(x_1 + x_2) - u(x_1 - x_2))}{2} \right)^2 + w(x_2) \partial_1^2 + w(x_1) \partial_2^2 \\ &+ w(x_1) w(x_2) + V(x_1) V(x_2) (u(x_1 + x_2) - u(x_1 - x_2)) \\ &- \frac{1}{2} \left( V(x_1) (u'(x_1 + x_2) + u'(x_1 - x_2)) + V(x_2) (u'(x_1 + x_2) - u'(x_1 - x_2)) \right). \end{aligned}$$

*Remark 3.4.* In Theorem 3.2 we have the following from (3.7) with complex numbers  $C_1, \dots$

i) If the fundamental half periods  $\omega_1$  and  $\omega_2$  of  $\wp$  are finite and  $c_4 = 0$ , then

$$\begin{cases} u(t) &= c_5 \wp(t) + c_6, \\ V(t) &= \sum_{j=1}^3 \frac{1}{2} C_j \frac{\wp'(t)}{\wp(t) - e_j}, \\ w(t) &= - \sum_{j=1}^4 (C_j + C_j^2) \wp(t + \omega_j) \\ &\quad - (C_1^2 - 2C_2 C_3) e_1 - (C_2^2 - 2C_3 C_1) e_2 - (C_3^2 - 2C_1 C_2) e_3, \\ C_4 &= -(C_1 + C_2 + C_3). \end{cases}$$

ii) If  $\omega_1$  and  $\omega_2$  are finite and  $c_2 = c_3 = 0$ , then

$$\begin{cases} u(t) &= C_2\wp(t) + C_3(\wp(\frac{t}{2} + \omega_1) + \wp(\frac{t}{2} + \omega_2)) + C_4, \\ V(t) &= \frac{1}{2}C_1\frac{\wp'(t)}{\wp(t)-e_3}, \\ w(t) &= -(C_1 + C_1^2)\wp(t + \omega_3) + (C_1 - C_1^2)\wp(t) - C_1^2e_3 \end{cases}$$

iii) If  $e_1 = e_2 = \frac{1}{3}\lambda^2 \neq 0$  and  $c_4 = 0$ , then

$$\begin{cases} u(t) &= C_4 \sinh^{-2} \lambda t + C_5, \\ V(t) &= C_1 \coth \lambda t + C_2 \tanh \lambda t + C_3 \sinh 2\lambda t, \\ w(t) &= -(C_1\lambda + C_1^2) \sinh^{-2} \lambda t + (C_2\lambda + C_2^2) \cosh^{-2} \lambda t \\ &\quad + 2(C_3\lambda - C_1C_3 - C_2C_3) \cosh 2\lambda t - C_3^2 \cosh^2 2\lambda t \\ &\quad - (C_1^2 + C_2^2 - C_3^2 + 2C_1C_2 + 2C_1C_3 - 2C_2C_3). \end{cases}$$

iv) If  $e_2 = e_3 = \frac{1}{3}\lambda^2 \neq 0$  and  $c_2 = c_3 = 0$ , then

$$\begin{cases} u(t) &= C_2 \sinh^{-2} \lambda t + C_3 \sinh^{-2} \frac{\lambda}{2} t + C_4, \\ V(t) &= C_1 \coth \lambda t, \\ w(t) &= -(C_1\lambda + C_1^2) \sinh^{-2} \lambda t - C_1^2. \end{cases}$$

v) If  $e_1 = e_2 = \frac{1}{3}\lambda^2 \neq 0$  and  $c_2 = c_3 = 0$ , then

$$\begin{cases} u(t) &= C_2 \sinh^{-2} \lambda t + C_3 \cosh 2\lambda t + C_4, \\ V(t) &= C_1 \sinh^{-1} 2\lambda t, \\ w(t) &= -C_1\lambda \sinh^{-2} \lambda t + (2C_1\lambda - C_1^2) \sinh^{-1} 2\lambda t. \end{cases}$$

vi) If  $e_1 = e_2 = c_4 = 0$ , then

$$\begin{cases} u(t) &= C_4 t^{-2} + C_5, \\ V(t) &= C_1 t^{-1} + C_2 t + C_3 t^3, \\ w(t) &= -(C_1 + C_1^2) t^{-2} - (2C_1C_2 - C_2) - (2C_1C_3 + C_2^2 - 3C_3) t^2 \\ &\quad - 2C_2C_3 t^4 - C_3^2 t^6. \end{cases}$$

vii) If  $e_1 = e_2 = c_2 = c_3 = 0$ , then

$$\begin{cases} u(t) &= C_2 t^{-2} + C_3 t^2 + C_4, \\ V(t) &= C_1 t^{-1}, \\ w(t) &= -(C_1 + C_1^2) t^{-2}. \end{cases}$$



**3.2.** To prove Theorem 3.2 we will translate the reducibility into a functional equation. The coefficients of  $\partial_1^2$  and  $\partial_2^2$  in (3.5) mean  $2\partial_1 a_2 = 2\partial_2 a_1 = 0$  and therefore

$$a_1 = V(x_2) \quad \text{and} \quad a_2 = V(x_1)$$

with a suitable odd function  $V(t)$ . The coefficient of  $\partial_1 \partial_2$  in (3.5) proves

$$(3.12) \quad b = 2(\partial_2 a_1 + \partial_1 a_2) = 2(V'(x_2) + V'(x_1)).$$

The coefficients of  $\partial_1$  and  $\partial_2$  in (3.5) are

$$\begin{cases} V''(x_2) + 2\partial_1 a_0 - \partial_2 R = 2V(x_2)(V'(x_1) + V'(x_2)), \\ V''(x_1) + 2\partial_2 a_0 - \partial_1 R = 2V(x_1)(V'(x_1) + V'(x_2)) \end{cases}$$

and equivalently

$$\begin{cases} (\partial_1 + \partial_2)(R - 2a_0 - V'(x_1) - V'(x_2) + (V(x_1) + V(x_2))^2) = 0, \\ (\partial_1 - \partial_2)(R + 2a_0 - V'(x_1) - V'(x_2) + (V(x_1) - V(x_2))^2) = 0. \end{cases}$$

Hence there exist functions  $u_+(t)$  and  $u_-(t)$  such that

$$(3.13) \quad \begin{aligned} R &= V'(x_1) + V'(x_2) - V(x_1)^2 - V(x_2)^2 + u_+(x_1 + x_2) + u_-(x_1 - x_2), \\ a_0 &= V(x_1)V(x_2) + \frac{1}{2}(u_+(x_1 + x_2) - u_-(x_1 - x_2)). \end{aligned}$$

Since  $g(a_0) = \varepsilon(g)a_0$ , we have

$$(3.14) \quad u_+(t) = u_-(t) = u(t)$$

with a suitable even function  $u(t)$ . This proves (3.6).

Let  $U(t)$  be the odd function with  $U'(t) = u(t)$ . Then

$$\begin{aligned} [P, Q] - bP &= -V(x_1)(u(x_1 + x_2) + u(x_1 - x_2)) - V(x_2)(u(x_1 + x_2) - u(x_1 - x_2)) \\ &\quad - (V'(x_1) + V'(x_2))(u(x_1 + x_2) - u(x_1 - x_2)) \\ &= -\partial_1 \partial_2 \left( V(x_1)(U(x_1 + x_2) + U(x_1 - x_2)) \right. \\ &\quad \left. + V(x_2)(U(x_1 + x_2) - U(x_1 - x_2)) \right). \end{aligned}$$

Thus we have

**Theorem 3.5.** *The operators  $P$  and  $Q$  satisfy (3.3), (3.4) and (3.5) if and only if there exist odd functions  $U$  and  $V$  and an even function  $H(t)$  such that*

$$(3.15) \quad \begin{aligned} V(x_1)(U(x_1 + x_2) + U(x_1 - x_2)) + V(x_2)(U(x_1 + x_2) - U(x_1 - x_2)) \\ = H(x_1) + H(x_2) \end{aligned}$$

and that the relation (3.6) holds with  $u = U'$ .

Hence we will concentrate the functional equation (3.15), which is a special case of (2.2).

**Lemma 3.6.** *Suppose  $(U, V, F, G)$  is a solution of (2.2) such that*

$$\begin{cases} U'(t) \text{ has a period } 2\omega, \\ W(t) := V(t + \omega) - V(t) - V(\omega) \text{ is an odd function.} \end{cases}$$

*Then  $(U, W, H)$  satisfies (3.15) with an appropriate  $H$ .*

*Proof.* Put  $U(t + 2\omega) = U(t) + \eta$ . Changing the variable  $(x_1, x_2)$  into  $(x_1 + \omega_1, x_2 + \omega_2)$  in the equation (2.2), we have

$$\begin{aligned} (U(x_1 + x_2) + \eta + U(x_1 - x_2))V(x_1 + \omega) + (U(x_1 + x_2) + \eta - U(x_1 - x_2))V(x_2 + \omega) \\ = F(x_1 + x_2 + 2\omega) + F(x_1 - x_2) + G(x_1 + \omega) + G(x_2 + \omega). \end{aligned}$$

Subtracting the original one from this, we obtain

$$\begin{aligned} (U(x_1 + x_2) + U(x_1 - x_2))W(x_1) + (U(x_1 + x_2) - U(x_1 - x_2))W(x_2) \\ = (F(x_1 + x_2 + 2\omega) - F(x_1 + x_2) - 2V(\omega)U(x_1 + x_2)) \\ + (G(x_1 + \omega) - G(x_1) - \eta V(x_1 + \omega)) \\ + (G(x_2 + \omega) - G(x_2) - \eta V(x_2 + \omega)). \end{aligned}$$

Since the left hand side is  $W(B_2)$ -invariant, the first term of the right hand side is constant. Then  $H(t) = G(t + \omega) - G(t) - \eta V(t + \omega) + C$  with a suitable constant number  $C$  and we have the lemma.  $\square$

**Corollary 3.7.** *Suppose the fundamental half periods  $\omega_1$  and  $\omega_2$  of  $\wp(t)$  are finite. Then for odd functions  $U(t)$  and  $V(t)$  given by*

$$\begin{cases} U'(t) = C_4\wp(t) + C_5, \\ V(t) = \sum_{j=1}^3 C_j(\zeta(t + \omega_j) - \zeta(t) - \zeta(\omega_j)) \end{cases}$$

or

$$\begin{cases} U'(t) = C_1(\wp(\frac{t}{2} + \omega_1) + \wp(\frac{t}{2} + \omega_2)) + C_2\wp(t) + C_3, \\ V(t) = \zeta(t + \omega_3) - \zeta(t) - \zeta(\omega_3), \end{cases}$$

*there exists a function  $H(t)$  so that (3.15) holds.*

*Proof.* Note that the equation (3.15) is bilinear for  $(U, V)$ . In the lemma put  $(U(t), V(t)) = (\zeta(t), \zeta(t))$  and  $\omega = \omega_j$  or put  $(U(t), V(t)) = (\zeta(\frac{t}{2} + \omega_1) + \zeta(\frac{t}{2} + \omega_2) - \zeta(\omega_1) - \zeta(\omega_2), \zeta(t))$  and  $\omega = \omega_3$ , we have the corollary.  $\square$

Now we will continue the proof of Theorem 3.2. Since

$$\zeta(t + \omega_j) - \zeta(t) - \zeta(\omega_j) = \frac{1}{2} \frac{\wp'(t)}{\wp(t) - e_j},$$

the last statement in Theorem 3.2 follows from Corollary 3.7 with the holomorphic continuation of the parameters  $e_1$  and  $e_2$  of  $\wp(t)$  and from the following Lemma 3.8 i). Thus we have proved that the operators given in Remark 3.4 satisfy (3.5).

- Lemma 3.8.** i) If  $U(t) = Ct$  ( $C \in \mathbb{C}$ ), then for any  $V$ ,  $H(t) := 2CtV(t)$  satisfies (3.15). If  $V(t) = 0$ , then  $H(t) = 0$  satisfies (3.15). We call these  $(U, V)$  trivial solutions of (3.15), which correspond to (3.10) and (3.11).  
ii) If  $(U, V, H)$  is a solution of (3.15), then  $(U + Ct, V, H + 2tV)$  is also a solution of (3.15)  
iii) If  $V'(t)$  has a period  $\omega$ , then  $U'(t)$  has a period  $2\omega$ .  
iv) If  $V(t + \omega) = V(t) + \eta$  with  $\eta \in \mathbb{C}$ , then  $\eta = 0$ .

*Proof.* The claims i) and ii) are clear and the claim iii) is also clear from the result in §2. Suppose  $V(t + \omega) = V(t) + \eta$ . Then  $U(t + 2\omega) = U(t) + \eta'$  with some  $\eta' \in \mathbb{C}$ . By the change of variable  $(x_1, x_2)$  into  $(x_1 + \omega, x_2 + \omega)$  in (3.15)

$$\begin{aligned} & (U(x_1 + x_2) + \eta' + U(x_1 - x_2))(V(x_1) + \eta) \\ & \quad + (U(x_1 + x_2) + \eta' - U(x_1 - x_2))(V(x_2) + \eta) \\ & \quad = H(x_1 + \omega) + H(x_2 + \omega). \end{aligned}$$

Subtracting (3.15) from the above,

$$\begin{aligned} & 2\eta U(x_1 + x_2) \\ & = (H(x_1 + \omega) - H(x_1) - \eta'V(x_1 + \omega)) + (H(x_2 + \omega) - H(x_2) - \eta'V(x_2 + \omega)). \end{aligned}$$

Since we have assumed  $U''(x_1 + x_2) \neq 0$ , we can conclude  $\eta = 0$ .  $\square$

Finally we will prove that there is no more solutions than we have already described in Remark 3.4 and Lemma 3.8 i).

From now on we consider only non-trivial solutions unless otherwise stated. Let  $(U, V, H)$  be a non-trivial solution of (3.15). Owing to Theorem 2.9 we see that  $V(t)$  is expressed by  $\wp(t)$ .

Suppose the fundamental half periods  $\omega_1$  and  $\omega_2$  of  $\wp$  are finite. Then

$$V(t) = \sum_{j=1}^4 C_j \zeta(t + \omega_j) + C_0 t$$

with  $C_j \in \mathbb{C}$ . By Lemma 3.8, we have

$$0 = V(t + 2\omega_i) - V(\omega_i) = \sum_{j=1}^4 2C_j \eta_i + 2C_0 \omega_i$$

for  $i = 1$  and  $2$ . Since  $\eta_2 \omega_1 - \eta_1 \omega_2 = \pm \frac{\pi\sqrt{-1}}{2} \neq 0$ ,  $C_0 = \sum_{j=1}^4 C_j = 0$  and we have the theorem.

When  $\omega_1 = \omega_2 = \infty$ , the theorem follows from

**Lemma 3.9.** *The rational solution of (3.15) is of the form in Remark 3.4 vi) or vii).*

*Proof.* Let  $(U, V, H)$  is a rational solution of (3.15). If  $U(t) = t^{-1}$ , the left hand side of (3.15) equals

$$2 \left( \frac{x_1 V(x_1) - x_2 V(x_2)}{x_1^2 - x_2^2} \right) = \sum_{n \geq 0} a_{2n} \frac{x_1^{2n} - x_2^{2n}}{x_1^2 - x_2^2}$$

for  $tV(t) = \sum_{n \geq 0} a_{2n} t^{2n}$ . Hence if  $u(t) = C_4 t^{-2} + C_5$  with  $C_4 \neq 0$ , the solution is of the form in Remark 3.4 vi).

Suppose  $U(t) = C_1 t^{-1} + C_2 t + C_3 t^3$  and  $V(t) = C_4 t^{-1} + C_5 t + C_6 t^3$  with  $C_3 \neq 0$ . We may assume  $C_1 = C_2 = C_4 = 0$ . Then the left hand side of (3.15) equals  $6C_3(x_1 V(x_1)x_2^2 + x_1^2 V(x_2)x_2)$  and hence  $C_5 = C_6 = 0$ . This is the case in Remark 3.4 vii).

Suppose  $V(t) = C_1 t^{-1} + C_2 t$  and  $U(t) = C_3 t^{-1} + C_4 t + C_5 t^3 + C_6 t^5 + C_7 t^7$ . We may assume  $C_3 = C_4 = 0$ . If  $C_1 \neq 0$  and  $C_2 = 0$ , the left hand side of (3.15) equals

$$C_1 \sum_{j=2}^4 C_{n+3} \frac{(x_1 + x_2)^{2n} - (x_1 - x_2)^{2n}}{x_1 x_2}$$

and therefore  $C_6 = C_7 = 0$ , which also corresponds to Remark 3.4 vii). Hence suppose  $C_2 \neq 0$ . Since  $(U(t), V(t)) = (t, t^n)$  does not satisfy (3.15) for  $n = 7, 5$  and  $3$ , we have  $C_7 = C_6 = C_5 = 0$  by considering the homogeneous parts of degree  $8, 6$  and  $4$ , successively. This is the case in Remark 3.4 vi).  $\square$

Lastly suppose  $\omega_1 = \infty$  and  $\omega_2$  is finite. We may assume

$$V(t) = C_1 \coth \lambda t + C_2 \tanh \lambda t + C_3 \sinh 2\lambda t + C_4 \sinh 4\lambda t + C_0 t.$$

Applying Lemma 3.9 to this, we have  $C_0 = 0$ .

If the dimension of the space  $\{V(t) | (\lambda \coth \lambda t, V(t)) \text{ is a solution of (3.15)}\}$  is larger than  $3$  for  $\lambda \neq 0$ , the dimension of the space  $\{V(t) | (t^{-1}, V(t)) \text{ is a solution of (3.15)}\}$  is proved to be larger than  $3$  by considering the limit to

$\lambda = 0$  (cf. [OSj, Proposition 2.21]), which contradicts to Lemma 3.9. Hence we have Theorem 3.2 if  $u(t) = C \sinh^{-2} \lambda t + C'$ .

In the same way we can prove that the space  $\{U(t) | (U(t), \lambda \coth \lambda t) \text{ is a solution of (3.15)}\}$  is of dimension 2 for  $\lambda \neq 0$ , which implies the theorem in the case  $V(t) = C_1 \coth \lambda t$ .

Thus we may assume that

$$(3.16) \quad \begin{cases} U(t) = C_1 \coth \lambda t + C_2 t + C_3 \coth \frac{\lambda}{2} t, \\ V(t) = C_4 \coth \lambda t + C_5 \sinh 2\lambda t. \end{cases}$$

or

$$(3.17) \quad \begin{cases} U(t) = C_1 \coth \lambda t + C_2 t + C_3 \sinh 2\lambda t, \\ V(t) = C_4 \sinh^{-1} 2\lambda t + C_5 \coth \lambda t. \end{cases}$$

The pairs  $(U, V)$  corresponding to  $C_3 C_5 = 0$  have been proved to be solutions. Suppose there exists a solution with  $C_3 C_5 \neq 0$  in (3.16) or (3.17). Then the bilinearity of the equation (3.15) implies that the pair  $(U, V)$  given by (3.16) or (3.17) is a solution for any complex numbers  $C_1, \dots, C_5$ , which similarly contradicts to Lemma 3.9 by taking the limit to  $\lambda = 0$ .

Thus we have completed the proof of Theorem 3.2.

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