Lower Estimates for the Growth of Painlevé Transcendents

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1 Introduction

Consider the first and the second Painlevé equations

(I) $\qquad w'' = 6w^2 + z,$

$(\mathrm{II})_\alpha$ $\qquad w'' = 2w^3 + zw + \alpha, \quad \alpha \in \boldsymbol{C}$

($' = d/dz$). All the solutions of these equations are meromorphic in the whole complex plane $\boldsymbol{C}$ ([5], [9]). Every solution of (I) is transcendental, and equation $(\mathrm{II})_\alpha$ admits a rational solution if and only if $\alpha \in \boldsymbol{Z}$ (e.g. [2], [8]); these equations define Painlevé transcendents.

The growth of a meromorphic function $f(z)$ is measured by the characteristic function defined by

$$T(r, f) = m(r, f) + N(r, f)$$

with

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \qquad \log^+ x = \max\{\log x, 0\},$$

$$N(r, f) = \int_0^r \big(n(t, f) - n(0, f)\big) \frac{dt}{t} + n(0, f) \log r;$$

here $n(r, f)$ denotes the number of poles in $|z| \le r$, each counted according to its multiplicity (for the notation of value distribution theory and basic facts, see [4], [6]). Also we use the notation $g(r) \ll h(r)$ if $g(r) = O(h(r))$ as $r \to \infty$.

The growth of each Painlevé transcendent is estimated as follows ([10], [11]):

Theorem A. *Let $w(z)$ be an arbitrary solution of* (I) (*resp.* $(\mathrm{II})_\alpha$). *Then,* $T(r, w) \ll r^{5/2}$ (*resp.* $T(r, w) \ll r^3$).

On the other hand, Mues and Redheffer [7] have shown the following:

Theorem B. *For every solution $w(z)$ of* (I), *we have $\sigma(w) \ge 5/2$, where* $\sigma(w) = \limsup_{r\to\infty} \log T(r, w)/\log r$.

By these results, the order of the first Painlevé transcendents is $5/2$.

In this paper we improve on the result of Theorem B, and under a certain condition, we give a lower estimate for $\sigma(w)$ of the second Painlevé transcendents. Our results are stated as follows:

Theorem 1.1. *For every solution $w(z)$ of* (I)*, we have*

$$r^{5/2}/\log r \ll T(r,w) \ll r^{5/2}.$$

An arbitrary solution of (I) is expressible in the form $w(z) = -(u'(z)/u(z))'$, where $u(z)$ is an entire function called a *τ-function.* Note that it is uniquely determined apart from the factor $\exp(a_0 z + a_1)$ $(a_0, a_1 \in \boldsymbol{C})$.

Theorem 1.2. *For every solution of* (I)*, its τ-function $u(z)$ satisfies*

$$r^{5/2}/\log r \ll T(r,u) \ll r^{5/2}.$$

Theorem 1.3. *Suppose that $2\alpha \in \boldsymbol{Z}$. Then, for every transcendental solution $w(z)$ of* $(\mathrm{II})_\alpha$*, we have $3/2 \le \sigma(w) \le 3$.*

Remark 1.1. The implicit coefficients of the relation in Theorem 1.1 are estimated as follows:

$$\liminf_{r\to\infty} T(r,w)(r^{5/2}/\log r)^{-1} \ge 4\cdot 10^{-11} K_0^{-5}, \quad \limsup_{r\to\infty} T(r,w) r^{-5/2} \le 2K_0/5,$$

where $K_0 = 1 + \limsup_{r\to\infty} n(r,w) r^{-5/2}$ $(< \infty)$.

Remark 1.2. For every solution $w(z)$ of (I), Boutroux [1] asserts the inequality $n(r,w) \gg r^{5/2}/\log r$, but his proof contains an incorrect part.

Remark 1.3. If $\alpha - 1/2 \in \boldsymbol{Z}$, then equation $(\mathrm{II})_\alpha$ admits a one-parameter family of solutions $\{v_c(z)\}_{c\in \boldsymbol{C}\cup\{\infty\}}$ such that $\sigma(v_c) = 3/2$ (see Section 4.3 and [2]).

2 Preliminaries

In this section, κ denotes an arbitrary positive numbere such that $\kappa > 1$. Let $\{c_j\}_{j=1}^{\infty}$ be a sequence satisfying $|c_1| \le |c_2| \le \cdots \le |c_j| \le \cdots$, where c_j $(j = 1, 2, ...)$ are not necessarily distinct. Now consider the summations

$$S_0(\{c_j\}, \kappa r, z) = \sum_{|c_j|<\kappa r} |z - c_j|^{-1}, \quad S_1(\{c_j\}, \kappa r, z) = \sum_{|c_j|<\kappa r} |z - c_j|^{-2}.$$

Put $\Delta_0(r) = \{z \mid |z| < r\}$. Let $\nu(r)$ denote the number of points c_j in $\Delta_0(r)$.

Lemma 2.1. *Suppose that $\nu(r) = O(r^\lambda)$ $(\lambda > 0)$. For every $r > r_0(\kappa)$, there exists a point $z_r \in \Delta_0(r) \setminus \Delta_0(r/\sqrt{2})$ with the properties*:

$$S_0(\{c_j\}, \kappa r, z_r) \le 32(\kappa+1)\nu(\kappa r) r^{-1}; \tag{2.1}$$

$$S_1(\{c_j\}, \kappa r, z_r) \le 6\lambda_0 \nu(\kappa r) r^{-2} \log r. \tag{2.2}$$

Here $r_0(\kappa)$ is a sufficiently large positive number, and $\lambda_0 = \max\{1, \lambda/2\}$.

Proof. Since

$$\iint_{|z-c_j|\le(\kappa+1)r} |z-c_j|^{-1}dxdy = \iint_{\substack{\rho\le(\kappa+1)r\\ 0\le\theta\le 2\pi}} d\rho d\theta = 2\pi(\kappa+1)r$$

with $|z-c_j| = \rho$, $z = x+yi$, we have

$$\iint_{\Delta_0(r)} S_0(\{c_j\}, \kappa r, z)dxdy \le 2\pi(\kappa+1)\nu(\kappa r)r. \tag{2.3}$$

Consider the set $F_r^0 = \{z \in \Delta_0(r) \mid S_0(\{c_j\}, \kappa r, z) \ge 32(\kappa+1)\nu(\kappa r)r^{-1}\}$. By (2.3), we have $32(\kappa+1)\nu(\kappa r)r^{-1}\mu(F_r^0) \le 2\pi(\kappa+1)\nu(\kappa r)r$, from which $\mu(F_r^0) \le \pi r^2/16$ follows. Here $\mu(\cdot)$ denotes the area of the set. Next consider the set $\Delta_\delta(r) = \Delta_0(r) \setminus D_\delta$, $D_\delta = \bigcup_{j=1}^\infty \{z \mid |z-c_j| < \delta\omega(|c_j|)\}$, $\delta < 1$, where $\omega(t) = \min\{1, t^{1-\lambda_0}\}$ $(t \ge 0)$. Then, $\Omega(r) = \sum_{1\le|c_j|\le r+1} \omega(|c_j|)^2$ is estimated as follows:

$$\int_1^{r+1} t^{2(1-\lambda_0)}d\nu(t) \ll r^{2(1-\lambda_0)}\nu(r+1) + \int_1^{r+1}(\lambda_0-1)t^{1-2\lambda_0}\nu(t)dt \ll r^2.$$

Since $\omega(t) \le 1$, we may choose δ so small that $\mu(D_\delta \cap \Delta_0(r)) < \pi\delta^2(\Omega(r)+O(1)) < \pi r^2/32$. Observing that

$$\iint_{\delta\omega(|c_j|)\le|z-c_j|\le(\kappa+1)r} |z-c_j|^{-2}dxdy = \iint_{\substack{\delta\omega(|c_j|)\le\rho\le(\kappa+1)r\\ 0\le\theta\le 2\pi}} \rho^{-1}d\rho d\theta \le 2\pi\big(\log(r/\omega(|c_j|)) + B_\kappa\big)$$

with $B_\kappa = \log((\kappa+1)/\delta)$, we obtain

$$\iint_{\Delta_\delta(r)} S_1(\{c_j\}, \kappa r, z)dxdy \le 2\pi\lambda_0\nu(\kappa r)\big(\log r + B_\kappa + \log\kappa\big).$$

Put $F_r^1 = \{z \in \Delta_\delta(r) \mid S_1(\{c_j\}, \kappa r, z) \ge 6\lambda_0\nu(\kappa r)r^{-2}\log r\}$. Then, we may take $r_0(\kappa)$ so large that $\mu(F_r^1) \le 3\pi r^2/8$ for $r \ge r_0(\kappa)$. Gathering the sets above, we get $H_r = F_r^0 \cup F_r^1 \cup (D_\delta \cap \Delta_0(r))$. Since $\mu(H_r) \le 15\pi r^2/32 < \pi r^2/2$, there exists a point $z_r \in (\Delta_0(r) \setminus \Delta_0(r/\sqrt{2})) \setminus H_r$ with the desired properties, provided that $r \ge r_0(\kappa)$. □

Lemma 2.2. *Let $g(z)$ be a meromorphic function, and let $\{\tilde{c}_j\}_{j=1}^\infty$ be the poles and the zeros of $g(z)$ such that $|\tilde{c}_1| \le |\tilde{c}_2| \le \cdots \le |\tilde{c}_j| \le \cdots$, each repeated according to its multiplicity. Then, for each $p \in \boldsymbol{N}$, we have*

$$\big|\big(g'(z)/g(z)\big)^{(p-1)}\big| \le C\Big(T(\kappa|z|, g)|z|^{-p} + \sum_{|\tilde{c}_j|<\kappa|z|} |z-\tilde{c}_j|^{-p} + 1\Big)$$

for every z, $|z| \ge 1$, where $C = C(p, \kappa)$ is some positive number.

Proof. Under the condition $g(0) \neq 0, \infty$, the lemma is derived from the Poisson-Jensen formula ([3],[4],[7]). The case where $g(0) = 0$ or ∞ is reduced to the former case by putting $h(z) = z^d g(z)$, $d \in \boldsymbol{Z}$, $h(0) \neq 0, \infty$. □

3 Proofs of Theorems 1.1 and 1.2

3.1. Lemmas

Let $w(z)$ be an arbitrary solution of (I). The following lemma is an immediate consequence of Theorem A with Clunie's reasoning [6, §2.4].

Lemma 3.1. *We have* $n(r, w) \ll r^{5/2}$ *and* $m(r, w) \ll \log r$.

Every pole of $w(z)$ is double and its residue is 0. Let $\{a_j\}_{j=1}^{\infty}$ be the distinct poles arranged as $|a_1| \leq |a_2| \leq \cdots \leq |a_j| \leq \cdots$. By the Mittag-Leffler theorem combined with Lemma 3.1, we write $w(z)$ in the form

$$w(z) = \varphi(z) + \Phi(z), \quad \Phi(z) = \sum_{j=1}^{\infty} \big((z - a_j)^{-2} - a_j^{-2} \big),$$

where $\varphi(z)$ is an entire function. In case $a_1 = 0$, the corresponding term in $\Phi(z)$ is to be replaced by z^{-2}. By Lemma 3.1, $K_0 = 1 + \limsup_{r \to \infty} n(r, w) r^{-5/2} < \infty$.

Lemma 3.2. *Suppose that* $\kappa \geq 8$. *If* $r > r_1$, *then, for every* $z \in \Delta_0(r)$,

$$\chi(\kappa r, z) = \sum_{|a_j| \geq \kappa r} |(z - a_j)^{-2} - a_j^{-2}| \leq 9 K_0 \kappa^{-1/2} r^{1/2}, \qquad \sum_{|a_j| \geq \kappa r} |z - a_j|^{-4} \leq K_0,$$

where r_1 *is a sufficiently large positive constant independent of* κ.

Proof. For $|a_j| \geq \kappa r$ and for $z \in \Delta_0(r)$, observing that $|z/a_j| \leq 1/8$, we have $|(z - a_j)^{-2} - a_j^{-2}| = 2|z||a_j|^{-3}|1 - (z/a_j)/2||1 - z/a_j|^{-2} \leq 3r|a_j|^{-3}$. Hence

$$\chi(\kappa r, z) \leq 3r \sum_{|a_j| \geq \kappa r} |a_j|^{-3} = 3r \int_{\kappa r}^{\infty} t^{-3} d\bar{n}(t) \leq 9r \int_{\kappa r}^{\infty} t^{-4} \bar{n}(t) dt \leq 9 K_0 \kappa^{-1/2} r^{1/2}$$

for $r \geq r_1$, where $\bar{n}(t) = n(t, w)/2 \leq K_0 t^{5/2}/2$ $(t \geq r_1)$, provided that r_1 is sufficiently large. Using $\sum_{|a_j| \geq \kappa r} |z - a_j|^{-4} \leq (8/7)^4 \sum_{|a_j| \geq \kappa r} |a_j|^{-4}$ for $z \in \Delta_0(r)$, we derive the second estimate in the same way. □

Lemma 3.3. *There exists a set* $E \subset (0, \infty)$ *with finite linear measure such that, for every* z *satisfying* $|z| \in (0, \infty) \setminus E$,

$$\sum_{0 < |a_j| < \infty} |(z - a_j)^{-2} - a_j^{-2}| \ll |z|^9.$$

Proof. Put $E = (0, |a_1| + 1) \cup \left(\bigcup_{j=2}^{\infty}(|a_j| - |a_j|^{-3}, |a_j| + |a_j|^{-3})\right)$. By Lemma 3.1, the linear measure of E is finite. By Lemmas 3.1 and 3.2, if $|z| \notin E$, then we have

$$\Big(\sum_{0<|a_j|<8|z|} + \sum_{|a_j|\geq 8|z|}\Big)|(z-a_j)^{-2} - a_j^{-2}| \ll (|z|^6+1)n(8|z|, w) + |z|^{1/2} \ll |z|^9.$$

□

Lemma 3.4. *Suppose that* $\kappa \geq 8$, *and that* $0 < \eta < 1$. *Then, for every* $r \geq 1$,

$$\gamma(\kappa r) = \sum_{0<|a_j|\leq \kappa r} |a_j|^{-2} \leq \eta^{-4} n(\kappa r, w) r^{-2} + L_0 \eta r^{1/2} + O(1),$$

where L_0 *is a positive constant independent of* κ *and* η.

Proof. Note that

$$\sum_{1\leq |a_j|\leq \kappa r} |a_j|^{-2} = \int_1^{\kappa r} t^{-2} d\bar{n}(t) \leq (\kappa r)^{-2}\bar{n}(\kappa r) + 2\Big(\int_1^{\eta^2 r} + \int_{\eta^2 r}^{\kappa r}\Big) t^{-3}\bar{n}(t)dt$$

$$\leq \Big((\kappa r)^{-2} + 2\int_{\eta^2 r}^{\kappa r} t^{-3}dt\Big)\bar{n}(\kappa r) + 2\int_1^{\eta^2 r} t^{-3}\bar{n}(t)dt = \eta^{-4}\bar{n}(\kappa r)r^{-2} + 2\int_1^{\eta^2 r} t^{-3}\bar{n}(t)dt.$$

By Lemma 3.1, we take a number L_0 such that $\bar{n}(t) = n(t, w)/2 \leq L_0 t^{5/2}/4$ for $t \geq 1$. Using these inequalities, we obtain the conclusion. □

3.2. Proof of Theorem 1.1

Note that $|\Phi(z)| \leq S_1(\{a_j\}, \kappa r, z) + \gamma(\kappa r) + \chi(\kappa r, z)$ $(r > 0)$. Take $\kappa = \kappa_0 = MK_0^2 > 8$, where M is a positive number determined later. By Lemma 3.2, $\chi(\kappa_0 r, z) \leq 9M^{-1/2}r^{1/2}$ for $z \in \Delta_0(r)$, if $r > r_1$. By Lemmas 3.1 and 2.1 ($\lambda = 5/2$, $\nu(r) = n(r, w)/2$), there exists a point $z_r \in \Delta_0(r) \setminus \Delta_0(r/\sqrt{2})$ such that $S_1(\{a_j\}, \kappa_0 r, z_r) \leq (15/4)n(\kappa_0 r, w)r^{-2}\log r$, if r is sufficiently large. From Lemma 3.4 with $\eta = \eta_0 = M^{-1/2}L_0^{-1}$, we have $\gamma(\kappa_0 r) \leq \eta_0^{-4} n(\kappa_0 r, w) r^{-2} + M^{-1/2}r^{1/2} + O(1)$. Hence,

$$|\Phi(z_r)| \leq 4n(\kappa_0 r, w)r^{-2}\log r + 10M^{-1/2}r^{1/2} + O(1) \ll r^{1/2}\log r \tag{3.1}$$

for some $z_r \in \Delta_0(r)\setminus\Delta_0(r/\sqrt{2})$, if $r \geq r_0^*$, where r_0^* is sufficiently large. Observing that $\left|\sum_{|a_j|<\kappa_0 r}(z_r - a_j)^{-4}\right| \leq S_1(\{a_j\}, \kappa_0 r, z_r)^2$, and using Lemma 3.2, we have

$$|\Phi''(z_r)| \leq 6\cdot 4^2 n(\kappa_0 r, w)^2 r^{-4}(\log r)^2 + 6K_0 \ll r(\log r)^2. \tag{3.2}$$

By Lemmas 3.1 and 3.3, $T(r, \varphi) = m(r, \varphi) \leq m(r, w) + m(r, \Phi) = O(\log r)$ as $r \to \infty$, $r \notin E$. By the finiteness of linear measure of E and [6, Lemma 1.1.1],

$\varphi(z)$ is a polynomial. Since $w(z_r) = (1/\sqrt{6})(w''(z_r) - z_r)^{1/2}$, by (3.1) and (3.2), we have $|\varphi(z_r)| \le |\Phi(z_r)| + (|\varphi''(z_r)| + |\Phi''(z_r)| + |z_r|)^{1/2} \ll r^{1/2}\log r + |\varphi''(z_r)|^{1/2}$, which implies $\varphi(z) \equiv C_0 \in \mathrm{C}$. Using (3.1) and (3.2), from $z_r = w''(z_r) - 6w(z_r)^2$, we obtain

$$
\begin{aligned}
&2^{-1/4} r^{1/2} \le |z_r|^{1/2} \le (|\Phi''(z_r)| + 6(|\Phi(z_r)| + |C_0|)^2)^{1/2} \\
&\le |\Phi''(z_r)|^{1/2} + \sqrt{6}(|\Phi(z_r)| + |C_0|) \le 8\sqrt{6}\, n(\kappa_0 r, w) r^{-2} \log r + 10\sqrt{6} M^{-1/2} r^{1/2} + O(1)
\end{aligned}
$$

for every $r \ge r_0^*$. Now we choose M so large that $2^{-1/4} - 10\sqrt{6}M^{-1/2} > 0$. Then we have $n(\kappa_0 r, w) \gg r^{5/2}/\log r$, which yields $N(r, w) \gg r^{5/2}/\log r$. Combining this with Theorem A, we arrive at the conclusion.

In the proof above, we put $M = 10^3$. Observing that $2^{-1/4} - 10\sqrt{6}M^{-1/2} > 0.066$, we obtain the estimates in Remark 1.1.

3.3. Proof of Theorem 1.2

Write the τ-function in the form $u(z) = e^{-h(z)}\Pi(z)$, where $h(z) = C_0 z^2/2 + C_1 z + C_2$ and $\Pi(z) = \prod_{j=1}^{\infty}(1 - z/a_j)\exp(z/a_j + z^2/(2a_j^2))$. By [4, Theorem 1.11] with $q = 3$, we have $\log^+ \Pi(z) \ll |z|^{5/2}$. Hence $T(r, u) \ll m(r, \Pi) + r^2 \ll r^{5/2}$. Also we have $n(r, 1/u) = n(r, w)/2 \gg r^{5/2}/\log r$, which implies $T(r, u) \gg r^{5/2}/\log r$.

4 Proof of Theorem 1.3

4.1. Lemma

Let $w(z)$ be an arbitrary transcendental solution of $(\mathrm{II})_\alpha$. The inequality $\sigma(w) \le 3$ is an immediate consequence of the lemma below, which follows from Theorem A.

Lemma 4.1. *We have $n(r, w) \ll r^3$ and $m(r, w) \ll \log r$.*

The proof of $\sigma(w) \ge 3/2$ is divided into two steps.

4.2. Proof for the case where $\alpha = 0$

It is sufficient to show that $\limsup_{r\to\infty} \log N(r, w)/\log r \ge 3/2$. To do so, suppose the contrary: $N(r, w) \ll r^{3/2-\varepsilon}$ for some $\varepsilon > 0$, which implies

$$
n(r, w) \ll n(r, w) \int_r^{2r} \frac{dt}{t} \le N(2r, w) \ll r^{3/2-\varepsilon}. \tag{4.1}
$$

Every pole of $w(z)$ is simple and its residue is ± 1. Let $\{b_j\}_{j=1}^{\infty}$ be the poles arranged as $|b_1| \le |b_2| \le \cdots \le |b_j| \le \cdots$. Then, we write $w(z)$ in the form

$$
w(z) = \psi(z) + \Psi(z), \quad \Psi(z) = \sum_{j=1}^{\infty} e(j)\big((z - b_j)^{-1} + b_j^{-1}\big),
$$

where $\psi(z)$ is an entire function, and $e(j)$ is the residue of the pole b_j. In case $b_1 = 0$, the term for $j = 1$ is to be replaced by $e(1)z^{-1}$. Observing the inequality $|(z - b_j)^{-1} + b_j^{-1}| = |z||b_j|^{-2}|1 - z/b_j|^{-1} \le 2|z||b_j|^{-2}$ for $|z/b_j| \le 1/2$, and putting $E^* = (0, |b_1| + 1) \cup \big(\bigcup_{j=2}^{\infty}(|b_j| - |b_j|^{-2}, |b_j| + |b_j|^{-2})\big)$, we obtain the following (cf. the proofs of Lemmas 3.2, 3.3 and 3.4).

Lemma 4.2. *Under* (4.1), *if* $r \ge 1$, *then, for every* $z \in \Delta_0(r)$,

$$\tilde{\chi}(2r, z) = \sum_{|b_j| \ge 2r} |(z - b_j)^{-1} + b_j^{-1}| \ll r^{1/2-\varepsilon}, \quad \tilde{\gamma}(2r) = \sum_{0<|b_j| \le 2r} |b_j|^{-1} \ll r^{1/2-\varepsilon}.$$

Lemma 4.3. *Under* (4.1), *there exists a set* $E^* \subset (0, \infty)$ *with finite linear measure such that, for every* z *satisfying* $|z| \in (0, \infty) \setminus E^*$,

$$\sum_{0<|b_j|<\infty} |(z - b_j)^{-1} + b_j^{-1}| \ll |z|^4.$$

Let $\{b'_j\}_{j=1}^{\infty}$ $(\supset \{b_j\}_{j=1}^{\infty})$ be the poles and the zeros of $w(z)$, each zero repeated according to its multiplicity. Put $n_*(r) = n(r, w) + n(r, 1/w)$. Using Lemma 4.1 and (4.1), we have $n_*(r) \ll r^{3/2-\varepsilon/2}$, since $n(r, 1/w) \ll N(2r, 1/w) \ll N(2r, w) + m(2r, w) \ll n(2r, w)\log r$. Then, by Lemma 2.1, for every sufficiently large r, we can find a point $z_r \in \Delta_0(r) \setminus \Delta_0(r/\sqrt{2})$ satisfying the relations $S_0(\{b'_j\}, 2|z_r|, z_r) \le S_0(\{b'_j\}, 2r, z_r) \ll n_*(2r)r^{-1} \ll r^{1/2-\varepsilon/2}$, $S_1(\{b'_j\}, 2|z_r|, z_r) \le S_1(\{b'_j\}, 2r, z_r) \ll n_*(2r)r^{-2}\log r \ll 1$ and $S_0(\{b_j\}, 2r, z_r) \le S_0(\{b'_j\}, 2r, z_r) \ll r^{1/2-\varepsilon/2}$; moreover, we have $T(2|z_r|, w) \ll r^{3/2-\varepsilon}$. These inequalities combined with Lemmas 2.2 and 4.2 yield

$$|\Psi(z_r)| \le S_0(\{b_j\}, 2r, z_r) + \tilde{\gamma}(2r) + \tilde{\chi}(2r, z_r) \ll r^{1/2-\varepsilon/2}, \tag{4.2}$$

$$|W(z_r)| \ll T(2|z_r|, w)|z_r|^{-1} + S_0(\{b'_j\}, 2|z_r|, z_r) + 1 \ll r^{1/2-\varepsilon/2}, \tag{4.3}$$

$$|W'(z_r)| \ll T(2|z_r|, w)|z_r|^{-2} + S_1(\{b'_j\}, 2|z_r|, z_r) + 1 \ll 1, \tag{4.4}$$

where $W(z) = w'(z)/w(z)$. From $(\mathrm{II})_0$, we have

$$W'(z_r) + W(z_r)^2 = 2w(z_r)^2 + z_r. \tag{4.5}$$

By Lemmas 4.1 and 4.3, $\psi(z) = w(z) - \Psi(z)$ is a polynomial. From (4.5) together with (4.2), (4.3) and (4.4), we obtain $|\psi(z_r)| \ll r^{1/2}$, so that $\psi(z) \equiv C_0 \in \boldsymbol{C}$. By (4.3), (4.4) and the fact that $|w(z_r)| \ll |\Psi(z_r)| \ll r^{1/2-\varepsilon/2}$, it follows from (4.5) that $r/\sqrt{2} \le |z_r| \ll |W'(z_r)| + |W(z_r)|^2 + |w(z_r)|^2 \ll r^{1-\varepsilon}$, which is a contradiction. Thus we conclude that $\sigma(w) \ge 3/2$.

4.3. Proof for the case where $2\alpha \in \boldsymbol{Z} \setminus \{0\}$

Note the lemmas below ([2, Propositions 2.5 and 2.7]) concerning Bäcklund transformations for $(\mathrm{II})_\alpha$ and a relation between $(\mathrm{II})_{\pm 1/2}$ and $(\mathrm{II})_0$.

Lemma 4.4. *Let w_α be a solution of* $(\mathrm{II})_\alpha$. *As far as $w'_\alpha - w_\alpha^2 - z/2 \not\equiv 0$ (resp. $w'_\alpha + w_\alpha^2 + z/2 \not\equiv 0$), the function*

$$w_{\alpha-1} = -w_\alpha + \frac{\alpha - 1/2}{w'_\alpha - w_\alpha^2 - z/2} \quad \left(\textit{resp. } w_{\alpha+1} = -w_\alpha - \frac{\alpha + 1/2}{w'_\alpha + w_\alpha^2 + z/2}\right)$$

satisfies $(\mathrm{II})_{\alpha-1}$ *(resp.* $(\mathrm{II})_{\alpha+1}$*). Equation* $(\mathrm{II})_\alpha$ *admits a solution satisfying the equation $w' - w^2 - z/2 = 0$ (resp. $w' + w^2 + z/2 = 0$) if and only if $\alpha = 1/2$ (resp. $\alpha = -1/2$).*

Lemma 4.5. *Let $w_{\pm 1/2}(z)$ be an arbitrary solution of* $(\mathrm{II})_{\pm 1/2}$. *Then there exists a solution $w_0(z)$ of* $(\mathrm{II})_0$ *such that $w'_{\pm 1/2}(z) = \pm\big(w_{\pm 1/2}(z)^2 + z/2 - 2^{1/3} w_0(\mp 2^{-1/3} z)^2\big)$.*

Suppose that $(\mathrm{II})_{1/2}$ admits a solution $w_{1/2}(z)$ satisfying $T(r, w_{1/2}) \ll r^{3/2-\varepsilon}$ for some $\varepsilon > 0$. Then, by Lemma 4.5, there exists a solution $w_0(z)$ of $(\mathrm{II})_0$ such that $-2^{1/3} w_0(-2^{-1/3} z)^2 = w'_{1/2}(z) - w_{1/2}(z)^2 - z/2$. From this relation we have $T(r, w_0) \ll r^{3/2-\varepsilon}$, which implies $w_0(z) \equiv 0$ (cf. Section 4.2), namely $w'_{1/2}(z) - w_{1/2}(z)^2 - z/2 \equiv 0$. Hence, $w_{1/2}(z) = -y'(z)/y(z)$, where $y(z)$ ($\not\equiv 0$) is a solution of $y'' + (z/2)y = 0$. This implies $\sigma(w_{1/2}) = 3/2$, which contradicts the supposition. Thus we conclude $\sigma(w_{1/2}) \geq 3/2$. The case where $\alpha = -1/2$ is treated in the same way. In the case where $2\alpha \in \boldsymbol{Z} \setminus \{0, \pm 1\}$, supposing the existence of a solution $w_\alpha(z)$ such that $T(r, w_\alpha) \ll r^{3/2-\varepsilon}$, and applying Bäcklund transformations of Lemma 4.4 finitely many times, we get a solution $v_{\alpha_0}(z)$, $\alpha_0 \in \{0, \pm 1/2\}$ of $(\mathrm{II})_{\alpha_0}$ such that $\sigma(v_{\alpha_0}) \leq 3/2 - \varepsilon$; which contradicts the fact shown above, except the case where $\alpha_0 = 0$, $v_0(z) \equiv 0$. In case $v_0(z) \equiv 0$, applying Bäcklund transformations reversely, we see that $w_\alpha(z)$ is a rational solution. In this way $\sigma(w) \geq 3/2$ has been proved for every transcendental solution of $(\mathrm{II})_\alpha$, $2\alpha \in \boldsymbol{Z}$.

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