EXISTENCE RESULTS FOR p-LAPLACIAN-LIKE SYSTEMS OF O.D.E.'S

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1. Introduction

In this paper we study the boundary value problem

$$(D_f) \qquad \begin{cases} (\phi(u'))' = f(t,u) & \text{a.e.} \quad \text{in} \quad (0,T), \\ u(0) = 0, \quad u(T) = 0, \end{cases}$$

where ϕ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ and the function $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory. Here $I := [0,T]$ and $' := \frac{d}{dt}$.

By a *solution* of (D_f) we understand a function $u : I \to \mathbb{R}^N$ of class C^1 with $\phi(u')$ absolutely continuous, which satisfies (D_f).

In most of the paper we shall ask ϕ to satisfy the following conditions:

(H_1) *For any* $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$,

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle > 0.$$

(H_2) *There exists a function* $\rho : [0,+\infty[\to [0,+\infty[$ *such that* $\rho(s) \to +\infty$ *as* $s \to +\infty$ *and*

$$\langle \phi(x), x \rangle \geq \rho(|x|)|x|, \quad \text{for all } x \in \mathbb{R}^N.$$

In (H_1) and (H_2), as in the rest of the paper, $\langle \cdot, \cdot \rangle$ denotes the inner product and $|\cdot|$ the Euclidean norm in $\mathbb{R}^N$. Throughout the paper $|\cdot|$ will also denote the absolute value in $\mathbb{R}$.

It is well known that conditions (H_1) and (H_2) ensure that ϕ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$. The vector version of the $p-$Laplace operator, namely the case when for $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$,

(1.1) $\phi(x) = \psi_p(x) \equiv |x|^{p-2}x, \quad \text{for} \quad x \neq 0, \quad \psi_p(0) = 0, \; p > 1,$

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as well as the cases

$$\phi(x) = |x|^{p-2}x\log(1+|x|), \quad p > 1, \quad x \in \mathbb{R}^N, \tag{1.2}$$

$$\phi(x) = |x|^{p-2}x + |x|^{q-2}x, \quad 1 < q < p, \quad x \in \mathbb{R}^N, \tag{1.3}$$

satisfy conditions (H_1) and (H_2). Further examples of functions satisfying these conditions can be found in [8].

When $N = 1$ it can be checked that the function ϕ as given in examples (1.1), (1.2) and (1.3) also satisfy the property

$$\lim_{|s|\to\infty} \frac{\phi(\sigma s)}{\phi(s)} = \sigma^{p-1} \qquad \text{for all } \sigma > 0. \tag{1.4}$$

In the scalar case, functions ϕ satisfying (1.4) have been called asymptotically homogeneous functions, see [1], [4], [5] and [6], where they were used in connection with the existence of solutions to quasilinear elliptic problems. They form an important class of non-homogeneous functions satisfying a suitable homogeneous behavior at infinity (or zero) without being necessarily asymptotic to any power at infinity or zero.

As we said before the function $f : I \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Carathéodory. This means that f satisfies the following conditions:

- (C_1) for almost every $t \in I$ the function $f(t,\cdot)$ is continuous;
- (C_2) for each $x \in \mathbb{R}^N$ the function $f(\cdot, x)$ is measurable on I;
- (C_3) for each $m > 0$ there is $\rho_m \in L^1(I,\mathbb{R})$ such that, for almost every $t \in I$ and every $x \in \mathbb{R}^N$ with $|x| \le m$, one has

$$|f(t,x)| \le \rho_m(t).$$

In case $f : I \times \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ (mapping (t, x, λ) into $f(t, x, \lambda)$), we shall say f is Carathéodory, if (C_1) and (C_2) are satisfied for each $\lambda \in [0,1]$ and the function ρ_m in (C_3) can be chosen independently of $\lambda \in [0,1]$, i.e.,

$$|f(t,x,\lambda)| \le \rho_m(t) \quad \text{for a.e } t \in I, \text{ all } \lambda \in [0,1] \text{ and all } |x| \le m.$$

We state a piece of notations used in this paper. For $N \ge 1$ we shall set $C = C(I,\mathbb{R}^N)$, $C^1 = C^1(I,\mathbb{R}^N)$, $C_0 = \{u \in C \mid u(0) = 0,\ u(T) = 0\}$, $C_0^1 = \{u \in C^1 \mid u(0) = 0,\ u(T) = 0\}$. The norm in C and C_0 will be denoted by $||\cdot||$, while the norm in C_0^1 by $||\cdot||_{C_0^1}$. $L^p = L^p((0,T),\mathbb{R})$, $L^p_N = \prod_{i=1}^N L^p((0,T),\mathbb{R})$, $p \ge 1$.

This paper is organized as follows. In section 2 we extend the concept of Asymptotically Homogeneous functions from the scalar to the vector case, and study some of their properties. In particular it is seen how this family is related to the vector p-Laplace function $|u|^{p-2}u$ at infinity.

In section 3 we study the eigenvalues of the weighted eigenvalue problem of the form

$$(1.5) \qquad (E_p) \quad \begin{cases} (\psi_p(u'))' + \mu\alpha(t)\psi_p(u) = 0, \\ u(0) = 0, \ u(T) = 0, \end{cases}$$

where α is positive a.e. and belongs to L^1. This is a key result to be used in later sections. Together with this result we also show in this section that an associated initial value problem has a unique solution.

In section 4 we give our first existence result for a system of ode's, under Dirichlet boundary conditions, containing a quasilinear operator generated by a vector function which is Asymptotically Homogeneous at infinity.

Section 5 is dedicated to the study of a system of ode's whose quasilinear operator is not Asymptotically Homogeneous, but its components are. A simple example of this situation is given by the function $\big(|x_1|^{p_1-2}x_1, \cdots, |x_N|^{p_N-2}x_N\big)$, which is not Asymptotically Homogeneous at infinity. We prove in this section our second existence result for a system of ode's, with Dirichlet boundary conditions, containing this type of quasilinear operator.

Finally in Section 6, we give some examples of vector functions which are Asymptotically Homogeneous at infinity.

2. Asymptotically homogeneous functions

As we said in the introduction, in this section we extend the class of Asymptotically Homogeneous (for short AH) functions to the vector case and study some of their properties.

We say that $\phi : \mathbb{R}^N \to \mathbb{R}^N$ continuous *is AH of order $p > 1$ at infinity* if it satisfies

(i) $\lim_{|y|\to\infty} |\phi(y)| = \infty$, and

(ii) *every time $\{y_n\}$ and $\{x_n\}$ are two sequences in $\mathbb{R}^N$ such that $|y_n| \to \infty$ and $x_n \to x$ as $n \to \infty$, then*

$$(2.1) \qquad \lim_{n\to\infty} \frac{\phi(|y_n|x_n)}{|\phi(y_n)|} = |x|^{p-2}x.$$

Remark 2.1. In case $x_n \to 0$, (2.1) is needed only when $\{|y_n||x_n|\}$ is not bounded.

It is not difficult to see that in the scalar case (with ϕ odd) (2.1) holds if and only if (1.4) is satisfied. Thus (2.1) is a natural generalization of condition (1.4) to the vector case.

On the other hand, if in the scalar case ϕ satisfies (1.4) then ϕ^{-1} also satisfies (1.4) with p replaced by $p^* = p/(p-1)$, i.e., if ϕ is AH of order p at infinity then ϕ^{-1} is AH of order p^* at infinity. We shall show in Proposition 2.3 below that this is also true for the vector case. First we need a couple of propositions.

Proposition 2.1. *Assume that $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is AH of order $p > 1$ at infinity. Let $\{y_n\}$ be a sequence in $\mathbb{R}^N$ and $\{\alpha_n\}$ a sequence in $\mathbb{R}$ such that $|y_n| \to \infty$ and $\alpha_n \to \alpha$ as $n \to \infty$. Then we have*

$$\lim_{n\to\infty} \frac{|\phi(\alpha_n y_n)|}{|\phi(y_n)|} = |\alpha|^{p-1}.$$

Proof. Set $x_n = \dfrac{y_n}{|y_n|}\alpha_n$, then there exists a subsequence x_{n_k} such that $x_{n_k} \to \tilde{y}\alpha$ with $|\tilde{y}| = 1$. Therefore, by (2.1), we get

$$\lim_{k\to\infty} \frac{\phi(\alpha_{n_k} y_{n_k})}{|\phi(y_{n_k})|} = \lim_{k\to\infty} \frac{\phi(|y_{n_k}| x_{n_k})}{|\phi(y_{n_k})|} = |\alpha|^{p-2}\alpha|\tilde{y}|^{p-2}\tilde{y}.$$

Hence we obtain

$$\lim_{k\to\infty} \frac{|\phi(\alpha_{n_k} y_{n_k})|}{|\phi(y_{n_k})|} = |\alpha|^{p-1}.$$

Since this argument does not depend on the choice of the subsequence, the original sequence $\dfrac{|\phi(\alpha_n y_n)|}{|\phi(y_n)|}$ converges to $|\alpha|^{p-1}$. □

Proposition 2.2. *Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be AH of order $p > 1$ at infinity, then ϕ satisfies the following growth condition: every time $\{y_n\}$ is a sequence in $\mathbb{R}^N$ and $\{\beta_n\}$ is a sequence in $\mathbb{R}$ such that $|y_n| \to \infty$ and $\beta_n \to \infty$ as $n \to \infty$, one has*

$$\lim_{n\to\infty} \frac{|\phi(\beta_n y_n)|}{|\phi(y_n)|} = \infty. \tag{2.2}$$

Proof. Let $\beta \neq 0$ and let $\{y_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{R}^N$ and $\mathbb{R}$ respectively such that $|y_n| \to \infty$ and $\beta_n \to \infty$. Set

$$A_n = \frac{|\phi(\beta_n y_n)|}{|\phi(y_n)|}, \qquad B_n = \frac{|\phi(\beta y_n)|}{|\phi(\beta_n y_n)|}.$$

Then, by Proposition 2.1,

$$\lim_{n\to\infty} A_n B_n = \lim_{n\to\infty} \frac{|\phi(\beta y_n)|}{|\phi(y_n)|} = |\beta|^{(p-1)} > 0. \tag{2.3}$$

On the other hand, since $|\beta_n y_n| \to \infty$ and $\dfrac{\beta}{\beta_n} \to 0$, Proposition 2.1 implies

$$\lim_{n\to\infty} B_n = \lim_{n\to\infty} \frac{|\phi(\frac{\beta}{\beta_n}(\beta_n y_n))|}{|\phi(\beta_n y_n)|} = 0. \tag{2.4}$$

Thus from (2.3) and (2.4) it follows that $\lim_{n\to\infty} A_n = +\infty$, which is what we wanted to prove. □

Now we can prove the following proposition.

Proposition 2.3. *Assume that ϕ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Then ϕ is AH of order $p > 1$ at infinity if and only if the inverse function ϕ^{-1} is AH of order p^* at infinity.*

Proof. Enough to prove one direction of the proposition. Thus we assume ϕ is AH of order $p > 1$ at infinity and note that (H_1) and (H_2) imply that $|\phi^{-1}(z)| \to \infty$ as $|z| \to \infty$. Next let $\{u_n\}$ and $\{z_n\}$ be two sequences in $\mathbb{R}^N$ such that $|z_n| \to \infty$ and $u_n \to u$ as $n \to \infty$, and set

$$w_n = \frac{\phi^{-1}(|z_n| u_n)}{|\phi^{-1}(z_n)|}. \tag{2.5}$$

We claim that the sequence $\{w_n\}$ is bounded. Otherwise, by passing to a subsequence if necessary, we can assume that $|w_n| \to \infty$ as $n \to \infty$. Noting that from (2.5)

$$u_n = \frac{\phi\big(w_n |\phi^{-1}(z_n)|\big)}{|\phi(\phi^{-1}(z_n))|},$$

by writing $w_n = |w_n| \hat{w}_n$ (with $|\hat{w}_n| = 1$), we find that

$$\frac{|\phi\big(\hat{w}_n |w_n|\, |\phi^{-1}(z_n)|\big)|}{|\phi\big(|w_n|\, \phi^{-1}(z_n)\big)|} = \frac{|\phi(\phi^{-1}(z_n))|}{|\phi\big(|w_n|\, \phi^{-1}(z_n)\big)|} |u_n|.$$

Then from (2.2) and the fact that $\{u_n\}$ is a convergent sequence, by letting $n \to \infty$ we obtain

$$\frac{\phi\big(\hat{w}_n |w_n|\, |\phi^{-1}(z_n)|\big)}{|\phi\big(|w_n|\, \phi^{-1}(z_n)\big)|} \quad \to \quad 0.$$

On the other hand, by passing to a subsequence if necessary, we can assume that $\hat{w}_n \to w_0$ in $\mathbb{R}^N$ with $|w_0| = 1$, and then by (2.1) we find that

$$\frac{\phi\big(\hat{w}_n |w_n|\, |\phi^{-1}(z_n)|\big)}{|\phi\big(|w_n|\, \phi^{-1}(z_n)\big)|} \quad \to \quad |w_0|^{p-2} w_0 = w_0,$$

which is a contradiction, showing the validity of the claim.

From this claim by passing to a subsequence if necessary, we can assume that $w_n \to w$ in $\mathbb{R}^N$. Then from (2.1), we find

$$u_n = \frac{\phi\big(w_n|\phi^{-1}(z_n)|\big)}{|\phi(\phi^{-1}(z_n))|} \quad \to \quad \psi_p(w) \quad \text{as} \quad n \to \infty,$$

and thus $w = \psi_{p^*}(u)$. Since we can apply this argument to any subsequence of the original sequence $\{w_n\}$ in (2.5), we have proved that the entire original sequence in (2.5) converges to $\psi_{p^*}(w)$, hence ϕ^{-1} is AH of order p^* at infinity. □

We say that ϕ is *Asymptotically Bounded* (for short AB) at infinity, *if every time* $\{y_n\}$ *and* $\{x_n\}$ *are two sequences in* $\mathbb{R}^N$ *such that* $|y_n| \to \infty$ *and* $\{x_n\}$ *is bounded, then the sequence* $\left\{\frac{\phi(|y_n|x_n)}{|\phi(y_n)|}\right\}$ *is bounded.* The AB property follows from the AH property in the following sense.

Proposition 2.4. *Suppose that ϕ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ which is AH of order $p > 1$ at infinity. Then ϕ and ϕ^{-1} are AB at infinity.*

Proof. We just show that ϕ is AB at infinity. Thus let $\{u_n\}$ and $\{z_n\}$ be two sequences in $\mathbb{R}^N$ such that $|z_n| \to \infty$ and $\{u_n\}$ is bounded. We want to show that the sequence $\{w_n\}$ defined by

$$w_n = \frac{\phi(|z_n|u_n)}{|\phi(z_n)|} \tag{2.6}$$

is bounded. Solving (2.6) for u_n, we get

$$u_n = \frac{\phi^{-1}(|\phi(z_n)|w_n)}{|\phi^{-1}(\phi(z_n))|}. \tag{2.7}$$

Assume the sequence $\{w_n\}$ is unbounded , then passing to a subsequence, renamed the same, we can assume that $|w_n| \to \infty$ as $n \to \infty$ and that the sequence $\{\hat{w}_n\}$ with $\hat{w}_n = \frac{w_n}{|w_n|} \to \hat{w}$ ($|\hat{w}| = 1$).

Let us set $v_n = \phi(z_n)$, then

$$|u_n| = \frac{|\phi^{-1}(|v_n||w_n|\hat{w}_n)|}{|\phi^{-1}(|w_n|v_n)|}\frac{|\phi^{-1}(|w_n|v_n)|}{|\phi^{-1}(v_n)|}. \tag{2.8}$$

Since ϕ^{-1} is AH at infinity by Proposition 2.3, we have from Proposition 2.1,

$$\frac{|\phi^{-1}(|v_n||w_n|\hat{w}_n)|}{|\phi^{-1}(|w_n|v_n)|} \quad \to 1,$$

and since ϕ^{-1} satisfies (2.2) (with ϕ replaced by ϕ^{-1}), it follows that

$$\frac{|\phi^{-1}(|w_n|v_n)|}{|\phi^{-1}(v_n)|} \quad \to \infty,$$

as $n \to \infty$. Hence from (2.8) the sequence $\{u_n\}$ is unbounded which is a contradiction, ending the proof of the proposition. $\square$

3. A vector eigenvalue problem

The purpose of this section is to study the vector eigenvalue problem

$$(3.1) \qquad (E_p) \quad \begin{cases} (\psi_p(u'))' + \mu\alpha(t)\psi_p(u) = 0, \\ u(0) = 0, \; u(T) = 0, \end{cases}$$

where α is positive a.e. and belongs to L^1. The results we shall obtain in this section will be used later in our main existence Theorems 4.1 and 5.1.

Suppose u is a nontrivial solution of (E_p). Multiplying the equation in (E_p) by u and integrating, we find that

$$\mu = \frac{\int_0^T |u'(t)|^p dt}{\int_0^T \alpha(t)|u(t)|^p dt},$$

which says that any possible eigenvalue of (E_p) is positive. Hence, without loss of generality, from now on we assume that $\mu > 0$ in (E_p).

A basic tool to study problem (E_p) is the initial value problem

$$(I_p) \quad \begin{cases} (\psi_p(u'))' + \mu\alpha(t)\psi_p(u) = 0, \\ u(t_0) = a, \; u'(t_0) = b. \end{cases}$$

Indeed assuming that for any $a, b \in \mathbb{R}^N$ and any $t_0 \in I$, the initial value problem has a unique solution defined in all I, we have immediately the following result:

Proposition 3.1. *If u is a solution of (E_p), it holds that*

$$(3.2) \qquad u(t) = w(t)d,$$

where $d = u'(0)$ and w is the unique solution of the scalar initial value problem

$$(S_p) \quad \begin{cases} (|w'|^{p-2}w')' + \mu\alpha(t)|w|^{p-2}w = 0, \\ w(0) = 0, \quad w'(0) = 1. \end{cases}$$

This proposition generalizes a result of Del Pino [2], see also Proposition 3.1 in [10] (for the case $\alpha \equiv 1$) and [12]. Since u in (3.2) also satisfies $u(T) = 0$, it must be that $w(T) = 0$ and thus w is a normalized solution to the scalar eigenvalue problem

$$(Z_p) \quad \begin{cases} (|w'|^{p-2}w')' + \mu\alpha(t)|w|^{p-2}w = 0, \\ w(0) = 0, \quad w(T) = 0. \end{cases}$$

Then, from the results of M. Zhang, see section 2 in [13], it follows that problem (Z_p) has a sequence of (positive) eigenvalues $\mu_k(\alpha)$ such that $\mu_k(\alpha) \to \infty$ as $k \to \infty$. Furthermore associated with each $\mu_k(\alpha)$ there is an eigenfunction w_k of (Z_p) which also satisfies (S_p) with μ replaced by $\mu_k(\alpha)$. Any other eigenfunctions corresponding to $\mu_k(\alpha)$ are of the form Cw_k, with C a constant. Also from the results in [13] one can easily see that $w_1(t) > 0$ for all $t \in (0,T)$ and that for $k \geq 2$ the eigenfunction w_k has exactly $k-1$ simple zeros in $(0,T)$.

It is then clear that if μ is an eigenvalue of (E_p) then it must be the case that $\mu = \mu_k$ for some $k \in \mathbb{N}$ with corresponding eigenfunction $u^k(t) = w_k(t)d$, for any $d \in \mathbb{R}^N$. In other words the eigenvalues of (E_p) are the same as those of the scalar case and to each eigenvalue corresponds an N-dimensional space of eigenvectors. Furthermore the eigenvector u^1 corresponding to μ_1 satisfies that each of its component is one signed in $(0,T)$.

We dedicate the rest of this section to the study of the following initial value problem slightly more general than (I_p). From this problem the results for problem (E_p) follow immediately.

Proposition 3.2. *Suppose that $\beta \in L^1_{\text{loc}}(\mathbb{R})$ with $\beta > 0$ a.e. in $\mathbb{R}$. Then for any $a \in \mathbb{R}^N$, $b \in \mathbb{R}^N$ and $t_0 \in \mathbb{R}$, the problem*

$$(IV) \qquad \begin{cases} (|u'|^{p-2}u')' + \beta(t)|u|^{p-2}u = 0, \\ u(t_0) = a, \quad u'(t_0) = b, \end{cases}$$

has a unique solution defined in all $\mathbb{R}$.

Proof. Setting $\tau = t - t_0$, we easily see that problem (IV) is transformed to one of the same form for the weight $\beta(\cdot + t_0)$, and with the initial conditions evaluated at $t = 0$. Because of this fact, without loss of generality, we assume in the rest of the proof that $t_0 = 0$ in problem (IV).

The existence of a local solution in a small interval around 0 is a direct application of Schauder's fixed point theorem, see for example [7]. We show next that this local solution is unique by examining the different possibilities. We do this for $t > 0$ only.

Case $a = 0,\ b = 0$. For small $\varepsilon > 0$, let $\delta > 0$ be such that

$$\delta\Big(\int_0^\delta \beta(s)ds\Big)^{\frac{1}{p-1}} \leq \varepsilon. \tag{3.3}$$

Integrating the equation of (IV) from 0 to $t \in (0, \delta]$, we find

$$u'(t) = -\psi_{p^*}\Big(\int_0^t \beta(s)\psi_p(u(s))ds\Big),$$

where ψ_{p^*} is defined in (1.1). This implies that

$$|u'(t)| \leq \Big(\int_0^\delta \beta(s)ds\Big)^{\frac{1}{p-1}} ||u||_{C_{\delta,N}}, \qquad t \in [0,\delta],$$

here and in the rest of this section $||u||_{C_{\delta,N}} := ||u||_{C([0,\delta],\mathbb{R}^N)}$. By a new integration

$$||u||_{C_{\delta,N}} \leq \delta\Big(\int_0^\delta \beta(s)ds\Big)^{\frac{1}{p-1}} ||u||_{C_{\delta,N}},$$

and thus

$$\Big(1 - \delta\Big(\int_0^\delta \beta(s)ds\Big)^{\frac{1}{p-1}}\Big) ||u||_{C_{\delta,N}} \leq 0,$$

yielding $u(t) = 0$ for $t \in [0,\delta]$, which is what we wanted to prove.

Case $a \neq 0,\ b \neq 0$. We write problem (IV) in the equivalent form

$$\begin{cases} u' = \psi_{p^*}(v), \quad v' = -\beta(t)\psi_p(u), \\ u(0) = a,\ v(0) = \psi_p(b). \end{cases} \tag{3.4}$$

Since ψ_{p^*} and ψ_p are of class C^1 in neighborhoods of $\psi_p(b)$ and a respectively, the solution is locally unique for this case applying classical theory, see [7].

Case $a \neq 0,\ b = 0$. Assume that u_1 and u_2 are two local solutions to (IV) with $u_i(0) = a$ and $u_i'(0) = 0$, $i = 1,2$. Let $\delta > 0$, be small, then there is a $\varepsilon_0 \in (0,|a|)$ such that $u_1(t)$ and $u_2(t)$ belong to $B(a,\varepsilon_0) \subset \mathbb{R}^N$, the ball centered at a with radius ε_0, for all $t \in [0,\delta]$. Thus $||u_i||_{C_{\delta,N}} \leq |a| + \varepsilon_0$, $i = 1,2$, and the convex combination $u_1(\tau) + r(u_2(\tau) - u_1(\tau))$ belongs to $B(a,\varepsilon_0)$, for all $\tau \in [0,\delta]$ and all $r \in [0,1]$.

Now, by integrating the equation of (IV) for each solution u_i and combining, we find,

$$\begin{aligned} (3.5)\quad & |u_1'(s) - u_2'(s)| \\ &= \Big|\psi_{p^*}\Big(\int_0^s \beta(\tau)\psi_p(u_1(\tau))d\tau\Big) - \psi_{p^*}\Big(\int_0^s \beta(\tau)\psi_p(u_2(\tau))d\tau\Big)\Big|. \end{aligned}$$

For fixed $0 < s \leq \delta$ and for $r \in [0,1]$, define

$$\gamma_s(r) := \psi_{p^*}(z_s(r)), \quad z_s(r) := \int_0^s \beta(\tau)w(\tau,r)d\tau, \tag{3.6}$$

with

$$w(\tau,r) = \psi_p(u_1(\tau) + r(u_2(\tau) - u_1(\tau))). \tag{3.7}$$

Since

$$v(\tau,r) := u_1(\tau) + r(u_2(\tau) - u_1(\tau)) \neq 0,$$

for all $\tau \in [0,\delta]$ and $r \in [0,1]$, we get $w \in C^1([0,\delta]\times[0,1])$. Then, from (3.6),

$$\begin{aligned}\frac{dz_s(r)}{dr} &= \int_0^s \beta(\tau)\frac{\partial w(\tau,r)}{\partial r}d\tau\\ &= \int_0^s \beta(\tau)\Big((p-2)|v(\tau,r)|^{p-4}\langle v(\tau,r),u_2(\tau)-u_1(\tau)\rangle v(\tau,r)\\ &\qquad +|v(\tau,r)|^{p-2}(u_2(\tau)-u_1(\tau))\Big)d\tau.\end{aligned}$$

Hence

$$(3.8)\quad |\frac{dz_s(r)}{dr}| \le (|a|+\varepsilon_0)^{p-2}(|p-2|+1)(\int_0^s \beta(\tau)d\tau\,)||u_1-u_2||_{C_{\delta,N}}.$$

Next, for $\varepsilon_1>0$, by the continuity of the function ψ_p, and by diminishing δ if necessary, we have

$$|\psi_p(u_1(\tau)+r(u_2(\tau)-u_1(\tau)))-\psi_p(a)| \le \varepsilon_1,$$

for all $\tau\in[0,\delta]$ and all $r\in[0,1]$. Hence

$$(3.9)\qquad |z_s(r)-(\int_0^s \beta(\tau)d\tau)\psi_p(a)| \le \varepsilon_1\int_0^s \beta(\tau)d\tau,$$

for all $r\in[0,1]$. Thus, taking ε_1 small enough (for example $\varepsilon_1 \le \frac{|\psi_p(a)|}{2}$), we see that for any fixed $s\in(0,\delta]$, $z_s(r)\neq 0$ for all $r\in[0,1]$. This implies that for any fixed $s\in(0,\delta]$, the function γ_s defined in (3.6) is of class $C^1([0,\delta])$. By differentiating

$$\frac{d\gamma_s(r)}{dr} = (p^*-2)|z_s(r)|^{p^*-4}\langle z_s(r),\frac{dz_s(r)}{dr}\rangle z_s(r)+|z_s(r)|^{p^*-2}\frac{dz_s(r)}{dr},$$

and hence

$$(3.10)\qquad |\frac{d\gamma_s(r)}{dr}| \le (|p^*-2|+1)|z_s(r)|^{p^*-2}|\frac{dz_s(r)}{dr}|.$$

From (3.9), we obtain

$$|z_s(r)| \le c_1\int_0^s \beta(\tau)d\tau,$$

with $c_1=(|\psi_p(a)|+\varepsilon_1)$. Replacing this expression and (3.8) into (3.10), we find

$$|\frac{d\gamma_s(r)}{dr}| \le c_3(\int_0^\delta \beta(\tau)d\tau)^{p^*-1}\;||u_1-u_2||_{C_{\delta,N}},$$

where c_2 and c_3 are positive constants independent of s. Then, from (3.5) and (3.6),

$$\begin{aligned} |u_1'(s) - u_2'(s)| = |\gamma_s(0) - \gamma_s(1)| &\leq \int_0^1 |\frac{d\gamma_s(r)}{dr}| dr \\ &\leq c_3 \Big(\int_0^\delta \beta(\tau) d\tau \Big)^{p^*-1} ||u_1 - u_2||_{C_{\delta,N}}, \end{aligned}$$

which implies

$$|u_1(s) - u_2(s)| \leq c_3 \delta \Big(\int_0^\delta \beta(\tau) d\tau \Big)^{p^*-1} ||u_1 - u_2||_{C_{\delta,N}},$$

for all $s \in [0, \delta]$. Since this implies

$$||u_1 - u_2||_{C_{\delta,N}} \big[1 - c_3 \delta \Big(\int_0^\delta \beta(\tau) d\tau \Big)^{p^*-1} \big] \leq 0,$$

we find, by diminishing δ if required, that $u_1(t) = u_2(t)$, for all $t \in [0, \delta]$.

Case $a = 0,\ b \neq 0$. As in the second case, we consider the equivalent system

$$\text{(3.11)} \qquad \begin{cases} u' = \psi_{p^*}(v), \quad v' = -\beta(t)\psi_p(u), \\ u(0) = 0, \quad v(0) = \psi_p(b) \neq 0. \end{cases}$$

Let (u_i, v_i) be two solutions of this system for $t \in [0, \delta]$, with δ small such that $v_i(t)$, $i = 1, 2$, belongs to the ball $B(\psi_p(b)), \varepsilon_0) \subset \mathbb{R}^N$, the ball centered at $\psi_p(b)$ and with small radius $\varepsilon_0 \in (0, |\psi_p(b)|)$, for all $t \in [0, \delta]$. Hence $v_1(t) + s(v_2(t) - v_1(t)) \in B(\psi_p(b), \varepsilon_0)$, for all $t \in [0, \delta]$ and all $s \in [0, 1]$. Integrating the first equation of (3.11), for each solution (u_i, v_i), substituting the result into the second equation and subtracting, we find

$$|v_1'(t) - v_2'(t)| = \beta(t) |\psi_p \Big(\int_0^t \psi_{p^*}(v_2(s)) ds \Big) - \psi_p \Big(\int_0^t \psi_{p^*}(v_1(s)) ds \Big)|,$$

expression entirely similar to (3.5). Thus, by the argument following (3.5) of the previous case, with obvious minor modifications, we find this time that

$$||v_1 - v_2||_{C_{\delta,N}} \big[1 - c\delta \Big(\int_0^\delta \beta(\tau) d\tau \Big) \big] \leq 0,$$

where c is a positive constant. Then, for δ small, $v_1(t) = v_2(t)$, for all $t \in [0, \delta]$, and by the first equation in (3.11), $u_1(t) = u_2(t)$, for all $t \in [0, \delta]$.

Thus in all the cases the solution to (IV) is locally unique. Clearly this argument also implies the global uniqueness to the solutions of

(IV) in their corresponding maximal interval of existence. We show next that any solution to (IV) is defined in all $\mathbb{R}$. As before we only prove this statement for $[0,\infty)$.

Let u be a solution to (IV) defined in an interval of the form $[0,\omega)$. We shall show there is a constant $\tilde{M}$ such that $|u(t)| \leq \tilde{M}$ and $|u'(t)| \leq \tilde{M}$ for all $t \in [0,\omega)$, implying that this solution can be extended to $[0,\omega]$ and hence to all $[0,\infty)$.

Integrating the equation in (IV) (with $t_0 = 0$) from 0 to $t \in (0,\omega)$, we obtain

$$u'(t) = \psi_{p^*}\big[\psi_p(b) - \int_0^t \beta(\tau)\psi_p(u(\tau))d\tau\big],$$

and hence

$$u(t) = a + \int_0^t \psi_{p^*}\big[\psi_p(b) - \int_0^s \beta(\tau)\psi_p(u(\tau))d\tau\big]ds.$$

Then

$$|u'(t)| \leq \big[|\psi_p(b)| + \int_0^t |\beta(\tau)||(u(\tau))|^{p-1}d\tau\big]^{\frac{1}{p-1}}, \tag{3.12}$$

and

$$|u(t)| \leq |a| + \int_0^t \big[|\psi_p(b)| + \int_0^s |\beta(\tau)||(u(\tau))|^{p-1}d\tau\big]^{\frac{1}{p-1}}ds. \tag{3.13}$$

Next by using the fact that for any $q > 0$ and $c, d \in [0,\infty)$, there is a constant $C = C(q)$ such that

$$(c+d)^q \leq C(c^q + d^q),$$

we find from (3.13) that

$$|u(t)|^{p-1} \leq C(1 + v(t)), \tag{3.14}$$

where

$$v(t) = \int_0^t |\beta(\tau)||(u(\tau))|^{p-1}d\tau$$

and $C = C(\omega)$ is a positive constant. Then, from (3.14), we obtain

$$v'(t) \leq C\beta(t)(1 + v(t)).$$

An application of Gronwall's inequality shows then that there is a constant $M = M(\omega)$ such that $v(t) \leq M$ for all $t \in [0,\omega)$, and hence from (3.12) and (3.13), that there is a constant $\tilde{M} = \tilde{M}(\omega)$ such that $|u(t)| \leq \tilde{M}$ and $|u'(t)| \leq \tilde{M}$, for all $t \in [0,\omega)$, which is what we wanted to show. $\square$

4. A first existence result

We begin this section by considering the problem

$$(P_h) \qquad \begin{cases} (\phi(w'))' = h(t), \\ w(0) = 0, \quad w(T) = 0, \end{cases}$$

where $h \in L^1_N$ and $\phi : \mathbb{R}^N \mapsto \mathbb{R}^N$ satisfies conditions (H_1) and (H_2). Let us define $H : L^1_N \to C$ by $H(h)(t) = \int_0^t h(s)ds$. From [9], see also [8], there is a function $Q_\phi : C \to \mathbb{R}^N$ that sends bounded sets into bounded sets and which is such that the equation

$$\int_0^T \phi^{-1}\left(a + H(h)(s)\right) ds = 0 \tag{4.1}$$

has a unique solution given by $a = -Q_\phi(H(h))$. Using this fact, one can prove (see [9] and [8]) that for each $h \in L^1_N$, problem (P_h) has a unique solution given by

$$w = \mathcal{K}^\phi_d(h),$$

where $\mathcal{K}^\phi_d : L^1_N \to C^1_0$ is given by

$$\mathcal{K}^\phi_d(h)(t) = \int_0^t \phi^{-1}\big(-Q_\phi(H(h)) + \int_0^s h(\tau)d\tau\big)ds. \tag{4.2}$$

The mapping $Q_\phi \circ H : L^1_N \to \mathbb{R}^N$ is continuous and sends bounded sets of L^1_N into bounded sets of $\mathbb{R}^N$, hence it is a completely continuous operator. Furthermore, from Lemma 2.1 in [8], $\mathcal{K}^\phi_d$ is continuous and sends equi-integrable sets in L^1_N into relatively compact sets in C^1_0.

We note, as it can be easily proved, that

$$w'(0) = \phi^{-1}(-Q_\phi(H(h))).$$

In the case $\phi = \psi_p$ in (P_h), (4.2) takes the form

$$\begin{aligned} \mathcal{K}^{\psi_p}_d(h)(t) &= \int_0^t \psi_{p^*}\big(\psi_p(w'(0)) + \int_0^s h(\tau)d\tau\big)ds \\ &= \int_0^t \psi_{p^*}\big(-Q_{\psi_p}(H(h)) + \int_0^s h(\tau)d\tau\big)ds, \end{aligned} \tag{4.3}$$

with

$$w'(0) = -\psi_{p^*}(Q_{\psi_p}(H(h))).$$

Lemma 4.1. *Assume $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (H_1), (H_2) and is AH of order $p > 1$. Let $\gamma : I \times \mathbb{R}^N \times [0,1] \to \mathbb{R}^N$ be a Carathéodory function and denote its associated Nemytskii operator by $\Gamma : C_0 \times [0,1] \to L^1_N$. Assume that γ satisfies the following two conditions.*

(i) *There is* $\beta \in L^1$ *such that*

$$\limsup_{|y|\to\infty} \frac{|\gamma(t,y,\lambda)|}{|\phi(y)|} \le \beta(t) \quad \text{for a.e.} \quad t \in I, \tag{4.4}$$

uniformly in $\lambda \in [0,1]$, $t \in [0,T]$.

(ii) *Every time* $\{\lambda_n\}$ *is a sequence in* $[0,1]$, $\{y_n\}$ *and* $\{x_n\}$ *are two sequences in* $\mathbb{R}^N$ *such that* $\lambda_n \to \lambda$, $|y_n| \to \infty$ *and* $x_n \to x$, *as* $n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{\gamma(t,|y_n|x_n,\lambda_n)}{|\phi(y_n)|} = \alpha(t)|x|^{p-2}x \quad \text{for a.e.} \quad t \in I, \tag{4.5}$$

where $\alpha \in L^1$.

Then, every time $\{\lambda_n\}$ *is a sequence in* $[0,1]$, $\{y_n\}$ *is a sequence in* $\mathbb{R}^N$ *such that* $|y_n| \to \infty$, *and* $\{u_n\}$ *is a sequence in* C_0 *such that* $u_n \to u$ *as* $n \to \infty$, *it holds that*

$$\lim_{n\to\infty} \frac{\mathcal{K}_d^{\phi}(\Gamma(|y_n|u_n,\lambda_n))}{|y_n|} = \mathcal{K}_d^{\psi_p}(\alpha(\cdot)\psi_p(u)). \tag{4.6}$$

Proof. Let $\{\lambda_n\}$, $\{y_n\}$, $\{u_n\}$ be sequences in $[0,1]$, $\mathbb{R}^N$ and C_0 respectively such that $\lambda_n \to \lambda$, $|y_n| \to \infty$ and $u_n \to u$ as $n \to \infty$. Define the sequence $\{w_n\}$ by

$$w_n = \mathcal{K}_d^{\phi}(\Gamma(|y_n|u_n,\lambda_n)), \tag{4.7}$$

then w_n satisfies

$$w_n(t) = \int_0^t \phi^{-1}\big[\phi(w_n'(0)) + \int_0^\tau \gamma(s,|y_n|u_n(s),\lambda_n)ds\big]d\tau, \tag{4.8}$$

together with

$$\int_0^T \phi^{-1}\big[\phi(w_n'(0)) + \int_0^\tau \gamma(s,|y_n|u_n(s),\lambda_n)ds\big]d\tau = 0. \tag{4.9}$$

From (4.7) and (4.8), we find that

$$\frac{\mathcal{K}_d^{\phi}(\Gamma(|y_n|u_n,\lambda_n))}{|y_n|} = \frac{w_n(t)}{|y_n|} = \frac{\int_0^t \phi^{-1}\big(z_n(\tau)|\phi(y_n)|\big)}{|\phi^{-1}(\phi(y_n))|}d\tau, \tag{4.10}$$

where

$$z_n(\tau) = \frac{\phi(w_n'(0))}{|\phi(y_n)|} + \frac{\int_0^\tau \gamma(s,|y_n|u_n(s),\lambda_n)ds}{|\phi(y_n)|}. \tag{4.11}$$

We note that if $\{z_n\}$ were convergent, say $z_n \to z$ as $n \to \infty$, then from (4.10) and since ϕ^{-1} is AH and AB and $|\phi(y_n)| \to \infty$, by Lebesgue we would obtain

$$\lim_{n\to\infty} \frac{\mathcal{K}_d^{\phi}(\Gamma(|y_n|u_n, \lambda_n))}{|y_n|} = \lim_{n\to\infty} \frac{w_n(t)}{|y_n|} = \int_0^t \psi_{p^*}(z(\tau))d\tau. \tag{4.12}$$

We also note that by defining the sequence $\{\chi_n\}$ by

$$\chi_n(\tau) = \frac{\phi^{-1}\big(|\phi(y_n)|z_n(\tau)\big)}{|\phi^{-1}(\phi(y_n))|}, \tag{4.13}$$

then from (4.9) and (4.11), we find that

$$\int_0^T \chi_n(\tau)\ d\tau = \int_0^T \frac{\phi^{-1}\big(|\phi(y_n)|z_n(\tau)\big)}{|\phi^{-1}(\phi(y_n))|}\ d\tau = 0, \text{ for all } n \in \mathbb{N}. \tag{4.14}$$

We next prove the sequence $\{z_n\}$ is convergent. We begin by studying the second term on the right of (4.11). Let us set

$$v_n(s) = \frac{\gamma(s, |y_n|u_n(s), \lambda_n)}{|\phi(y_n)|}.$$

By (4.5) it is immediate that

$$v_n(s) \to \alpha(s)\psi_p(u(s)) \quad \text{for a.e.} \quad s \in (0,T). \tag{4.15}$$

Also since $\{u_n\}$ is a convergent sequence in C it follows that $u_n(s) \in B(0,R) \subset \mathbb{R}^N$ for some $R > 0$, for all $s \in I$ and all $n \in \mathbb{N}$.

Next observe that from (4.4), there is a $m_0 > 0$ such that for all $|y| \geq m_0$

$$\frac{|\gamma(s,y,\lambda)|}{|\phi(y)|} \leq \beta(s) + 1 \quad \text{for a.e.} \quad s \in I \quad \text{and all} \quad \lambda \in [0,1]. \tag{4.16}$$

Thus if $|u_n(s)||y_n| > m_0$, then

$$\frac{|\gamma(s, |y_n|u_n(s), \lambda_n)|}{|\phi(y_n)|} \leq (\beta(s)+1)\frac{|\phi(|y_n|u_n(s))|}{|\phi(y_n)|} \text{ for a.e. } s \in I. \tag{4.17}$$

On the other hand, since γ is Carathéodory if $|y| \leq m_0$, there is a $\rho \in L^1(I, \mathbb{R}^N)$ such that $|\gamma(s,y,\lambda)| \leq \rho(s)$ for a.e. $s \in I$. Thus, if it is the case that $|u_n(s)||y_n| \leq m_0$, then

$$|\gamma(s, |y_n|u_n(s), \lambda_n)| \leq \rho(s) \quad \text{for a.e.} \quad s \in I,$$

and hence, for sufficiently large n,

$$\frac{|\gamma(s, |y_n|u_n(s), \lambda_n)|}{|\phi(y_n)|} \leq \frac{\rho(s)}{|\phi(y_n)|} \leq C\rho(s) \quad \text{for a.e.} \quad s \in I, \tag{4.18}$$

where C is a positive constant.

Now since ϕ is AB, it is not difficult to see that the sequence $\{\dfrac{|\phi(|y_n|u_n(s))|}{|\phi(y_n)|}\}$ in (4.17) is a bounded sequence (with a bound independent of $s \in I$). From (4.15), (4.17), (4.18), and the Lebesgue dominated convergence theorem we find that

$$(4.19)\quad \lim_{n\to\infty} \int_0^\tau v_n(s)ds = \int_0^\tau \alpha(s)\psi_p(u(s))ds, \text{ uniformly on } [0,T].$$

Indeed, we have

$$\Big|\int_0^\tau v_n(s)ds - \int_0^\tau \alpha(s)\psi_p(u(s))ds\Big| \leq \int_0^T |R_n(s)|ds,$$

where

$$R_n(s) = v_n(s) - \alpha(s)\psi_p(u(s)).$$

By (4.15), $R_n(s) \to 0$ a.e. in $(0,T)$ and from (4.17), (4.18) it is immediate to see that R_n is integrable, satisfying in this form the hypotheses of the Lebesgue dominated convergence theorem implying the validity of (4.19).

Next we study the sequence $\{\frac{\phi(w_n'(0))}{|\phi(y_n)|}\}$.

Claim. The sequence $\{\frac{\phi(w_n'(0))}{|\phi(y_n)|}\}$ is bounded.
Otherwise by passing to a subsequence, if necessary, we can assume that $\frac{|\phi(w_n'(0))|}{|\phi(y_n)|} \to \infty$, as $n \to \infty$. From (4.11) and (4.19), this implies

$$(4.20)\quad |z_n(\tau)| \to \infty \quad \text{as} \quad n \to \infty, \quad \text{uniformly on} \quad [0,T].$$

But this cannot be, since together with (4.13) and Proposition 2.2, this would imply that $|\chi_n(\tau)| \to \infty$ uniformly in $[0,T]$ as $n \to \infty$, contradicting (4.14). Thus the claim is proved.

From this claim, by passing to a subsequence if necessary, we can suppose that $\frac{\phi(w_n'(0))}{|\phi(y_n)|} \to c \in \mathbb{R}^N$ and therefore that

$$z_n(\tau) = \frac{\phi(w_n'(0))}{|\phi(y_n)|} + \int_0^\tau v_n(s)ds \to z(\tau) := c + \int_0^\tau \alpha(s)\psi_p(u(s))ds,$$

as $n \to \infty$, uniformly for $\tau \in [0,T]$.

Letting $n \to \infty$ in (4.14), and applying the Lebesgue dominated convergence theorem, we obtain

$$(4.21)\qquad \int_0^T \psi_{p^*}\big(c + \int_0^\tau \alpha(s)\psi_p(u(s))ds\big)d\tau = 0,$$

expression which should be compared with (4.3), for $t = T$. In this situation proposition 2.2 of [8], tells us that necessarily

$$c = \psi_p(w'(0)) = -Q_{\psi_p}(H(\alpha\psi_p(u))),$$

where w is the solution to the problem

$$(P_u) \qquad \begin{cases} (\psi_p(w'))' = \alpha(s)\psi_p(u) & \text{a.e. in} \quad (0,T), \\ w(0) = 0, \quad w(T) = 0, \end{cases}$$

i.e.,

$$w(t) = \int_0^t \psi_{p^*}\big(\psi_p(w'(0)) + \int_0^\tau \alpha(s)\psi_p(u(s))ds\big)d\tau.$$

Since the complete argument, started after the proof of the claim, can be applied to any subsequence of $\{\frac{\phi(w_n'(0))}{|\phi(y_n)|}\}$ we have indeed showed that

$$(4.22) \qquad \lim_{n\to\infty} \frac{\phi(w_n'(0))}{|\phi(y_n)|} = \psi_p(w'(0)) = -Q_{\psi_p}(H(\alpha\psi_p(u))).$$

Hence, from (4.11), (4.19), and (4.22), we obtain

$$\lim_{n\to\infty} z_n(\tau) = z(\tau) = -Q_{\psi_p}(H(\alpha\psi_p(u))) + \int_0^\tau \alpha(s)\psi_p(u(s))ds.$$

Thus from (4.12), we conclude that

$$(4.23) \quad \lim_{n\to\infty} \frac{\mathcal{K}_d^\phi(\Gamma(|y_n|u_n, \lambda_n))}{|y_n|} = \int_0^t \psi_{p^*}(-Q_{\psi_p}(H(\alpha\psi_p(u))) + \int_0^\tau \alpha(s)\psi_p(u(s))ds)d\tau.$$

We end the proof by noting that (4.6) follows from (4.23) and (4.3). □

Remark 4.1. Under the conditions of Lemma 4.1 it is immediate to see that

$$(4.24) \quad \lim_{n\to\infty} \frac{w_n'(t)}{|y_n|} = \psi_{p^*}\big(-Q_{\psi_p}(H(\alpha\psi_p(u))) + \int_0^\tau \alpha(s)\psi_p(u(s))ds\big).$$

Theorem 4.1. *Consider the problem*

$$(\mathcal{P}) \qquad \begin{cases} (\phi(u'))' + f(t,u) = g(t,u) & a.e. \quad t \in (0,T), \\ u(0) = 0, \quad u(T) = 0, \end{cases}$$

where $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (H_1), (H_2) and is AH of order $p > 1$. In addition we assume that ϕ is an odd function. The functions $f, g : I \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory, by this we mean they satisfy (C_1), (C_2), and (C_3) of the Introduction.

Furthermore assume that f and g satisfy the following condition: every time $\{y_n\}$ and $\{x_n\}$ are two sequences in $\mathbb{R}^N$ such that $|y_n| \to \infty$ and $x_n \to x$, as $n \to \infty$, then

$$(4.25) \quad \lim_{n\to\infty} \frac{f(t,|y_n|x_n)}{|\phi(y_n)|} = \mu\alpha(t)|x|^{p-2}x \quad \text{uniformly a.e. in } t \in I,$$

where $\alpha \in L^1$ *and* $\mu \in \mathbb{R}$, *and*

$$\frac{|g(t,|y_n|x_n)|}{|\phi(y_n)|} \to 0 \quad \textit{uniformly a.e. in} \quad t \in I, \tag{4.26}$$

Then, if μ *is not an eigenvalue of problem* (E_p), *it follows that problem* $(\mathcal{P})$ *has a solution.*

Remark 4.2. Condition (4.26) is satisfied if for example we assume that

$$\frac{|g(t,y)|}{|\phi(y)|} \to 0 \quad \text{as} \quad |y| \to \infty \quad \text{uniformly a.e. in} \quad t \in I,$$

and that for each $m > 0$ there is $\rho_m \in L^\infty(I, \mathbb{R})$ such that, for almost every $t \in I$ and every $x \in \mathbb{R}^N$ with $|x| \le m$, one has

$$|g(t,x)| \le \rho_m(t).$$

Proof of Theorem 4.1. For $\lambda \in [0,1]$, let us define a family of problems as follows

$$(P_\lambda) \qquad \begin{cases} (\phi(u'))' = \gamma(t,u,\lambda) \quad \text{a.e.} \quad t \in (0,T), \\ u(0) = 0, \quad u(T) = 0, \end{cases}$$

where

$$\gamma(t,y,\lambda) = -(1-\lambda)\mu\alpha(t)\phi(y) - \lambda f(t,y) + \lambda g(t,y). \tag{4.27}$$

We show first that γ satisfies conditions (4.5) and (4.4) of Lemma 4.1. To do this let $\{\lambda_n\}$ be a sequence in $[0,1]$ with $\lambda_n \to \lambda$, and $\{y_n\}$ and $\{x_n\}$ be two sequences in $\mathbb{R}^N$ such that $|y_n| \to \infty$ and $x_n \to x$ as $n \to \infty$. Then, since ϕ is AH of order p, by using (4.25) and (4.26), we find that

$$\begin{aligned} \lim_{n\to\infty} \frac{\gamma(t,|y_n|x_n,\lambda_n)}{|\phi(y_n)|} &= -\mu\alpha(t) \lim_{n\to\infty}(1-\lambda_n)\frac{\phi(|y_n|x_n)}{|\phi(y_n)|} \\ &\quad - \lim_{n\to\infty} \lambda_n \frac{f(t,|y_n|x_n)}{|\phi(y_n)|} + \lim_{n\to\infty} \lambda_n \frac{g(t,|y_n|x_n)}{|\phi(y_n)|} \\ &= -\mu\alpha(t)\psi_p(x) \quad \text{for a.e.} \quad t \in I, \end{aligned}$$

and thus (4.5) is satisfied. Now it is not difficult to see from (4.25) and (4.26) that

$$\lim_{|y|\to\infty} \frac{|f(t,y)|}{|\phi(y)|} = \mu\alpha(t) \qquad \text{uniformly a.e. in} \quad t \in I,$$

and

$$\lim_{|y|\to\infty} \frac{|g(t,y)|}{|\phi(y)|} = 0 \qquad \text{uniformly a.e. in} \quad t \in I.$$

Hence we obtain that

$$\limsup_{|y|\to\infty} \frac{|\gamma(t,y,\lambda)|}{|\phi(y)|} \le (1-\lambda)\mu\alpha(t) + \lambda \lim_{|y|\to\infty} \frac{|f(t,y)|}{|\phi(y)|} + \lambda \lim_{|y|\to\infty} \frac{|g(t,y)|}{|\phi(y)|}$$
$$= \mu\alpha(t), \qquad \text{for a.e. } t\in I,$$

and uniformly for $\lambda \in [0,1]$ and $t \in I$. Thus (4.4) holds true with $\beta(t) = \mu\alpha(t)$.

We now show there exists a positive constant M (independent of λ) such that if the pair (u,λ) satisfies (P_λ), then

$$||u||_{C_0^1} \le M. \tag{4.28}$$

We argue by contradiction and thus we assume there is a sequence $(u_n,\lambda_n) \in C_0^1 \times [0,1]$ such that for each $n \in \mathbb{N}$, (u_n,λ_n) satisfies

$$(P_{\lambda_n}) \qquad \begin{cases} (\phi(u_n'))' = \gamma(t,u_n(t),\lambda_n) & t\in(0,T),\\ u_n(0)=0, \quad u_n(T)=0, \end{cases}$$

and $||u_n||_{C_0^1} \to \infty$ as $n\to\infty$.

We claim this implies that $||u_n||_{C_0} \to \infty$ as $n\to\infty$. Otherwise, we can choose a subsequence of $\{u_n\}$ denoted again by $\{u_n\}$ such that $|u_n(t)| \le ||u_n||_{C_0} \le M_1$, for all $t\in[0,T]$ and all $n\in\mathbb{N}$, where M_1 is a positive constant. Then, since the functions f and g satisfy condition (C_3), there is a function $\rho \in L^1(I,\mathbb{R})$ such that

$$|\gamma(t,u_n(t),\lambda_n)| \le \rho(t), \tag{4.29}$$

for a.e. $t\in[0,T]$ and all $n\in\mathbb{N}$. Now from (P_{λ_n}),

$$u_n'(t) = \phi^{-1}\big(\phi(u_n'(0)) + \int_0^t \gamma(s,u_n(s),\lambda_n)ds\big), \tag{4.30}$$

and then

$$0 = \int_0^T \phi^{-1}\big(\phi(u_n'(0)) + \int_0^t \gamma(s,u_n(s),\lambda_n)ds\big)dt.$$

Hence taking into account (4.29) we find that $\phi(u_n'(0))$ is bounded, therefore by (4.30), there is a positive constant M_2 such that $||u_n'||_{C_0} \le M_2$ for all $n\in\mathbb{N}$. Since this contradicts that $||u_n||_{C_0^1} \to\infty$, it must be that $||u_n||_{C_0}\to\infty$ as $n\to\infty$.

Now problem (P_{λ_n}) is equivalent to (u_n,λ_n) satisfies

$$u_n = \mathcal{K}_d^\phi(\Gamma(u_n,\lambda_n)), \tag{4.31}$$

where $\Gamma : C_0\times[0,1] \to L_N^1$ is the Nemistki operator corresponding to γ as given in (4.27) and where u_n in the right hand side of (4.31) is seen as an element of C_0.

For $n \in \mathbb{N}$ let us set $v_n = \dfrac{u_n}{||u_n||_{C_0}}$, then $\{v_n\}$ is a bounded sequence in C_0. We show next that $\{v_n\}$ is also an equicontinuous sequence. To do this it is enough to prove that $\{v_n'\}$ is a bounded sequence in C_0.

Let next $\hat{a}$ denote an arbitrary but fixed unit vector in $\mathbb{R}^N$, and set

$$y_n = ||u_n||_{C_0}\hat{a} \in \mathbb{R}^N, \tag{4.32}$$

then $|y_n| = ||u_n||_{C_0}$ and (4.30) can be written as

$$v_n'(t) = \frac{\phi^{-1}\big(\phi(|y_n|v_n'(0)) + \int_0^t \gamma(s, |y_n|v_n(s), \lambda_n)ds\big)}{|y_n|},$$

which in turn can be written as

$$v_n'(t) = \frac{\phi^{-1}(|\phi(y_n)|w_n(t))}{|\phi^{-1}(\phi(y_n))|}, \tag{4.33}$$

where

$$w_n(t) = \frac{\phi(|y_n|v_n'(0))}{|\phi(y_n)|} + \int_0^t \frac{\gamma(s, |y_n|v_n(s), \lambda_n)}{|\phi(y_n)|}ds\,. \tag{4.34}$$

To show that $\{v_n'\}$ is bounded it is enough to prove that $\{w_n\}$ is a bounded sequence in C_0, since ϕ^{-1} is AB and $|\phi(y_n)| \to \infty$, as $n \to \infty$. By repeating the argument that begins just before (4.16) in the proof of Lemma 4.1 and ends the line following (4.18), we find that there is $\delta \in L^1(0,T)$ such that

$$\frac{|\gamma(s, |y_n|v_n(s), \lambda_n)|}{|\phi(y_n)|} \le \delta(s), \qquad \text{for a.e. s} \in (0,T),$$

and for all $n \in \mathbb{N}$, hence

$$\Big|\int_0^t \frac{\gamma(s, |y_n|v_n(s), \lambda_n)}{|\phi(y_n)|}ds\Big| \le \int_0^T \delta(s)\,ds. \tag{4.35}$$

It only remains to show that the sequence $\{\dfrac{\phi(|y_n|v_n'(0))}{|\phi(y_n)|}\}$ is bounded. We follow the proof of the Claim in Lemma 4.1. If the sequence $\{\dfrac{\phi(|y_n|v_n'(0))}{|\phi(y_n)|}\}$ were not bounded, then by (4.34) and (4.35) and by passing to a subsequence, if necessary, we would have that $|w_n(t)| \to \infty$ uniformly in $t \in I$ as $n \to \infty$. Therefore Proposition 2.2 says that $\dfrac{\phi^{-1}(|\phi(y_n)|w_n(t))}{|\phi^{-1}(\phi(y_n))|} \to \infty$ uniformly in $[0,T]$ as $n \to \infty$. This contradicts the fact

$$\int_0^T \frac{\phi^{-1}(|\phi(y_n)|w_n(t))}{|\phi^{-1}(\phi(y_n))|}dt = 0.$$

Thus the sequence $\{v_n'\}$ is bounded and hence the sequence $\{v_n\}$ is equicontinuous. This together with the fact that $\{v_n\}$ is also bounded implies by Ascoli-Arzela's theorem and by passing to a subsequence, if necessary, that $v_n \to v \in C_0$ as $n \to \infty$. Since from (4.31)

$$\begin{aligned} v_n &= \frac{\mathcal{K}_d^{\phi}(\Gamma(u_n, \lambda_n))}{||u_n||_{C_0}} \\ &= \frac{\mathcal{K}_d^{\phi}(\Gamma(||u_n||_{C_0} v_n, \lambda_n))}{||u_n||_{C_0}} = \frac{\mathcal{K}_d^{\phi}(\Gamma(|y_n| v_n, \lambda_n))}{|y_n|}, \end{aligned} \tag{4.36}$$

with y_n as in (4.32), we are in a position to apply Lemma 4.1 to find

$$v = \lim_{n\to\infty} \frac{\mathcal{K}_d^{\phi}(\Gamma(|y_n| v_n, \lambda_n))}{|y_n|} = \mathcal{K}_d^{\psi_p}(-\mu\alpha(\cdot)\psi_p(v)).$$

But this is a contradiction because it tells us that v (of class C_0^1) is a non-trivial solution to problem (E_p) and therefore μ is an eigenvalue of (E_p) which is not true, thus (4.28) holds.

Now let $\mathcal{T} : C_0^1(0,T) \times [0,1] \to C_0^1(0,T)$ be given by

$$\mathcal{T}(u, \lambda) = \mathcal{K}_d^{\phi}(\Gamma(u, \lambda)). \tag{4.37}$$

Since f and g in (4.27) are Carathéodory, it is known that Γ is continuous and sends bounded sets of $C_0^1(0,T) \times [0,1]$ into equi-integrable sets in L_N^1, implying that $\mathcal{T}$ is a completely continuous operator. From (4.28) by increasing M if necessary, we see that there are no solutions to

$$u = \mathcal{T}(u, \lambda)$$

for $u \in \partial B(0, M)$ and $\lambda \in [0,1]$, where $B(0,M)$ is the ball centered at the origin with radius M in $C_0^1(0,T)$. In this form the Leray-Schauder degree $d_{LS}(I - \mathcal{T}(\cdot, \lambda), B(0,M), 0)$ is defined and from the properties of this degree it follows that

$$d_{LS}(I - \mathcal{T}(\cdot, 1), B(0,M), 0) = d_{LS}(I - \mathcal{T}(\cdot, 0), B(0,M), 0). \tag{4.38}$$

Since $\mathcal{T}(y, 0) = \mathcal{K}_d^{\phi}(-\mu\alpha(\cdot)\phi(y))$, we observe that $I - \mathcal{T}(\cdot, 0)$ is an odd operator. Then by Borsuk's theorem (see for example [11]) assures that

$$d_{LS}(I - \mathcal{T}(\cdot, 0), B(0,M), 0) \neq 0.$$

Hence from (4.38) the operator $\mathcal{T}(\cdot, 1)$ has a fixed point, which is equivalent to say that problem (P) has a solution. □

Before stating our next result let us consider the vector eigenvalue problem with Dirichlet boundary conditions

$$(E) \qquad \begin{cases} (\psi_p(u'))' + \lambda\psi_p(u) = 0, & t \in (0,T), \\ u(0) = 0, \ u(T) = 0. \end{cases}$$

It follows from [2], see also [10], and Section 3 of this paper, that the set of eigenvalues of problem (E), denoted by $\epsilon_D(p,N)$, is actually independent of N. For this case we have the following result.

Proposition 4.1. *For each* $N \geq 1$,

$$\epsilon_D(p,N) = \epsilon_D(p,1) = \left\{ \left(\frac{k\pi_p}{T} \right)^p : k \in \mathbb{N} = \{1,2,3,\ldots\} \right\},$$

with corresponding eigenfunctions given by

$$u_k(t) = \sin_p \left(\frac{k\pi_p t}{T} \right) d, \quad d \in \mathbb{R}^N, \quad k \in \mathbb{N},$$

where

$$(4.39) \qquad \pi_p = 2(p-1)^{1/p} \int_0^1 (1-t^p)^{-1/p}\, dt = 2(p-1)^{1/p} \frac{(\pi/p)}{\sin(\pi/p)}.$$

(For a definition of the function $\sin_p$ *and some of its properties we refer to* [3]*.)*

The following result is an immediate corollary of Theorem 4.1.

Corollary 4.1. *Suppose that* $\phi : \mathbb{R}^N \to \mathbb{R}^N$ *satisfies conditions* (H_1), (H_2) *and is AH of order* $p > 1$. *If* $\mu \neq (\frac{k\pi_p}{T})^p$ *for all* $k \in \mathbb{N}$, *then the problem*

$$(P) \qquad \begin{cases} (\phi(u'))' + \mu\phi(u) = e(t), & t \in (0,T), \\ u(0) = 0 \quad u(T) = 0, \end{cases}$$

has a solution for each $e \in L^1_N$.

5. An existence result for a weakly coupled system

As a motivation for this section let us begin by considering the function $\chi : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\chi(x) = \big(|x_1|^{p_1-2}x_1, \cdots, |x_N|^{p_N-2}x_N\big),$$

where $p_i > 1$, $i = 1, \cdots, N$, and $x = (x_1, \cdots, x_N)$. It is known, see [8] and [9], that χ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Nevertheless, it is immediately seen that χ is not an AH function at infinity, in the sense that it does not satisfy (2.1). It is the purpose of this section to show that in spite of this fact AH functions allow us to deal with vector Dirichlet boundary value problems whose quasilinear operator is generated by functions like χ.

Next let us state the hypotheses we shall use in our existence theorem. For $i = 1, \cdots, k$, let $N_i \in \mathbb{N}$ and $\theta_i : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ be a function satisfying the following conditions.

(i) $\langle \theta_i(z) - \theta_i(y), z - y\rangle_i \geq 0$, for any $z, y \in \mathbb{R}^{N_i}$, with equality holding true if and only if $z = y$, and where $\langle \cdot, \cdot \rangle_i$ denotes the inner product in $\mathbb{R}^{N_i}$;

(ii) there exists a function $\rho_i : [0, +\infty) \to [0, +\infty)$, $\rho_i(s) \to +\infty$ as $s \to +\infty$, such that
$$\langle \theta_i(z), z\rangle_i \geq \rho_i(|z|)|z|, \quad \text{for all } z \in \mathbb{R}^{N_i}.$$

Thus for each $i = 1, \cdots k$, the function θ_i is a homeomorphism from $\mathbb{R}^{N_i}$ onto $\mathbb{R}^{N_i}$, which we further assume to be odd. Define the function

$$\begin{aligned} \theta : \quad \prod_{i=1}^{k} \mathbb{R}^{N_i} \quad &\to \quad \prod_{i=1}^{k} \mathbb{R}^{N_i}; \\ x = (x_1, \cdots, x_k) \quad &\mapsto \quad \theta(x) = (\theta_1(x_1), \cdots, \theta_k(x_k)). \end{aligned} \tag{5.1}$$

It is shown in [8] that θ satisfies conditions (H_1) and (H_2) with $N = \sum\limits_{i=1}^{k} N_i$ and therefore it is a homeomorphism from $\prod\limits_{i=1}^{k} \mathbb{R}^{N_i}$ onto $\prod\limits_{i=1}^{k} \mathbb{R}^{N_i}$, which is odd because all the components are odd.

We also assume that for each $i = 1, \cdots, k$, θ_i is AH of order $p_i > 1$ at infinity (AH in the sense of section 2). The inverse homeomorphism of θ is given by $\theta^{-1}(x) = (\theta_1^{-1}(x_1), \cdots, \theta_k^{-1}(x_k))$, where $\theta_i^{-1} : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ the inverse of θ_i is AH of order $p_i^* > 1$, with $p_i^* = \frac{p_i}{p_i - 1}$.

Concerning $g = (g_1, \cdots, g_k)$, we assume the following hypotheses. For $x = (x_1, \cdots, x_k)$, $x_i \in \mathbb{R}^{N_i}$, the function $g_i : I \times \prod\limits_{i=1}^{k} \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$, $i = 1, \cdots, k$, has the form

$$g_i(t, x) = \tilde{g}_i(t, x) h_i(t, x_i), \tag{5.2}$$

where $\tilde{g}_i$ and h_i satisfy one of the following two hypotheses, $B_i(0, 1)$ denotes the unit ball centered at zero in $\mathbb{R}^{N_i}$ and $\overline{B_i(0, 1)}$ its closure:

(j) $\tilde{g}_i : I \times \prod\limits_{i=1}^{k} \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ is continuous and bounded and $h_i : I \times \mathbb{R}^{N_i} \to \mathbb{R}$ is Carathéodory and satisfies

$$\lim_{|x_i| \to \infty} \frac{h_i(t, |x_i| b)}{|\theta_i(x_i)|} \to 0 \tag{5.3}$$

uniformly for a.e. $t \in (0, T)$ and $b \in \overline{B_i(0, 1)} \subset \mathbb{R}^{N_i}$.

(jj) $\tilde{g}_i : I \times \prod_{i=1}^{k} \mathbb{R}^{N_i} \to \mathbb{R}$ is continuous and bounded and $h_i : I \times \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ is Carathéodory and satisfies

$$\lim_{|x_i|\to\infty} \frac{|h_i(t,|x_i|b)|}{|\theta_i(x_i)|} \to 0 \tag{5.4}$$

uniformly for a.e. $t \in (0,T)$ and $b \in \overline{B_i(0,1)} \subset \mathbb{R}^{N_i}$.

Theorem 5.1. *Let us consider the problem*

$$(S)\begin{cases}(\theta(u'))' + \big(\mu_1\alpha_1(t)\theta_1(u_1),\cdots,\mu_k\alpha_k(t)\theta_k(u_k)\big) = g(t,u),\ t\in(0,T),\\ u(0)=0,\quad u(T)=0,\end{cases}$$

where $u = (u_1,\cdots u_k) : I \to \prod_{i=1}^{k}\mathbb{R}^{N_i}$ *and* $g = (g_1,\cdots,g_k) : I \times \prod_{i=1}^{k}\mathbb{R}^{N_i} \to \prod_{i=1}^{k}\mathbb{R}^{N_i}$. *The components* g_i, $i = 1,\cdots,k$, *of* g *are given by (5.2) and satisfy either* (j) *or* (jj). *Also, the function* $\theta = (\theta_1,\cdots,\theta_k) : \prod_{i=1}^{k}\mathbb{R}^{N_i} \to \prod_{i=1}^{k}\mathbb{R}^{N_i}$ *and its components satisfy* (i) *and* (ii) *in this section and are odd functions. In addition, for* $i = 1,\cdots,k$, *we assume that the coefficient* $\mu_i \in (0,\infty)$ *and the function* $\alpha_i \in L^1$ *is positive a.e. Under these conditions, if for each* $i = 1,\cdots,k$, μ_i *is not an eigenvalue of the problem*

$$(E_i)\qquad \begin{cases}(\psi_{p_i}(u_i'))' + \mu_i\alpha_i(t)\psi_{p_i}(u_i) = 0,\quad t\in(0,T),\\ u_i(0)=0,\quad u_i(T)=0,\end{cases}$$

where $\psi_{p_i}(z) = |z|^{p_i-2}z$, $z \in \mathbb{R}^{N_i}$, *then problem* (S) *has a solution.*

Proof. We consider the family of problems

$$(S_\lambda)\begin{cases}(\theta(u'))' + \big(\mu_1\alpha_1(t)\theta_1(u_1),\cdots,\mu_k\alpha_k(t)\theta_k(u_k)\big) = \lambda g(t,u),\ t\in(0,T),\\ u(0)=0,\qquad u(T)=0,\end{cases}$$

for $\lambda \in [0,1]$.

Claim. There exists a positive constant M (independent of λ) such that if the pair (u,λ) satisfies (S_λ), then

$$||u||_{C_0^1} \le M.$$

Here and in the rest of the section $C_0 = \prod_{i=1}^k C_0(I,\mathbb{R}^{N_i})$ and $C_0^1 = \prod_{i=1}^k C_0^1(I,\mathbb{R}^{N_i})$ with norms $||u||_{C_0} = \sum_{i=1}^k ||u_i||_{C_0(I,\mathbb{R}^{N_i})}$ and $||u||_{C_0^1} = \sum_{i=1}^k ||u_i||_{C_0^1(I,\mathbb{R}^{N_i})}$, respectively.

We argue by contradiction and thus we assume there is a sequence $(u_n, \lambda_n) \in C_0^1 \times [0,1]$, $u_n = (u_{1,n}, \cdots, u_{k,n})$, of solutions of (S_λ) with $\lambda = \lambda_n$ such that $||u_n||_{C_0^1} \to \infty$ as $n \to \infty$. As in the proof of Theorem 4.1, we shall prove this implies that $||u_n||_{C_0} \to \infty$ as $n \to \infty$. If not, by choosing a subsequence renamed the same, we may suppose that there is a positive constant M_1 such that $||u_n||_{C_0} \leq M_1$ for all $n \in \mathbb{N}$. Writing (S_λ) with $\lambda = \lambda_n$ in components, we obtain the problem

$$(S^i_{\lambda_n}) \qquad \begin{cases} (\theta_i(u'_{i,n}))' = \gamma_i(t, u_n, \lambda_n), & \text{a.e.} \quad t \in (0,T), \\ u_{i,n}(0) = 0, \quad u_{i,n}(T) = 0, \end{cases}$$

for each $i = 1, \cdots, k,\ n \in \mathbb{N}$, and where

$$\gamma_i(t, u, \lambda) = -\mu_i \alpha_i(t) \theta_i(u_i) + \lambda g_i(t, u).$$

Since $|u_{i,n}(t)| \leq M_1$, for all $t \in [0,T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$, it follows that there is $M_2 > 0$ such that $|\theta_i(u_{i,n}(t))| \leq M_2$, for all $t \in [0,T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$. Assuming condition (j) holds (the case where (jj) holds can be handled similarly), we see that there is $M_3 > 0$ such that $|g_i(t, u_n(t))| \leq M_3$ and since h_i is Carathéodory there is $\rho \in L^1(0,T,\mathbb{R})$ such that $|h_i(t, u_{i,n}(t))| \leq \rho(t)$, for a.e. $t \in [0,T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$. Hence $|g_i(t, u_n(t))| \leq M_3 \rho(t)$, for a.e. $t \in [0,T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$, and then

$$|\gamma_i(t, u_n(t), \lambda)| \leq \mu_i \alpha_i(t) M_2 + M_3 \rho(t) \equiv \delta(t).$$

for a.e. $t \in [0,T]$, all $\lambda \in [0,1]$ and all $n \in \mathbb{N}$. This implies that

$$|(\theta_i(u'_{i,n}))'(t)| \leq \delta(t),$$

for a.e. $t \in (0,T)$ and all $n \in \mathbb{N}$. At this point the argument is a repetition of that starting at (4.29), hence we obtain from those arguments that there is $M_4 > 0$ such that $|u'_{i,n}(t)| \leq M_4$, for all $t \in [0,T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$, implying that $\{u'_n\}$ is a bounded sequence in C_0 and hence that $\{u_n\}$ is a bounded sequence in C_0^1, which is a contradiction. Thus it must be that $||u_n||_{C_0} \to \infty$ as $n \to \infty$. Let next $\hat{a} = (a_1, \cdots, a_k)$, $a_i \in \mathbb{R}^{N_i}$, $a_i \neq 0$, $i = 1, \cdots, k$, be a unit vector in $\prod_{i=1}^{k} \mathbb{R}^{N_i}$, and set $y_n = (y_{1,n}, \cdots, y_{k,n}) = ||u_n||_{C_0} \hat{a}$. Then $y_n \in \prod_{i=1}^{k} \mathbb{R}^{N_i}$, $y_{i,n} \in \mathbb{R}^{N_i}$, $i = 1, \cdots, k$, and $|y_n| = ||u_n||_{C_0}$. Also for each $n \in \mathbb{N}$ let us set $v_n = (v_{1,n}, \cdots, v_{k,n}) := \dfrac{u_n}{||u_n||_{C_0}}$.

We show next that $\{v'_n\}$ is a bounded sequence in C_0, which will imply that $\{v_n\}$ is equicontinuous. By integrating the equation of $(S^i_{\lambda_n})$

and using the boundary conditions, we first find that

$$u'_{i,n}(t) = \theta_i^{-1}\big[\theta_i(u'_{i,n}(0)) + \int_0^t \gamma_i(s, u_n(s), \lambda_n)ds\big], \tag{5.5}$$

and then, by a new integration,

$$u_{i,n}(t) = \int_0^t \theta_i^{-1}\big[\theta_i(u'_{i,n}(0)) + \int_0^\tau \gamma_i(s, u_n(s), \lambda_n)ds\big]d\tau, \tag{5.6}$$

together with

$$\int_0^T \theta_i^{-1}\big[\theta_i(u'_{i,n}(0)) + \int_0^\tau \gamma_i(s, u_n(s), \lambda_n)ds\big]d\tau = 0. \tag{5.7}$$

Dividing (5.5) and (5.6) by $||u_n||_{C_0}$, we obtain

$$v'_{i,n}(t) = |a_i|\frac{\theta_i^{-1}\big[\theta_i(|y_{i,n}|\dfrac{v'_{i,n}(0)}{|a_i|}) + \int_0^t \gamma_i(s, |y_n|v_n(s), \lambda_n)ds\big]}{|y_{i,n}|} \tag{5.8}$$

and

$$v_{i,n}(t) = |a_i|\frac{\int_0^t \theta_i^{-1}\big[\theta_i(|y_{i,n}|\dfrac{v'_{i,n}(0)}{|a_i|}) + \int_0^\tau \gamma_i(s, |y_n|v_n(s), \lambda_n)ds\big]d\tau}{|y_{i,n}|}, \tag{5.9}$$

where we have used that $|y_{i,n}| = ||u_n||_{C_0}|a_i|$. We note that $|y_{i,n}| \to \infty$ as $n \to \infty$, for all $i = 1, \dots, k$. Now (5.8) can be written as

$$v'_{i,n}(t) = |a_i|\frac{\theta_i^{-1}\big[\theta_i(|y_{i,n}|)\xi_{i,n}(t)\big]}{|\theta_i^{-1}(\theta_i(y_{i,n}))|},$$

where

$$\xi_{i,n}(t) = \frac{\theta_i(|y_{i,n}|\dfrac{v'_{i,n}(0)}{|a_i|})}{|\theta_i(y_{i,n})|} + \int_0^t \frac{\gamma_i(s, |y_n|v_n(s), \lambda_n)}{|\theta_i(y_{i,n})|}ds. \tag{5.10}$$

As in the proof of Theorem 4.1 the boundedness of the sequence $\{v'_{i,n}\}$ and hence that of $\{v'_n\}$ will follow if we can prove that the sequence $\{\xi_{i,n}\}$ is bounded in $C(I, \mathbb{R}^{N_i})$, for each $i = 1, \dots, k$.

The integrand of the second term on the right hand side of (5.10) is given by

$$\frac{\gamma_i(s, |y_n|v_n(s), \lambda_n)}{|\theta_i(y_{i,n})|} = -\mu_i\alpha_i(s)\frac{\theta_i(|y_n|v_{i,n}(s))}{|\theta_i(y_{i,n})|} + \lambda_n\frac{g_i(s, |y_n|v_n(s))}{|\theta_i(y_{i,n})|}, \tag{5.11}$$

then noticing that the sequence $\{v_{i,n}\}$ is bounded and that θ_i is AB for each $i = 1, \cdots, k$, we find that there is a $A > 0$ such that

$$\frac{|\theta_i(|y_n|v_{i,n}(s))|}{|\theta_i(y_{i,n})|} \le A, \tag{5.12}$$

for all $s \in [0, T]$, all $i = 1, \cdots, k$, and all $n \in \mathbb{N}$. Next we observe that

$$\begin{aligned} \frac{g_i(s, |y_n|v_n(s))}{|\theta_i(y_{i,n})|} &= \tilde{g}_i(s, |y_n|v_n(s))\frac{h_i(s, |y_n|v_{i,n}(s))}{|\theta_i(y_{i,n})|} \\ &= \tilde{g}_i(s, |y_n|v_n(s))\frac{h_i(s, |y_{i,n}|\frac{v_{i,n}(s)}{|a_i|})}{|\theta_i(y_{i,n})|}, \end{aligned} \tag{5.13}$$

where $|a_i| < 1$. Let us choose $r > 0$ such that $r|a_i| \ge 1$, $i = 1, \cdots, k$. Then by letting $n \to \infty$ in (5.13) and taking into account (5.3), we find that

$$\frac{h_i(s, |y_{i,n}|\frac{v_{i,n}(s)}{|a_i|})}{|\theta_i(y_{i,n})|} = \frac{h_i(s, |ry_{i,n}|\frac{v_{i,n}(s)}{r|a_i|})}{|\theta_i(ry_{i,n})|}\frac{|\theta_i(ry_{i,n})|}{|\theta_i(y_{i,n})|} \to 0, \tag{5.14}$$

uniformly for a.e. $s \in (0, T)$, because the second factor on the right hand side is bounded by Proposition 2.1. From these results and the fact that $\tilde{g}$ is a bounded function, we find that

$$\frac{g_i(s, |y_n|v_n(s))}{|\theta_i(y_{i,n})|} \to 0, \quad \text{uniformly} \quad \text{for a.e.} \quad s \in I, \tag{5.15}$$

and hence by (5.11) and (5.12) that there is $\overline{\delta} \in L^1(0, \mathbb{R})$ such that

$$\frac{|\gamma_i(s, |y_n|v_n(s), \lambda_n)|}{|\theta_i(y_{i,n})|} \le \overline{\delta}(s), \quad \text{for a.e. } s \in [0, T], \text{ all } i = 1, \cdots, k \text{ and all } n \in \mathbb{N}. \tag{5.16}$$

By taking into account that (5.7) can be written as

$$\int_0^T \frac{\theta_i^{-1}(\xi_{i,n}(\tau)|\theta_i(y_{i,n})|)}{|\theta_i^{-1}(\theta_i(y_{i,n}))|}d\tau = 0, \tag{5.17}$$

we easily see that the rest of the argument is entirely similar to the one following (4.35) in the proof of Theorem 4.1. In this form we obtain that $\{v_n'\}$ is a bounded sequence in C_0, which implies that $\{v_n\}$ is equicontinuous. Since this sequence is also bounded, by Ascoli-Arzela, and by passing to a subsequence (renamed the same) if necessary, we can assume that $v_n \to z \in C_0$ as $n \to \infty$, with $z(s) = (z_1(s), \cdots, z_k(s))$, $s \in I$. The subsequence $\{v_n\}$ can be chosen such that $v_n'(0) \to d =$

$(d_1, \cdots, d_k) \in \prod_{i=1}^{k} \mathbb{R}^{N_i}$. We note here that at least one of the z_i, $i = 1, \cdots, k$ must be nontrivial. Next (5.9) can be written as

$$v_{i,n}(t) = |a_i| \int_0^t \frac{\theta_i^{-1}(\xi_{i,n}(\tau)|\theta_i(y_{i,n})|)}{|\theta_i^{-1}(\theta_i(y_{i,n}))|} d\tau, \tag{5.18}$$

where $\xi_{i,n}$ is given by (5.10). Since $v'_{i,n}(0) \to d_i$, we immediately obtain that the first term on the right hand side of (5.10) satisfies

$$\frac{\theta_i(|y_{i,n}| \frac{v'_{i,n}(0)}{|a_i|})}{|\theta_i(y_{i,n})|} \longrightarrow \psi_{p_i}(\frac{d_i}{|a_i|}) \quad \text{as} \quad n \to \infty. \tag{5.19}$$

Furthermore by (5.15),

$$\frac{\gamma_i(s, |y_n| v_n(s), \lambda_n)}{|\theta_i(y_{i,n})|} \to -\mu_i \alpha_i(s) \psi_{p_i}(\frac{z_i(s)}{|a_i|}) \text{ for a.e. in } s \in I. \tag{5.20}$$

Then by virtue of (5.11), (5.16), (5.19), (5.20) and the Lebesgue dominated convergence theorem, by letting $n \to \infty$ in (5.10), we find that $\xi_{i,n} \to \xi_i$ a.e. in I, where

$$\xi_i(\tau) = \psi_{p_i}(\frac{d_i}{|a_i|}) - \mu_i \int_0^\tau \alpha_i(s) \psi_{p_i}(\frac{z_i(s)}{|a_i|}) ds, \ \tau \in I, i = 1, \cdots, k. \tag{5.21}$$

This argument also tells us that $\{\xi_{i,n}(\tau)\}$ is bounded uniformly in $\tau \in I$, then by using the fact that θ^{-1} is AB, a new application of the Lebesgue dominated convergence theorem to (5.18) yields

$$\begin{aligned} v_{i,n}(t) \to z_i(t) &= |a_i| \int_0^t \psi_{p_i^*}\big[\psi_{p_i}(\frac{d_i}{|a_i|}) - \mu_i \int_0^\tau \alpha_i(s) \psi_{p_i}(\frac{z_i(s)}{|a_i|}) ds\big] d\tau \\ &= \int_0^t \psi_{p_i^*}\big[\psi_{p_i}(d_i) - \mu_i \int_0^\tau \alpha_i(s) \psi_{p_i}(z_i(s)) ds\big] d\tau, \end{aligned}$$

for all $t \in I$. Differentiating this expression, we get

$$z_i'(t) = \psi_{p_i^*}\big[\psi_{p_i}(d_i) - \mu_i \int_0^t \alpha_i(s) \psi_{p_i}(z_i(s)) ds\big] d\tau,$$

and hence $d_i = z_i'(0)$. Thus

$$z_i'(t) = \psi_{p_i^*}\big[\psi_{p_i}(z_i'(0)) - \mu_i \int_0^t \alpha_i(s) \psi_{p_i}(z_i(s)) ds\big] d\tau, \tag{5.22}$$

and

$$z_i(t) = \int_0^t \psi_{p_i^*}\big[\psi_{p_i}(z_i'(0)) - \mu_i \int_0^\tau \alpha_i(s) \psi_{p_i}(z_i(s)) ds\big] d\tau. \tag{5.23}$$

We note that this argument with (5.17) implies

$$\int_0^T \psi_{p_i^*}\big[\psi_{p_i}(z_i'(0)) - \mu_i \int_0^\tau \alpha_i(s)\psi_{p_i}(z_i(s))ds\big]d\tau = 0. \tag{5.24}$$

Thus $z_i(0) = 0$, $z_i(T) = 0$, implying that $z_i \in C_0^1$. Now from (5.22)

$$\psi_{p_i}(z_i'(t)) = \psi_{p_i}(z_i'(0)) - \mu_i \int_0^t \alpha_i(s)\psi_{p_i}(z_i(s))ds,$$

and differentiating this expression, we find that indeed z_i is a solution of (E_i). Since this argument holds for $i = 1, \cdots, k$ and necessarily z_i must be non trivial for at least one index $i \in \{1, \cdots, k\}$, we obtain a contradiction since μ_j is not an eigenvalue of (E_j), for any $j = 1, \cdots, k$. This ends the proof of the claim.

We recall at this point that for each $h = (h_1, \cdots, h_k)$, $h_i \in L^1_{N_i}$, $i = 1, \cdots, k$, the unique solution of problem (P_h) with $\phi = \theta$ and $N = \sum_1^k N_i$ is given by $\mathcal{K}_d^\theta(h)$ with the operator $\mathcal{K}_d^\theta$ given from (4.2) as follows :

$$\mathcal{K}_d^\theta(h) = \Big(\mathcal{K}_{d,1}^{\theta_1}(h_1), \cdots, \mathcal{K}_{d,k}^{\theta_k}(h_k)\Big),$$

where for $i = 1, \cdots, k$,

$$\mathcal{K}_{d,i}^{\theta_i}(h_i)(t) = \int_0^t \theta_i^{-1}\big(-Q_{\theta_i}(H_i(h_i)) + \int_0^s h_i(\tau)d\tau\big)ds. \tag{5.25}$$

Here $H_i : L^1_{N_i} \to C(I, \mathbb{R}^{N_i})$ is given by $H_i(l)(t) = \int_0^t l(s)ds$ with $l \in L^1_{N_i}$. Also as before for each $i = 1, \cdots, k$, from [9] or [8], the function $Q_{\theta_i} : C(I, \mathbb{R}^{N_i}) \to \mathbb{R}^{N_i}$ sends bounded sets into bounded sets and is such that the equation

$$\int_0^T \theta_i^{-1}\left(a_i + H_i(h)(s)\right)\, ds = 0 \tag{5.26}$$

has a unique solution given by $a_i = -Q_{\theta_i}(H_i(h_i))$.

Furthermore for each $i = 1, \cdots, k$ the operator $\mathcal{K}_{d,i}^{\theta_i}$ is continuous and sends equi-integrable sets in $L^1_{N_i}$ into relatively compact sets in $C_0^1(I, \mathbb{R}^{N_i})$, hence the operator K_d^θ is continuous and sends equi-integrable sets in L^1_N into relatively compact sets in C_0^1.

Next let $\gamma : I \times \prod_{i=1}^k \mathbb{R}^{N_i} \times [0,1] \to \prod_{i=1}^k \mathbb{R}^{N_i}$ be given by

$$\gamma(t, x, \lambda) = \lambda g(t, x) - \big(\mu_1\alpha_1(t)\theta_1(x_1), \cdots, \mu_k\alpha_k(t)\theta_k(x_k)\big).$$

Then γ is Carathéodory, and if we denote by $\Gamma : C_0^1 \times [0,1] \to L^1_N$ the *Nemistki operator associated to* γ defined by

$$\Gamma(u, \lambda)(t) = (\Gamma_1(u, \lambda)(t), \cdots, \Gamma_k(u, \lambda)(t)),$$

$$\Gamma_i(u,\lambda)(t) = \gamma_i(t,u(t),\lambda), (i=1,\cdots,k) \text{ a.e. on } I,$$

then it is known that Γ is continuous and sends bounded sets into equi-integrable sets. In this form, problem (S_λ) is equivalent to the abstract problem

$$u = \mathcal{T}(u,\lambda), \tag{5.27}$$

where $\mathcal{T} : C_0^1 \times [0,1] \to C_0^1$, given by $\mathcal{T} = K_d^\theta \circ \Gamma$, is completely continuous. In particular problem (S) is equivalent to the fixed point problem

$$u = \mathcal{T}(u,1). \tag{5.28}$$

We shall prove that this problem has a fixed point by applying Leray-Schauder's degree to (5.27). We observe from the claim just proved that there is a ball $B(0,M)$ in C_0^1 such that for any $u \in \partial B(0,M)$ and $\lambda \in [0,1]$, it holds that $u \neq \mathcal{T}(u,\lambda)$.

Then the Leray-Schauder degree $d_{LS}(I - \mathcal{T}(\cdot,\lambda), B(0,M), 0)$ is defined and from the properties of this degree it follows that

$$d_{LS}(I - \mathcal{T}(\cdot,1), B(0,M), 0) = d_{LS}(I - \mathcal{T}(\cdot,0), B(0,M), 0). \tag{5.29}$$

Since the operator $\mathcal{T}(\cdot,0) : C_0^1 \to C_0^1$ is an odd operator, by applying Borsuk's theorem we find that the degrees involved in (5.29) are different from zero. This implies that there is a solution of (5.28), equivalently that problem (S) has a solution, ending the proof of the theorem. □

6. Some examples

We begin this section by showing some examples of vector functions ϕ that are AH and satisfy conditions (H_1)-(H_2).

Example 1. Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$\phi(x) = |x|^{p-2} x \theta(|x|), \tag{6.1}$$

where $p > 1$, $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, and the function θ is a homeomorphism from $[0,\infty)$ onto $[0,\infty)$ such that for all $\sigma > 0$ it satisfies

$$\lim_{s\to\infty} \frac{\theta(\sigma s)}{\theta(s)} = 1. \tag{6.2}$$

Since for $x, y \in \mathbb{R}^N$, it holds that

$$\langle \phi(x) - \phi(y), x - y\rangle \geq \big(|x|^{p-1}\theta(|x|) - |y|^{p-1}\theta(|y|)\big)(|x| - |y|), \tag{6.3}$$

where the equality holds if and only if $x/|x| = y/|y|$. Hence (H_1) follows. Also by setting $y = 0$ in (6.3) and $\rho(s) = s^{p-1}\theta(s)$, for $s \geq 0$, we find that (H_2) is satisfied.

Now let $\{y_n\}$ and $\{x_n\}$ be two sequences in $\mathbb{R}^N$ such that $|y_n| \to \infty$ and $x_n \to x$ as $n \to \infty$. Then, by using (6.2), it can be easily verified that

$$\limsup_{n\to\infty} \frac{\theta(|x_n|\,|y_n|)}{\theta(|y_n|)} \leq 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\theta(|x_n|\,|y_n|)}{\theta(|y_n|)} = 1 \quad \text{if} \quad |x| > 0.$$

Hence

$$\frac{\phi(|y_n|x_n)}{|\phi(y_n)|} = \psi_p(x_n)\frac{\theta(|x_n|\,|y_n|)}{\theta(|y_n|)} \quad \to \quad \psi_p(x),$$

as $n \to \infty$, showing that (2.1) is satisfied and that ϕ is AH of order p at infinity.

We note that from Proposition 2.3 the homeomorphism ϕ^{-1} is AH of order p^* at infinity and it satisfies (2.1), with ϕ replaced by ϕ^{-1}.

Remark 6.1. It is then clear that examples (1.1), (1.2), and (1.3) of the Introduction provide AH functions at infinity that satisfy (H_1) and (H_2). An additional example that satisfies these conditions is

$$\phi(x) = (|x|^{p-2}x + |x|^{q-2}x)\log(1 + \log(1 + |x|)), \quad x \in \mathbb{R}^N,$$

where $p, q > 1$ and $p \neq q$.

We observe that so far all the examples of AH homogeneous functions we have shown have the form given in (6.1). It is our purpose to show next an example of an AH function at infinity that does not have that form. To do this we need a proposition.

Proposition 6.1. *Let $\phi_1 : \mathbb{R}^N \to \mathbb{R}^N$ and $\phi_2 : \overline{B(0,R)} \to \mathbb{R}^N$ be two continuous functions such that $\phi_1(x) = \phi_2(x)$ for all $x \in \partial B(0,R)$, where $B(0,R)$ denotes the ball in $\mathbb{R}^N$ centered at 0 and with radius $R > 0$ and $\overline{B(0,R)}$ its closure. Suppose that ϕ_1 satisfies (H_1) and that ϕ_2 satisfies*

$(\tilde{H}_1)$ *For any $y_1, y_2 \in \overline{B(0,R)}$, $y_1 \neq y_2$,*

$$\langle \phi_2(y_1) - \phi_2(y_2), y_1 - y_2 \rangle > 0.$$

Define $\phi : \mathbb{R}^N \to \mathbb{R}^N$ by

$$\phi(x) = \begin{cases} \phi_1(x) & \text{for} \quad x \in \mathbb{R}^N \setminus \overline{B(0,R)}, \\ \phi_2(x) & \text{for} \quad x \in \overline{B(0,R)}. \end{cases} \tag{6.4}$$

Then ϕ satisfies (H_1).

Proof. We only have to consider the case $x \in \mathbb{R}^N \setminus \{\overline{B(0,R)}\}$ and $y \in B(0,R)$ and show that

$$\langle \phi(x) - \phi(y), x - y \rangle > 0.$$

To do this, let z be the point of intersection of the line joining x and y with $\partial B(0,R)$. Then there is a $\tau\in(0,1)$ such that $z-y=\tau(x-y)$ and $x-z=(1-\tau)(x-y)$. We have

$$\begin{aligned}\langle\phi(x)-\phi(y),x-y\rangle&=\langle\phi(x)-\phi(z),x-y\rangle+\langle\phi(z)-\phi(y),x-y\rangle\\&=\frac{1}{1-\tau}\langle\phi(x)-\phi(z),(1-\tau)(x-y)\rangle+\frac{1}{\tau}\langle\phi(z)-\phi(y),\tau(x-y)\rangle\\&=\frac{1}{1-\tau}\langle\phi(x)-\phi(z),x-z\rangle+\frac{1}{\tau}\langle\phi(z)-\phi(y),z-y\rangle\ >\ 0,\end{aligned}$$

ending the proof. □

With the help of this proposition we shall construct our next example.

Example 2. Let $\phi_1(x)=|x|^{p-2}x\ \theta(|x|)$, where $p>1$, and θ is the function defined in Example 1. For $x\in\overline{B(0,R)}$, $R>0$, let us define

$$\phi_2(x)=mx+\varepsilon\eta(x)(R^2-|x|^2),$$

where $\varepsilon>0$, $\eta:B(0,\tilde R)\to\mathbb{R}^N$, $\tilde R>R$, is a C^1 function with $\eta(0)=0$, and $m=R^{p-2}\theta(R)$.

Under these circumstances it is well known that for ε small enough ϕ_2 is a diffeomorphism from $B(0,R)$ onto itself. We shall show that ϕ_2 satisfies $(\tilde H_1)$ of Proposition 6.1.

For any $y_1,y_2\in\overline{B(0,R)}$, $y_1\neq y_2$, we have

$$\begin{aligned}&\langle\phi_2(y_1)-\phi_2(y_2),y_1-y_2\rangle=m|y_1-y_2|^2\\&\qquad+\varepsilon\langle(R^2-|y_1|^2)\eta(y_1)-(R^2-|y_2|^2)\eta(y_2),y_1-y_2\rangle,\end{aligned}\tag{6.5}$$

and since the function $(R^2-|y|^2)\eta(y)$ is Lipschitz in $\overline{B(0,R)}$, say with K the Lipschitz constant, it follows that

$$|\langle(R^2-|y_1|^2)\eta(y_1)-(R^2-|y_2|^2)\eta(y_2),y_1-y_2\rangle|\le K|y_1-y_2|^2.$$

Then combining with (6.5), we find that

$$\langle\phi_2(y_1)-\phi_2(y_2),y_1-y_2\rangle\ge(m-\varepsilon K)|y_1-y_2|^2,\tag{6.6}$$

and hence for ε small enough, ϕ_2 satisfies $(\tilde H_1)$. Since ϕ_1 satisfies (H_1), then defining $\phi:\mathbb{R}^N\to\mathbb{R}^N$ as in (6.4) it follows from Proposition 6.1 that ϕ satisfies (H_1). We next show that this ϕ also satisfies (H_2) and (2.1). From (6.6), and taking into account that $\eta(0)=0$, we first get

$$\langle\phi_2(y),y\rangle\ge(m-\varepsilon K)|y|^2.$$

Combining this expression with the fact that ϕ_1 satisfies (H_2), we find that ϕ satisfies (H_2). To see that ϕ satisfies (2.1), we note that ϕ coincides with $\phi=\phi_1$ for large $|x|$, thus since ϕ_1 satisfies (2.1) so does ϕ.

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