Quadratic Relations for Generalized Hypergeometric Functions ${}_pF_{p-1}$

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1 Introduction

Let ${}_pF_{p-1}(a_1, \ldots, a_p, b_2, \ldots, b_p; z)$ be the generalized hypergeometric function

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_2)_n \cdots (b_p)_n} z^n, \quad (a)_n = a(a+1)\cdots(a+n-1).$$

Theorem 1.1 *The generalized hypergeometric function ${}_pF_{p-1}$ satisfies the following quadratic relation*

$$(1) \qquad \sum_{i=1,j=1}^{p} \left(\theta^{i-1} {}_pF_{p-1}(A,B;z)\right) \frac{c_{ij}}{c_{11}} \left(\theta^{j-1} {}_pF_{p-1}(-A, 2-B; z)\right) = 1$$

for generic values of parameters a_i and b_j where $\theta = zd/dz$, $A = (a_1, \ldots, a_p)$, $B = (b_2, \ldots, b_p)$, $-A = (-a_1, \ldots, -a_p)$, $2 - B = (2 - b_2, \ldots, 2 - b_p)$. The number c_{ij} is the (i,j)-element of the transposed inverse of the intersection matrix of cocycles associated to ${}_pF_{p-1}$ and the intersection matrix is inductively determined with respect to p by the formula given in Theorem 8.2. For example, these relations for $p = 2, 3$ are as follows.

(a) $p = 2$

$$\begin{aligned} & {}_2F_1(a_1, a_2, b_2; z)\, {}_2F_1(-a_1, -a_2, 2 - b_2; z) \\ + & \frac{z}{e_2}\, {}_2F_1'(a_1, a_2, b_2; z)\, {}_2F_1(-a_1, -a_2, 2 - b_2; z) \\ - & \frac{z}{e_2}\, {}_2F_1(a_1, a_2, b_2; z)\, {}_2F_1'(-a_1, -a_2, 2 - b_2; z) \\ - & \frac{a_1 + a_2 - e_2}{a_1 a_2 e_2} z^2\, {}_2F_1'(a_1, a_2, b_2; z)\, {}_2F_1'(-a_1, -a_2, 2 - b_2; z) = 1 \end{aligned}$$

where $e_2 = b_2 - 1$ *and* $a_1a_2 \neq 0, e_2, \notin \mathbf{Z}$.
(b) $p = 3$

$$
\begin{aligned}
& {}_3F_2(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+\ & \frac{(-t_1+1)z}{t_2}\,{}_3F_2(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+\ & \frac{z^2}{t_2}\,{}_3F_2(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
+\ & \frac{(t_1+1)z}{t_2}\,{}_3F_2'(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+\ & \frac{((t_2+1)s_1 - t_1s_2 - s_3 - t_1)z^2}{t_2s_3}\,{}_3F_2'(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+\ & \frac{(s_1+s_2-t_1-t_2)z^3}{t_2s_3}\,{}_3F_2'(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
+\ & \frac{z^2}{t_2}\,{}_3F_2''(A,B;z)\,{}_3F_2(-A,2-B;z) \\
+\ & \frac{(s_1-s_2-t_1+t_2)z^3}{t_2s_3}\,{}_3F_2''(A,B;z)\,{}_3F_2'(-A,2-B;z) \\
+\ & \frac{(s_1-t_1)z^4}{t_2s_3}\,{}_3F_2''(A,B;z)\,{}_3F_2''(-A,2-B;z) \\
=\ & 1
\end{aligned}
$$

where $s_1 = a_1 + a_2 + a_3,\ s_2 = a_1a_2 + a_2a_3 + a_3a_1,\ s_3 = a_1a_2a_3,\ t_1 = e_2 + e_3,$ $t_2 = e_2e_3,\ e_2 = b_2 - 1, e_3 = b_3 - 1,$ *and* $A = (a_1, a_2, a_3, a_4),\ B = (b_2, b_3, b_4),$ $-A = (-a_1, -a_2, -a_3, -a_4),\ 2 - B = (2 - b_2, 2 - b_3, 2 - b_4)$. *Parameters must satisfy the condition* $a_1a_2a_3 \neq 0, b_2, b_3 \notin \mathbf{Z}$.

Some identities for hypergeometric functions have geometric meaning behind. Aomoto proposed a method to study hypergeometric functions as pairings of cycles and cocycles about 30 years ago [1]. This ingenious point of view has enabled us to yield a lot of formulas for hypergeometric functions.

The identities we have presented above are quadratic relations for ${}_pF_{p-1}$. We will see that it also has a geometric meaning based on a work of Cho and Matsumoto. They proved an analog of Riemann's period relation for intersection numbers of cycles and cocycles and associated period integral, of which entries are nothing but hypergeometric functions [2]. The period relation yields a quadratic relation of hypergeometric functions. Therefore, the problem of deriving quadratic relations is reduced to evaluation of intersection numbers. Cho, Kita, Matsumoto and Yoshida gave formulas to evaluate intersection numbers for a class of hypergeometric functions expressed by a definite integral of which integrant has a normally crossing singular locus [2], [5], [6], [7]. The GHF ${}_pF_{p-1}$ has a multiple integral representation on $\mathbf{C}^{p-1}$ but the singular locus of the integrant is not normally crossing. Hence, the generalized hypergeometric functions are out of the class for general p, because their method requires a construction of resolutions of singularities and it is difficult in general. We will introduce a different approach to study the GHF. The GHF is expressed in terms of the single integral

$$\begin{aligned} & \frac{\Gamma(a_p)\Gamma(b_p-a_p)}{\Gamma(b_p)}\,{}_pF_{p-1}(a_1,\ldots,a_p,b_2,\ldots,b_p;z) \\ = \; & z^{-b_p+1}\int_0^z t^{a_p}(z-t)^{b_p-1-a_p}\,{}_{p-1}F_{p-2}(a_1,\ldots,a_{p-1},b_2,\ldots,b_{p-1};z)\frac{dt}{t} \end{aligned}$$

The kernel function ${}_{p-1}F_{p-2}$ defines a locally constant sheaf of which rank is $p-1$. We will evaluate intersection numbers by utilizing this integral representation. Cho, Kita, Matsumoto and Yoshida's formulas are those for locally constant sheaves associated to a product of linear forms, which are rank one sheaves. The kernel function ${}_{p-1}F_{p-2}$ defines a locally constant sheaf of which rank is more than 1. Hence, we cannot apply their formulas to the single integral representation of ${}_pF_{p-1}$.

The first author studied a method to evaluate intersection numbers for cocycles with coefficients in locally constant sheaves of which rank is more than one. He applied the method for evaluating intersection numbers for cocycles associated to the Selberg type integrals [9], [10].

In this paper, we first reexamine the definition of intersection number in view of the topological cup product, because discussions by Kita and Yoshida [5] are not satisfactory to apply for our problem of ${}_pF_{p-1}$ and those by the first author in [10] are not satisfactory to be a rigorous foundation. We will see that the method of the first author in [9], [10] is not only useful for computation, but also is a consequence of a general theory of duality. Next, we apply the method to evaluate intersection numbers for ${}_pF_{p-1}$. The twisted cohomology and homology groups associated to the single integral representation of ${}_pF_{p-1}$ are direct sums of *primary* parts and *degenerate* parts. Only the primary parts stand for ${}_pF_{p-1}$ [8]. This degeneration makes the evaluation of intersection numbers more complicated than the evaluation problem for the Selberg type integrals.

2 Cup product

Let X be an n-dimensional topological manifold and let f be a map from X to the final set $\{\mathrm{pt}\}$. Set $\omega_X^{\cdot} := f^! \mathbf{C}_{\{\mathrm{pt}\}}$. Then, $\omega_X^{\cdot}$ is called the dualizing complex on X and is concentrated in degree $-n$. By putting $\mathrm{or}_X := H^{-n}(\omega_X^{\cdot})$, we get the orientation sheaf or_X on X. Remark that or_X is a locally constant sheaf of rank 1.

Let $\mathcal{M}$ be a locally constant sheaf of rank m. Then, applying the Poincaré-Verdier duality, we get

$$\begin{aligned} \mathrm{R}\Gamma(X; R\mathcal{H}om\,(\mathcal{M}, \omega_X^{\cdot})) &= \mathrm{RHom}\,(\mathcal{M}, \omega_X^{\cdot}) \\ &\simeq \mathrm{RHom}\,(\mathcal{M}, f^! \mathbf{C}_{\{\mathrm{pt}\}}) \\ &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(X; \mathcal{M}), \mathbf{C}). \end{aligned} \tag{2.1}$$

Lemma 2.1 *The complex $R\mathcal{H}om\,(\mathcal{M}, \omega_X^{\cdot})$ is concentrated in degree $-n$.*

Proof. By the hypothesis of $\mathcal{M}$, we may assume that $\mathcal{M} \simeq \mathbf{C}_X^{\oplus m}$. Then, we have

$$R\mathcal{H}om\,(\mathcal{M}, \omega_X^{\cdot}) \simeq R\mathcal{H}om\,(\mathbf{C}_X^{\oplus m}, \omega_X^{\cdot}) \simeq (\omega_X^{\cdot})^{\oplus m}.$$

Since $\omega_X^{\cdot}$ is concentrated in degree $-n$, so is $R\mathcal{H}om\,(\mathcal{M}, \omega_X^{\cdot})$. Q.E.D.

We put $\mathcal{M}' := \mathcal{H}om\,(\mathcal{M}, \mathbf{C}_X)$. Then, $\mathcal{M}'$ is a locally constant sheaf of rank m, and we have

$$\begin{aligned} H^{-n}(R\mathcal{H}om\,(\mathcal{M}, \omega_X^{\cdot})) &\simeq \mathcal{H}om\,(\mathcal{M}, H^{-n}(\omega_X^{\cdot})) \\ &\simeq \mathcal{H}om\,(\mathcal{M}, \mathrm{or}_X) \\ &\simeq \mathcal{M}' \otimes \mathrm{or}_X. \end{aligned}$$

So, (2.1) gives rise to

$$H^l(X; \mathcal{M}' \otimes \mathrm{or}_X) \simeq \mathrm{Hom}\,(H_c^k(X; \mathcal{M}), \mathbf{C}),$$

where $k + l = n$. Hence, we get the cup product

$$H_c^k(X; \mathcal{M}) \otimes H^l(X; \mathcal{M}' \otimes \mathrm{or}_X) \to \mathbf{C}. \tag{2.2}$$

Assume that X is a C^∞-manifold of dimension n. We denote by $\mathcal{E}_X^j$ the sheaf on X of j-forms with coefficients that are C^∞-class functions. Consider the de Rham complex on X

$$0 \to \mathbf{C}_X \to \mathcal{E}_X^0 \to \cdots \to \mathcal{E}_X^n \to 0.$$

This induces two soft resolution

$$0 \to \mathcal{M} \to \mathcal{M} \otimes \mathcal{E}_X^0 \to \cdots \to \mathcal{M} \otimes \mathcal{E}_X^n \to 0 \tag{2.3}$$

and

$$0 \to \mathcal{M}' \otimes \mathrm{or}_X \to \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^0 \to \cdots \to \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^n \to 0. \tag{2.4}$$

Hence, we obtain the following isomorphisms

$$\begin{aligned} H^j(X; \mathcal{M}' \otimes \mathrm{or}_X) &\simeq H^j(\Gamma(X; \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^{\cdot})), \\ H_c^j(X; \mathcal{M}) &\simeq H^j(\Gamma_c(X; \mathcal{M} \otimes \mathcal{E}_X^{\cdot})). \end{aligned}$$

When $k + l = n$, we shall define the morphism

$$\Gamma_c(X; \mathcal{M} \otimes \mathcal{E}_X^k) \otimes \Gamma(X; \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^l) \to \mathbf{C} \tag{2.5}$$

as follows. For any point $x \in X$, we can choose a neighborhood U of x such that $\Gamma(U, \mathcal{M})$ is an n-dimensional vector space. It is sufficient to define the morphism

$$\Gamma_c(U; \mathcal{M} \otimes \mathcal{E}_X^k) \otimes \Gamma(U; \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^l) \to \mathbf{C}. \tag{2.6}$$

Note that $\Gamma(U; \mathcal{M}')$ is the dual space of $\Gamma(U; \mathcal{M})$. If we take a basis $\{f_1, \cdots, f_m\}$ of $\Gamma(U; \mathcal{M})$, each element of $\Gamma_c(U; \mathcal{M} \otimes \mathcal{E}_X^p)$ is represented as a direct sum $\psi_1 f_1 + \cdots + \psi_m f_m$, where $\psi_j \in \Gamma_c(U; \mathcal{E}_X^p)$ $(j = 1, \cdots, m)$. Similarly, if $\{g_1, \cdots, g_m\}$ is the dual basis of $\{f_1, \cdots, f_m\}$, we can write each element of $\Gamma(U; \mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}_X^q)$ as a direct sum $\varphi_1 g_1 + \cdots + \varphi_m g_m$, where $\varphi_j \in \Gamma(U; \mathrm{or}_X \otimes \mathcal{E}_X^q)$ $(j = 1, \cdots, m)$. We define the morphism (2.6) as

$$(\sum_{j=1}^m \psi_j f_j) \otimes (\sum_{j=1}^m \varphi_j g_j) \mapsto \sum_{j=1}^m \int_U \psi_j \wedge \varphi_j. \tag{2.7}$$

We will prove that (2.5) induces the cup product (2.2). We need the following lemma.

Lemma 2.2 *Let S be a c-soft sheaf on X.*

(i) *the presheaf*

$$S^* : U \mapsto \mathrm{Hom}\,(\Gamma_c(U; S), \mathbf{C})$$

is an injective sheaf.

(ii) *If F is a sheaf on X, then there is an isomorphism*

$$\text{(2.8)} \qquad \mathrm{Hom}\,(\Gamma_c(X; F \otimes S), \mathbf{C}) \xrightarrow{\sim} \mathrm{Hom}\,(F, S^*).$$

Although we do not prove this lemma here, we note the construction of the morphism (2.8). For any open subset U of X, we have a sequence of morphisms

$$\Gamma(U; F) \otimes \Gamma_c(U; S) \to \Gamma_c(U; F \otimes S) \to \Gamma_c(X; F \otimes S).$$

The sequence induces the morphism (2.8).

Now we set $\omega_X^{-j} := (\mathcal{E}_X^j)^*$. We define the morphism $\omega_X^{-j-1}(U) \to \omega_X^j(U)$ as

$$\begin{array}{rcl} \mathrm{Hom}\,(\Gamma_c(U; \mathcal{E}_X^{j+1}), \mathbf{C}) & \to & \mathrm{Hom}\,(\Gamma_c(U; \mathcal{E}_X^j), \mathbf{C}) \\ f & \mapsto & (-1)^j f \circ d, \end{array}$$

where $d : \mathcal{E}_X^j \to \mathcal{E}_X^{j+1}$ is the exterior differential operator. Then, we get a complex $\omega_X^\cdot$ of sheaves. Applying (2.8), there is the following isomorphism of complexes

$$\mathrm{Hom}\,(\Gamma_c(X; \mathcal{M} \otimes \mathcal{E}_X^\cdot), \mathbf{C}) \simeq \mathrm{Hom}\,(\mathcal{M}, \omega_X^\cdot).$$

This is just the Poincaré-Verdier duality (2.1).

Let $\mathcal{D}_X^j$ be the sheaf on X of j-forms with coefficients that are distributions. Suppose that U is an open subset of X. Since each element $\varphi \in \Gamma(U; \mathrm{or}_X \otimes \mathcal{D}_X^j)$ is a certain continuous map from $\Gamma_c(U; \mathcal{E}_X^{n-j})$ to $\mathbf{C}$, we can define the morphism $(\mathrm{or}_X \otimes \mathcal{D}_X^{-j+n})(U) \to \omega_X^{-j}(U)$ as

$$\begin{array}{rcl} \Gamma(U; \mathrm{or}_X \otimes \mathcal{D}_X^{-j+n}) & \to & \mathrm{Hom}\,(\Gamma_c(U; \mathcal{E}_X^j), \mathbf{C}) \\ \varphi & \mapsto & (\psi \mapsto \displaystyle\int_X \psi \wedge \varphi). \end{array}$$

Then, we get a morphism $\mathrm{or}_X \otimes \mathcal{D}_X^\cdot[n] \to \omega_X^\cdot$ of complexes. Consider the following commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{or}_X & \to & \mathrm{or}_X \otimes \mathcal{D}_X^0 & \to & \cdots & \to & \mathrm{or}_X \otimes \mathcal{D}_X^n & \to & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \\ 0 & \to & H^{-n}(\omega_X^\cdot) & \to & \omega_X^{-n} & \to & \cdots & \to & \omega_X^0 & \to & 0. \end{array}$$

Since the two rows are exact, the morphism $\mathrm{or}_X \otimes \mathcal{D}^{\cdot}_X[n] \to \omega^{\cdot}_X$ is a quasi-isomorphism. Moreover, the natural embedding $\mathcal{E}^{\cdot}_X \to \mathcal{D}^{\cdot}_X$ is also a quasi-isomorphism, so we get the following sequence of morphisms

$$\begin{aligned}
\Gamma(X;\mathcal{M}' \otimes \mathrm{or}_X \otimes \mathcal{E}^{\cdot}_X[n]) &\simeq \Gamma(X;\mathcal{H}om\,(\mathcal{M},\mathbf{C}_X) \otimes \mathrm{or}_X \otimes \mathcal{E}^{\cdot}_X[n]) \\
&\simeq \Gamma(X;\mathcal{H}om\,(\mathcal{M},\mathrm{or}_X \otimes \mathcal{E}^{\cdot}_X[n])) \\
&\simeq \mathrm{Hom}\,(\mathcal{M},\mathrm{or}_X \otimes \mathcal{E}^{\cdot}_X[n]) \\
&\to \mathrm{Hom}\,(\mathcal{M},\mathrm{or}_X \otimes \mathcal{D}^{\cdot}_X[n]) \\
&\to \mathrm{Hom}\,(\mathcal{M},\omega^{\cdot}_X) \\
&\simeq \mathrm{Hom}\,(\Gamma_c(X;\mathcal{M} \otimes \mathcal{E}^{\cdot}_X),\mathbf{C}).
\end{aligned}$$

This sequence induces (2.5). Therefore, we can calculate the cup product (2.2) by using (2.5).

3 Intersection number

Let Y be an smooth algebraic variety of dimension q. Since Y is orientable, we may suppose that $\mathrm{or}_Y \simeq \mathbf{C}_Y$. If there is no risk of confusion, we sometimes write $\mathcal{E}^{\cdot}$ instead of $\mathcal{E}^{\cdot}_Y$; that is, we omit the base space Y. We denote by $\mathcal{O}$ the sheaf of holomorphic functions on Y. Consider an integrable connection ∇_+ on the trivial vector bundle $\mathcal{O} \otimes \mathbf{C}^m$ on Y. Then, we define the sheaf $\mathrm{Ker}\,\nabla_+$ as

$$U \mapsto (\mathrm{Ker}\,\nabla_+)(U) := \{f \in \mathcal{O}(U) \otimes \mathbf{C}^m \mid \nabla_+ f = 0\}.$$

Note that $\mathrm{Ker}\,\nabla_+$ is a locally constant sheaf of rank m.

Let S be a non-degenerate complex metric on the bundle $\mathcal{O} \otimes \mathbf{C}^m$. Then, there is a unique integrable connection ∇_- on $\mathcal{O} \otimes \mathbf{C}^m$ with the following condition; for any vector field v and any two sections ψ, φ of $\mathcal{O} \otimes \mathbf{C}^m$, we have

$$S((\nabla_+)_v\psi,\varphi) + S(\psi,(\nabla_-)_v\varphi) = vS(\psi,\varphi). \tag{3.1}$$

We call ∇_- the *conjugate connection* (adjoint connection) of ∇_+ by S. Note that it is a connection on the same vector bundle $\mathcal{O} \otimes \mathbf{C}^m$. The definition induces an isomorphism

$$\mathrm{Ker}\,\nabla_- \xrightarrow{\sim} (\mathrm{Ker}\,\nabla_+)' \tag{3.2}$$

$$g \mapsto S(\cdot,g). \tag{3.3}$$

We put $\mathcal{M} := \operatorname{Ker} \nabla_+$ and let us apply results in the previous section. By (3.2), the cup product (2.2) can be rewritten as

$$H_c^k(Y; \operatorname{Ker} \nabla_+) \otimes H^l(Y; \operatorname{Ker} \nabla_-) \to \mathbf{C}, \tag{3.4}$$

where $k + l = 2q$.

The two integrable connections ∇_+ and ∇_- give rise to two exact sequence

$$0 \to \operatorname{Ker} \nabla_+ \to \mathcal{E}^0 \otimes \mathbf{C}^m \stackrel{\nabla_+}{\to} \cdots \stackrel{\nabla_+}{\to} \mathcal{E}^{2q} \otimes \mathbf{C}^m \to 0 \tag{3.5}$$

$$0 \to \operatorname{Ker} \nabla_- \to \mathcal{E}^0 \otimes \mathbf{C}^m \stackrel{\nabla_-}{\to} \cdots \stackrel{\nabla_-}{\to} \mathcal{E}^{2q} \otimes \mathbf{C}^m \to 0. \tag{3.6}$$

Since these sequence are c-soft resolutions of $\operatorname{Ker} \nabla_+$ and $\operatorname{Ker} \nabla_-$, we obtain

$$\begin{array}{rcl} H_c^r(Y; \operatorname{Ker} \nabla_+) & \simeq & H^r(\Gamma_c(Y; \mathcal{E}^{\cdot} \otimes \mathbf{C}^m), \nabla_+) \\ H^s(Y; \operatorname{Ker} \nabla_-) & \simeq & H^s(\Gamma(Y; \mathcal{E}^{\cdot} \otimes \mathbf{C}^m), \nabla_-), \end{array}$$

where $r, s = 0, \cdots, 2q$. We extend S as the morphism

$$S : (\mathcal{E}^r \otimes \mathbf{C}^m) \otimes (\mathcal{E}^s \otimes \mathbf{C}^m) \to \mathcal{E}^{r+s}.$$

Proposition 3.1 *We can calculate the cup product (3.4) by*

$$[\psi] \otimes [\varphi] \ \mapsto \ \int_Y S(\psi, \varphi), \tag{3.7}$$

where $\psi \in \Gamma_c(Y; \mathcal{E}^k \otimes \mathbf{C}^m)$ *and* $\varphi \in \Gamma(Y; \mathcal{E}^l \otimes \mathbf{C}^m)$.

Proof. We remark that the two sequences (3.5) and (3.6) are equivalent to the sequences which are given by putting $\mathcal{M} := \operatorname{Ker} \nabla_+$ under (2.3) and (2.4) respectively. In other words, we have the following commutative diagrams

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ker} \nabla_+ & \to & \operatorname{Ker} \nabla_+ \otimes \mathcal{E}^0 & \stackrel{d}{\to} \cdots \stackrel{d}{\to} & \operatorname{Ker} \nabla_+ \otimes \mathcal{E}^{2q} & \to & 0 \\ & & \| & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & \operatorname{Ker} \nabla_+ & \to & \mathcal{E}^0 \otimes \mathbf{C}^m & \stackrel{\nabla_+}{\to} \cdots \stackrel{\nabla_+}{\to} & \mathcal{E}^{2q} \otimes \mathbf{C}^m & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & (\operatorname{Ker} \nabla_+)' & \to & (\operatorname{Ker} \nabla_+)' \otimes \mathcal{E}^0 & \stackrel{d}{\to} \cdots \stackrel{d}{\to} & (\operatorname{Ker} \nabla_+)' \otimes \mathcal{E}^{2n} & \to & 0 \\ & & \wr| & & \wr\downarrow & & \wr\downarrow & & \\ 0 & \to & \operatorname{Ker} \nabla_- & \to & \mathcal{E}^0 \otimes \mathbf{C}^m & \stackrel{\nabla_-}{\to} \cdots \stackrel{\nabla_-}{\to} & \mathcal{E}^{2n} \otimes \mathbf{C}^m & \to & 0, \end{array}$$

where the vertical arrows are defined as follows

$$\begin{array}{rcl} \mathrm{Ker}\,\nabla_+ \otimes \mathcal{E}^j & \to & \mathcal{E}^j \otimes \mathbf{C}^m \\ f \otimes \psi & \mapsto & \psi f \\ (\mathrm{Ker}\,\nabla_+)' \otimes \mathcal{E}^j & \to & \mathcal{E}^j \otimes \mathbf{C}^m \\ S(\,\cdot\,, g) \otimes \varphi & \mapsto & \varphi g. \end{array}$$

Take a local basis $\{f_1, \cdots, f_m\}$ of $\mathrm{Ker}\,\nabla_+$. Then, there is a collection $\{g_1, \cdots, g_m\}$ of sections on $\mathcal{O} \otimes \mathbf{C}^m$ with the following condition; for any $r, s = 1, \cdots, m$, we have $S(f_r, g_s) = \delta_{rs}$. The collection $\{g_1, \cdots, g_m\}$ is a basis of $\mathrm{Ker}\,\nabla_-$.

Let $\psi \in \Gamma_c(Y; \mathcal{E}^k \otimes \mathbf{C}^m)$ and $\varphi \in \Gamma(Y; \mathcal{E}^l \otimes \mathbf{C}^m)$. Then, we can represent ψ and φ as

$$\begin{array}{rcl} \psi & = & \psi_1 f_1 + \cdots + \psi_m f_m \\ \varphi & = & \varphi_1 g_1 + \cdots + \varphi_m g_m, \end{array}$$

where $\psi_j \in \Gamma(Y; \mathcal{E}^k)$ and $\varphi_j \in \Gamma(Y; \mathcal{E}^l)$. Hence, we get

$$S(\psi, \varphi) = S(\sum_{i=1}^m \psi_i f_i, \sum_{j=1}^m \varphi_j g_j) = \sum_{i=1}^m \sum_{j=1}^m \psi_i \wedge \varphi_j S(f_i, g_j) = \sum_{j=1}^m \psi_j \wedge \varphi_j.$$

Therefore, (3.7) correspond to (2.7). Q.E.D.

Definition 3.1 *Let* $[\psi] \in H^q(\Gamma_c(Y, \mathcal{E}^{\cdot} \otimes \mathbf{C}^m), \nabla_+)$ *and* $[\varphi] \in H^q(\Gamma(Y, \mathcal{E}^{\cdot} \otimes \mathbf{C}^m), \nabla_-)$*. Then, we define the intersection number of the cocycles* ψ *and* φ *as the following value*

$$[\psi] \cdot [\varphi] := \int_Y S(\psi, \varphi). \tag{3.8}$$

4 Intersection numbers of twisted cycles

From now on, we assume that Y is the complement of an algebraic hypersurface in $\mathbf{C}^q$. We fix the frame $\{1 \otimes e_i\}$ of the trivial vector bundle $\mathcal{O} \otimes \mathbf{C}^m$. We are given a complex metric S which is expressed by a non-degenerate $m \times m$-matrix with respect to the frame $\{1 \otimes e_i\}$. We denote the matrix by S if no confusion arises. As in the previous section, we consider two integrable connections $\nabla_+ = d + \Omega_+$ and $\nabla_- = d - \Omega_-$ which are conjugate connections in the sense of (3.1). Here, $\Omega_\pm$ is an $m \times m$-matrix of which entries are holomorphic one forms and we have $d\Omega_\pm + \Omega_\pm \wedge \Omega_\pm = 0$. The condition (3.1) can be written in terms of $\Omega_\pm$ and S as

$$S(\Omega_+ f, g) = S(f, \Omega_- g) + (dS)(f, g) \tag{4.1}$$

for any $f, g \in \mathcal{O} \otimes \mathbf{C}^m$, which means

$${}^t\Omega_+ S - S\Omega_- - dS = 0.$$

Example 4.1 Let E be the complex metric on the bundle $\mathcal{O} \otimes \mathbf{C}^m$ defined by the standard inner product $E(1 \otimes e_i, 1 \otimes e_j) = \delta_{ij}$. Then the conjugate connection of $\nabla = d + \Omega$ is $\nabla^* = d - {}^t\Omega$. The sheaf $\mathrm{Ker}\,\nabla^*$ is isomorphic to the dual sheaf $(\mathrm{Ker}\,\nabla)'$ by $\mathrm{Ker}\,\nabla^*(U) \ni g \mapsto E(\cdot, g) \in (\mathrm{Ker}\,\nabla)'(U)$.

The next lemma, which is well-known, is a key to evaluate intersection numbers and derive quadratic relations associated to ${}_pF_{p-1}$.

Note that we consider two metric E and S. The metric E will be used to define a period matrix later and the metric S is used to define intersection numbers as we have seen.

Lemma 4.1 *Fix the frame $\{1 \otimes e_i\}$ of the vector bundle $\mathcal{O} \otimes \mathbf{C}^m$. Assume that $\mathrm{Ker}\,\nabla^*_\pm$ is the conjugate connection of $\mathrm{Ker}\,\nabla_\pm$ with respect to the complex metric E. If $\mathrm{Ker}\,\nabla_-$ is the conjugate connection of $\mathrm{Ker}\,\nabla_+$ by the complex metric S, then $\mathrm{Ker}\,\nabla^*_-$ is the conjugate connection of $\mathrm{Ker}\,\nabla^*_+$ by the complex metric $S^* := {}^tS^{-1}$;*

$$\begin{array}{ccc} \mathrm{Ker}\,\nabla_+ & \overset{S}{\longleftrightarrow} & \mathrm{Ker}\,\nabla_- \\ E \updownarrow & & E \updownarrow \\ \mathrm{Ker}\,\nabla^*_+ & \overset{{}^tS^{-1}}{\longleftrightarrow} & \mathrm{Ker}\,\nabla^*_- \end{array}$$

where $M \overset{B}{\longleftrightarrow} N$ means that the matrix B gives a non-degenerate locally $\mathbf{C}$-bilinear sheaf homomorphism $M \times N \overset{B}{\longrightarrow} \mathbf{C}$.

Proof. By taking the differential of $SS^{-1} = E$, we have $(dS)S^{-1} + Sd(S^{-1}) = 0$. Since $S^t(\Omega_+{}^tS^{-1} - {}^tS^{-1t}\Omega_- + d({}^tS^{-1}))S = {}^t\Omega_+ S - S\Omega_- + S(dS^{-1})S = 0$, we have $\Omega_+{}^tS^{-1} - {}^tS^{-1t}\Omega_- + d({}^tS^{-1}) = 0$. Q.E.D.

For the frame $\{1\otimes e_i\}$ of $\mathcal{O}\otimes \mathbf{C}^m$, we introduce the dual frame $\{1\otimes e_j^*\}$ of the dual vector bundle $\mathcal{O}\otimes \mathrm{Hom}(\mathbf{C}^m, \mathbf{C})$ such that $\langle 1\otimes e_i, 1\otimes e_j^*\rangle = \delta_{ij}$. We identify $\mathcal{O}\otimes \mathbf{C}^m$ and $\mathcal{O}\otimes \mathrm{Hom}(\mathbf{C}^m, \mathbf{C})$ by the dual frame. We regard $\nabla_\pm^*$ as the dual connection of $\nabla_\pm$ on the dual vector bundle by the identification. In this geometric picture, the matrix S^* is the transposed dual metric on the dual vector bundle.

In order to apply our discussions for the generalized hypergeometric function ${}_pF_{p-1}$, we will assume that the cohomology group $H^i(Y, \mathrm{Ker}\,\nabla_\pm)$ vanishes when $i \neq q$ in the sequel. Hypergeometric functions are pairings between cycles with the coefficients in $\mathrm{Ker}\,\nabla_\pm^*$ and cocycles with the coefficients in $\mathrm{Ker}\,\nabla_\pm$ [1]. To explain this meaning, let us recall the definition of twisted homology groups. Let Δ be an r-dimensional oriented smooth simplex in Y. Here, we regard the smooth algebraic variety as a C^∞ real manifold. We denote by $C_r(Y, \mathrm{Ker}\,\nabla_-^*)$ the space of formal finite sums of $\Delta\otimes u_\Delta^-$ where $u_\Delta^- \in \lim_{\Delta\subset U}(\mathrm{Ker}\,\nabla_-^*)(U)$ and by $C_r^{lf}(Y, \mathrm{Ker}\,\nabla_+^*)$ the space of formal locally finite sums of $\Delta\otimes u_\Delta^+$ where $u_\Delta^+ \in \lim_{\Delta\subset U}(\mathrm{Ker}\,\nabla_+^*)(U)$. The pairing between an r-chain and a vector valued r-form φ is given by the linear extension of the following pairing between $\Delta\otimes u_\Delta^+$ and φ :

$$(\varphi, \Delta\otimes u_\Delta^+) = \int_\Delta (u_\Delta^+, \varphi).$$

Here $(\cdot,\ \cdot)$ is the standard inner product.

We define the boundary operator $\partial_{\nabla_\pm^*}$ by the $\mathbf{C}$-linear extension of the boundary operator

$$\partial_{\nabla_\pm^*}\left(\Delta\otimes u_\Delta^\pm\right) = (\partial\Delta)\otimes (u_\Delta^\pm)_{|\partial\Delta}.$$

The homology groups of the complexes $C_\cdot(Y, \mathrm{Ker}\,\nabla_\pm^*)$ and $C_\cdot^{lf}(Y, \mathrm{Ker}\,\nabla_\pm^*)$ are denoted by $H_\cdot(Y, \mathrm{Ker}\,\nabla_\pm^*)$ and $H_\cdot^{lf}(Y, \mathrm{Ker}\,\nabla_\pm^*)$ respectively.

Put $q = \dim Y$. We can regard twisted cycles as a current as in the work of de Rham. In fact, for $\sigma\in C_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*)$, the functional

$$F_\sigma\ :\ \varphi \longmapsto (\varphi, \sigma), \quad \varphi\in \Gamma_c(Y, \mathcal{E}^q)\otimes \mathbf{C}^m$$

defines a vector valued current of degree q. We denote by $\langle F_\sigma, \varphi\rangle$ the evaluation by φ. For $\psi \in \Gamma_c(\mathcal{E}^{q-1}) \otimes \mathbf{C}^m$, we have

$$\begin{aligned}\langle \nabla_+^* F_\sigma, \psi\rangle &= \langle dF_\sigma - {}^t\Omega_+ F_\sigma, \psi\rangle \\ &= (-1)^{q+1}\langle F_\sigma, d\psi\rangle - (-1)^q \langle F_\sigma, \Omega_+\psi\rangle \\ &= (-1)^{q+1}(\nabla_+\psi, \sigma) \\ &= (-1)^{q+1}(\partial_{\nabla_+^*}\sigma, \psi) = 0.\end{aligned}$$

Therefore, we have a morphism

(4.2)

$$\theta \,:\, H_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*) \ni [\sigma] \longmapsto [F_\sigma] \in H^q(\Gamma(Y, \mathcal{D}^{\cdot} \otimes \mathbf{C}^m), \nabla_+^*) = H^q(Y, \nabla_+^*).$$

Here, we denote by $\mathcal{D}^q$ the space of currents of degree q. Similarly, we have

(4.3)

$$\theta_c \,:\, H_q(Y, \mathrm{Ker}\,\nabla_-^*) \ni [\tau] \longmapsto [F_\tau] \in H_c^q(\Gamma_c(Y, \mathcal{D}^{\cdot} \otimes \mathbf{C}^m), \nabla_-^*) = H_c^q(Y, \nabla_-^*).$$

Let us take a cycle

$$\gamma \in C_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*)$$

and a cocycle

$$\varphi \in \mathrm{Ker}\,(\nabla_+ \,:\, \Gamma_c(Y, \mathcal{E}^q \otimes \mathbf{C}^m) \to \Gamma_c(Y, \mathcal{E}^{q+1} \otimes \mathbf{C}^m)).$$

The pairing

$$(\varphi, \gamma) = \sum_\Delta \int_\Delta (u_\Delta^+, \varphi), \quad \gamma = \sum \Delta \otimes u_\Delta^+,\ u_\Delta^+ \in \mathrm{Ker}\,\nabla_+^*(\Delta)$$

is called *the hypergeometric integral*. By virtue of the twisted Stokes theorem, the value of the pairing depends only on the cohomology and homology classes.

Example 4.2 Let us consider the hypergeometric integral

$$I(\alpha, \beta, \gamma) = \int_{s\geq 0, t\geq 0, s+t\leq 1} s^\alpha t^\beta (1-s-t)^\gamma \frac{dsdt}{st(1-s-t)} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}.$$

Put $X = \{(s,t) \in \mathbf{C}^2 \,|\, st(1-s-t) \neq 0,\, t \neq 1\}$, $Y = \{t \in \mathbf{C} \,|\, t \neq 0, 1\}$, $f\,:\,(s,t) \mapsto t$. Although the function $s^\alpha t^\beta (1-s-t)^\gamma$ is holomorphic on $st(1-s-t) \neq 0$, the condition $t \neq 1$ is added so that $R^1 f_* \mathcal{F}_+$ is a locally constant sheaf where $\mathcal{F}_+ = \mathrm{Ker}\,(d - \alpha\frac{ds}{s} - \beta\frac{dt}{t} - \gamma\frac{d(1-s-t)}{1-s-t})$. We will regard the integral $I(\alpha,\beta,\gamma)$ defined as a pairing of homology and cohomology on the space Y. The integral can be written as an iteration of single integrals;

$$I(\alpha,\beta,\gamma) = \int_0^1 u^+(\alpha,\beta,\gamma;t)\frac{dt}{(1-t)t},$$

where

$$\begin{aligned} u^+(t) &= t^\beta \int_0^{1-t} s^\alpha (1-s-t)^\gamma \frac{(1-t)ds}{s(1-s-t)} \\ &= t^\beta (1-t)^{\alpha+\gamma} \int_0^1 \xi^\alpha (1-\xi)^\gamma \frac{d\xi}{\xi(1-\xi)}. \end{aligned}$$

The function $u^+(t)$ satisfies $\nabla_+^* u^+ = 0$ where $\nabla_+^* = d - \left(\frac{\beta}{t} - \frac{\alpha+\gamma}{1-t}\right)dt$. We regard $\frac{dt}{(1-t)t}$ as an element of $H^1(Y, \mathrm{Ker}\,\nabla_+)$. Thus, we have

$$I(\alpha,\beta,\gamma) = \left(\frac{dt}{(1-t)t}, \mathrm{reg}\,([0,1]\otimes u^+(t))\right),$$

which is a pairing of cycle and cocycle on the space Y. Here, we denote by "reg" the regularization map of twisted cycles [1, p.28].

Following Kita and Yoshida [5] and Lemma 4.1, we define intersection numbers for cycles as follows.

Definition 4.1 For $[\sigma] \in H_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*)$ and $[\tau] \in H_q(Y, \mathrm{Ker}\,\nabla_-^*)$,

$$[\sigma]\cdot[\tau] := \int_Y S^*(\theta(F_\sigma), \theta_c(F_\tau))$$

is called *the intersection number* of the cycles $[\sigma]$ and $[\tau]$. Here, we regard elements of twisted homology groups as those of twisted cohomology groups by the Poincare dualities (4.2) and (4.3).

We assume that the dimension of homology and cohomology groups $H^q(Y, \mathrm{Ker}\,\nabla_\pm)$ and $H_q(Y, \mathrm{Ker}\,\nabla_\pm^*)$ is s. Let us take bases c_i, c_i', h_i, h_i' of $H^q(Y, \mathrm{Ker}\,\nabla_\pm)$ and $H_q(Y, \mathrm{Ker}\,\nabla_\pm^*)$ respectively as follows:

$$\begin{array}{ccc} c_i \in H_c^q(Y, \mathrm{Ker}\,\nabla_+) & \overset{S}{\longleftrightarrow} & c_i' \in H^q(Y, \mathrm{Ker}\,\nabla_-) \\ E \updownarrow \text{H.G.} & & E \updownarrow \text{H.G.} \\ h_i \in H_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*) & \overset{{}^tS^{-1}}{\longleftrightarrow} & h_i' \in H_q(Y, \mathrm{Ker}\,\nabla_-^*) \end{array}$$

We define four $s \times s$-matrices

$$P_+ = (c_i, h_j)_{ij},\ P_- = (c_i', h_j')_{ij},\ I_{ch} = ([c_i] \cdot [c_j'])_{ij},\ I_h = ([h_i] \cdot [h_j'])_{ij}.$$

The matrices P_+ and P_- are called period matrices. We have the following twisted period relation under our definition of intersection numbers.

Theorem 4.1 (cf. Cho and Matsumoto [**2**])

$$I_h = {}^tP_+ {}^tI_{ch}^{-1} P_-.$$

Proof. Our proof is similar to that of Cho and Matsumoto, but we have to make a minor modification because of the metric S. Before starting proof, we note some preparatory facts. We fix the frame $\{1 \otimes e_i\}$ of $\mathcal{O} \otimes \mathbf{C}^m$.

It follows from the relation ${}^t\Omega_+ S = S\Omega_- + dS$ that

$$\begin{aligned} \nabla_+^* S &= S\nabla_- \\ \nabla_-^* {}^tS &= {}^tS\nabla_+ \end{aligned}$$

Hence, the maps

$$\begin{aligned} S &: \mathrm{Ker}\,\nabla_- \longrightarrow \mathrm{Ker}\,\nabla_+^* \\ {}^tS &: \mathrm{Ker}\,\nabla_+ \longrightarrow \mathrm{Ker}\,\nabla_-^* \end{aligned}$$

are isomorphisms of sheaves.

For $\sigma \in H_q^{lf}(Y, \mathrm{Ker}\,\nabla_+^*)$, $S^{-1}\sigma$ can be regarded as an element of $H_q^{lf}(Y, \mathrm{Ker}\,\nabla_-)$. The current $F_{S^{-1}\sigma}$ is equal to the current $S^{-1}F_\sigma$. Therefore the Poincare dual map θ and θ_c commute with S^{-1}.

Let us prove the theorem. Let $(H_{j\ell})$ be the intersection matrix of homology groups. We claim that

$$H_{j\ell} = (-1)^{q^2} \int_Y S(\theta_c {}^tS^{-1} h_\ell', \theta S^{-1} h_j). \tag{4.4}$$

In fact, it can be shown as follows by using the commutatively of S^{-1} and θ.

$$\begin{aligned} & \int_Y S(\theta_c {}^tS^{-1} h_\ell', \theta S^{-1} h_j) \\ = & \int_Y {}^t(\theta_c h_\ell')\, S^{-1} S S^{-1}\, (\theta h_j) \\ = & \int_Y {}^t\left({}^tS^{-1} \theta_c h_\ell'\right) \cdot (\theta h_j) \\ = & (-1)^{q^2} \int_Y {}^t(\theta h_j)\ {}^tS^{-1} \theta_c h_\ell'. \end{aligned}$$

Here, we regard h_i, h'_i as column vectors.

Since $\{c'_k\}$ is a basis of the cohomology group, there exist constants T_{jk} such that

$$\theta S^{-1} h_j = \sum_k T_{jk} c'_k.$$

Replacing $\theta S^{-1} h_j$ in (4.4) by the right hand side of above, we have

$$H_{j\ell} = (-1)^{q^2} \sum_k T_{jk} \int_Y S(\theta_c\, {}^t S^{-1} h'_\ell, c'_k) = \sum_k T_{jk} (c'_k, h'_\ell). \tag{4.5}$$

Thus, we have obtained a relation among the intersection matrix of cycles and the period matrix. Next, let us consider the another period matrix (c_ℓ, h_j). This is equal to

$$\begin{aligned} & \int_Y {}^t c_\ell \cdot \theta h_j \\ = & \int_Y S(c_\ell, \theta S^{-1} h_j) \\ = & \sum_k T_{jk} \int_Y S(c_\ell, c'_k). \end{aligned}$$

Hence, we obtain a relation among the intersection matrix of cocycles and the period matrix:

$$(c_\ell, h_j) = \sum T_{jk} (c_\ell, c_k). \tag{4.6}$$

Since the intersection pairing is nondegenerate, the intersection matrix of cocycles has an inverse. Therefore, we can eliminate the matrix (T_{jk}) from (4.5) and (4.6) and we obtain the twisted period relation. Q.E.D.

As we noted, twisted cohomology and homology group associated to the single integral representation of ${}_pF_{p-1}$ are sums of primary parts and degenerate parts [8]. In order to apply the twisted period relation to the generalized hypergeometric functions ${}_pF_{p-1}$, we need the following corollary.

Corollary 4.1 *Take m cocycles and cycles as follows ($m \leq s$):*

$$\begin{array}{ll} \varphi_1, \ldots, \varphi_m \in H^q_c(Y, \mathrm{Ker}\, \nabla_+) & \varphi'_1, \ldots, \varphi'_m \in H^q(Y, \mathrm{Ker}\, \nabla_-) \\ \gamma_1, \ldots, \gamma_m \in H^{lf}_q(Y, \mathrm{Ker}\, \nabla^*_+) & \gamma'_1, \ldots, \gamma'_m \in H_q(Y, \mathrm{Ker}\, \nabla^*_-) \end{array}$$

If the determinant of the $m \times m$ subperiod matrices (φ_i, γ_j) and (φ'_i, γ'_j) do not vanish, then we have the twisted period relation for the subperiod matrices:

$$\left([\gamma_i] \cdot [\gamma'_j]\right) = {}^t\left((\varphi_i, \gamma_j)\right) {}^t\left([\varphi_i] \cdot [\varphi'_j]\right)^{-1} \left((\varphi'_i, \gamma'_j)\right).$$

Proof. The m cycles γ_i and m cocycles φ_j are linearly independent, because the determinant of the subperiod matrix (φ_i, γ_j) does not vanish. Since the pairing of $H^q_c(Y, \operatorname{Ker} \nabla_+)$ and $H^{lf}_q(Y, \operatorname{Ker} \nabla^*_+)$ is perfect (see, e.g., [1, p.45]), we can find cocycles $\varphi_{m+1}, \ldots, \varphi_s \in H^q_c(Y, \operatorname{Ker} \nabla_+)$ which are orthogonal to $\gamma_1, \ldots, \gamma_s$ by the hypergeometric pairing (φ_i, γ_j) and that φ_i $(i = 1, \ldots, s)$ span $H^q_c(Y, \operatorname{Ker} \nabla_+)$. We can extend the cycles γ_i similarly so that the $s \times s$ period matrix (φ_i, γ_j) is block diagonal.

We extend the cocycles φ'_i and cycles γ'_j so that the $s \times s$ period matrix (φ'_i, γ'_j) is also block diagonal. The conclusion follows from the twisted period relation. Q.E.D.

5 Evaluation of intersection numbers of cocycles

Matsumoto [7] gave a formula evaluating intersection numbers of cocycles for arrangements in general position and locally constant sheaves of rank 1. Since our point configuration on $\mathbf{P}^1$ is in general position, we can generalize his formula to study our hypergeometric integrals with a careful treatment of locally constant sheaves of more than 1 dimension. It has been done in [9]. For reader's convenience, we will include results of [9] here.

We will generalize Theorem 2.1 [7] to that for twisted cohomology groups with locally constant sheaf whose rank is more than 1.

Let $L_1, \ldots, L_n$ be constant $m \times m$-matrices, $L_{n+1} = -(L_1 + \cdots + L_n)$, $T = \mathbf{P}^1 \setminus \{s_1, \ldots, s_n, s_{n+1} = \infty\}$ and

$$\Omega_+ = \frac{dt}{t - s_1} L_1 + \cdots + \frac{dt}{t - s_n} L_n.$$

Let $V_1, \ldots, V_{n+1}$ be neighborhoods of $s_1, \ldots, s_{n+1}$ respectively and U_i a neighborhood of s_i which contains V_i. Then there exists a smooth function

$h_i(t)$ satisfying

$$\begin{aligned} h_i(t) = 1 &\qquad t \in V_i \\ 0 \leq h_i(t) \leq 1 &\qquad t \in U_i \setminus V_i \\ h_i(t) = 0 &\qquad t \notin U_i \end{aligned}$$

Proofs of the lemmas and the theorem below are analogous to those given in [7], once we properly set conditions on eigenvalues of coefficient matrices of ∇.

Lemma 5.1 ([7], Lemma 4.1) *Let v be an eigenvector of L_i with an eigenvalue λ. If $\lambda \notin \mathbf{Z}_{\leq 0}$, then there exists a holomorphic function $\psi = \lambda^{-1}v + \sum_{k=1}^{\infty} v_k (t - s_i)^k$ such that*

$$\nabla_+ \psi = \frac{dt}{t - s_i} v \qquad \text{on } U_i.$$

Lemma 5.2 ([7], Lemma 4.2) *Let v be an eigenvector of L_i with an eigenvalue λ. Suppose that all eigenvalues of L_{n+1} are not non-positive integers and $\lambda \notin \mathbf{Z}_{\leq 0}$. For $\varphi = \frac{dt}{t-s_i} v \in H^1(Y, \nabla_+)$, we put*

$$\operatorname{coreg}(\varphi) = \varphi - \nabla_+(h_i \psi_i + h_{n+1} \psi_{n+1}),$$

where

$$\psi_i = \frac{1}{\lambda} v + \sum_{k=1}^{\infty} v_k (t - s_i)^k, \qquad \psi_{n+1} = -L_{n+1}^{-1} v + \sum_{k=1}^{\infty} v'_k (1/t)^k.$$

Then, under a suitable choice of v'_k's, the C^∞-form $\operatorname{coreg}(\varphi)$ *is cohomologous to φ in $H^1(Y, \nabla_+)$ and has a compact support. Note that the form* $\operatorname{coreg}(\varphi)$ *can be regarded as an element of $H^1_c(Y, \nabla_+)$.*

Proof. From the hypothesis and the linearity of L_{n+1}^{-1}, we can choose v'_k such that $\nabla_+ \psi_{n+1} = \varphi$ on U_{n+1}. The remainder of the proof is analogous to [7]. Q.E.D.

Theorem 5.1 ([7], Theorem 2.1) *Suppose that the bilinear form S is holomorphic on Y. Let $w \in \mathbf{C}^m$. Under the hypothesis of Lemma 5.2, the intersection number of cocycles $\varphi = \frac{dt}{t-s_i} v \in H^1(Y, \nabla_+)$ and $\phi = \frac{dt}{t-s_j} w \in H^1(Y, \nabla_-)$ is*

$$[\varphi] \cdot [\phi] = \int_{\partial V_i} S(\psi_i, \phi) + \int_{\partial V_{n+1}} S(\psi_{n+1}, \phi).$$

Corollary 5.1 *When S is a constant, we have*

$$[\varphi]\cdot[\phi] = 2\pi\sqrt{-1}\left\{\delta_{ij}\ S(\frac{1}{\lambda}v,\ w) + S(L_{n+1}^{-1}v,\ w)\right\},$$

where δ_{ij} is Kronecker's delta.

6 Evaluation of intersection numbers of cycles

Let K_+ be a smooth triangulation of Y and K_- the dual cell decomposition. We can give a formula to evaluate intersection numbers of cycles in an analogous way to [5].

Theorem 6.1 *For*

$$\sigma = \sum c_\Delta \Delta \otimes u_\Delta^+ \in H_q^{lf}(K_+, \mathrm{Ker}\,\nabla_+^*)$$

and

$$\tau = \sum C_{\Delta'}\Delta' \otimes u_{\Delta'}^- \in H_q(K_-, \mathrm{Ker}\,\nabla_-^*),$$

the intersection number $[\sigma]\cdot[\tau]$ is given by

$$\sum_{\Delta,\Delta',\{v\}=\Delta\cap\Delta'} c_\Delta c_{\Delta'} S^*(u_\Delta^+, u_{\Delta'}^-) I_v(\Delta,\Delta')$$

where $I_v(\Delta,\Delta')$ is the topological intersection number of Δ and Δ' at v and S^* is the bilinear form defined by the matrix ${}^tS^{-1}$.

Proof. Let δ_Δ be the delta q-current which has support on Δ. Then, we have $F_\sigma = \sum c_\Delta \delta_\Delta u_\Delta^+$ and $F_\tau = \sum c_{\Delta'}\delta_{\Delta'}u_{\Delta'}^-$. We note that it is not always possible to take a wedge product for currents. For example, $\delta(t)\delta(t)$ is not well-defined. Since Δ and Δ' cross transversally, the wedge product of currents $F_\sigma \wedge F_\tau$ can be defined. If we regard the operator $\nabla_\pm$ as an operator on the $2q$-dimensional real manifold Y, it is holonomic at degree $q-1$ and hypo-elliptic on Y; for any current F of degree q and G of degree $q-1$, if $\nabla_\pm G = F$ and F is smooth at the point p, then G is also smooth at the point p. Hence, when $\mathrm{reg}\,(F_\sigma) = F_\sigma + \nabla_+ G_\sigma$ and $\mathrm{reg}\,(F_\tau) = F_\tau + \nabla_- G_\tau$, the wedge product of G_σ and G_τ is well-defined. Therefore, we may compute the intersection number by evaluating the integral of currents $\int_Y S^*(F_\sigma, F_\tau)$, which is equal to

$$\sum_{\Delta,\Delta',\{v\}=\Delta\cap\Delta'} c_\Delta c_{\Delta'} S^*(u^+_\Delta, u^-_{\Delta'}) \int_Y \delta_\Delta \wedge \delta_{\Delta'}.$$

Q.E.D.

Example 6.1 Let us evaluate an intersection number of twisted cycles associated to the integral of Example 4.2. We consider two connections

$$\nabla^*_+ = d - \left(\frac{\beta}{t} - \frac{\alpha+\gamma}{1-t}\right) dt \quad \text{and} \quad \nabla^*_- = d + \left(\frac{\beta}{t} - \frac{\alpha+\gamma}{1-t}\right) dt.$$

The functions u^+ and

$$\begin{aligned} u^-(t) &= t^{-\beta}(1-t)^2 \int_0^{1-t} s^{-\alpha}(1-s-t)^{-\gamma} \frac{ds}{s(1-s-t)} \\ &= t^{-\beta}(1-t)^{-\alpha-\gamma} \int_0^1 \xi^{-\alpha}(1-\xi)^{-\beta} \frac{d\xi}{\xi(1-\xi)} \end{aligned}$$

belong to $\mathrm{Ker}\,\nabla^*_+$ and $\mathrm{Ker}\,\nabla^*_-$ respectively.

We take a nonzero complex number c and we put $S = 2\pi\sqrt{-1}\,(1/\alpha + 1/\gamma)\,c$ and $S^* = S^{-1}$. Here, the number S/c is the intersection number of $(1-t)ds/(s(1-s-t)) \in H^1_c(f^{-1}(t), \mathcal{F}_+|_{f^{-1}(t)})$ and $(1-t)ds/(s(1-s-t)) \in H^1(f^{-1}(t), \mathcal{F}_-|_{f^{-1}(t)})$. The reason that we do not simply put $S = c$ is to state an interesting observation later. It is easy to see that ∇^*_- is the conjugate connection of ∇^*_+ with respect to the complex metric S^* in the sense of (3.1). Note that for the given connection ∇^*_+, the conjugate connection ∇^*_- in the sense of (3.1) is invariant for any change of the constant $c \neq 0$.

From the twisted period relation of the Beta function (or from a well-known identity of the Beta function), we have

$$u^+ \cdot S^* \cdot u^- = d_1/c, \; d_1 := \frac{e^{2\pi\sqrt{-1}(\alpha+\gamma)} - 1}{(e^{2\pi\sqrt{-1}\alpha} - 1)(e^{2\pi\sqrt{-1}\gamma} - 1)}.$$

Put $\sigma = (0,1) \otimes u^+$ which belongs to $H^{lf}_1(\mathbf{C} \setminus \{0,1\}, \mathrm{Ker}\,\nabla^*_+)$. Consider $\tau = (0,1) \otimes u^- \in H^{lf}_1(\mathbf{C} \setminus \{0,1\}, \mathrm{Ker}\,\nabla^*_-)$. The regularization $\mathrm{reg}\,(\tau)$ of τ is

$$\frac{1}{e^{-2\pi\sqrt{-1}\beta} - 1} S^1_\varepsilon(0;\varepsilon) \otimes u^- + [\varepsilon, 1-\varepsilon] \otimes u^- - \frac{1}{e^{-2\pi\sqrt{-1}(\alpha+\gamma)} - 1} S^1_\varepsilon(1; 1-\varepsilon) \otimes u^-$$

where $S^1_r(c;b)$ is the circle with the counterclockwise orientation which has the center c, the radius r, and the base point b. By applying Theorem 6.1, we have

$$\begin{aligned}[\sigma]\cdot[\mathrm{reg}\,(\tau)] &= \frac{1}{e^{-2\pi\sqrt{-1}\beta}-1}\frac{d_1}{c}+\frac{d_1}{c}-\frac{1}{e^{-2\pi\sqrt{-1}(\alpha+\gamma)}-1}(-1)\frac{d_1}{c}\\ &= \frac{d_2}{c},\quad d_2:=\frac{1-e^{2\pi\sqrt{-1}(\alpha+\beta+\gamma)}}{(e^{2\pi\sqrt{-1}\alpha}-1)(e^{2\pi\sqrt{-1}\beta}-1)(e^{2\pi\sqrt{-1}\gamma}-1)}.\end{aligned}$$

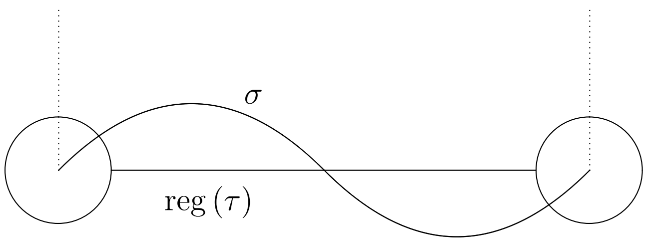

Figure 1: Intersection of $[\sigma]$ and $[\mathrm{reg}\,(\tau)]$

Evaluating the intersection number of cocycles, we have the following twisted period relation:

$$I(\alpha,\beta,\gamma)\frac{1}{2\pi\sqrt{-1}}\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)^{-1}I(-\alpha,-\beta,-\gamma)=d_2.$$

Finally let us state an observation on a relation between intersection numbers associated to multiple integral representations and those associated to single integral representations. Put $\Delta=\{(s,t)\,|\,s,t\geq 0,\ s+t<1\}$. Then, our function $I(\alpha,\beta,\gamma)$ has a multiple integral representation

$$\left(\frac{dsdt}{st(1-s-t)},\mathrm{reg}\,\left(\Delta\otimes s^{\alpha}t^{\beta}(1-s-t)^{\gamma}\right)\right).$$

We note that the intersection number of one dimensional cycles $(0,1)\otimes u^+$ and $\mathrm{reg}\,((0,1)\otimes u^-)$ on the Y space agrees with the intersection number of the two dimensional cycles $\Delta\otimes s^{\alpha}t^{\beta}(1-s-t)^{\gamma}$ and $\mathrm{reg}\,\left(\Delta\otimes s^{-\alpha}t^{-\beta}(1-s-t)^{-\gamma}\right)$ on the X-space when $c=1$. In fact, we can check this equality by evaluating the intersection number of the two dimensional cycles by the method of Kita and Yoshida [6]. We conjecture that *this coincidence holds for a large class of hypergeometric integrals.* Sugiki gives a partial answer to our conjecture [11] by the relative Verdier duality theorem.

7 Generalized hypergeometric function ${}_3F_2$

In this section, we derive intersection matrices for ${}_3F_2$ by using intersection matrices for ${}_2F_1$ to clarify our idea. The evaluation method for general p will be discussed in the next section.

We follow the notation of the paper computing monodromy group of ${}_pF_{p-1}$ [8]. Put $\alpha_{3j-2} = a_j$, $\alpha_{3j-1} = b_j - a_j$, $\alpha_{3j} = -b_j$ and put $b_1 = 1$. Let $c_i = \exp(2\pi\sqrt{-1}\alpha_i)$. Since $\alpha_{3j-2}+\alpha_{3j-1}+\alpha_{3j} = 0$, we have $c_{3j-2}c_{3j-1}c_{3j} = 1$. Put $e_i = b_i - 1$.

In case of ${}_3F_2$, the relation of the constants c_i and parameters a_j and e_k are as follows.

p	a_1	$-a_1$	0	a_2	$e_2 - a_2$	$-e_2$	a_3	$e_3 - a_3$	$-e_3$
$e^{2\pi\sqrt{-1}p}$	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9

7.1 Cycles and cocycles for the Gauss hypergeometric function

The Gauss hypergeometric function has the integral representation

$$ {}_2F_1(a_1, a_2, b_2, t) = \frac{\Gamma(b_2)}{\Gamma(a_2)\Gamma(b_2 - a_2)} t^{-e_2} \int_0^t s^{a_2}(t-s)^{e_2-a_2}(1-s)^{-a_1}\frac{ds}{s}. $$

We assume $a_1, a_2, e_2, e_2 - a_2 \notin \mathbf{Z}$ in the sequel. We define a locally constant sheaf which stands for this integration as follows. Put the holomorphic 1-form

$$ {}^t\Omega = a_2\frac{ds}{s} + (e_2 - a_2)\frac{d(t-s)}{t-s} + (-a_1)\frac{d(1-s)}{1-s}. $$

on $Y = \mathbf{C}\setminus\{0, t, 1\}$ and define $\nabla^* = d - {}^t\Omega$. Put $u = s^{a_2}(t-s)^{e_2-a_2}(1-s)^{-a_1}$. Let $\nu_1 = \mathrm{reg}\,((t,1)\otimes u)$ and $\nu_2 = \mathrm{reg}\,((0,t)\otimes u)$. Then, $\{\nu_1, \nu_2\}$ is a basis of $H_1(Y, \mathrm{Ker}\,\nabla^*)$.

Put

$$ \begin{aligned} \omega_1 &= \frac{ds}{s} \\ \omega_2 &= a_1\frac{ds}{s-1} \end{aligned} $$

Then, we have

$$ t\frac{d}{dt}\left(t^{\alpha_6}\int u\omega_1\right) = t^{\alpha_6}\int u\omega_2 $$

and $\{\omega_1,\omega_2\}$ is a basis of $H^1(Y,\mathrm{Ker}\,\nabla)$. In the sequel, we put $\nabla_+=\nabla$ and $\nabla_-=\nabla_{|a_i\mapsto -a_i, e_i\mapsto -e_i}$.

To define the covariant derivation for the case ${}_3F_2$ in the next section, we will use the fact that the pairing $t^{-e_2}\begin{pmatrix}(\omega_1,\nu_j)\\(\omega_2,\nu_j)\end{pmatrix}$ satisfies the formula

$$\frac{d}{ds}\begin{pmatrix}f\\ \theta f\end{pmatrix}=\left\{\frac{1}{s}\begin{pmatrix}0&1\\0&-B_1\end{pmatrix}+\frac{1}{s-1}\begin{pmatrix}0&0\\A_0&A_1\end{pmatrix}\right\}\begin{pmatrix}f\\ \theta f\end{pmatrix}\tag{7.1}$$

where $B_1=e_2,\ A_0=-a_1a_2,\ A_1=e_2-a_2-a_1$ and $f=t^{-e_2}(\omega_1,\nu_j)$ and $\theta f=t\frac{df}{dt}=t^{-e_2}(\omega_2,\nu_j)$.

Let ω be a rational differential form in $\mathrm{Ker}\,(\nabla_+ : \Gamma(Y,\mathcal{E}^r)\longrightarrow\Gamma(Y,\mathcal{E}^{r+1}))$. We denote by $\mathrm{coreg}\,(\omega)$ a smooth r-form with compact support which is cohomologous to ω modulo $\nabla_+\Gamma(Y,\mathcal{E}^{r-1})$ if such r-form exists.

Define

$$\varphi_1=\mathrm{coreg}\,(\omega_1),\ \varphi_2=\mathrm{coreg}\,(\omega_2)\in H^1_c(Y,\mathrm{Ker}\,\nabla_+),$$

$$h_1=(t,1)\otimes u,\ h_2=(0,t)\otimes u\in H_1^{lf}(Y,\mathrm{Ker}\,\nabla_+^*)$$

and

$$\varphi_1'=\omega_{1|a_i\mapsto -a_i,e_i\mapsto -e_i},\ \varphi_2'=\omega_{2|a_i\mapsto -a_i,e_i\mapsto -e_i}\in H^1(Y,\mathrm{Ker}\,\nabla_-),$$

$$h_1'=(t,1)\otimes u_{|a_i\mapsto -a_i,e_i\mapsto -e_i},\ h_2'=(0,t)\otimes u_{|a_i\mapsto -a_i,e_i\mapsto -e_i}\in H_1(Y,\mathrm{Ker}\,\nabla_-^*)$$

Then, we have the following twisted period relation due to Cho and Matsumoto.

$$I_h=\begin{pmatrix}(\varphi_1,h_1)&(\varphi_2,h_1)\\(\varphi_1,h_2)&(\varphi_2,h_2)\end{pmatrix}{}^tS^{-1}\begin{pmatrix}(\varphi_1',h_1')&(\varphi_1',h_2')\\(\varphi_2',h_1')&(\varphi_2',h_2')\end{pmatrix}\tag{7.2}$$

where

$$S=I_{ch}=2\pi\sqrt{-1}\begin{pmatrix}\frac{a_1+a_2-e_2}{a_2(a_1-e_2)}&\frac{a_1}{a_1-e_2}\\-\frac{a_1}{a_1-e_2}&-\frac{a_1e_2}{a_1-e_2}\end{pmatrix},\tag{7.3}$$

and

$$I_h=\begin{pmatrix}\frac{-(c_5c_2-1)}{(c_5-1)(c_2-1)}&\frac{c_5}{c_5-1}\\\frac{1}{c_5-1}&\frac{-(c_5c_4-1)}{(c_5-1)(c_4-1)}\end{pmatrix}\tag{7.4}$$

7.2 ${}_3F_2$ as a pairing of homology and cohomology groups with coefficients in a locally constant sheaf of the rank 2

As we have seen, the generalized hypergeometric function ${}_3F_2$ has the following integral representation

$$ {}_3F_2(a_1,a_2,a_3;b_2,b_3;t)=\frac{\Gamma(b_3)}{\Gamma(a_3)\Gamma(b_3-a_3)}t^{-e_3}\int_0^t s^{a_3}(t-s)^{e_3-a_3}{}_2F_1(a_1,a_2;b_2;s)\frac{ds}{s} $$

We put the matrix-valued holomorphic 1-form

$$ \begin{aligned} \Omega_+ &= \frac{ds}{s}\begin{pmatrix} a_3 & 0 \\ 1 & a_3-B_1\end{pmatrix}+\frac{ds}{s-1}\begin{pmatrix} 0 & A_0 \\ 0 & A_1\end{pmatrix}+\frac{ds}{s-t}\begin{pmatrix} e_3-a_3 & 0 \\ 0 & e_3-a_3\end{pmatrix} \\ &= M_0\frac{ds}{s}+M_1\frac{ds}{s-1}+M_t\frac{ds}{s-t}. \end{aligned} $$

Then, by (7.1), the vector-valued function $q_i(s):=s^{a_3}(t-s)^{e_3-a_3}\begin{pmatrix} t^{-e_2}(\omega_1,\nu_i) \\ t^{-e_2}(\omega_2,\nu_i)\end{pmatrix}_{|t\to s}$ is annihilated by $\nabla_+^*=d-{}^t\Omega_+$.

Define Ω_- by replacing a_i by $-a_i$ and e_i by $-e_i$ in $-\Omega_+$ respectively. The connection ∇_- is the conjugate connection of ∇_+ with respect to S defined by (7.3) in the sense of (3.1). We assume

(7.5) $\quad a_3, a_3-B_1, A_1, e_3-a_3, a_1-e_3, a_2-e_3\notin \mathbf{Z}$ and $a_1,a_2,b_2,b_3\notin\mathbf{Z}$

in the sequel. These conditions are necessary for the vanishing of cohomology groups, for the existence of coreg $(\cdot)$ and reg $(\cdot)$, and for the existence of power series expansion of the integral.

Proposition 7.1

$$ \begin{aligned} &\dim H_c^1(Y,\operatorname{Ker}\nabla_+)=\dim H^1(Y,\operatorname{Ker}\nabla_-) \\ =\ &\dim H_1^{lf}(Y,\operatorname{Ker}\nabla_+^*)=\dim H_1(Y,\operatorname{Ker}\nabla_-^*)=4. \end{aligned} $$

Proof. By applying a method to construct a basis of cycles for ${}_pF_{p-1}$ [8, Th. 2.1], we have $\dim H_1(Y,\operatorname{Ker}\nabla_-^*)=4$. It follows from the Poincare duality theorem, we have $H_c^1(Y,\operatorname{Ker}\nabla_+)\simeq H_1(Y,\operatorname{Ker}\nabla_+)$. Hence, we have $\dim H_c^1(Y,\operatorname{Ker}\nabla_+)=4$.

By applying the vanishing theorem of twisted cohomology groups [4], we get $\dim H^1(Y,\operatorname{Ker}\nabla_-)=4$. It follows from the Poincare duality theorem, we have $H^1(Y,\operatorname{Ker}\nabla_+^*)\simeq H_1(Y,\operatorname{Ker}\nabla_+^*)$. Hence, we have $\dim H_1^{lf}(Y,\operatorname{Ker}\nabla_+^*)=4$. Q.E.D.

Replace a_i by $-a_i$ and e_i by $-e_i$ in the vector valued function q_j. The new function is denoted by q'_j. Define the following four bounded cycles

$$h'_1 = \mathrm{reg}\,((t,1)\otimes q'_1), \quad h'_2 = \mathrm{reg}\,((0,t)\otimes q'_1),$$
$$h'_3 = \mathrm{reg}\,((0,t)\otimes q'_2), \quad h'_4 = S_\varepsilon(1)\otimes(-(c_2^{-1}-1)q'_1 - (c_2^{-1}c_5^{-1}-1)q'_2)$$

where $S_\varepsilon(1)$ is the circle of radius ε with center 1. It follows from [8, p.115] that h'_i, $i = 1,\ldots,4$ span $H_1(Y, \mathrm{Ker}\,\nabla^*_-)$ and the subspace spanned by h'_1, h'_2, h'_3 is called the primary parts and the subspace spanned by h'_4 is called the degenerate part. Note that the function $(-(c_2^{-1}-1)q'_1 - (c_2^{-1}c_5^{-1}-1)q'_2)$ is meromorphic at $t = 1$.

Define the following four locally finite cycles

$$h_1 = (t,1)\otimes q_1, \quad h_2 = (0,t)\otimes q_1,$$
$$h_3 = (0,t)\otimes q_2, \quad h_4 = (t,1)\otimes q_2.$$

They span $H_1^{lf}(Y, \mathrm{Ker}\,\nabla^*_+)$. Note that h'_4 is 0 in the locally finite homology group.

Let us determine a basis of the cohomology group $H^1(Y, \mathrm{Ker}\,\nabla_+)$. Noting the integral representation

$$\theta_t^k {}_3F_2(a;b;t) = \frac{\Gamma(b_3)}{\Gamma(a_3)\Gamma(b_3-a_3)} t^{-e_3} \int_0^t s^{a_3}(t-s)^{e_3-a_3}\{\theta_s^k {}_2F_1(a';b';s)\}\frac{ds}{s}$$

we will determine cocycles which stand for $\theta^k {}_3F_2$. For a cycle $\sigma = D\otimes\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and a cocycle $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, the hypergeometric pairing is defined by $(\psi,\sigma) = \int_D (g_1\psi_1 + g_2\psi_2)$. Put

$$\varphi_1 = \frac{ds}{s}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \varphi_2 = \frac{ds}{s}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \varphi_3 = \frac{ds}{s}\begin{pmatrix} 0 \\ -B_1 \end{pmatrix} + \frac{ds}{s-1}\begin{pmatrix} A_0 \\ A_1 \end{pmatrix}$$

and c_i is $\mathrm{coreg}\,(\varphi_i)$. The smooth one form with compact support $\mathrm{coreg}\,(\varphi_i)$ can be explicitly constructed by Matsumoto's method [7]. We can see $\theta_k f = (c_{k+1}, h_1)$, $k = 1, 2$, for the function $f = (c_1, h_1)$ by utilizing the differential equation for ${}_3F_2$

$$\theta^2 f = -B_1\theta f + \frac{s}{s-1}(A_0 f + A_1\theta f).$$

Linearly independence of c_i follows from the fact that the 3×3 Wronskian determinant for ${}_3F_2$ does not vanish. The cocycles $\varphi_1, \varphi_2, \varphi_3$ span the primary part of the cohomology group.

7.3 Intersection matrix

By differentiating the quadratic relation (7.2) for ${}_2F_3$ with respect to t and using the differential equation for the period matrix, we have

$$
{}^t\Omega_+ S - S\Omega_- = 0. \tag{7.6}
$$

Let v_1, v_2, w_1, w_2 be two dimensional complex column vectors. The intersection number of coreg $\left(\frac{ds}{s}w_1 + \frac{ds}{s-1}w_2\right)$ and $\frac{ds}{s}v_1 + \frac{ds}{s-1}v_2$ is equal to $2\pi\sqrt{-1}$ times

$$
S(M_0^{-1}w_1, v_1) + S(w_2', v_2) + S(M_\infty^{-1}(w_1+w_2), v_1+v_2) \tag{7.7}
$$

Here, w_2' is a vector such that $M_1 w_2' = w_2$. Although M_1 is not invertible, when $\frac{ds}{s}w_1 + \frac{ds}{s-1}w_2 = c_i$, $(i=1,2,3)$, w_2 is an eigenvector of M_1 or a zero vector. So, we can evaluate intersection numbers of c_i and c_j for $1 \le i, j \le 3$ by the formula above. The formula (7.7) can be proved by Matsumoto's method [7] that explicitly constructs a cocycle with compact support from a rational cocycle. See Section 5 and [9] for details. Thus, we have the following proposition.

Proposition 7.2 *The 3×3 submatrix $([c_i] \cdot [\varphi_j'])$ of the intersection matrix of cocycles is equal to*

$$
2\pi\sqrt{-1}\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ -c_{12} & -c_{13} & c_{23} \\ c_{13} & -c_{23} & c_{33} \end{pmatrix}
$$

where $c_{11} = ((a_2^2a_1^2 + a_3^2a_1^2 + a_3^2a_2^2) + a_1a_2a_3(a_1+a_2+a_3) - (e_2+e_3)((a_1+a_2+a_3)(a_1a_2+a_2a_3+a_3a_1) - a_1a_2a_3) + (e_2+e_3)^2(a_1a_2+a_2a_3+a_3a_1) - e_2e_3(e_2+e_3)(a_1+a_2+a_3) + e_2e_3(a_1^2+a_2^2+a_3^2) + e_2^2e_3^2)/(a_2a_3(a_1-e_3)(a_1-e_2)(a_3-e_2)(a_2-e_3))$,

$c_{12} = \frac{a_1((a_1a_2+a_2a_3+a_3a_1)-(e_2+e_3)(a_1+a_2+a_3)-e_2e_3+(e_2+e_3)^2)}{(a_1-e_3)(a_1-e_2)(a_3-e_2)(a_2-e_3)}$,

$c_{13} = \frac{a_1(a_1a_2a_3-e_2e_3(a_1+a_2+a_3)+e_2e_3(e_2+e_3))}{(a_1-e_3)(a_1-e_2)(a_3-e_2)(a_2-e_3)}$,

$c_{23} = -\frac{a_1((e_2+e_3)a_1a_2a_3-e_2e_3(a_1a_2+a_2a_3+a_3a_1)+e_2^2e_3^2)}{(a_1-e_3)(a_1-e_2)(a_3-e_2)(a_2-e_3)}$,

$c_{33} = \frac{a_1(((e_2+e_3)^2-e_2e_3)a_1a_2a_3-e_2e_3(e_2+e_3)(a_1a_2+a_2a_3+a_3a_1)+e_3^2e_2^2(a_1+a_2+a_3))}{(a_1-e_3)(a_1-e_2)(a_3-e_2)(a_2-e_3)}$

Proposition 7.3 *The* 3×3 *submatrix* $([h_i]\cdot[h'_j])$ *of the intersection matrix of cycles is equal to*

$$\begin{pmatrix} \frac{(c_8c_5c_2-1)}{(c_8-1)(c_5-1)(c_2-1)} & \frac{-c_8(c_5c_2-1)}{(c_8-1)(c_5-1)(c_2-1)} & \frac{c_5c_8}{(c_8-1)(c_5-1)} \\ \frac{-(c_5c_2-1)}{(c_8-1)(c_5-1)(c_2-1)} & h_{22} & \frac{-c_5(c_8c_7-1)}{(c_8-1)(c_7-1)(c_5-1)} \\ \frac{1}{(c_8-1)(c_5-1)} & \frac{-(c_8c_7-1)}{(c_8-1)(c_7-1)(c_5-1)} & \frac{(c_8c_7-1)(c_5c_4-1)}{(c_8-1)(c_7-1)(c_5-1)(c_4-1)} \end{pmatrix}$$

where $h_{22} = ((((c_7-1)c_5^2+(c_8-1)c_7c_5)c_4+(-c_8c_7^2+c_7)c_5)c_2+(-c_8c_7+1)c_5c_4+(c_8-1)c_7c_5+c_8c_7^2-c_8c_7)/((c_8-1)(c_7-1)(c_5-1)(c_5c_4-c_7)(c_2-1))$.

Proof. The proof can be done by applying Theorem 6.1 and the formula of intersection numbers for ${}_2F_1$. For example, let us evaluate $[h_1]\cdot[h'_2]$. Applying the formula in Theorem 6.1, we have

$$[h_1]\cdot[h'_2] = \frac{1}{1-e^{2\pi\sqrt{-1}(a_3-e_3)}}S^*(q_1,q'_1).$$

The twisted period relation (7.2) for ${}_2F_1$ yields $S^*(q_1,q'_2)=\frac{c_5}{c_5-1}$ and hence $[h_1]\cdot[h'_2]=\frac{c_8(c_5c_2-1)}{(-1)(c_8-1)(c_5-1)(c_2-1)}$. Q.E.D.

The quadratic relation for ${}_3F_2$ is nothing but the identities of the $(3,3)$ element of the twisted period relation for a subperiod matrix. We note that the function ${}_3F_2(a_1,a_2,a_3,b_2,b_3;z)$ is holomorphic with respect to the parameters a_i and b_j when $b_j\notin \mathbf{Z}_{<0}$. Therefore, once we obtain the identity, the condition for parameters (7.5) is no longer necessary as far as the expression is well defined. Thus, we have proved Theorem 1.1 for ${}_3F_2$.

We can inductively apply the procedure explained in this section, which will be explained in the next section.

8 Generalized Hypergeometric Function ${}_pF_{p-1}$

The GHF ${}_pF_{p-1}$ is expressed as

$$\begin{aligned}&{}_pF_{p-1}(a_1,\ldots,a_p;b_2,\ldots,b_p;t)\\ &\quad= \frac{\Gamma(b_p)}{\Gamma(a_p)\Gamma(b_p-a_p)}t^{-e_p}\int_0^t s^{a_p}(t-s)^{e_p-a_p}{}_{p-1}F_{p-2}(a_1,\ldots,a_{p-1};b_2,\ldots,b_{p-1};s)\frac{ds}{s},\end{aligned}$$

where $e_k=b_k-1$ and $b_1=1$. This representation is regarded as a recursion formula among ${}_pF_{p-1}$ and ${}_{p-1}F_{p-2}$. We may expect that formulas for ${}_pF_{p-1}$ can be derived from those for ${}_{p-1}F_{p-2}$. In fact, it is true that there exists a recursion formula of intersection numbers associated GHF.

8.1 Connection form for ${}_pF_{p-1}$.

Let B_k, C_k be elementary symmetric polynomials of e_i, a_j defined respectively as follows:

$$x\prod_{i=2}^{p-1}(x+e_i) = B_0 + B_1x + \cdots + B_{p-2}x^{p-2} + x^{p-1}$$
$$\prod_{i=1}^{p-1}(x+a_i) = C_0 + C_1x + \cdots + C_{p-2}x^{p-2} + x^{p-1}$$

We put $A_k = B_k - C_k$.

For an expression $g = g(a,e)$ with parameters $a = (a_1, \dots), e = (e_1, \dots)$, we put

$$g^\vee := g|_{a_i\mapsto -a_i, e_j\mapsto -e_j}.$$

In the sequel, we will use this notation to avoid to write a complicated replacement rule.

Put

$$\begin{aligned}
T &= \mathbf{P}^1 \setminus \{0,1,t,\infty\} \\
\Omega &= \frac{ds}{s}\begin{pmatrix} a_p & \dots & 0 & 0 \\ 1 & \ddots & & -B_1 \\ \vdots & \ddots & a_p & \vdots \\ 0 & \dots & 1 & a_p - B_{p-2} \end{pmatrix} \\
&\qquad + \frac{ds}{s-1}\begin{pmatrix} 0 & \dots & 0 & A_0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & A_{p-2} \end{pmatrix} + \frac{ds}{s-t}\begin{pmatrix} e_p - a_p & & 0 \\ & \ddots & \\ 0 & & e_p - a_p \end{pmatrix} \\
&= \frac{ds}{s}L_0 + \frac{ds}{s-1}L_1 + \frac{ds}{s-t}L_t \\
\Omega_+ &= \Omega \\
\Omega_- &= -\Omega^\vee.
\end{aligned}$$

Then, it follows from the p-th order ordinary differential equation satisfied by ${}_pF_{p-1}$ that we have

$$\nabla_+^* \left(s^{a_p}(s-t)^{e_p-a_p} \begin{pmatrix} f \\ \theta f \\ \vdots \\ \theta^{p-2} f \end{pmatrix} \right) = 0, \tag{8.1}$$

where $f = {}_{p-1}F_{p-2}(a_1, \dots, a_{p-1}; b_2, \dots, b_{p-1}; s)$ and $\theta = s\frac{d}{ds}$ is the Euler operator for s.

Quadratic relations of GHF ${}_pF_{p-1}$ is obtained by evaluating intersection numbers for twisted (co)homology groups. In this section, we will derive a recursive formula of I_{ch} and I_h and will prove Theorem 1.1.

8.2 Eigenvalues of coefficients of Ω

In this subsection, we evaluate eigenvalues of coefficient matrices of Ω_+ to specify an exceptional set for regularization.

Put

$$L_\infty = -(L_0 + L_1 + L_t) = - \begin{pmatrix} e_p & \dots & 0 & -C_0 \\ 1 & \ddots & & -C_1 \\ \vdots & \ddots & e_p & \vdots \\ 0 & \dots & 1 & e_p - C_{p-2} \end{pmatrix}.$$

and we note that L_∞ is the coefficient matrix of Ω_+ at ∞. The following theorem can be shown by an elementary calculation.

Theorem 8.1 *The eigenvalues of L_i are*

L_0	a_p	$a_p - e_2$	$\cdots$	$a_p - e_{p-1}$
L_1	0	$\cdots$	0	A_{p-2}
L_t	$e_p - a_p$	$\cdots$	$\cdots$	$e_p - a_p$
L_∞	$-e_p + a_1$	$\cdots$	$\cdots$	$-e_p + a_{p-1}$

Note that $A_{p-2} = (e_1 + \cdots + e_{p-1}) - (a_1 + \cdots + a_{p-1})$.

Definition 8.1 We define two sets E_p, E'_p by

$$\begin{aligned} E_p = &\{a_p, a_p - e_2, \dots, a_p - e_{p-1}\} \cup \{a_1 - e_p, \dots, a_{p-1} - e_p\} \cup \{e_p - a_p\} \\ &\cup \{(e_1 + \cdots + e_{p-1}) - (a_1 + \cdots + a_{p-1})\} \\ E'_p = &\{e_i - a_j \mid 1 \le i, j \le p\} \cup \left\{ \sum_{i=1}^{k} (e_i - a_i) \mid 2 \le k \le p-1 \right\}. \end{aligned}$$

8.3 Intersection numbers for cocycles for ${}_pF_{p-1}$

Definition 8.2 Put

$$\varphi_1 = \frac{ds}{s}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \quad \varphi_2 = \frac{ds}{s}\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \quad \dots, \quad \varphi_{p-1} = \frac{ds}{s}\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix},$$

$$\varphi_p = \frac{ds}{s}\begin{pmatrix}0\\-B_0\\\vdots\\-B_{p-2}\end{pmatrix} + \frac{ds}{s-1}\begin{pmatrix}A_0\\A_1\\\vdots\\A_{p-2}\end{pmatrix}$$

Lemma 8.1 *Assume $a_1, \dots, a_p, b_2, \dots, b_p, b_p - a_p \notin \mathbf{Z}_{\le 0}$. For cocycles $\varphi_1, \dots, \varphi_p \in H^1(Y, \nabla_+)$, it holds that*

$$\theta^k {}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; t) = \frac{\Gamma(b_p)}{\Gamma(a_p)\Gamma(b_p - a_p)} t^{-e_p} (\varphi_k, \tau),$$

where τ is a twisted cycle defined by

$$\tau = (0, t) \otimes \left\{ s^{a_p}(t-s)^{e_p - a_p} \begin{pmatrix} f \\ \theta f \\ \vdots \\ \theta^{p-1} f \end{pmatrix} \right\}$$

$$f = {}_{p-1}F_{p-2}(a_1, \dots, a_{p-1}; b_2, \dots, b_{p-1}; s).$$

Proof. By the well-known formula

$$\theta_t^k {}_pF_{p-1}(a_1, \dots, a_p; b_2, \dots, b_p; t) = \frac{\Gamma(b_p)}{\Gamma(a_p)\Gamma(b_p - a_p)} t^{-e_p} \int_0^t s^{a_p}(t-s)^{e_p - a_p} (\theta_s^k f) \frac{ds}{s},$$

the conclusion is clear for $1 \le k < p$. For $k = p$, by the differential system (8.1), we have

$$\theta^{p-1} f = -\sum_{k=1}^{p-2} B_k \theta^k f + \frac{s}{s-1} \sum_{j=0}^{p-2} A_j \theta^j f.$$

Q.E.D.

Lemma 8.2 *Let $F(t) = \sum_{n=0}^{\infty} c_n t^n$ be a holomorphic function around $t = 0$. Define $W(t) = \det \left(\frac{d^m}{dt^m} \theta^k F \right)_{0 \le k, m \le p-1}$, which is the Wronskian of $F, \theta F, \ldots, \theta^{p-1} F$. Then, the value of $W(t)$ at $t = 0$ is equal to $\prod_{j=0}^{p-1} c_j$.*

Proof. Noting $\left(\frac{d^m}{dt^m} \theta^k F \right)(0) = (c_m/m!) m^k$, we have

$$\begin{aligned} W(0) &= \det((c_m/m!) m^k)_{0 \le k, m \le p-1} \\ &= c_0 \cdot \det((c_m/m!) m^k)_{1 \le k, m \le p-1} \\ &= c_0 \left(\prod_{m=1}^{p-1} (c_m/m!) m \right) \det(m^{k-1})_{1 \le k, m \le p-1} \\ &= c_0 (p-1)! \left(\prod_{j=1}^{p-1} (c_j/j!) \right) \left(\prod_{1 \le i < j \le p-1} (j - i) \right) \\ &= \prod_{j=1}^{p-1} c_j \end{aligned}$$

Here, we used Vandermonde's determinant. Q.E.D.

Lemma 8.3 *Suppose that $a_1, \ldots, a_p, b_2, \ldots, b_p, b_p - a_p \notin \mathbf{Z}_{\le 0}$. Then cocycles $\varphi_1, \ldots, \varphi_p \in H^1(Y, \nabla_+)$ are linearly independent.*

Proof. It is enough to prove that (φ_k, τ), $k = 1, \ldots, p$ are linearly independent over the field of complex numbers. By Lemma 8.1, we have

$$(\varphi_k, \tau) = \frac{\Gamma(a_p)\Gamma(b_p - a_p)}{\Gamma(b_p)} t^{e_p} \theta^k {}_pF_{p-1}(a_1, \ldots, a_p; b_2, \ldots, b_p; t).$$

By using Lemma 8.2, we have $W(0) \ne 0$ for $F = {}_pF_{p-1}(t)$. Q.E.D.

Lemma 8.4 *If $E_p \cap \mathbf{Z}_{\le 0} = \emptyset$, then there exists* coreg (φ_i)

Proof. By Lemma 5.2, it is clear. Q.E.D.

Let $\mathbf{e}_k$ be unit vectors, $u = -B_1\mathbf{e}_2 - \cdots - B_{p-2}\mathbf{e}_{p-1}$, and $v = A_0\mathbf{e}_1 + \cdots + A_{p-2}\mathbf{e}_{p-1}$. The vector v is an eigenvector of L_1 with the eigenvalue A_{p-2} and that $u + v = -C_0\mathbf{e}_1 - \cdots - C_{p-2}\mathbf{e}_{p-1}$.

Lemma 8.5

$$\begin{aligned}
L_0\mathbf{e}_{p-1} &= a_p\mathbf{e}_{p-1} + u \\
L_0^{-1}u &= \mathbf{e}_{p-1} - a_pL_0^{-1}\mathbf{e}_{p-1} \\
u &= (L_0 - a_pI)\mathbf{e}_{p-1} \\
(-L_\infty)\mathbf{e}_{p-1} &= e_p\mathbf{e}_{p-1} + (u+v) \\
L_\infty^{-1}(u+v) &= -\mathbf{e}_{p-1} - e_pL_\infty^{-1}\mathbf{e}_{p-1}
\end{aligned}$$

We take a constant matrix S such that ${}^t\Omega_+S - S\Omega_- = 0$. The connections $\nabla_+ = d + \Omega_+$ and $\nabla_- = d - \Omega_-$ are conjugate with respect to the metric defined by S. The existence of such matrix S for any p is inductively proved by Theorem 4.1 (the twisted period relation) and Theorem 8.2; we take the intersection matrix for ${}_{p-1}F_{p-2}$ as our S.

Lemma 8.6 *We have*

$$S(Lw_1, w_2) = S(w_1, L^\vee w_2)$$

for any $L = L_0, L_1, L_\infty$ *and any* $w_1, w_2 \in \mathbf{C}^m$.

Lemma 8.7 *For any* $w \in \mathbf{C}^m$, *we have*

$$\begin{aligned}
-S(w, \mathbf{e}_{p-1}) + a_pS(L_0^{-1}w, \mathbf{e}_{p-1}) &= S(L_0^{-1}w, u^\vee) \\
S(w, \mathbf{e}_{p-1}) + e_pS(L_\infty^{-1}w, \mathbf{e}_{p-1}) &= S(L_\infty^{-1}w, (u+v)^\vee) \\
S(v, \mathbf{e}_{p-1}) + A_{p-2}^{-1}S(v, v^\vee) &= 0
\end{aligned}$$

Theorem 8.2 *Suppose a condition on parameters* $E_p' \cap \mathbf{Z} = \emptyset$. *Then the intersection matrix*

$$I_{ch} = \begin{pmatrix} [\varphi_1]\cdot[\varphi_1^\vee] & \dots & [\varphi_1]\cdot[\varphi_p^\vee] \\ \vdots & & \vdots \\ [\varphi_p]\cdot[\varphi_1^\vee] & \dots & [\varphi_p]\cdot[\varphi_p^\vee] \end{pmatrix}$$

is determined by the following formula

$$\frac{1}{2\pi\sqrt{-1}}[\varphi_i]\cdot[\varphi_j^\vee] = \begin{cases} ({}^tP_0S)_{ij} & (1\le i,j\le p-1) \\ ({}^tP_1S)_{i,p-1} & (1\le i\le p-1,\ j=p) \\ -({}^tP_1S)_{p-1,j} = (SP_1^\vee)_{p-1,j} & (i=p,\ 1\le j\le p-1) \\ -({}^tP_2S)_{p-1,p-1} & (i=j=p) \end{cases},$$

where

$$\begin{aligned} P_0 &= L_0^{-1} + L_\infty^{-1} \\ P_1 &= a_pL_0^{-1} + e_pL_\infty^{-1} \\ P_2 &= a_p^2L_0^{-1} + e_p^2L_\infty^{-1} - (a_p - e_p)I \end{aligned}$$

We call I_{ch} the intersection matrix of cocycles of ${}_pF_{p-1}$. The intersection matrix I_{ch} for $p=2$ is given in (7.3). Since, the matrix S is the intersection matrix standing for ${}_{p-1}F_{p-2}$, the formula above gives a recursion formula for intersection numbers for ${}_pF_{p-1}$ and ${}_{p-1}F_{p-2}$.

Proof. Under our preparatory lemmas in this section, the proof can be done by applying Theorem5.1. We note that

$$\begin{aligned} \varphi_i &= \frac{ds}{s}\mathbf{e}_i, \qquad (i=1,\dots,p-1) \\ \varphi_p &= \frac{ds}{s}u + \frac{ds}{s-1}v. \end{aligned}$$

By Theorem 5.1, intersection numbers are evaluated as

$$\begin{aligned}
\frac{1}{2\pi\sqrt{-1}}[\varphi_i]\cdot[\varphi_j^\vee] &= S(L_0^{-1}\mathbf{e}_i,\mathbf{e}_j)+S(L_\infty^{-1}\mathbf{e}_i,\mathbf{e}_j)\\
&= S((L_0^{-1}+L_\infty^{-1})\mathbf{e}_i,\mathbf{e}_j)\\
&= ({}^tP_0S)_{ij}\\
\frac{1}{2\pi\sqrt{-1}}[\varphi_i]\cdot[\varphi_p^\vee] &= S(L_0^{-1}\mathbf{e}_i,u^\vee)+S(L_\infty^{-1}\mathbf{e}_i,(u+v)^\vee)\\
&= -S(\mathbf{e}_i,\mathbf{e}_{p-1})+a_pS(L_0^{-1}\mathbf{e}_i,\mathbf{e}_{p-1})+S(\mathbf{e}_i,\mathbf{e}_{p-1})+e_pS(L_\infty^{-1}\mathbf{e}_i,\mathbf{e}_{p-1})\\
&= a_pS(L_0^{-1}\mathbf{e}_i,\mathbf{e}_{p-1})+e_pS(L_\infty^{-1}\mathbf{e}_i,\mathbf{e}_{p-1})\\
&= ({}^tP_1S)_{i,p-1}\\
\frac{1}{2\pi\sqrt{-1}}[\varphi_p]\cdot[\varphi_j^\vee] &= S(L_0^{-1}u,\mathbf{e}_j)+S(L_\infty^{-1}(u+v),\mathbf{e}_j)\\
&= S(u,(L_0^{-1})^\vee\mathbf{e}_j)+S(u+v,(L_\infty^{-1})^\vee\mathbf{e}_j)\\
&= S(\mathbf{e}_{p-1},\mathbf{e}_j)-a_pS(\mathbf{e}_{p-1},(L_0^{-1})^\vee\mathbf{e}_j)-S(\mathbf{e}_{p-1},\mathbf{e}_j)-e_pS(\mathbf{e}_{p-1},(L_\infty^{-1})^\vee\mathbf{e}_j)\\
&= -a_pS(\mathbf{e}_{p-1},(L_0^{-1})^\vee\mathbf{e}_j)-e_pS(\mathbf{e}_{p-1},(L_\infty^{-1})^\vee\mathbf{e}_j)\\
&= -a_pS(L_0^{-1}\mathbf{e}_{p-1},\mathbf{e}_j)-e_pS(L_\infty^{-1}\mathbf{e}_{p-1},\mathbf{e}_j)\\
&= -({}^tP_1S)_{p-1,j}\\
\frac{1}{2\pi\sqrt{-1}}[\varphi_p]\cdot[\varphi_p^\vee] &= S(L_0^{-1}u,u^\vee)+S(A_{p-2}^{-1}v,v^\vee)+S(L_\infty^{-1}(u+v),(u+v)^\vee)\\
&= -S(u,\mathbf{e}_{p-1})+a_pS(L_0^{-1}u,\mathbf{e}_{p-1})+A_{p-2}^{-1}S(v,v^\vee)\\
&\qquad +S(u+v,\mathbf{e}_{p-1})+e_pS(L_\infty^{-1}(u+v),\mathbf{e}_{p-1})\\
&= S(v,\mathbf{e}_{p-1})+A_{p-2}^{-1}S(v,v^\vee)+a_pS(L_0^{-1}u,\mathbf{e}_{p-1})+e_pS(L_\infty^{-1}(u+v),\mathbf{e}_{p-1})\\
&= a_pS(L_0^{-1}u,\mathbf{e}_{p-1})+e_pS(L_\infty^{-1}(u+v),\mathbf{e}_{p-1})\\
&= a_p\{S(\mathbf{e}_{p-1}-a_pL_0^{-1}\mathbf{e}_{p-1},\mathbf{e}_{p-1})\}-e_p\{S(\mathbf{e}_{p-1}+e_pL_\infty^{-1}\mathbf{e}_{p-1},\mathbf{e}_{p-1})\}\\
&= (a_p-e_p)S(\mathbf{e}_{p-1},\mathbf{e}_{p-1})-a_p^2S(L_0^{-1}\mathbf{e}_{p-1},\mathbf{e}_{p-1})-e_p^2S(L_\infty^{-1}\mathbf{e}_{p-1},\mathbf{e}_{p-1})\\
&= (a_p-e_p)S_{p-1,p-1}-({}^t(a_p^2L_0^{-1}+e_p^2L_\infty^{-1})S)_{p-1,p-1}\\
&= -({}^tP_2S)_{p-1,p-1}
\end{aligned}$$

Q.E.D.

Example 8.1 $p=2$

$$I_{ch}^{-1} = \frac{1}{2\pi\sqrt{-1}(-a_1)(e_2-a_2)} \begin{pmatrix} -B_1C_0 & -C_0 \\ C_0 & C_1-B_1 \end{pmatrix}$$
$$= \frac{1}{2\pi\sqrt{-1}(-a_1)(e_2-a_2)} \begin{pmatrix} -a_1a_2e_2 & -a_1a_2 \\ a_1a_2 & a_1+a_2-e_2 \end{pmatrix},$$

where $C_0 = a_1a_2$, $C_1 = a_1+a_2$, $B_1 = e_2$.

$p=3$

$$I_{ch}^{-1} = \frac{1}{(2\pi\sqrt{-1})^2(-a_1)(e_2-a_2)(e_3-a_3)} \times \begin{pmatrix} -B_1C_0 & -B_2C_0 & -C_0 \\ B_2C_0 & C_0+B_2C_1-B_1C_2 & C_1-B_1 \\ -C_0 & -(C_1-B_1) & -(C_2-B_2) \end{pmatrix},$$

where $C_0 = a_1a_2a_3$, $C_1 = a_1a_2+a_2a_3+a_3a_1$, $C_2 = a_1+a_2+a_3$, $B_1 = e_2e_3$, $B_2 = e_2+e_3$.

$p=4$

$$I_{ch}^{-1} = \frac{1}{(2\pi\sqrt{-1})^3(-a_1)(e_2-a_2)(e_3-a_3)(e_4-a_4)} \times \begin{pmatrix} -B_1C_0 & -B_2C_0 & -B_3C_0 & -C_0 \\ B_2C_0 & B_3C_0+B_2C_1-B_1C_2 & C_0+B_3C_1-B_1C_3 & C_1-B_1 \\ -B_3C_0 & -(C_0+B_3C_1-B_1C_3) & -C_1-B_3C_2+B_2C_3+B_1 & -(C_2-B_2) \\ C_0 & C_1-B_1 & C_2-B_2 & C_3-B_3 \end{pmatrix}$$

where $C_0 = a_1a_2a_3a_4$, $C_1 = a_1a_2a_3+a_2a_3a_4+a_3a_4a_1+a_4a_1a_2$, $C_2 = a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4$, $C_3 = a_1+a_2+a_3+a_4$, $B_1 = e_2e_3e_4$, $B_2 = e_2e_3+e_3e_4+e_4e_2$, $B_3 = e_2+e_3+e_4$.

8.4 Intersection numbers for cycles of ${}_pF_{p-1}$

Definition 8.3 For $p>1$, we recursively define $(p-1)$-dimensional vector valued functions $q_1^{(p)}(s),\dots,q_{p-1}^{(p)}(s)$ and cycles $\sigma_1^{(p)},\dots,\sigma_p^{(p)}$ of the locally finite homology group $H_1^{lf}(Y,\nabla_+^*)$ associated with ${}_pF_{p-1}$ as follows:

1. For any $p>1$, put

$$\sigma_1^{(p)} = (t,1)\otimes q_1^{(p)},\ \sigma_2^{(p)} = (0,t)\otimes q_1^{(p)},\ \dots,\ \sigma_p^{(p)} = (0,t)\otimes q_{p-1}^{(p)}$$

2. When $p=2$, put $q_1^{(2)}(s) = s^{a_2}(t-s)^{e_2-a_2}(1-s)^{-a_1}$.

3. When $p>2$, put

$$q_i^{(p)} = s^{a_p}(t-s)^{e_p-a_p} \begin{pmatrix} (\varphi_1, \sigma_i^{(p-1)}) \\ \vdots, \\ (\varphi_{p-1}, \sigma_i^{(p-1)}) \end{pmatrix} \qquad (i=1,\ldots,p-1),$$

where φ_k are cocycles in $H^1(Y, \nabla_+)$ defined by Definition 8.2 for the case $(p-1)$.

We will put $q_k = q_k^{(p)}$ and $\sigma_k = \sigma_k^{(p)}$.

Definition 8.4 We define $p \times p$ matrices $M_1^{(p)} = (u_{ij}^{(p)}), M_2^{(p)} = (v_{ij}^{(p)})$ by

$$u_{ij}^{(p)} = \begin{cases} (-1)^{i+j}\left(c_{3p-3i+3} + \frac{\delta_{ij}-1}{c_{3p-3i+4}}\right) & (j \leq i), \\ 0 & (\text{otherwise}), \end{cases}$$

$$v_{ij}^{(p)} = \delta_{ij} + \delta_{i1}(-1)^{j+1}\left(\prod_{k=0}^{p-j} c_{3k+2} - 1\right).$$

Theorem 8.3 ([8], Theorem 5.1) *Suppose that* $e_j - a_k, \sum_{i=2}^{k}(e_i - a_i) \notin \mathbf{Z}$ *for all* $1 \leq j \leq k < p$. *Then the monodromy matrices of cycles* $\sigma_1, \ldots, \sigma_p$ *around* 0 *and* 1 *are* $c_{3p}^{-1} M_1^{(p)}$ *and* $M_2^{(p)}$ *respectively.*

Corollary 8.1 *Suppose that* $E'_p \cap \mathbf{Z} \neq \emptyset$, *then cycles* $\sigma_1, \ldots, \sigma_p$ *in* $H_1^{lf}(Y, \nabla_+^*)$ *are linearly independent.*

Proof. If cycles above are linearly dependent, then monodromy matrices are degenerate. Therefore, it is enough to show that $\det M_i^{(p)} \neq 0$. Since matrices $M_1^{(p)}, M_2^{(p)}$ are lower and upper triangular, we have

$$\det c_{3p}^{-1} M_1^{(p)} = c_{3p}^{-p} \prod_{i=1}^{p} u_{ii}^{(p)} = c_{3p}^{-p} \prod_{i=1}^{p} c_{3i} \neq 0,$$

$$\det M_2^{(p)} = \prod_{i=1}^{p} v_{ii}^{(p)} = \prod_{k=1}^{p} c_{3k-1} \neq 0.$$

Here we used the fact that the values c_i are not zero because the c_i is written as $\exp(\,\cdot\,)$. Q.E.D.

We will put $M_1 = M_1^{(p-1)}$ and $M_2 = M_2^{(p-1)}$.

Definition 8.5 We define linear maps $\tilde{M}_1, \tilde{M}_2$ on $\sum_k \mathbf{C}q_k$ by

$$\tilde{M}_1 : (q_1, \ldots, q_{p-1}) \mapsto (q_1, \ldots, q_{p-1})(c_{3p-2}M_1),$$
$$\tilde{M}_2 : (q_1, \ldots, q_{p-1}) \mapsto (q_1, \ldots, q_{p-1})M_2.$$

We note that the linear map $\tilde{M}_2$ has an eigenvector $q_1(s)$ with an eigenvalue $\nu = \prod_{i=1}^{p-1} c_{3i-1}$.

Lemma 8.8 *If the condition $E_p \cap \mathbf{Z} = \emptyset$ holds, then cycles σ_k are regularizable. The regularized cycle* $\mathrm{reg}\,(\sigma_k)$ *in $H_1(Y, \nabla_+^*)$ has the following representation;*

$$\mathrm{reg}\,(\sigma_k) = \begin{cases} C_\varepsilon(t) \otimes \frac{1}{c_{3p-1}-1} q_1 + [t+\varepsilon, 1-\varepsilon] \otimes q_1 - C_\varepsilon(1) \otimes \frac{1}{\nu - 1} q_1 & (k=1), \\ C_\varepsilon(0) \otimes (\tilde{M}_1 - \mathrm{id}\,)^{-1} q_{k-1} + [\varepsilon, t-\varepsilon] \otimes q_{k-1} - C_\varepsilon(t) \otimes \frac{1}{c_{3p-1}-1} q_{k-1} & (k \geq 2), \end{cases}$$

where $C_\varepsilon(P)$ is the circle with the radius ε and the center P.

Proof. From the definition of q_i and Theorem 8.3, the linear maps $\tilde{M}_1, \tilde{M}_2$ agree with the monodromy of $q_i(s)$. The set of eigenvalues of $\tilde{M}_1$ consists of $\{\exp(2\pi\sqrt{-1}(e_i - a_p)) \mid 1 \leq i < p\}$. Therefore, the inverse of $\tilde{M}_1 - \mathrm{id}$ exists under the assumption. It is easy to see that $\partial_{\nabla_+^*} \mathrm{reg}\,\sigma_k = 0$ and the remaining part of the lemma can be shown by the standard argument [1, p.28, p.128]. Q.E.D.

We put $S_{ij}^{**} = S^*(q_i, q_j^\vee)$ and $S^{**} = (S_{ij}^{**})_{ij}$. As we have seen in Section 8.3, the value S_{ij}^{**} above can be regard as the intersection number $[\sigma_i^{(p-1)}] \cdot [\sigma_j^{(p-1)\vee}]$ of cycles associated to ${}_{p-1}F_{p-2}$.

Theorem 8.4 *The intersection matrix*

$$I_h = \begin{pmatrix} [\sigma_1] \cdot [\sigma_1^\vee] & \ldots & [\sigma_1] \cdot [\sigma_p^\vee] \\ \vdots & & \vdots \\ [\sigma_p] \cdot [\sigma_1^\vee] & \ldots & [\sigma_p] \cdot [\sigma_p^\vee] \end{pmatrix}$$

satisfies the following formula:

$$[\sigma_i] \cdot [\sigma_j^\vee] = [\sigma_i] \cdot [\mathrm{reg}\,(\sigma_j^\vee)] = \begin{cases} S_{11}^{**} N_2^\vee & (i=j=1) \\ \frac{c_{3p-1}}{c_{3p-1}-1} S_{1,j-1}^{**} & (i=1,\ 1<j\leq p) \\ \frac{1}{c_{3p-1}-1} S_{i-1,1}^{**} & (1<i\leq p,\ j=1) \\ (S^{**}N_1^\vee)_{i-1,j-1} & (1<i,j\leq p) \end{cases},$$

where

$$N_1 := (c_{3p-2}M_1 - I)^{-1} + I + (c_{3p-1}I - I)^{-1} = (c_{3p-2}M_1 - I)^{-1} + \frac{c_{3p-1}}{c_{3p-1}-1}I$$

$$N_2 := (\prod_{i=1}^{p-1} c_{3i-1} - 1)^{-1} + 1 + (c_{3p-1} - 1)^{-1} = \frac{\prod_{k=1}^{p} c_{3k-1} - 1}{\left(\prod_{i=1}^{p-1} c_{3i-1} - 1\right)(c_{3p-1} - 1)}.$$

Here N_1 is a lower triangular matrix.

We call the matrix I_h the intersection matrix for cycles of ${}_pF_{p-1}$. The intersection matrix I_h for $p = 2$ is given in (7.4), hence we can inductively obtain the intersection matrix I_h for any p.

Proof. By Theorem 6.1 and Lemma 8.8, intersection numbers are given as follows.

$$\begin{aligned}
[\sigma_1]\cdot[\sigma_1^\vee] &= (+1)S^*\left(q_1, \frac{1}{c_{3p-1}^\vee - 1}q_1^\vee\right) + (+1)S^*(q_1, q_1^\vee) + (-1)S^*\left(q_1, -\frac{1}{\nu^\vee - 1}q_1^\vee\right)\\
&= S^*(q_1, q_1^\vee)\left(\frac{1}{c_{3p-1}^\vee - 1} + 1 + \frac{1}{\nu^\vee - 1}\right)\\
&= S_{11}^{**}N_2^\vee\\
[\sigma_1]\cdot[\sigma_{1+j}^\vee] &= (+1)S^*\left(q_1, -\frac{1}{c_{3p-1}^\vee - 1}q_j^\vee\right)\\
&= -\frac{1}{c_{3p-1}^{-1} - 1}S^*(q_1, q_j^\vee)\\
&= \frac{c_{3p-1}}{c_{3p-1} - 1}S_{1j}^{**}\\
[\sigma_{1+i}]\cdot[\sigma_1^\vee] &= (-1)S^*\left(q_i, \frac{c_{3p-1}^\vee}{c_{3p-1}^\vee - 1}q_1^\vee\right)\\
&= -\frac{c_{3p-1}^{-1}}{c_{3p-1}^{-1} - 1}S^*(q_i, q_1^\vee)\\
&= \frac{1}{c_{3p-1} - 1}S_{i1}^{**}\\
[\sigma_{1+i}]\cdot[\sigma_{1+j}^\vee] &= (+1)S^*(q_i, (\tilde{M}_1^\vee - \mathrm{id}\,)^{-1}q_j^\vee) + (+1)S^*(q_i, q_j^\vee) + (-1)S^*\left(q_i, -\frac{1}{c_{3p-1}^\vee - 1}q_j^\vee\right)\\
&= S^*\left(q_i, \left((\tilde{M}_1^\vee - \mathrm{id}\,)^{-1} + \mathrm{id}\, + \frac{1}{c_{3p-1}^\vee - 1}\right)q_j^\vee\right)\\
&= (S^{**}N_1^\vee)_{ij},
\end{aligned}$$

where $1 \leq i, j \leq p-1$. Q.E.D.

Theorem 8.5

$$(I_h)_{11} = (-1)^{p+1} \frac{(\prod_{k=1}^{p} c_{3k-1}) - 1}{\prod_{k=1}^{p}(c_{3k-1} - 1)}$$

$$(I_h)_{1j} = (-1)^{p-j} \frac{\left\{(\prod_{k=1}^{p-j+1} c_{3k-1}) - 1\right\} (\prod_{k=p-j+2}^{p} c_{3k-1})}{\prod_{k=1}^{p}(c_{3k-1} - 1)}$$

$$(I_h)_{i1} = (-1)^{p-i} \frac{(\prod_{k=1}^{p-i+1} c_{3k-1}) - 1}{\prod_{k=1}^{p}(c_{3k-1} - 1)}$$

Proof. We can prove the theorem by an induction on p. We denote by $I_h^{(p)}$ the intersection matrix for p, that is $S^{**} = I_h^{(p-1)}$. We note that $I_h^{(1)} = (1)$ is the 1×1-matrix.

The first identity can be shown by the recurrence relation

$$(I_h^{(p)})_{11} = (I_h^{(p-1)})_{11} \frac{\prod_{k=1}^{p} c_{3k-1}^{-1} - 1}{\left(\prod_{k=1}^{p-1} c_{3k-1}^{-1} - 1\right)(c_{3p-1}^{-1} - 1)}$$

Let us consider $(I_h)_{1j}$. We note the recurrence relation

$$(I_h^{(p)})_{1j} = \frac{c_{3p-1}}{c_{3p-1} - 1} (I_h^{(p-1)})_{1,j-1}.$$

Rewriting the righthand side by the same relation for smaller p, we have

$$\begin{aligned}
(I_h^{(p)})_{1j} &= \prod_{k=p-j+2}^{p} \frac{c_{3k-1}}{c_{3k-1} - 1} (I_h^{(p-j+1)})_{11} \\
&= \prod_{k=p-j+2}^{p} \frac{c_{3k-1}}{c_{3k-1} - 1} (-1)^{p-j} \frac{(\prod_{k=1}^{p-j+1} c_{3k-1}) - 1}{\prod_{k=1}^{p-j+1}(c_{3k-1} - 1)} \\
&= (-1)^{p-j} \frac{\left\{(\prod_{k=1}^{p-j+1} c_{3k-1}) - 1\right\} (\prod_{k=p-j+2}^{p} c_{3k-1})}{\prod_{k=1}^{p}(c_{3k-1} - 1)}
\end{aligned}$$

The formula for $(I_h)_{i1}$ can be shown in an analogous way. Q.E.D.

8.5 Proof of Theorem 1.1

Proof. We consider the period matrices P_+ and P_- defined by the cocycles introduced in Section 8.3 and by the cycles introduced in Section 8.4. Let $\mathbf{g}$ be the p-th column vector of the period matrix P_- and $\mathbf{f}$ the p-th row vector of the period matrix ${}^tP_+$. It follows from Lemma 8.1 that the i-th element of $\mathbf{f}$ is equal to

$$\left(\prod_{k=2}^{p} z^{b_k-1}\frac{\Gamma(a_k)\Gamma(b_k-a_k)}{\Gamma(b_k)}\right)\theta^{i-1}{}_pF_{p-1}(A,B;z)$$

and the j-th element of $\mathbf{g}$ is equal to

$$\left(\prod_{k=2}^{p} z^{(2-b_k)-1}\frac{\Gamma(-a_k)\Gamma(2-b_k+a_k)}{\Gamma(2-b_k)}\right)\theta^{j-1}{}_pF_{p-1}(-A,2-B;z).$$

Hence, we have the quadratic relation

$$g_p\sum_{i=1,j=1}^{p}\left(\theta^{i-1}{}_pF_{p-1}(A,B;z)\right)c_{ij}\left(\theta^{j-1}{}_pF_{p-1}(-A,2-B;z)\right)=h_{pp}$$

where c_{ij} is the (i,j)-element of the transposed inverse of the intersection matrix I_{ch} and h_{pp} is the (p,p)-element of the intersection matrix I_h. The constant g_p is equal to

$$(2\pi\sqrt{-1})^{p-1}\prod_{k=2}^{p}\frac{(1-b_k+a_k)}{a_k(b_k-1)}\frac{(c_{3k-2}c_{3k-1}-1)}{(c_{3k-2}-1)(c_{3k-1}-1)},$$

which is obtained by reducing the product of Γ functions appearing in $\mathbf{f}$ and $\mathbf{g}$ by utilizing the formula $\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}$. Since $\frac{h_{pp}}{c_{11}g_p}$ must be equal to 1 by taking the limit $z=0$, we obtain the formula. Q.E.D.

A program to obtain intersection matrices for ${}_pF_{p-1}$: A computer program to derive intersection numbers for ${}_pF_{p-1}$ is obtainable as a library of Risa/Asir contrib project (`http://www.openxm.org`). The package names are `pfphom.rr` and `pfpcoh.rr`. The program is written by the user language of the computer algebra system Risa/Asir. The recurrence relations given in Theorems 8.2 and 8.4 are used in this program.

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