Evaluation of Stokes multipliers for a certain system of differential equations corresponding to a rigid local system

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0. Introduction.

In [BJL2] Balser, Jurkat and Lutz studied a system of linear differential equations of Birkhoff canonical form of Poincaré rank one, i.e.

$$\frac{dZ}{dx} = \left(\Lambda + \frac{A}{x}\right)Z, \tag{0.1}$$

where A is a constant $n \times n$ matrix and $\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]$. Under the assumuption that Λ has all distinct diagonal elements, they showed that the Stokes multipliers for (0.1) can be expressed by using the connection coefficients for the associated system of linear differential equations

$$(\Lambda - tI_n)\frac{dY}{dt} = (A + I_n)Y, \tag{0.2}$$

which is Fuchsian and has regular singular points $\lambda_1, \ldots, \lambda_n, \infty$. Balser [Ba] studied the same problem for the system (0.1) with $\Lambda = \operatorname{diag}[0, \ldots, 0, 1]$ whose associated system (0.2) is equivalent to the differential equation satisfied by the generalized hypergeometric series ${}_nF_{n-1}$, and evaluated the Stokes multipliers for (0.1) explicitly. As Balser's result suggests, we do not need the assumption that Λ has all distinct diagonal elements to establish the relation between the Stokes multipliers for (0.1) and the connection coefficients for (0.2) (Theorem 5.2). Then the essential part of his result is the evaluation of the connection coefficients for the associated system.

Recently we have shown that, if the monodromy representation of the system (0.2) defines a rigid local system (we call such a system (0.2) *rigid*), the solutions of the system have an integral representation ([Ha4]). Then, for rigid systems, by using the integral representation we can evaluate the connection coefficients and hence the Stokes multipliers for the corresponding system (0.1). (Note that Balser's system (0.2) is also rigid.) In this paper we take a rigid system of rank 4 whose solution can be represented by a double integral, evaluate the connection coefficients by using the integral, and evaluate the Stokes multipliers for the corresponding system (0.1). It seems hard to describe the connection coefficients for a general rigid system; however, the computation in this paper will be applied to each rigid system.

1. A Fuchsian system of rank 4.

Let λ_1, λ_2 be distinct complex numbers, and set $\Lambda = \operatorname{diag}[\lambda_1, \lambda_1, \lambda_2, \lambda_2]$. We consider the system of linear differential equations

$$(tI_4 - \Lambda)\frac{dY}{dt} = AY \tag{1.1}$$

with a constant 4×4 matrix A. The system (1.1) is Fuchsian with regular singular points $\lambda_1, \lambda_2, \infty$. The residue matrices at $\lambda_1, \lambda_2, \infty$ are

$$A_1 := \begin{pmatrix} I_2 & \\ & O \end{pmatrix} A, \quad A_2 := \begin{pmatrix} O & \\ & I_2 \end{pmatrix} A, \quad \text{and} \quad -A$$

respectively. We assume that

$$A_1 \sim \mathrm{diag}[a_1, a_2, 0, 0], \ A_2 \sim \mathrm{diag}[0, 0, b_1, b_2], \ A \sim \mathrm{diag}[\mu_1, \mu_1, \mu_2, \mu_3], \tag{1.2}$$

where $a_1, a_2, b_1, b_2, \mu_1, \mu_2, \mu_3$ are complex numbers satisfying $a_1+a_2+b_1+b_2 = 2\mu_1+\mu_2+\mu_3$. Moreover we assume

$$a_j, b_j, \mu_\ell, a_1 - a_2, b_1 - b_2, \mu_\ell - \mu_m, a_j - \mu_1, b_j - \mu_1, a_j + b_k - \mu_1 - \mu_2 \notin \mathbf{Z} \tag{1.3}$$

for $j, k = 1, 2$ and $\ell, m = 1, 2, 3$ with $\ell \neq m$. Then it is shown that the system (1.1) is irreducible and rigid, and that the matrix A can be determined uniquely up to gauge transformations $Y \mapsto PY$ with $P \in \mathrm{GL}(4, \mathbf{C})$ ([ST],[Ha1]). This system first appeared in the classification of rigid systems ([O2], [Y], [ST]), and Mimachi [M] found an integral representation of its solutions. This integral representation is rediscovered in [Ha4] as an example of the general theory. In the sequel we fix the matrix A as

$$A = \begin{pmatrix} a_1 & 0 & a_{13} & a_{14} \\ 0 & a_2 & a_{23} & a_{24} \\ a_{31} & a_{32} & b_1 & 0 \\ a_{41} & a_{42} & 0 & b_2 \end{pmatrix}, \tag{1.4}$$

where

$$\begin{cases} a_{13} = \dfrac{(\mu_1 - a_1)(a_1 + b_2 - \mu_1 - \mu_2)}{a_1 - a_2}, & a_{14} = \dfrac{(\mu_1 - a_1)(a_1 + b_1 - \mu_1 - \mu_2)}{a_1 - a_2}, \\ a_{23} = \dfrac{(\mu_1 - a_2)(a_2 + b_2 - \mu_1 - \mu_2)}{a_2 - a_1}, & a_{24} = \dfrac{(\mu_1 - a_2)(a_2 + b_1 - \mu_1 - \mu_2)}{a_2 - a_1}, \\ a_{31} = \dfrac{(\mu_1 - b_1)(a_2 + b_1 - \mu_1 - \mu_2)}{b_1 - b_2}, & a_{32} = \dfrac{(\mu_1 - b_1)(a_1 + b_1 - \mu_1 - \mu_2)}{b_1 - b_2}, \\ a_{41} = \dfrac{(\mu_1 - b_2)(a_2 + b_2 - \mu_1 - \mu_2)}{b_2 - b_1}, & a_{42} = \dfrac{(\mu_1 - b_2)(a_1 + b_2 - \mu_1 - \mu_2)}{b_2 - b_1} \end{cases} \tag{1.5}$$

([Ha1],[Ha4]). Then we have

Theorem 1.1. ([Ha4, Proposition 5.10]) *Every solution of (1.1) can be represented by the integral*

$$Y(t) = \int_\Delta \Phi U d\tau_1 \wedge d\tau_2, \tag{1.6}$$

where

$$\Phi = \left(1 - \frac{\lambda_2 - t}{\lambda_2 - \lambda_1}\tau_2\right)^{\mu_1} {\tau_2}^{-\mu_2}(1-\tau_2)^{a_1-\mu_1}(1-\tau_1-\tau_2)^{\mu_1+\mu_2-a_1-b_1}$$
$$\times {\tau_1}^{a_2+b_1-\mu_1-\mu_2}(1-\tau_1)^{\mu_1+\mu_2-a_2-b_2},$$
$$U = \begin{pmatrix} \dfrac{-a_{13}}{\tau_2(1-\tau_2)(1-\tau_1)} \\ \dfrac{a_{24}}{\tau_2(1-\tau_1-\tau_2)\tau_1} \\ \dfrac{a_{31}}{\tau_2\tau_1(1-\tau_1)} \\ \dfrac{-a_{42}}{\tau_2(1-\tau_1-\tau_2)} \end{pmatrix}. \tag{1.7}$$

For the convergence of the integral (1.6), we assume that the real part of every exponent of the power functions in the integrand is greater than -1*. Note that this condition can be relaxed by the analytic continuation with respect to the exponents.*

The multivalued function Φ *defines a local system* $\mathcal{L}$ *on* $X := \mathbf{C}^2 \setminus S$*, where* S *is the singular locus of* Φ*:*

$$\begin{aligned} S = \{1 - \frac{\lambda_2 - t}{\lambda_2 - \lambda_1}\tau_2 = 0\} \cup \{\tau_2 = 0\} \cup \{1 - \tau_2 = 0\} \\ \cup \{1 - \tau_1 - \tau_2 = 0\} \cup \{\tau_1 = 0\} \cup \{1 - \tau_1 = 0\}. \end{aligned} \tag{1.8}$$

Then the chain Δ *in (1.6) can be understood as a cycle in the twisted homology group* $H_2^{\ell f}(X, \mathcal{L})$*.*

By using (1.4) and (1.2) we see the local behaviors of solutions of (1.1) at each singular point. Let e_k *be the* k*-th unit vector in* $\mathbf{C}^4$ *for* $1 \le k \le 4$*.*

Proposition 1.1. (i) *At* $t = \lambda_1$ *there is a set* $(Y_1^{(1)}, Y_2^{(1)}, Y_3^{(1)}, Y_4^{(1)})$ *of linearly independent solutions of (1.1) of the form*

$$\begin{aligned} &Y_1^{(1)} = (t-\lambda_1)^{a_1}[e_1 + O(t-\lambda_1)], \quad Y_2^{(1)} = (t-\lambda_1)^{a_2}[e_2 + O(t-\lambda_1)], \\ &Y_3^{(1)} = v_3^{(1)} + O(t-\lambda_1), \quad Y_4^{(1)} = v_4^{(1)} + O(t-\lambda_1), \end{aligned} \tag{1.9}$$

where $v_3^{(1)}, v_4^{(1)}$ *lie in the 2-dimensional vector space*

$$\{v \in \mathbf{C}^4; Av = \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix}\}.$$

$Y_1^{(1)}, Y_2^{(1)}$ *are uniquely determined if a branch of* $\log(t-\lambda_1)$ *is fixed.*

(ii) *At* $t=\lambda_2$ *there is a set* $(Y_1^{(2)},Y_2^{(2)},Y_3^{(2)},Y_4^{(2)})$ *of linearly independent solutions of (1.1) of the form*

$$\begin{aligned} Y_1^{(2)} &= (t-\lambda_2)^{b_1}[e_3+O(t-\lambda_2)], \quad Y_2^{(2)} = (t-\lambda_2)^{b_2}[e_4+O(t-\lambda_2)], \\ Y_3^{(2)} &= v_3^{(2)}+O(t-\lambda_2), \quad Y_4^{(2)} = v_4^{(2)}+O(t-\lambda_2), \end{aligned} \tag{1.10}$$

where $v_3^{(2)},v_4^{(2)}$ *lie in the 2-dimensional vector space*

$$\{v\in\mathbf{C}^4; Av=\begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}\}.$$

$Y_1^{(2)},Y_2^{(2)}$ *are uniquely determined if a branch of* $\log(t-\lambda_2)$ *is fixed.*
(iii) *At* $t=\infty$ *there is a set* $(Y_1^{(\infty)},Y_2^{(\infty)},Y_3^{(\infty)},Y_4^{(\infty)})$ *of linearly independent solutions of (1.1) of the form*

$$\begin{aligned} Y_1^{(\infty)} &= t^{\mu_1}[v_1^{(\infty)}+O(t^{-1})], \quad Y_2^{(\infty)} = t^{\mu_1}[v_2^{(\infty)}+O(t^{-1})], \\ Y_3^{(\infty)} &= t^{\mu_2}[v_3^{(\infty)}+O(t^{-1})], \quad Y_4^{(\infty)} = t^{\mu_3}[v_4^{(\infty)}+O(t^{-1})], \end{aligned} \tag{1.11}$$

where $v_1^{(\infty)},v_2^{(\infty)}$ *are* μ_1*-eigenvectors,* $v_3^{(\infty)}$ *is a* μ_2*-eigenvector and* $v_4^{(\infty)}$ *is a* μ_3*-eigenvector of* A*.*

The connection problem is to describe the linear relations among these sets of solutions. In §2 we shall represent each solution $Y_j^{(k)}$ *by the integral (1.6) by taking an appropriate cycle* Δ*. Next in §3 we shall give linear relations among the cycles* Δ*. These relations yield the linear relations among the* $Y_j^{(k)}$*'s, which solves the connection problem (§4). The Stokes multipliers for the corresponding system are evaluated in §5.*

2. Local analysis.

In this and the next sections we assume that λ_1,λ_2 *and* t *are real numbers. In this case the singular locus* S *given by (1.8) becomes a complexfied real arrangement, and we consider the underlying real arrangement* $S_{\mathbf{R}}$*.*

First we assume

$$\lambda_1<t<\lambda_2. \tag{2.1}$$

We define the cycles $\Delta_j\in H_2^{\ell f}(X,\mathcal{L})$ $(1\le j\le 16)$ *as chambers given by Figure 1. We fix the branch of* Φ *on each cycle* Δ_j *by assigning*

$$\arg\ell(\tau_1,\tau_2)=0 \quad \text{or} \quad \arg\ell(\tau_1,\tau_2)=\pi$$

according to $\ell(\tau_1,\tau_2)>0$ *or* $\ell(\tau_1,\tau_2)<0$*, where* $\ell(\tau_1,\tau_2)$ *stands for a linear function constituting* Φ*.*

Figure 1.

We set

$$Y_j(t) := \int_{\Delta_j} \Phi U d\tau_1 \wedge d\tau_2 \tag{2.2}$$

for $1 \leq j \leq 16$.

Proposition 2.1. (i) *When* t *lies in the interval (2.1), we have*

$$Y_8(t) = b_1^{(1)} Y_1^{(1)}(t), \quad Y_6(t) = b_2^{(1)} Y_2^{(1)}(t) \tag{2.3}$$

with

$$\begin{aligned} b_1^{(1)} &= \frac{e^{\pi i(\mu_1+2\mu_2-a_2-b_1-b_2)}}{(\lambda_2-\lambda_1)^{a_1}} \\ &\quad \times \frac{\Gamma(a_1-\mu_1+1)\Gamma(\mu_1+1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_2-b_2)}{\Gamma(a_1+1)\Gamma(a_1-a_2+1)}, \\ b_2^{(1)} &= \frac{e^{\pi i(a_2-\mu_1-1)}}{(\lambda_2-\lambda_1)^{a_2}} \\ &\quad \times \frac{\Gamma(a_2-\mu_1+1)\Gamma(\mu_1+1)\Gamma(a_2+b_1-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_1-b_1)}{\Gamma(a_2+1)\Gamma(a_2-a_1+1)}, \end{aligned}$$

where we take the branches of $Y_1^{(1)}$ *and* $Y_2^{(1)}$ *so that* $\arg(t-\lambda_1) = \arg(\lambda_2-\lambda_1)$.
(ii) *When* t *lies in the interval (2.1), we have*

$$Y_3(t) = b_1^{(2)} Y_1^{(2)}(t), \quad Y_1(t) = b_2^{(2)} Y_2^{(2)}(t) \tag{2.4}$$

with

$$
\begin{aligned}
b_1^{(2)} &= \frac{e^{\pi i(\mu_1+\mu_2-b_1-1)}}{(\lambda_1-\lambda_2)^{b_1}} \\
&\quad \times \frac{\Gamma(b_1-\mu_1+1)\Gamma(\mu_1+1)\Gamma(a_2+b_1-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_2-b_2)}{\Gamma(b_1+1)\Gamma(b_1-b_2+1)}, \\
b_2^{(2)} &= \frac{e^{\pi i(a_1+a_2+b_1-\mu_1-\mu_2-1)}}{(\lambda_1-\lambda_2)^{b_2}} \\
&\quad \times \frac{\Gamma(b_2-\mu_1+1)\Gamma(\mu_1+1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_1-b_1)}{\Gamma(b_2+1)\Gamma(b_2-b_1+1)},
\end{aligned}
$$

where we take the branches of $Y_1^{(2)}$ *and* $Y_2^{(2)}$ *so that* $\arg(t-\lambda_2)=\arg(\lambda_1-\lambda_2)$.
(iii) $Y_j(t)$ $(13\le j\le 16)$ *are holomorphic in any simply connected domain in* $\mathbf{C}\setminus\{\lambda_2\}$, *and* $Y_j(t)$ $(9\le j\le 12)$ *are holomorphic in any simply connected domain in* $\mathbf{C}\setminus\{\lambda_1\}$.

Proof. *We calcualte the asymptotic behavior of* $Y_8(t)$ *as* $t\to\lambda_1$. *Consider the first element*

$$y(t) = -a_{13}\int_{\Delta_8}\frac{\Phi}{\tau_2(1-\tau_2)(1-\tau_1)}d\tau_1\wedge d\tau_2$$

of $Y_8(t)$. *By the change of variables*

$$\tau_1=\frac{1}{\sigma_1+\sigma_2},\qquad \tau_2=\frac{\frac{\lambda_2-\lambda_1}{\lambda_2-t}\sigma_1+\sigma_2}{\sigma_1+\sigma_2},$$

Δ_8 *is transformed into*

$$\Delta_0 := \{(\sigma_1,\sigma_2);\sigma_1>0,\ \sigma_2>0,\ 1-\sigma_1-\sigma_2>0\}.$$

Then we get

$$
\begin{aligned}
y(t) &= -a_{13}\int_{\Delta_0}\left(\frac{\frac{t-\lambda_1}{\lambda_2-\lambda_1}\sigma_2}{\sigma_1+\sigma_2}\right)^{\mu_1}\left(\frac{\frac{\lambda_2-\lambda_1}{\lambda_2-t}\sigma_1+\sigma_2}{\sigma_1+\sigma_2}\right)^{-\mu_2-1}\left(e^{\pi i}\frac{\frac{t-\lambda_1}{\lambda_2-t}\sigma_1}{\sigma_1+\sigma_2}\right)^{a_1-\mu_1-1} \\
&\quad\times\left(e^{\pi i}\frac{1+\frac{t-\lambda_1}{\lambda_2-t}\sigma_1}{\sigma_1+\sigma_2}\right)^{\mu_1+\mu_2-a_1-b_1}\left(\frac{1}{\sigma_1+\sigma_2}\right)^{a_2+b_1-\mu_1-\mu_2} \\
&\quad\times\left(e^{\pi i}\frac{1-\sigma_1-\sigma_2}{\sigma_1+\sigma_2}\right)^{\mu_1+\mu_2-a_2-b_2-1}\frac{\frac{t-\lambda_1}{\lambda_2-t}}{(\sigma_1+\sigma_2)^3}d\sigma_1\wedge d\sigma_2 \\
&\sim -a_{13}e^{\pi i(\mu_1+2\mu_2-a_2-b_1-b_2)}\left(\frac{t-\lambda_1}{\lambda_2-\lambda_1}\right)^{a_1} \\
&\quad\times\int_{\Delta_0}{\sigma_1}^{a_1-\mu_1-1}{\sigma_2}^{\mu_1}(1-\sigma_1-\sigma_2)^{\mu_1+\mu_2-a_2-b_2-1}(\sigma_1+\sigma_2)^{b_2-\mu_1-\mu_2-1}d\sigma_1\wedge d\sigma_2,
\end{aligned}
$$

and the integral in the last member can be evaluated by using Lemma 2.1 (ii) below. Thus we get $y(t)=(t-\lambda_1)^{a_1}[b_1^{(1)}+O(t-\lambda_1)]$ *as* $t\to\lambda_1$. *In a similar manner we see that the* *behaviors of the rest of the elements of* $Y_8(t)$ *are* $(t-\lambda_1)^{a_1+1}O(1)$. *Hence the asymptotic behaviors of* $Y_8(t)/b_1^{(1)}$ *and* $Y_1^{(1)}(t)$ *coincide, which implies that these two solutions coincide by virtue of the uniqueness of* $Y_1^{(1)}(t)$.

The assertions for $Y_6(t), Y_3(t), Y_1(t)$ *are shown similarly. We note only changes of integral variables used in the calculation. For the calculation of* $Y_6(t)$ *we use the change of variables*

$$\tau_1 = e^{\pi i}\frac{t-\lambda_1}{\lambda_2-t}\sigma_1, \quad \tau_2 = \frac{\lambda_2-\lambda_1}{\lambda_2-t}\left(1-\frac{t-\lambda_1}{\lambda_2-\lambda_1}\sigma_2\right).$$

For $Y_3(t)$ *and* $Y_1(t)$ *we use*

$$\tau_1 = \frac{\sigma_2}{\sigma_1+\sigma_2}, \quad \tau_2 = \frac{\frac{\lambda_2-\lambda_1}{\lambda_2-t}}{\sigma_1+\sigma_2},$$

and

$$\tau_1 = e^{\pi i}\frac{t-\lambda_1}{\lambda_2-t}\cdot\frac{1}{\sigma_2}, \quad \tau_2 = \frac{t-\lambda_1}{\lambda_2-t}\cdot\frac{\sigma_1+\frac{\lambda_2-\lambda_1}{t-\lambda_2}\sigma_2}{\sigma_2},$$

respectively.

When t *tends to* λ_1, *the chains* Δ_{13}, Δ_{14}, Δ_{15} *and* Δ_{16} *do not change, so that the solutions* $Y_j(t)$ $(13 \le j \le 16)$ *are holomorphic at* $t=\lambda_1$. *Similarly* $Y_j(t)$ $(9 \le j \le 12)$ *are holomorphic at* $t=\lambda_2$. *Hence the assertion (iii) follows.* ∎

Lemma 2.1.

(i)
$$\int_{\Delta_0} {\sigma_1}^{\beta-1}{\sigma_2}^{\beta'-1}(1-\sigma_1-\sigma_2)^{\gamma-\beta-\beta'-1}(1-\sigma_2)^{-\alpha}d\sigma_1\wedge d\sigma_2 = \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta-\beta')\Gamma(\gamma-\alpha-\beta')}{\Gamma(\gamma-\beta')\Gamma(\gamma-\alpha)}.$$

(ii)
$$\int_{\Delta_0} {\sigma_1}^{\beta-1}{\sigma_2}^{\beta'-1}(1-\sigma_1-\sigma_2)^{\gamma-\beta-\beta'-1}(\sigma_1+\sigma_2)^{-\alpha}d\sigma_1\wedge d\sigma_2 = \frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\beta+\beta'-\alpha)\Gamma(\gamma-\beta-\beta')}{\Gamma(\beta+\beta')\Gamma(\gamma-\alpha)}.$$

Proof. *We use the integral representation of the generalized hypergeometric series* ${}_3F_2$:

$$\begin{aligned} {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};x\right) &:= \sum_{n=0}^{\infty}\frac{(\alpha_1,n)(\alpha_2,n)(\alpha_3,n)}{(\beta_1,n)(\beta_2,n)(1,n)}x^n \\ &= \frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)} \\ &\quad\times\int_{\Delta_0}{\sigma_1}^{\beta_1-\alpha_1-1}(1-\sigma_1)^{\alpha_1-\beta_2}{\sigma_2}^{\alpha_2-1} \\ &\qquad\times(1-\sigma_1-\sigma_2)^{\beta_2-\alpha_2-1}(1-x\sigma_2)^{-\alpha_3}d\sigma_1\wedge d\sigma_2. \end{aligned} \tag{2.5}$$

Put $x = 0$ into (2.5) to show the assertion (i). Assertion (ii) follows from (i) by the change of variables $\sigma_1 \mapsto \sigma_1, 1 - \sigma_1 - \sigma_2 \mapsto \sigma_2$. ∎

Next we study the behavior of holomorphic solutions. To state the result we use the special value at $x = 1$ of ${}_3F_2$. Note that the value ${}_3F_2\left({\alpha_1,\alpha_2,\alpha_3 \atop \beta_1,\beta_2};1\right)$ generally has not known to be expressed elementarily, say, in terms of Γ-factors ([A2], [L]).

Proposition 2.2. (i) *As $t \to \lambda_1$ on the condition (2.1), we have*

$$Y_{14}(t) = v_3^{(1)} + O(t - \lambda_1), \quad Y_{16}(t) = v_4^{(1)} + O(t - \lambda_1), \tag{2.6}$$

where $v_3^{(1)} = {}^t(v_1, v_2, v_3, v_4)$ and $v_4^{(1)} = {}^t(v_1', v_2', v_3', v_4')$ are given by

$$\begin{aligned}
v_3 &= e^{-\pi i \mu_2} \frac{\mu_1 - b_1}{b_2 - b_1} \cdot \frac{\Gamma(-\mu_2)\Gamma(\mu_3)\Gamma(a_2 + b_1 - \mu_1 - \mu_2 + 1)\Gamma(\mu_1 + \mu_2 - a_2 - b_2)}{\Gamma(\mu_1 - a_1 - b_2)\Gamma(\mu_1 - a_2 - b_2)} \\
&\quad \times {}_3F_2\left({-\mu_3, -\mu_2, \mu_1 - b_2 \atop \mu_1 - a_1 - b_2, \mu_1 - a_2 - b_2}; 1 \right), \\
v_4 &= e^{-\pi i \mu_2} a_{42} \frac{\Gamma(-\mu_2)\Gamma(\mu_3)\Gamma(a_2 + b_1 - \mu_1 - \mu_2 + 1)\Gamma(\mu_1 + \mu_2 - a_2 - b_2 + 1)}{\Gamma(\mu_1 - a_1 - b_2 + 1)\Gamma(\mu_1 - a_2 - b_2 + 1)} \\
&\quad \times {}_3F_2\left({-\mu_3, -\mu_2, \mu_1 - b_2 + 1 \atop \mu_1 - a_1 - b_2 + 1, \mu_1 - a_2 - b_2 + 1}; 1 \right), \\
v_1 &= -\frac{a_{13}}{a_1} v_3 - \frac{a_{14}}{a_1} v_4, \quad v_2 = -\frac{a_{23}}{a_2} v_3 - \frac{a_{24}}{a_2} v_4, \\
v_3' &= e^{-\pi i \mu_3} a_{31} \frac{\Gamma(-\mu_2)\Gamma(-\mu_3)\Gamma(b_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_1 - b_1 + 1)}{\Gamma(b_2 - \mu_1 - \mu_3 + 1)\Gamma(\mu_1 - a_1 - b_1 + 1)} \\
&\quad \times {}_3F_2\left({-\mu_3, \mu_1 + \mu_2 - a_1 - b_1 + 1, -a_1 \atop b_2 - \mu_1 - \mu_3 + 1, \mu_1 - a_1 - b_1 + 1}; 1 \right), \\
v_4' &= e^{-\pi i \mu_3} \frac{a_1 + b_3 - \mu_1 - \mu_2}{b_2 - b_1} \cdot \frac{\Gamma(-\mu_2)\Gamma(-\mu_3)\Gamma(b_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_1 - b_1)}{\Gamma(b_2 - \mu_1 - \mu_3)\Gamma(\mu_1 - a_1 - b_1)} \\
&\quad \times {}_3F_2\left({-\mu_3, \mu_1 + \mu_2 - a_1 - b_1, -a_1 \atop b_2 - \mu_1 - \mu_3, \mu_1 - a_1 - b_1}; 1 \right) \\
v_1' &= -\frac{a_{13}}{a_1} v_3' - \frac{a_{14}}{a_1} v_4', \quad v_2' = -\frac{a_{23}}{a_2} v_3' - \frac{a_{24}}{a_2} v_4'.
\end{aligned}$$

(ii) *As $t \to \lambda_2$ on the condition (2.1), we have*

$$Y_9(t) = v_3^{(2)} + O(t - \lambda_2), \quad Y_{11}(t) = v_4^{(2)} + O(t - \lambda_2), \tag{2.7}$$

where $v_3^{(2)} = {}^t(v_1, v_2, v_3, v_4)$ *and* $v_4^{(2)} = {}^t(v_1', v_2', v_3', v_4')$ *are given by*

$$\begin{aligned}
v_1 &= e^{-\pi i(a_2+b_1-\mu_1-\mu_2)} \\
&\quad\times \frac{\Gamma(-\mu_2)\Gamma(a_1-\mu_1+1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)\Gamma(a_2+b_1-\mu_1-\mu_2+1)}{\Gamma(\mu_3-\mu_2+1)\Gamma(a_1-\mu_1-\mu_2)} \\
&\quad\times \frac{1}{a_1-a_2}\,{}_3F_2\left(\begin{matrix} a_1+b_2-\mu_1-\mu_2, -\mu_2, a_1+b_1-\mu_1-\mu_2 \\ \mu_3-\mu_2+1, a_1-\mu_1-\mu_2 \end{matrix}; 1\right), \\
v_2 &= e^{-\pi i(a_2+b_1-\mu_1-\mu_2)} \\
&\quad\times \frac{\Gamma(-\mu_2)\Gamma(a_1-\mu_1+1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)\Gamma(a_2+b_1-\mu_1-\mu_2+1)}{\Gamma(\mu_3-\mu_2+1)\Gamma(a_1-\mu_1-\mu_2+1)} \\
&\quad\times \frac{\mu_1-a_2}{a_1-a_2}\,{}_3F_2\left(\begin{matrix} a_1+b_2-\mu_1-\mu_2+1, -\mu_2, a_1+b_1-\mu_1-\mu_2+1 \\ \mu_3-\mu_2+1, a_1-\mu_1-\mu_2+1 \end{matrix}; 1\right), \\
v_3 &= -\frac{a_{31}}{b_1}v_1 - \frac{a_{32}}{b_1}v_2, \quad v_4 = -\frac{a_{41}}{b_2}v_1 - \frac{a_{42}}{b_2}v_2, \\
v_1' &= e^{-\pi i(\mu_1+\mu_2-a_1-b_1)} \\
&\quad\times \frac{\Gamma(a_1-\mu_1+1)\Gamma(a_2-\mu_1+1)\Gamma(\mu_1+\mu_2-a_1-b_1+1)\Gamma(\mu_1+\mu_2-a_2-b_2)}{\Gamma(\mu_2-b_1+1)\Gamma(\mu_2-b_2+1)} \\
&\quad\times \frac{a_1+b_2-\mu_1-\mu_2}{a_1-a_2}\,{}_3F_2\left(\begin{matrix} a_2-\mu_1+1, a_1-\mu_1, \mu_2+1 \\ \mu_2-b_2+1, \mu_2-b_1+1 \end{matrix}; 1\right), \\
v_2' &= e^{-\pi i(\mu_1+\mu_2-a_1-b_1)} \\
&\quad\times \frac{\Gamma(a_1-\mu_1+1)\Gamma(a_2-\mu_1+1)\Gamma(\mu_1+\mu_2-a_1-b_1)\Gamma(\mu_1+\mu_2-a_2-b_2+1)}{\Gamma(\mu_2-b_1+1)\Gamma(\mu_2-b_2+1)} \\
&\quad\times \frac{a_2+b_1-\mu_1-\mu_2}{a_1-a_2}\,{}_3F_2\left(\begin{matrix} a_2-\mu_1, a_1-\mu_1, \mu_2+1 \\ \mu_2-b_2+1, \mu_2-b_1+1 \end{matrix}; 1\right) \\
v_3' &= -\frac{a_{31}}{b_1}v_1' - \frac{a_{32}}{b_1}v_2', \quad v_4' = -\frac{a_{41}}{b_2}v_1' - \frac{a_{42}}{b_2}v_2'.
\end{aligned}$$

This proposition can be proved similarly to the proof of Proposition 2.1.

In order to study the solutions at $t = \infty$*, we assume*

$$t < \lambda_1 < \lambda_2 \tag{2.8}$$

instead of (2.1). Then the arrangement $S_{\mathbf{R}}$ *changes into a new one, and we define the cycles* Δ_j' *for this new arrangement by Figure 2. The branch of* Φ *on each cycle is fixed in the same manner as before.*

Figure 2.

We set

$$Y_{j'}(t) := \int_{\Delta'_j} \Phi U d\tau_1 \wedge d\tau_2 \tag{2.9}$$

for $1 \leq j \leq 16$.

Proposition 2.3. (i) *As* $t \to \infty$ *on the condition (2.8), we have*

$$\begin{aligned} Y_{9'}(t) = & e^{\pi i(a_2+b_1-\mu_1-\mu_2)} \\ & \times \frac{\Gamma(-\mu_2)\Gamma(\mu_1+1)\Gamma(a_2+b_1-\mu_1-\mu_2+1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)}{\Gamma(\mu_1-\mu_2+1)\Gamma(\mu_3-\mu_2+1)} \\ & \times \left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)^{-\mu_2} \left[v_3^{(\infty)} + O\left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)\right], \end{aligned}$$

where

$$v_3^{(\infty)} = \begin{pmatrix} \frac{a_1-\mu_1}{a_1-a_2} \\ \frac{a_2-\mu_1}{a_2-a_1} \\ \frac{b_1-\mu_1}{b_1-b_2} \\ \frac{b_2-\mu_1}{b_2-b_1} \end{pmatrix},$$

and

$$\begin{aligned} Y_{11'}(t) = & e^{\pi i(\mu_1+\mu_2-a_1-b_1)} \frac{\Gamma(-\mu_3)\Gamma(\mu_1+1)\Gamma(\mu_1+\mu_2-a_1-b_1)\Gamma(\mu_1+\mu_2-a_2-b_2)}{\Gamma(\mu_1-\mu_3+1)\Gamma(\mu_2-\mu_3+1)} \\ & \times \left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)^{-\mu_3} \left[v_4^{(\infty)} + O\left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)\right], \end{aligned}$$

where

$$v_4^{(\infty)} = \begin{pmatrix} -\frac{(a_1-\mu_1)(a_1+b_1-\mu_1-\mu_2)(a_1+b_2-\mu_1-\mu_2)}{a_1-a_2} \\ -\frac{(a_2-\mu_1)(a_2+b_1-\mu_1-\mu_2)(a_2+b_2-\mu_1-\mu_2)}{a_2-a_1} \\ \frac{(b_1-\mu_1)(a_1+b_1-\mu_1-\mu_2)(a_2+b_1-\mu_1-\mu_2)}{b_1-b_2} \\ \frac{(b_2-\mu_1)(a_1+b_2-\mu_1-\mu_2)(a_2+b_2-\mu_1-\mu_2)}{b_2-b_1} \end{pmatrix}.$$

The vector $v_3^{(\infty)}$ *(resp.* $v_4^{(\infty)}$*) is a* μ_2 *(resp.* μ_3*)-eigenvector of* A*.*

(ii) *For* $1 \leq j \leq 4$, $Y_{j'}(t)$ *is a solution of exponent* $-\mu_1$ *at* $t = \infty$*, and* $Y_{1'}(t)$ *and* $Y_{3'}(t)$ *make a basis of such solutions. As* $t \to \infty$ *on the condition (2.8), we have*

$$Y_{1'}(t) = e^{\pi i(a_1+a_2+b_1-\mu_1-\mu_2)} \left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)^{-\mu_1} \left[v_1^{(\infty)} + O\left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)\right],$$

$$Y_{3'}(t) = e^{\pi i(\mu_1+\mu_2-b_1)} \left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)^{-\mu_1} \left[v_2^{(\infty)} + O\left(\frac{\lambda_2-\lambda_1}{\lambda_2-t}\right)\right],$$

where $v_1^{(\infty)} = {}^t(v_1, v_2, v_3, v_4)$ *and* $v_2^{(\infty)} = {}^t(v_1', v_2', v_3', v_4')$ *are given by*

$$
\begin{aligned}
v_1 &= \frac{a_1 + b_2 - \mu_1 - \mu_2}{a_1 - a_2} \\
&\quad \times \frac{\Gamma(a_1 - \mu_1 + 1)\Gamma(a_2 - \mu_1 + 1)\Gamma(b_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_1 - b_1 + 1)}{\Gamma(a_2 + b_2 - 2\mu_1 + 2)\Gamma(\mu_2 - b_1 + 1)} \\
&\quad \times {}_3F_2\left(\begin{matrix} a_2 - \mu_1 + 1, \mu_1 + \mu_2 - a_1 - b_1 + 1, \mu_2 - \mu_1 + 1 \\ a_2 + b_2 - 2\mu_1 + 2, \mu_2 - b_1 + 1 \end{matrix} ; 1 \right), \\
v_3 &= \frac{(\mu_1 - b_1)(a_2 + b_1 - \mu_1 - \mu_2)}{b_1 - b_2} \\
&\quad \times \frac{\Gamma(a_1 - \mu_1 + 1)\Gamma(a_2 - \mu_1 + 1)\Gamma(b_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_1 - b_1 + 1)}{\Gamma(a_2 + b_2 - 2\mu_1 + 2)\Gamma(\mu_2 - b_1 + 2)} \\
&\quad \times {}_3F_2\left(\begin{matrix} a_2 - \mu_1 + 1, \mu_1 + \mu_2 - a_1 - b_1 + 1, \mu_2 - \mu_1 + 1 \\ a_2 + b_2 - 2\mu_1 + 2, \mu_2 - b_1 + 2 \end{matrix} ; 1 \right), \\
v_2 &= \frac{a_2 + b_1 - \mu_1 - \mu_2}{\mu_1 + \mu_2 - a_1 - b_1} v_1 + \frac{b_2 - b_1}{\mu_1 + \mu_2 - a_1 - b_1} v_3, \\
v_4 &= \frac{a_2 - a_1}{\mu_1 + \mu_2 - a_1 - b_1} v_1 + \frac{a_1 + b_2 - \mu_1 - \mu_2}{\mu_1 + \mu_2 - a_1 - b_1} v_3, \\
v_1' &= \frac{\mu_1 + \mu_2 - a_1 - b_1}{a_1 - a_2} \\
&\quad \times \frac{\Gamma(a_1 - \mu_1 + 1)\Gamma(a_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_2 - b_2)\Gamma(a_2 + b_1 - \mu_1 - \mu_2 + 1)}{\Gamma(\mu_2 - b_2 + 1)\Gamma(\mu_3 - b_2 + 1)} \\
&\quad \times {}_3F_2\left(\begin{matrix} a_2 - \mu_1 + 1, a_1 - \mu_1, \mu_1 - b_2 \\ \mu_2 - b_2 + 1, \mu_3 - b_2 + 1 \end{matrix} ; 1 \right), \\
v_2' &= \frac{\mu_1 + \mu_2 - a_2 - b_1}{a_1 - a_2} \\
&\quad \times \frac{\Gamma(a_1 - \mu_1 + 1)\Gamma(a_2 - \mu_1 + 1)\Gamma(\mu_1 + \mu_2 - a_2 - b_2 + 1)\Gamma(a_2 + b_1 - \mu_1 - \mu_2)}{\Gamma(\mu_2 - b_2 + 1)\Gamma(\mu_3 - b_2 + 1)} \\
&\quad \times {}_3F_2\left(\begin{matrix} a_2 - \mu_1, a_1 - \mu_1 + 1, \mu_1 - b_2 \\ \mu_2 - b_2 + 1, \mu_3 - b_2 + 1 \end{matrix} ; 1 \right), \\
v_3' &= \frac{a_2 + b_1 - \mu_1 - \mu_2}{b_1 - b_2} v_1' + \frac{a_1 + b_1 - \mu_1 - \mu_2}{b_1 - b_2} v_2', \\
v_4' &= \frac{\mu_1 + \mu_2 - a_2 - b_2}{b_1 - b_2} v_1' + \frac{\mu_1 + \mu_2 - a_1 - b_2}{b_1 - b_2} v_2'.
\end{aligned}
$$

The vectors $v_1^{(\infty)}$ *and* $v_2^{(\infty)}$ *are* μ_1*-eigenvectors of* A.

This proposition can be proved similarly to the proof of Propositions 2.1.

3. Linear relations among the cycles.

In the following we use the notation

$$e(a) := e^{2\pi i a} \tag{3.1}$$

for $a \in \mathbf{C}$. *We set*

$$\begin{aligned} &e_1 = e(\mu_1), \quad e_2 = e(-\mu_2), \quad e_3 = e(a_1 - \mu_1), \quad e_4 = e(\mu_1 + \mu_2 - a_1 - b_1), \\ &e_5 = e(a_2 + b_1 - \mu_1 - \mu_2), \quad e_6 = e(\mu_1 + \mu_2 - a_2 - b_2). \end{aligned} \tag{3.2}$$

First we assume (2.1), and consider the arrangement given in Figure 1. Take a real line ℓ *parallel to the* τ_1*-axis which passes through* Δ_1, Δ_2, Δ_3 *and* Δ_4 *(see Figure 3).*

Figure 3.

In the complexification $\ell_{\mathbf{C}}$ *of* ℓ *we take paths* γ_1, γ_2 *which are given in Figure 4, where* $p_1 = \ell \cap \{1 - \tau_1 - \tau_2 = 0\}$, $p_2 = \{\tau_1 = 0\}$ *and* $p_3 = \{\tau_1 = 1\}$.

Figure 4.

Integrating the branch of Φ *on* Δ_1 *along* γ_1 *and* γ_2, *we obtain two linear relations among the twisted 1-chains* $\ell \cap \Delta_j$ $(1 \leq j \leq 4)$ *by Cauchy's theorem. Then we vary the line* ℓ *in the vertical direction so as to sweep the chains* $\Delta_1, \Delta_2, \Delta_3$ *and* Δ_4. *In this way we obtain the following relations among* $\Delta_1, \Delta_2, \Delta_3$ *and* Δ_4*:*

$$\begin{cases} \Delta_1 + {e_4}^{-1}\Delta_2 + {e_4}^{-1}\Delta_3 + {e_4}^{-1}{e_6}^{-1}\Delta_4 = 0, \\ \Delta_1 + \Delta_2 + e_5\Delta_3 + e_5\Delta_4 = 0. \end{cases} \tag{3.3}$$

In a similar way we get the following relations.

$$\begin{cases} \Delta_5 + {e_4}^{-1}\Delta_6 + {e_4}^{-1}\Delta_7 + {e_4}^{-1}{e_6}^{-1}\Delta_8 = 0, \\ \Delta_5 + \Delta_6 + e_5\Delta_7 + e_5\Delta_8 = 0, \\ \Delta_9 + \Delta_{10} + {e_4}^{-1}\Delta_{11} + {e_4}^{-1}{e_6}^{-1}\Delta_{12} = 0, \\ \Delta_9 + e_5\Delta_{10} + e_5\Delta_{11} + e_5\Delta_{12} = 0, \\ \Delta_{13} + \Delta_{14} + {e_6}^{-1}\Delta_{15} + {e_4}^{-1}{e_6}^{-1}\Delta_{16} = 0, \\ \Delta_{13} + e_5\Delta_{14} + e_5\Delta_{15} + e_5\Delta_{16} = 0, \\ \Delta_2 + \Delta_1 + \Delta_6 + \Delta_5 + \Delta_9 + {e_2}^{-1}\Delta_{13} = 0, \\ \Delta_2 + e_4\Delta_1 + e_1\Delta_6 + e_1e_4\Delta_5 + e_1e_3e_4\Delta_9 + e_1e_3e_4\Delta_{13} = 0, \\ \Delta_3 + \Delta_7 + \Delta_{11} + \Delta_{10} + {e_2}^{-1}\Delta_{14} = 0, \\ \Delta_3 + e_1\Delta_7 + e_1e_3\Delta_{11} + e_1e_3e_4\Delta_{10} + e_1e_3e_4\Delta_{14} = 0, \end{cases} \tag{3.4}$$

These 12 relations in (3.3) and (3.4) are linearly independent.

As we have seen in §2, the cycles Δ_6, Δ_8, Δ_{14} and Δ_{16} give a fundamental solution at $t = \lambda_1$, and Δ_1, Δ_3, Δ_9 and Δ_{11} give one at $t = \lambda_2$. In order to obtain the connection relation among these two fundamental solutions, we solve the relations (3.3) and (3.4) with respect to $\Delta_6, \Delta_8, \Delta_{14}, \Delta_{16}$ and $\Delta_1, \Delta_3, \Delta_9, \Delta_{11}$. Thus we have the following proposition.

Proposition 3.1. *(i) The twisted cycles $\Delta_6, \Delta_8, \Delta_{14}, \Delta_{16}$ can be written as linear combinations of the twisted cycles $\Delta_1, \Delta_3, \Delta_9, \Delta_{11}$ as follows:*

$$\begin{aligned} \Delta_6 = & \frac{(1-e_1e_2e_3e_4e_5e_6)(1-e_4e_5e_6)}{e_1(1-e_2e_3e_4e_5e_6)(1-e_5e_6)}\Delta_1 + \frac{e_5(1-e_1e_2e_3e_4)(1-e_6)}{e_1(1-e_2e_3e_4)(1-e_5e_6)}\Delta_3 \\ & - \frac{e_3e_4(1-e_2)(1-e_2e_3e_4e_6)(1-e_4e_5e_6)}{(1-e_2e_3e_4)(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_9 \\ & - \frac{e_3e_4e_5(1-e_2e_3)(1-e_2e_4e_6)(1-e_6)}{(1-e_2e_3e_4)(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_{11}, \\ \Delta_8 = & -\frac{e_6(1-e_1e_2e_3e_4e_5e_6)(1-e_4)}{e_1(1-e_2e_3e_4e_5e_6)(1-e_5e_6)}\Delta_1 + \frac{e_6(1-e_1e_2e_3e_4)(1-e_5)}{e_1(1-e_2e_3e_4)(1-e_5e_6)}\Delta_3 \\ & - \frac{e_3e_4e_6(1-e_2)(1-e_2e_3{e_4}^2e_5e_6)(1-e_5)}{e_5(1-e_2e_3e_4)(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_9 \\ & + \frac{e_3e_6(1-e_2e_3e_4e_5)(1-e_2e_4e_6)(1-e_4)}{(1-e_2e_3e_4)(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_{11}, \\ \Delta_{14} = & \frac{e_2(1-e_1)}{e_1(1-e_2e_3e_4)}\Delta_3 + \frac{e_2(1-e_3e_4)(1-e_4e_5e_6)}{e_5(1-e_2e_3e_4)(1-e_4e_6)}\Delta_9 \\ & + \frac{e_2(1-e_4)(e_3-e_6)}{(1-e_2e_3e_4)(1-e_4e_6)}\Delta_{11}, \\ \Delta_{16} = & -\frac{e_2e_4e_6(1-e_1)}{e_1(1-e_2e_3e_4e_5e_6)}\Delta_1 - \frac{e_2e_4e_6(1-e_3e_4e_5e_6)(1-e_5)}{e_5(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_9 \\ & - \frac{e_2e_4e_6(1-e_3e_5)(1-e_6)}{(1-e_2e_3e_4e_5e_6)(1-e_4e_6)}\Delta_{11}. \end{aligned}$$

(ii) *The twisted cycles $\Delta_1, \Delta_3, \Delta_9, \Delta_{11}$ can be written as linear combinations of the twisted cycles $\Delta_6, \Delta_8, \Delta_{14}, \Delta_{16}$ as follows:*

$$
\begin{aligned}
\Delta_1 = {} & \frac{e_1(1-e_3e_4e_5)(1-e_5)}{(1-e_1e_3e_4e_5)(1-e_4e_5)}\Delta_6 - \frac{e_1e_5(1-e_3)(1-e_6)}{e_6(1-e_1e_3)(1-e_4e_5)}\Delta_8 \\
& - \frac{e_1e_3e_5(1-e_1e_2e_3e_4)(1-e_5)(1-e_6)}{e_2(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{14} \\
& - \frac{e_1e_3e_5 f(e_1,\ldots,e_6)}{e_2e_4e_6(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{16}, \\
\Delta_3 = {} & \frac{e_1(1-e_3e_4e_5)(1-e_4)}{(1-e_1e_3e_4e_5)(1-e_4e_5)}\Delta_6 + \frac{e_1(1-e_3)(1-e_4e_5e_6)}{e_6(1-e_1e_3)(1-e_4e_5)}\Delta_8 \\
& + \frac{e_1e_3 g(e_1,\ldots,e_6)}{e_2(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{14} \\
& + \frac{e_1e_3(1-e_1e_2e_3e_4e_5e_6)(1-e_4)(1-e_4e_5e_6)}{e_2e_4e_6(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{16}, \\
\Delta_9 = {} & -\frac{e_5(1-e_1)(1-e_4)}{(1-e_1e_3e_4e_5)(1-e_4e_5)}\Delta_6 - \frac{e_5(1-e_1)(1-e_6)}{e_6(1-e_1e_3)(1-e_4e_5)}\Delta_8 \\
& + \frac{e_5(1-e_1e_2e_3e_4)(1-e_1e_3e_5)(1-e_6)}{e_2(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{14} \\
& + \frac{e_5(1-e_1e_2e_3e_4e_5e_6)(e_1e_3-e_6)(1-e_4)}{e_2e_4e_6(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{16}, \\
\Delta_{11} = {} & -\frac{e_4(1-e_1)(1-e_5)}{(1-e_1e_3e_4e_5)(1-e_4e_5)}\Delta_6 + \frac{(1-e_1)(1-e_4e_5e_6)}{e_6(1-e_1e_3)(1-e_4e_5)}\Delta_8 \\
& - \frac{(1-e_1e_2e_3e_4)(1-e_1e_3e_4e_5e_6)(1-e_5)}{e_2(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{14} \\
& - \frac{(1-e_1e_2e_3e_4e_5e_6)(1-e_1e_3e_4)(1-e_4e_5e_6)}{e_2e_4e_6(1-e_1e_3)(1-e_1e_3e_4e_5)(1-e_5e_6)}\Delta_{16},
\end{aligned}
$$

where

$$
\begin{aligned}
f(e_1,\ldots,e_6) = {} & 1 - e_6 + e_4e_6 - e_1e_3e_4 - e_2e_4e_6 - e_4e_5e_6 \\
& + e_1e_2e_3e_4e_6 + e_1e_3e_4e_5e_6 + e_2e_4e_5{e_6}^2 \\
& - e_1e_2e_3e_4e_5e_6 + e_1e_2e_3{e_4}^2e_5e_6 - e_1e_2e_3{e_4}^2e_5{e_6}^2, \\
g(e_1,\ldots,e_6) = {} & 1 - e_5 - e_2e_4 + e_4e_5 - e_4e_5e_6 - e_1e_3e_4e_5 \\
& + e_2e_4e_5e_6 + e_1e_2e_3e_4e_5 - e_1e_2e_3e_4e_5e_6 \\
& + e_1e_3e_4{e_5}^2e_6 + e_1e_2e_3{e_4}^2e_5e_6 - e_1e_2e_3{e_4}^2{e_5}^2e_6.
\end{aligned}
$$

Next we study the relations among the two sets $\{\Delta_j\}, \{\Delta'_j\}$ of twisted cycles given in Figure 1 and Figure 2. We consider the path in the t-plane

$$C: \; t = t(\theta) = \lambda_1 + re^{i\theta} \quad (0 \le \theta \le \pi)$$

with $0 < r < \lambda_2 - \lambda_1$.

Figure 5.

We start from $\theta = 0$, *i.e. from the arrangement in Figure 1, and deform the arrangement by varying* θ *from* 0 *to* π *to obtain the arrangement in Figure 2. We denote by* $\bar{\Delta}_j$ *the result of the continuation along* C *of the twisted cycle* Δ_j $(1 \leq j \leq 16)$. *Then every* $\bar{\Delta}_j$ *can be expressed as a linear combination of the* Δ'_j*'s.*

Let us examine the continuation of Δ_2 *along* C. *Since* Δ_2 *was surrounded by the lines* $\tau_1 = 0, 1 - \frac{\lambda_2 - t}{\lambda_2 - \lambda_1}\tau_2 = 0, 1 - \tau_1 - \tau_2 = 0$ *and the line at* ∞, *after the deformation* $\bar{\Delta}_2$ *is still surrounded by the same lines with a new* t. *Thus we see that* $\bar{\Delta}_2$ *becomes a linear combination of* Δ'_2 *and* Δ'_6. *Taking account of the orientations and the branches, we get*

$$\bar{\Delta}_2 = \Delta'_2 - e_3 e_4 e_5 \Delta'_6.$$

In a similar manner we can show the following proposition.

Proposition 3.2. *Let* $\bar{\Delta}_j$ *be the result of the continuation of* Δ_j *along the curve* C $(1 \leq j \leq 16)$. *Then they are expressed as linear combinations of the* Δ'_j*'s as follows:*

$$\begin{array}{ll}
\bar{\Delta}_1 = \Delta'_1 + e_3\Delta'_5 + e_3e_5\Delta'_6, & \bar{\Delta}_2 = \Delta'_2 - e_3e_4e_5\Delta'_6, \\
\bar{\Delta}_3 = \Delta'_3 + e_3e_4\Delta'_6 + e_3\Delta'_7, & \bar{\Delta}_4 = \Delta'_4 + e_3\Delta'_8, \\
\bar{\Delta}_5 = -e_3\Delta'_5 - e_3e_5\Delta'_6, & \bar{\Delta}_6 = e_3e_4e_5\Delta'_6, \\
\bar{\Delta}_7 = -e_3e_4\Delta'_6 - e_3\Delta'_7, & \bar{\Delta}_8 = -e_3\Delta'_8, \\
\bar{\Delta}_9 = \Delta'_5 + \Delta'_9, & \bar{\Delta}_{10} = \Delta'_6 + \Delta'_{10}, \\
\bar{\Delta}_{11} = \Delta'_7 + \Delta'_{11}, & \bar{\Delta}_{12} = \Delta'_8 + \Delta'_{12}, \\
\bar{\Delta}_{13} = \Delta'_{13}, & \bar{\Delta}_{14} = \Delta'_{14}, \\
\bar{\Delta}_{15} = \Delta'_{15}, & \bar{\Delta}_{16} = \Delta'_{16}.
\end{array}$$

4. Connection coefficients.

Combining Propositions 2.1, 2.2, 2.3, 3.1 and 3.2, we get connection relations among any two sets of solutions given in Proposition 1.1. In this section we focus on the two point connection problem between λ_1 *and* λ_2, *which is related to the evaluation of the Stokes multipliers for the corresponding system*

$$\frac{dZ}{dx} = \left(\Lambda - \frac{A + I_4}{x}\right)Z. \tag{4.1}$$

Let D be a simply connected domain in $\mathbf{C}\setminus\{\lambda_1,\lambda_2\}$ which contains the point $(\lambda_1+\lambda_2)/2$. Fix branches of $\log(t-\lambda_1)$ and $\log(t-\lambda_2)$ in D. Then the solutions $Y_1^{(1)}, Y_2^{(1)}, Y_1^{(2)}$, $Y_2^{(2)}$ are uniquely determined in D by (1.9) and (1.10). We see that there exist unique constants c_{jk} $(1\le j,k\le 2)$ such that

$$Y_k^{(1)}(t)=c_{1k}Y_1^{(2)}(t)+c_{2k}Y_2^{(2)}(t)+\mathrm{reg}(t-\lambda_2)\quad (k=1,2) \tag{4.2}$$

for $t\in D$, where by $\mathrm{reg}(t-\lambda_2)$ we denote a holomorphic solution at $t=\lambda_2$. By Propositions 2.1 and 3.1 with the help of the formula

$$e(\alpha)-1=\frac{2\pi i e^{\pi i\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)},$$

we can evaluate the connection coefficients c_{jk}.

Theorem 4.1. *Let c_{jk} $(1\le j,k\le 2)$ be the constants in the relation (4.2). Then we have*

$$\begin{aligned}
c_{11}&=-\frac{(\lambda_2-\lambda_1)^{a_1}}{(\lambda_1-\lambda_2)^{b_1}}\\
&\quad\times\frac{\Gamma(a_1+1)\Gamma(-b_1)\Gamma(a_1-a_2+1)\Gamma(b_2-b_1)}{\Gamma(a_1-\mu_1+1)\Gamma(\mu_1-b_1)\Gamma(a_1+b_2-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_2-b_1)},\\
c_{21}&=\frac{(\lambda_2-\lambda_1)^{a_1}}{(\lambda_1-\lambda_2)^{b_2}}\\
&\quad\times\frac{\Gamma(a_1+1)\Gamma(-b_2)\Gamma(a_1-a_2+1)\Gamma(b_1-b_2)}{\Gamma(a_1-\mu_1+1)\Gamma(\mu_1-b_2)\Gamma(a_1+b_1-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_2-b_2)},\\
c_{12}&=-\frac{(\lambda_2-\lambda_1)^{a_2}}{(\lambda_1-\lambda_2)^{b_1}}\\
&\quad\times\frac{\Gamma(a_2+1)\Gamma(-b_1)\Gamma(a_2-a_1+1)\Gamma(b_2-b_1)}{\Gamma(a_2-\mu_1+1)\Gamma(\mu_1-b_1)\Gamma(a_2+b_2-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_1-b_1)},\\
c_{22}&=\frac{(\lambda_2-\lambda_1)^{a_2}}{(\lambda_1-\lambda_2)^{b_2}}\\
&\quad\times\frac{\Gamma(a_2+1)\Gamma(-b_2)\Gamma(a_2-a_1+1)\Gamma(b_1-b_2)}{\Gamma(a_2-\mu_1+1)\Gamma(\mu_1-b_2)\Gamma(a_2+b_1-\mu_1-\mu_2+1)\Gamma(\mu_1+\mu_2-a_1-b_2)},
\end{aligned} \tag{4.3}$$

where the branches of $(\lambda_2-\lambda_1)^{a_j}$ and $(\lambda_1-\lambda_2)^{b_k}$ are determined by $\arg(t-\lambda_1)=\arg(\lambda_2-\lambda_1)$ and $\arg(t-\lambda_2)=\arg(\lambda_1-\lambda_2)$ at $t=(\lambda_1+\lambda_2)/2$.

We can also calculate the monodromy matrices for the system (1.1). We consider the following three fundamental solutions

$$\mathcal{Y}=(Y_8,Y_6,Y_3,Y_1),\quad \mathcal{Y}_1=(Y_8,Y_6,Y_{14},Y_{16}),\quad \mathcal{Y}_2=(Y_9,Y_{11},Y_3,Y_1).$$

Then we have the relations

$$\mathcal{Y}_1 = \mathcal{Y}\begin{pmatrix} I_2 & P_1 \\ O & Q_1 \end{pmatrix}, \quad \mathcal{Y}_2 = \mathcal{Y}\begin{pmatrix} P_2 & O \\ Q_2 & I_2 \end{pmatrix}, \tag{4.4}$$

where P_1, P_2, Q_1, Q_2 *are* 2×2*-matrices whose entries can be calculated by using Proposition 3.1.*

Take a point t_0 *near* $(\lambda_1+\lambda_2)/2$. *Let* γ_1 *(resp.* γ_2*) be a loop in* $\mathbf{P}^1\mathbf{C}\setminus\{\lambda_1,\lambda_2,\infty\}$ *starting from and ending at* t_0 *which encircles* λ_1 *(resp.* λ_2*) once in the positive direction. We denote the analytic continuation along* γ_k *by* γ_{k*}. *By virtue of Proposition 2.1 we have*

$$\gamma_{1*}\mathcal{Y}_1 = \mathcal{Y}_1\begin{pmatrix} E_1 & O \\ O & I_2 \end{pmatrix}, \quad \gamma_{2*}\mathcal{Y}_2 = \mathcal{Y}_2\begin{pmatrix} I_2 & O \\ O & E_2 \end{pmatrix}, \tag{4.5}$$

where

$$E_1 = \begin{pmatrix} e(a_1) & \\ & e(a_2) \end{pmatrix}, \quad E_2 = \begin{pmatrix} e(b_1) & \\ & e(b_2) \end{pmatrix}. \tag{4.6}$$

Combining (4.4) and (4.5), we get the following.

Theorem 4.2. *Let* C_1, C_2 *be the matrices determined by*

$$\gamma_{1*}\mathcal{Y} = \mathcal{Y}C_1, \quad \gamma_{2*}\mathcal{Y} = \mathcal{Y}C_2.$$

Then we have

$$C_1 = \begin{pmatrix} E_1 & (I_2 - E_1)P_1{Q_1}^{-1} \\ O & I_2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} I_2 & O \\ (I_2 - E_2)Q_2{P_2}^{-1} & E_2 \end{pmatrix},$$

where

$$P_1{Q_1}^{-1} = \begin{pmatrix} \frac{e(a_2)(e(a_1)-e(\mu_1))(e(a_1+b_2)-e(\mu_1+\mu_2))}{e(\mu_1+\mu_2)(e(a_1)-1)(e(a_2)-e(a_1))} & \frac{e(a_1+a_2+b_1)(e(a_1)-e(\mu_1))(e(a_2+b_2)-e(\mu_1+\mu_2))}{e(2\mu_1+2\mu_2)(e(a_1)-1)(e(a_1)-e(a_2))} \\ \frac{(e(a_2)-e(\mu_1))(e(a_1+b_1)-e(\mu_1+\mu_2))}{e(b_1)(e(a_2)-1)(e(a_2)-e(a_1))} & \frac{e(a_1)(e(a_2)-e(\mu_1))(e(a_2+b_1)-e(\mu_1+\mu_2))}{e(\mu_1+\mu_2)(e(a_2)-1)(e(a_1)-e(a_2))} \end{pmatrix},$$

$$Q_2{P_2}^{-1} = \begin{pmatrix} \frac{(e(b_1)-e(\mu_1))(e(a_2+b_1)-e(\mu_1+\mu_2))}{e(a_2+\mu_1)(e(b_1)-1)(e(b_2)-e(b_1))} & \frac{e(b_1)(e(b_1)-e(\mu_1))(e(a_2+b_2)-e(\mu_1+\mu_2))}{e(2\mu_1+\mu_2)(e(b_1)-1)(e(b_1)-e(b_2))} \\ \frac{e(\mu_2)(e(b_2)-e(\mu_1))(e(a_1+b_1)-e(\mu_1+\mu_2))}{e(a_1+a_2+b_1)(e(b_2)-1)(e(b_2)-e(b_1))} & \frac{(e(b_2)-e(\mu_1))(e(a_1+b_2)-e(\mu_1+\mu_2))}{e(a_1+\mu_1)(e(b_2)-1)(e(b_1)-e(b_2))} \end{pmatrix},$$

and E_1, E_2 *are given in (4.6). The matrices* C_1, C_2 *are generators of the monodromy group of (1.1) with respect to the fundamental solution* $\mathcal{Y}$.

The generators of the monodromy group of (1.1) are calculated in [ST] and [Ha2] by using the behaviors of solutions at $t=\infty$. *We note that, in these references, the generators are determined only up to diagonal transformations.*

5. Stokes multipliers.

Let η *be a real number such that* $\arg(\lambda_2-\lambda_1)\not\equiv\eta$ *mod* 2π. *We consider a* t*-plane together with parallel cuts from each* λ_j *to* ∞ *along the ray* $\arg(t-\lambda_j)=\eta$. *We denote the* t*-plane* *with these cuts by* D_η, *and fix the branches of* $\log(t-\lambda_j)$ *on* D_η *by* $\eta - 2\pi < \arg(t-\lambda_j) < \eta$ $(j = 1, 2)$. *Then the branch of any solution* $Y(t)$ *of (1.1) is determined on* D_η.

Let $\gamma_\eta^{(j)}$ *be a path in* D_η *around the cut* $\arg(t-\lambda_j) = \eta$ *which starts from* $\infty e^{i(\eta-2\pi)}$, *encircles* λ_j *once in the positive dirction and ends at* $\infty e^{i\eta}$ *(see Figure 6).*

Figure 6.

Then, for any solution $Y(t)$ *of (1.1), the Laplace integral*

$$Z(x) = \int_{\gamma_\eta^{(j)}} e^{xt} Y(t) dt \tag{5.1}$$

converges for $\pi/2 - \eta < \arg x < 3\pi/2 - \eta$, *and gives a solution of the system (4.1).*

We set

$$Z_k^{(j)}(x) := \int_{\gamma_\eta^{(j)}} e^{xt} Y_k^{(j)}(t) dt \tag{5.2}$$

for $j = 1, 2$ *and* $k = 1, 2$. *Then, by using (1.9) and (1.10), we see that the* $Z_k^{(j)}$*'s are asymptotically developable as*

$$\begin{aligned} Z_1^{(1)}(x) &\sim (1 - e(-a_1))\Gamma(a_1+1) e^{\pi i(a_1+1)} e^{\lambda_1 x} x^{-a_1-1}[e_1 + O(x^{-1})], \\ Z_2^{(1)}(x) &\sim (1 - e(-a_2))\Gamma(a_2+1) e^{\pi i(a_2+1)} e^{\lambda_1 x} x^{-a_2-1}[e_2 + O(x^{-1})], \\ Z_1^{(2)}(x) &\sim (1 - e(-b_1))\Gamma(b_1+1) e^{\pi i(b_1+1)} e^{\lambda_2 x} x^{-b_1-1}[e_3 + O(x^{-1})], \\ Z_2^{(2)}(x) &\sim (1 - e(-b_2))\Gamma(b_2+1) e^{\pi i(b_2+1)} e^{\lambda_2 x} x^{-b_2-1}[e_4 + O(x^{-1})] \end{aligned} \tag{5.3}$$

as $x \to \infty$ *in the sector* $\pi/2 - \theta_{12} < \arg x < 7\pi/2 - \theta_{12}$, *where* θ_{12} *is the value of* $\arg(\lambda_2 - \lambda_1)$ *specified by* $\eta < \theta_{12} < \eta + 2\pi$. *Since these asymptotic expansions are valid in a sector with the opening greater than* π, *(5.3) determines the solutions* $Z_k^{(j)}$*'s of (4.1) uniquely ([Hu2], see also [BJL1], [Hu1]).*

Let $\tilde{Z}_k^{(1)}(x)$ $(k = 1, 2)$ *be the solution of (4.1) determined by the asymptotic expansions*

$$\begin{aligned} \tilde{Z}_1^{(1)}(x) &\sim (1 - e(-a_1))\Gamma(a_1+1) e^{\pi i(a_1+1)} e^{\lambda_1 x} x^{-a_1-1}[e_1 + O(x^{-1})], \\ \tilde{Z}_2^{(1)}(x) &\sim (1 - e(-a_2))\Gamma(a_2+1) e^{\pi i(a_2+1)} e^{\lambda_1 x} x^{-a_2-1}[e_2 + O(x^{-1})] \end{aligned} \tag{5.4}$$

as $x \to \infty$ *in the sector* $-3\pi/2 - \theta_{12} < \arg x < 3\pi/2 - \theta_{12}$. *Namely* $\tilde{Z}_k^{(1)}(x)$ *has the same asymptotic expansion as* $Z_k^{(1)}(x)$ *in a different sector. Then they are related as follows.*

Theorem 5.1. *In the sector $\pi/2 - \theta_{12} < \arg x < 3\pi/2 - \theta_{12}$, the linear relations*

$$Z_k^{(1)}(x) - \tilde{Z}_k^{(1)}(x) = (1 - e(-a_k))c_{1k}Z_1^{(2)}(x) + (1 - e(-a_k))c_{2k}Z_2^{(2)}(x) \tag{5.5}$$

hold for $k = 1, 2$, where the constants c_{jk} are given in (4.3).

Proof. *Note that $\tilde{Z}_k^{(1)}(x)$ can be given by the integral*

$$\tilde{Z}_k^{(1)}(x) = \int_{\gamma_{\tilde{\eta}}^{(1)}} e^{xt} Y_k^{(1)}(t)dt,$$

where $\tilde{\eta}$ is a real number satisfying $\theta_{12} < \tilde{\eta} < \theta_{12} + 2\pi$. For a moment we assume that $\mathrm{Re}(a_1) > -1, \mathrm{Re}(a_2) > -1$. *Then the path $\gamma_\eta^{(1)}$ of integration can be replaced by the line from λ_1 to $\infty e^{i\eta}$ as*

$$\int_{\gamma_\eta^{(1)}} e^{xt} Y_k^{(1)}(t)dt = (1 - e(-a_k)) \int_{\lambda_1}^{\infty e^{i\eta}} e^{xt} Y_k^{(1)}(t)dt,$$

where the branch on the line is determined by $\arg(t - \lambda_1) = \eta$. *By deforming the path of integration (Figure 7, below), we have*

$$\begin{aligned} Z_k^{(1)}(x) - \tilde{Z}_k^{(1)}(x) &= (1 - e(-a_k)) \int_{\lambda_1}^{\infty e^{i\eta}} e^{xt} Y_k^{(1)}(t)dt - (1 - e(-a_k)) \int_{\lambda_1}^{\infty e^{i\tilde{\eta}}} e^{xt} Y_k^{(1)}(t)dt \\ &= (1 - e(-a_k)) \int_{\gamma_{\hat{\eta}}^{(2)}} e^{xt} Y_k^{(1)}(t)dt, \end{aligned} \tag{5.6}$$

where $\hat{\eta}$ is a real number between η and $\tilde{\eta}$. Put the right hand side of (4.2) into $Y_k^{(1)}(t)$ in the last hand side of (5.6). Since the integral of $\mathrm{reg}(t - \lambda_2)$ *over $\gamma_{\hat{\eta}}^{(2)}$ vanishes by Cauchy's theorem, we get (5.5) by using (5.2). By considering the analytic continuation, we see that the relations (5.5) hold without the assumption of the real parts of a_1 and a_2.* ∎

Figure 7.

The coefficients $(1 - e(-a_k))c_{jk}$ in the relation (5.5) are the Stokes multipliers for the system (4.1). In this way we obtain the Stokes multipliers directly from the connection coefficients for the associated system (1.1).

The above argument can be applied to a general system (0.1). We consider the system (0.1) with

$$\Lambda = \mathrm{diag}[\overbrace{\lambda_1, \ldots, \lambda_1}^{n_1}, \overbrace{\lambda_2, \ldots, \lambda_2}^{n_2}, \ldots, \overbrace{\lambda_p, \ldots, \lambda_p}^{n_p}],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_p$ *are distinct complex numbers, and* $n_1 + n_2 + \cdots + n_p = n$. *We assume that no three points of the* λ_j*'s lie on a line. Set* $B := -A - I_n$. *Denote the zero matrix of size* $m \times m$ *by* O_m. *Then the residue matrix at each regular singular point* λ_j *of the associated system (0.2) is*

$$B_j := \begin{pmatrix} O_{n_1+\cdots+n_{j-1}} & & \\ & I_{n_j} & \\ & & O_{n_{j+1}+\cdots+n_p} \end{pmatrix} B,$$

and that at ∞ *is* $-B$. *We assume that* $B_1, \ldots, B_p$ *and* B *are diagonalizable. For each* j *the matrix* B_j *has eignvalue* 0 *of multiplicity* $n - n_j$. *We denote by* $b_\ell^{(j)}$ $(1 \leq \ell \leq n_j)$ *the rest of the eigenvalues, and by* $v_\ell^{(j)}$ *the corresponding eigenvectors. We assume that* $b_\ell^{(j)} \notin \mathbf{Z}$ *for* $1 \leq j \leq p$, $1 \leq \ell \leq n_j$.

Let η *be a real number such that* $\arg(\lambda_k - \lambda_j) \not\equiv \eta$ *mod* 2π *for* $1 \leq j \neq k \leq p$. *We consider the* t*-plane* D_η *with the parallel cuts* $\arg(t - \lambda_j) = \eta$ $(1 \leq j \leq p)$, *where the branches of* $\log(t - \lambda_j)$ *are fixed as before. Then we see that there is a unique solution* $Y_\ell^{(j)}(t)$ *of (0.2) on* D_η *such that*

$$Y_\ell^{(j)}(t) = (t - \lambda_j)^{b_\ell^{(j)}}[v_\ell^{(j)} + O(t - \lambda_j)]$$

for $1 \leq j \leq p$, $1 \leq \ell \leq n_j$. *Among these solutions there hold connection relations; namely, there exist unique constants* $c_{m\ell}^{kj}$ *such that*

$$Y_\ell^{(j)}(t) = \sum_{m=1}^{n_k} c_{m\ell}^{kj} Y_m^{(k)}(t) + \mathrm{reg}(t - \lambda_k), \tag{5.7}$$

where by $\mathrm{reg}(t - \lambda_k)$ *we denote a holomorphic solution of (0.2) at* $t = \lambda_k$.

Now we take two points λ_j, λ_k, *and fix a value* θ_{jk} *of* $\arg(\lambda_k - \lambda_j)$. *In the following we choose* $\eta \in (\theta_{jk} - 2\pi, \theta_{jk})$ *so that the sector* $\{t; \eta < \arg(t - \lambda_j) < \theta_{jk}\}$ *contains none of the* λ_m*'s.*

Figure 8.

We set

$$Z_\ell^{(j)}(x) := \int_{\gamma_\eta^{(j)}} e^{xt} Y_\ell^{(j)}(t) dt, \tag{5.8}$$

where the path $\gamma_\eta^{(j)}$ *is defined as before (Figure 6). Then the integral converges, and gives a solution of (0.1). Moreover there exists a positive number* δ *such that*

$$Z_\ell^{(j)}(x) \sim (1 - e(-b_\ell^{(j)}))\Gamma(b_\ell^{(j)} + 1) e^{\pi i (b_\ell^{(j)}+1)} e^{\lambda_j x} x^{-b_\ell^{(j)}-1} [v_\ell^{(j)} + O(x^{-1})] \tag{5.9}$$

as $x \to \infty$ *in* S_{jk}*, where*

$$S_{jk} = \{x; \frac{\pi}{2} - \theta_{jk} < \arg x < \frac{3\pi}{2} - \theta_{jk} + \delta\}.$$

Since the opening of S_{jk} *is greater than* π*, we see that the asymptotic expansion (5.9) determines the solution* $Z_\ell^{(j)}(x)$ *uniquely ([Hu2], see also [BJL1], [Hu1]).*

We may assume that δ *is so small that the sector* $\{t; \theta_{jk} < \arg(t - \lambda_j) < \theta_{jk} + \delta\}$ *contains none of the* λ_m*'s. Then by considering* $\tilde{\eta} \in (\theta_{jk}, \theta_{jk} + \delta)$ *in place of* η*, we have a unique solution* $\tilde{Z}_\ell^{(j)}(x)$ *such that*

$$\tilde{Z}_\ell^{(j)}(x) \sim (1 - e(-b_\ell^{(j)}))\Gamma(b_\ell^{(j)} + 1) e^{\pi i (b_\ell^{(j)}+1)} e^{\lambda_j x} x^{-b_\ell^{(j)}-1} [v_\ell^{(j)} + O(x^{-1})] \tag{5.10}$$

as $x \to \infty$ *in* $\tilde{S}_{jk}$*, where*

$$\tilde{S}_{jk} = \{x; \frac{\pi}{2} - \theta_{jk} - \delta < \arg x < \frac{3\pi}{2} - \theta_{jk}\}.$$

Then in a similar manner as Theorem 5.1 we can show the following.

Theorem 5.2. *In the sector* $\pi/2 - \theta_{jk} < \arg x < 3\pi/2 - \theta_{jk}$*, the linear relations*

$$Z_\ell^{(j)}(x) - \tilde{Z}_\ell^{(j)}(x) = \sum_{m=1}^{n_k} (1 - e(-a_k)) c_{m\ell}^{kj} Z_m^{(k)}(x) \tag{5.11}$$

hold for $1 \le \ell \le n_j$*, where the constants* $c_{m\ell}^{kj}$ *are given in (5.7).*

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