Asymptotic Behaviors of a Linear Difference System with Multiple Delays

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1 Introduction.

Consider the linear difference system including N delays

$$y_{n+1} - y_n + A\sum_{j=1}^{N} y_{n-k_j} = 0, \quad n \in \mathbf{Z}_+ = \{0, 1, 2, \ldots\}, \tag{1.1}$$

where A is a real $m \times m$ constant matrix and the delays $k_j\,(j = 1, \ldots, N)$ are positive integers satisfying the conditions $k_1 \leq k_2 \leq \cdots \leq k_N$. Here we assume that the sequence $\{k_j\}$ is arithmetic, that is,

$$k_{j+1} = k_j + d, \qquad j = 1, 2, \ldots, N-1$$

hold for some nonnegative integer d. We are concerned with the asymptotic stability of the system (1.1). Furthermore, we are concerned with the asymptotic periodic behavior of solutions when the system meets some critical conditions. Here we call that a linear system is asymptotically stable if all solutions of the system approach the zero solution as n tends to infinity.

By an appropriate linear transformation, we can obtain from (1.1) a system whose coefficient matrix is given in a Jordan form. Thus it is sufficient to discuss the problems above for the two-dimensional system

$$x_{n+1} - x_n + B\sum_{j=1}^{N} x_{n-k_j} = 0, \qquad n \in \mathbf{Z}_+, \tag{1.2}$$

where B is a 2×2 matrix given by either

$$\text{(i)} \quad p\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{or} \quad \text{(ii)} \quad \begin{pmatrix} p_1 & q \\ 0 & p_2 \end{pmatrix}$$

with real constants p, θ, p_1, p_2, q, and θ satisfying $0 < |\theta| \leq \pi/2$.

Recently, Matsunaga and Hara [5] (in the case of $N = 1$) and then the author [7] (in the case of $N = 2$) obtained necessary and sufficient conditions for (1.2) to be asymptotically stable. Their results can be viewed as generalizations of the well-known criterion due to Levin and May [4] (see also [3, 10]) for the scalar difference equation which originally appeared in mathematical biology:

$$u_{n+1} - u_n + pu_{n-k} = 0, \qquad n \in \mathbf{Z}_+.$$

Also, Ogita et.al.[9] considered the scalar equation corresponding to (1.2):

$$u_{n+1} - u_n + p\sum_{j=1}^{N} u_{n-k_j} = 0, \qquad n \in \mathbf{Z}_+ \tag{1.3}$$

and gave a necessary and sufficient condition for the asymptotic stability, when $\{k_j\}$ is arithmetic.

The first purpose of this paper is to establish necessary and sufficient conditions for (1.2) to be asymptotically stable, which are described explicitly in terms of the components of B and the delays $\{k_j\}$ and our results extend ones mentioned above [4, 5, 7, 9].

In addition, we investigate the behavior of solutions of (1.2) in the critical case where the system loses its asymptotic stability. In such a case, numerical simulations seem to show that every solution approaches some periodic solution depending on its initial data as n tends to infinity. Actually, the author [8] proved such an asymptotic periodic behavior of solutions of (1.2) with $N = 1$, and obtained explicit representations of those periodic solutions in the critical case. Related to this kind of problem, Matsunaga et. al [6] studied the asymptotic periodicity and the asymptotic constancy for a certain type of linear difference systems with one delay. The second purpose of this paper is to discuss the asymptotic periodicity of (1.2) for general N when the system is critical in the above sense.

This paper is outlined as follows: In section 2 we state our main results for asymptotic stability, give their proofs and also discuss the asymptotic stability condition of (1.1). In section 3 we prove that solutions of (1.2) show asymptotic periodic behavior in the critical case and moreover we give explicit expressions of those periodic solutions.

2 Asymptotic stability.

In this section we shall establish necessary and sufficient conditons for (1.2) to be asymptotically stable. We note the fact that a linear difference system is asymptotically stable if and only if all the characteristic roots of the system lie in the interior of the unit disk $D^2 \subset \mathbf{C}$ for the complex plane $\mathbf{C}$.

2.1 Statement of the results.

Our main results on asymptotic stability are the following.

Theorem 2.1 *Suppose that the matrix B is given by* (i). *Then the system* (1.2) *is asymptotically stable if and only if*

$$0 < p < \frac{2\sin\left(\dfrac{\pi/2-|\theta|}{k_N+k_1+1}\right)\sin\left(\dfrac{d(\pi/2-|\theta|)}{k_N+k_1+1}\right)}{\sin\left(\dfrac{Nd(\pi/2-|\theta|)}{k_N+k_1+1}\right)}.$$

Theorem 2.2 *Suppose that the matrix B is given by* (ii). *Then the system* (1.2) *is asymptotically stable if and only if*

$$0 < p_1,\, p_2 < \frac{2\sin\left(\dfrac{\pi}{2(k_N+k_1+1)}\right)\sin\left(\dfrac{d\pi}{2(k_N+k_1+1)}\right)}{\sin\left(\dfrac{Nd\pi}{2(k_N+k_1+1)}\right)}.$$

Theorems 2.1 and 2.2 are extensions of the results cited in the previous section.

The characteristic equation of (1.2) is given by

$$\Delta(\lambda) = \det\Big((\lambda^{k_N+1} - \lambda^{k_N})I_2 + T(\lambda)B\Big) = 0, \tag{2.1}$$

where I_2 is the identity matrix of degree 2 and $T(\lambda)$ is a polynomial of degree $(N-1)d$ defined by $T(\lambda) = \sum_{j=0}^{N-1} \lambda^{jd}$.

We discuss the asymptotic stability of (1.2) in the case that B is of the form (i), since similar arguments hold for the other case. In this case, the characteristic polynomial is

$$\begin{aligned}
\Delta(\lambda) &= \begin{vmatrix} \lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)\cos\theta & -pT(\lambda)\sin\theta \\ pT(\lambda)\sin\theta & \lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)\cos\theta \end{vmatrix} \\
&= \{\lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)\cos\theta\}^2 + \{pT(\lambda)\sin\theta\}^2 \\
&= \{\lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)e^{i\theta}\}\{\lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)e^{-i\theta}\}.
\end{aligned}$$

Put $F_\theta(\lambda) := \lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)e^{i\theta}$. Then we can see $\Delta(\lambda) = F_\theta(\lambda) \cdot \overline{F_\theta(\overline{\lambda})}$, which implies that

$$\Delta^{-1}(0) = F_\theta^{-1}(0) \cup \overline{F_\theta^{-1}(0)} \quad \text{and} \quad (F_{-\theta})^{-1}(0) = \overline{F_\theta^{-1}(0)}.$$

As a consequence, we have

$$\Delta^{-1}(0) = F_{|\theta|}^{-1}(0) \cup \overline{F_{|\theta|}^{-1}(0)}. \tag{2.2}$$

The analysis of characteristic roots is reduced to that of the roots of the equation $F_\theta(\lambda) = 0$ with $0 < \theta \le \pi/2$.

2.2 Analysis of characteristic roots and proof of Theorem 2.1.

We observe the values of p which admit a root of $F_\theta(\lambda) = 0$ on the unit circle $S^1 = \partial D^2$. Obviously, when $p = 0$, the roots of $F_\theta(\lambda) = 0$ consist of 0 (multiplicity k_N) and 1 (simple). Admissible values of p, other than 0, are determined exactly in the following lemma.

Lemma 2.1 *$F_\theta(\lambda) = 0$ has a root on S^1 if and only if p is one of $\{p_\nu\}$ defined by*

$$p_\nu = (-1)^\nu 2 \frac{\sin(\omega_\nu/2)\sin(d\omega_\nu/2)}{\sin(Nd\omega_\nu/2)},$$

where $\omega_\nu = \dfrac{2\theta + (2\nu+1)\pi}{k_N + k_1 + 1}$ with $\nu = -[(k_N + k_1)/2 + \theta/\pi] - 1, \ldots, -1, 0, 1, \ldots,$ $[(k_N + k_1)/2 - \theta/\pi]$, and $[\,\cdot\,]$ denotes the greatest-integer function. Moreover, for each p_ν, $F_\theta(\lambda) = 0$ has only one root $\lambda_\nu = e^{i\omega_\nu}$ on S^1, which is simple.

Proof. Let $\lambda \in S^1$ satisfy $F_\theta(\lambda) = \lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)e^{i\theta} = 0$. We note that $T(\lambda)$ does not vanish for such a λ and we have

$$-p = \frac{\lambda^{k_N+1} - \lambda^{k_N}}{e^{i\theta}T(\lambda)}. \tag{2.3}$$

In virtue of $\overline{\lambda} = 1/\lambda$ and $T(1/\lambda) = T(\lambda)/\lambda^{(N-1)d}$, the conjugate of (2.3) is written by

$$-p = \frac{e^{i\theta}(1/\lambda^{k_N+1} - 1/\lambda^{k_N})}{T(1/\lambda)} = \frac{e^{i\theta}(1-\lambda)}{\lambda^{k_N+1-(N-1)d}\,T(\lambda)} = \frac{e^{i\theta}(1-\lambda)}{\lambda^{k_1+1}T(\lambda)}. \tag{2.4}$$

It follows from (2.3) and (2.4) that $\lambda^{k_N+k_1+1} = -e^{2i\theta}$, which yields

$$\lambda = \lambda_\nu := e^{i\omega_\nu} \qquad \text{where} \qquad \omega_\nu = \frac{2\theta + (2\nu+1)\pi}{k_N + k_1 + 1}.$$

Under the condition $-\pi < \omega_\nu \le \pi$, the index ν runs over the range

$$\nu = -[(k_N + k_1)/2 + \theta/\pi] - 1, \ldots, -1, 0, 1, \ldots, [(k_N + k_1)/2 - \theta/\pi].$$

Also the value of p with $F_\theta(\lambda_\nu) = 0$ is determined by (2.3) as follows: If $\lambda_\nu^d \ne 1$,

$$\begin{aligned} p &= p_\nu := \frac{e^{i\theta}(\lambda_\nu - 1)(\lambda_\nu^d - 1)}{\lambda_\nu^{k_1+1}(\lambda_\nu^{Nd} - 1)} = \frac{(\lambda_\nu^{1/2} - \lambda_\nu^{-1/2})(\lambda_\nu^{d/2} - \lambda_\nu^{-d/2})}{e^{-i\theta}\lambda_\nu^{-(d+1)/2}(\lambda_\nu^{k_N+d+1} + e^{2i\theta}\lambda_\nu^{-k_N})} \\ &= \frac{(e^{i\omega_\nu/2} - e^{-i\omega_\nu/2})(e^{id\omega_\nu/2} - e^{-id\omega_\nu/2})}{e^{i\{(k_N+(d+1)/2))\omega_\nu - \theta\}} + e^{-i\{(k_N+(d+1)/2)\omega_\nu - \theta\}}} \\ &= -2\frac{\sin(\omega_\nu/2)\,\sin(d\omega_\nu/2)}{\cos\{(2k_N + d + 1)\omega_\nu/2 - \theta\}}. \end{aligned}$$

By noting that

$$\begin{aligned} (2k_N + d + 1)\frac{\omega_\nu}{2} - \theta &= (k_N + k_1 + 1 + Nd)\frac{\omega_\nu}{2} - \theta \\ &= \frac{2\nu + 1}{2}\pi + \frac{Nd\omega_\nu}{2}, \end{aligned}$$

we have

$$p_\nu = (-1)^\nu 2\frac{\sin(\omega_\nu/2)\sin(d\omega_\nu/2)}{\sin(Nd\omega_\nu/2)}. \tag{2.5}$$

If $\lambda_\nu^d = 1$, then $T(\lambda_\nu) = N$ and $d\omega_\nu = 2\ell\pi$ for some integer ℓ with $|\ell| \le d/2$. Hence from (2.3) we have

$$\begin{aligned} p_\nu &= -\frac{\lambda_\nu^{k_N+1/2}(\lambda_\nu^{1/2} - \lambda_\nu^{-1/2})}{e^{i\theta}N} \\ &= -2ie^{i\{(k_N+1/2)\omega_\nu - \theta\}}\frac{\sin(\omega_\nu/2)}{N}. \end{aligned}$$

It is easy to see that

$$\Big(k_N + \frac{1}{2}\Big)\omega_\nu - \theta = \Big(k_N + \frac{1}{2}\Big)\frac{2\ell\pi}{d} - \theta = (N-1)\ell\pi + \frac{(2\nu+1)\pi}{2},$$

so that

$$p_\nu = (-1)^{\nu+(N-1)\ell}\, 2\,\frac{\sin(\omega_\nu/2)}{N},$$

which may be regarded as a special case of (2.5) by setting $d\omega_\nu = 2\ell\pi$, because the function $\sin x/\sin Nx$ has a unique continuous extension to the whole real line.

Now, let us verify that each λ_ν is a simple root. It is easily seen that if $p = p_\nu$, then

$$\left.\frac{\partial F_\theta(\lambda)}{\partial\lambda}\right|_{\lambda=\lambda_\nu} = (k_N + 1)\lambda_\nu^{k_N} - k_N\lambda_\nu^{k_N-1} - (\lambda_\nu^{k_N+1} - \lambda_\nu^{k_N})\frac{T'(\lambda_\nu)}{T(\lambda_\nu)},$$

because $F_\theta(\lambda_\nu) = 0$ with $p = p_\nu$. Assume that $\partial F_\theta(\lambda)/\partial\lambda\,|_{\lambda=\lambda_\nu} = 0$. Then, we have

$$G(\lambda_\nu) := (k_N + 1)\lambda_\nu - k_N - (\lambda_\nu^2 - \lambda_\nu)\frac{\sum_{j=1}^{N-1} jd\lambda_\nu^{jd-1}}{T(\lambda_\nu)} = 0. \tag{2.6}$$

Taking its conjugate and multiplying λ_ν, we also see

$$\begin{aligned} H(\lambda_\nu) :=\ & \lambda_\nu \overline{G(\lambda_\nu)} \\ =\ & k_N + 1 - k_N\lambda_\nu - (\lambda_\nu - \lambda_\nu^2)\frac{\sum_{j=0}^{N-2}(N-j-1)d\lambda_\nu^{jd-1}}{T(\lambda_\nu)} = 0, \end{aligned} \tag{2.7}$$

where we used again $\overline{T(\lambda)} = T(\lambda)/\lambda^{(N-1)d}$ for $\lambda \in S^1$. Therefore it follows from (2.6) and (2.7) that

$$\begin{aligned} G(\lambda_\nu) - H(\lambda_\nu) &= (2k_N + 1)(\lambda_\nu - 1) - \frac{\lambda_\nu - 1}{T(\lambda_\nu)} \cdot (N-1)d\,T(\lambda_\nu) \\ &= (k_N + k_1 + 1)(\lambda_\nu - 1) = 0, \end{aligned} \tag{2.8}$$

which is impossible since $\lambda_\nu \neq 1$. Thus we deduce $\partial F_\theta(\lambda)/\partial\lambda\,|_{\lambda=\lambda_\nu} \neq 0$. This implies that $F_\theta(\lambda)$ maps a neighborhood of $\lambda = \lambda_\nu$ diffeomorphically onto a neighborhood of 0 in $\mathbf{C}$, which shows that λ_ν is simple, as desired. *Q.E.D.*

We next discuss how the roots on S^1 of $F_\theta(\lambda) = 0$ behave when the parameter p is perturbed. For this, let $\Lambda_\nu(p)$ denote the root of $F_\theta(\lambda) = 0$ around $p = p_\nu$ satisfying $\Lambda_\nu(p_\nu) = \lambda_\nu$. (It is clear from Lemma 2.1 that $\Lambda_\nu(p)$ depends continuously on p.) Analogously, we denote by $\Lambda(p)$ the root around $p = 0$ with $\Lambda(0) = 1$.

On the behavior of $\Lambda_\nu(p)$ near p_ν, we have:

Lemma 2.2 *For small $|\,p - p_\nu\,|$, the absolute value $|\,\Lambda_\nu(p)\,|$ increases as $|\,p\,|$ increases.*

Proof. We regard F_θ as a function on $\mathbf{C}^2$ and denote

$$F_\theta(p,\,\lambda) = \lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)e^{i\theta}.$$

It is obvious that $F_\theta(p_\nu,\,\lambda_\nu) = 0$ and, as in the proof of the previous lemma, that

$$\frac{\partial F_\theta}{\partial\lambda}(p_\nu,\,\lambda_\nu) \neq 0.$$

Then, implicit function theorem assures that $\Lambda_\nu(p)$ is holomorphic around $p = p_\nu$. Particularly, from the identity $F_\theta(p,\,\Lambda_\nu(p)) = 0$, we see that

$$\frac{d\Lambda_\nu}{dp} = -\frac{\partial F_\theta/\partial p}{\partial F_\theta/\partial\lambda} \tag{2.9}$$

holds for small $|p - p_\nu|$.

Furthermore, by the polar form $\Lambda_\nu(p) = re^{i\omega}$, it is also represented as

$$\frac{d\Lambda_\nu}{dp} = \frac{\partial \Lambda_\nu}{\partial\, \Re p} = \frac{\Lambda_\nu}{r}\left(\frac{\partial\, r}{\partial\, \Re p} + ir\frac{\partial\, \omega}{\partial\, \Re p}\right).$$

Hence restricting p within the real values, we have

$$\frac{dr}{dp} = \Re\left\{\frac{r}{\Lambda_\nu}\frac{d\Lambda_\nu}{dp}\right\} = \frac{r}{|\partial F_\theta/\partial\lambda|^2}\Re\left\{-\frac{1}{\Lambda_\nu}\frac{\partial F_\theta}{\partial p}\overline{\frac{\partial F_\theta}{\partial \lambda}}\right\} \tag{2.10}$$

by (2.9). By

$$\left.\frac{\partial F_\theta}{\partial p}\right|_{p=p_\nu} = e^{i\theta}T(\lambda)\,|_{\lambda=\Lambda_\nu(p_\nu)} = e^{i\theta}T(\lambda_\nu) = -\frac{1}{p_\nu}(\lambda_\nu^{k_N+1} - \lambda_\nu^{k_N}),$$

and $\partial F_\theta/\partial\lambda|_{p=p_\nu} = \lambda_\nu^{k_N-1}G(\lambda_\nu)$, we have

$$-\frac{1}{\Lambda_\nu}\frac{\partial F_\theta}{\partial p}\left.\overline{\frac{\partial F_\theta}{\partial \lambda}}\right|_{p=p_\nu} = \frac{\lambda_\nu - 1}{p_\nu}\overline{G(\lambda_\nu)} = \frac{\lambda_\nu - 1}{p_\nu\lambda_\nu}H(\lambda_\nu),$$

and hence

$$\begin{aligned}
\Re\left\{-\frac{1}{\Lambda_\nu}\frac{\partial F_\theta}{\partial p}\overline{\frac{\partial F_\theta}{\partial \lambda}}\right\}\Bigg|_{p=p_\nu} &= \frac{1}{2}\left\{\frac{\lambda_\nu - 1}{p_\nu\lambda_\nu}H(\lambda_\nu) + \frac{\overline{\lambda_\nu} - 1}{p_\nu}G(\lambda_\nu)\right\} \\
&= \frac{\overline{\lambda_\nu} - 1}{2p_\nu}(G(\lambda_\nu) - H(\lambda_\nu)) \\
&= \frac{(k_N + k_1 + 1)}{2p_\nu}|\lambda_\nu - 1|^2
\end{aligned}$$

by (2.8). This, together with (2.10), implies

$$\left.\frac{dr}{dp}\cdot \operatorname{sgn} p\right|_{p=p_\nu} > 0,$$

which shows that $r = |\Lambda_\nu(p)|$ is an increasing function of $|\,p\,|$ around $p = p_\nu$. *Q.E.D.*

With respect to the behavior of the root $\Lambda(p)$ satisfying

$$F_\theta(\Lambda(p)) = 0, \quad \Lambda(0) = 1,$$

we have the next lemma.

Lemma 2.3 *For small* $|p|$, *followings hold:*

(i) *If* $0 < \theta < \pi/2$, *then* $(|\Lambda(p)| - 1) \cdot \operatorname{sgn} p < 0$ *for* $p \neq 0$.

(ii) *If* $\theta = \pi/2$, *then* $|\Lambda(p)| > 1$ *for* $p \neq 0$.

Proof. (i) As in the proof of Lemma 2.2, we can verify

$$\left.\frac{d\Lambda}{dp}\right|_{p=0} = -\left.\frac{\partial F_\theta/\partial p}{\partial F_\theta/\partial \lambda}\right|_{(p,\,\lambda)=(0,\,1)} = -Ne^{i\theta},$$

and hence

$$\begin{aligned}\left.\frac{dr}{dp}\right|_{p=0} &= \Re\left.\left\{\frac{r}{\Lambda}\frac{d\Lambda}{dp}\right\}\right|_{p=0} = \Re(-Ne^{i\theta}) \\ &= -N\cos\theta < 0 \quad \text{for} \quad 0 < \theta < \pi/2.\end{aligned}$$

It follows from $\Lambda(0) = 1$ that

$$(|\Lambda(p)| - 1) \cdot \operatorname{sgn} p < 0 \quad (p \neq 0).$$

(ii) $\theta = \pi/2$ implies $dr/dp|_{p=0} = 0$. We shall check the sign of $d^2r/dp^2|_{p=0}$. From $F_\theta(p,\, \Lambda(p)) = 0$, it follows that

$$\frac{\partial^2 F_\theta}{\partial p^2} + 2\frac{\partial^2 F_\theta}{\partial p \partial \lambda}\frac{d\Lambda}{dp} + \frac{\partial^2 F_\theta}{\partial \lambda^2}\left(\frac{d\Lambda}{dp}\right)^2 + \frac{\partial F_\theta}{\partial \lambda}\frac{d^2\Lambda}{dp^2} = 0.$$

By using direct computations, we deduce

$$\left.\frac{d^2\Lambda}{dp^2}\right|_{p=0} = -N^2(k_N + k_1)e^{\pi i} = N^2(k_N + k_1).$$

Also, by means of the polar representation,

$$\frac{d^2 r}{dp^2} = \Re\left\{\frac{r}{\Lambda}\frac{d^2\Lambda}{dp^2}\right\} + r\left(\frac{d\omega}{dp}\right)^2 \geq \Re\left\{\frac{r}{\Lambda}\frac{d^2\Lambda}{dp^2}\right\}.$$

In particular,

$$\left.\frac{d^2 r}{dp^2}\right|_{p=0} \geq \Re\left.\left\{\frac{r}{\Lambda}\frac{d^2\Lambda}{dp^2}\right\}\right|_{p=0} \geq N^2(k_N + k_1) > 0.$$

Therefore we conclude that

$$|\Lambda(p)| > 1 \quad \text{for} \quad p \neq 0.$$

Q.E.D.

Now let us denote by p^* the positive minimum of $\{p_\nu\}$ in Lemma 2.1, that is,

$$p^* = \min\{p_\nu > 0 \,|\, -[(k_N + k_1)/2 + \theta/\pi] - 1 \leq \nu \leq [(k_N + k_1)/2 - \theta/\pi]\}.$$

Our next objective is to determine the value of p^*.

Lemma 2.4 *The value of p^* is given by*

$$p^* = \begin{cases} p_{-1}, & 0 < \theta \le \pi/2, \\ p_0, & -\pi/2 \le \theta < 0. \end{cases}$$

Remark 2.1 *It is straightforward to check that*

$$p_{-1} = \frac{2\sin\left((\pi/2-\theta)/M\right)\sin\left(d(\pi/2-\theta)/M\right)}{\sin\left(Nd(\pi/2-\theta)/M\right)}$$

and

$$p_0 = \frac{2\sin\left((\pi/2+\theta)/M\right)\sin\left(d(\pi/2+\theta)/M\right)}{\sin\left(Nd(\pi/2+\theta)/M\right)},$$

where $M = k_N + k_1 + 1$. Hence we can see

$$p^* = \frac{2\sin\left((\pi/2-|\theta|)/M\right)\sin\left(d(\pi/2-|\theta|)/M\right)}{\sin\left(Nd(\pi/2-|\theta|)/M\right)}, \tag{2.11}$$

which coincides with the critical value in Theorem 2.1.

Postponing the proof of Lemma 2.4 to the next subsection, we shall give the proof of Theorem 2.1.

Proof of Theorem 2.1. If p decreases from 0, then the root $\Lambda(p)$ moves outside D^2 (Lemma 2.3). It is also seen from Lemma 2.2 and the continuous dependence on p of the roots that $F_\theta(\lambda) = 0$ has at least one root in $\mathbf{C} \setminus \mathrm{Int}\, D^2$ as long as p is negative. As a consequence, $p > 0$ is necessary for the asymptotic stability of (1.2). Conversely let us assume $p > 0$. It follows from Lemma 2.3 (i) and the continuity of the roots in p that if $0 < \theta < \pi/2$, then all roots of $F_\theta(\lambda) = 0$ are in $\mathrm{Int}\, D^2$ for small $p > 0$. On the other hand, in the case of $\theta = \pi/2$, Lemma 2.3 (ii), combining with Lemma 2.2, implies that if p increases from 0, the root $\Lambda(p)$ moves outside D^2 and that $F_\theta(\lambda) = 0$ has at least one root in $\mathbf{C} \setminus \mathrm{Int}\, D^2$ as long as p is positive. Thus we conclude that all roots of $F_\theta(\lambda) = 0$ belong to $\mathrm{Int}\, \mathrm{D}^2$ if and only if $0 < \theta < \pi/2$ and $0 < p < p^*$. In virtue of the relation (2.2), it turns out that a necessary and sufficient condition for (1.2) to be asymptotically stable is that $0 < |\theta| < \pi/2$ and $0 < p < p^*$. *Q.E.D.*

Thus, it remains to prove Lemma 2.4.

2.3 Proof of Lemma 2.4.

Lemma 2.4 holds true in the cases of $N = 1$ and $N = 2$ (see [5, 7]). So it is sufficient to prove the lemma for $N \geq 3$. We shall consider the case $0 < \theta \leq \pi/2$, the other case being similar. The proof will be devided into two steps. Let M be $k_N + k_1 + 1$.

Step I. $|p_\nu| \geq p_{-1}$ for $\nu = 0,\ 1,\ 2, \ldots$.

Case (Ia): $Nd(\pi/2 - \theta)/M \leq \pi/2$. Note that the following inequality is valid:

$$\left|\frac{\sin Nu}{\sin u}\right| \leq N \quad \text{for} \quad u \in \mathbf{R}. \tag{2.12}$$

Since $0 < \omega_\nu/2 \leq \pi/2$ ($\nu = 0,\ 1,\ 2, \ldots$), we can see

$$\begin{aligned} |p_\nu| &= 2\left|\sin(\omega_\nu/2)\frac{\sin(d\omega_\nu/2)}{\sin Nd\omega_\nu/2}\right| \\ &\geq 2\cdot\frac{2}{\pi}\,\frac{\omega_\nu}{2}\cdot\frac{1}{N} \\ &= \frac{2(2\theta + (2\nu+1)\pi)}{\pi MN}. \end{aligned} \tag{2.13}$$

By the assumption (Ia), we have $0 < Nd(-\omega_{-1})/2 = Nd(\pi/2 - \theta)/M \leq \pi/2$, so that

$$\begin{aligned} p_{-1} &= \frac{2\sin(-\omega_{-1}/2)\sin(d(-\omega_{-1}/2))}{\sin(Nd(-\omega_{-1})/2)} \leq \frac{2\cdot\left(\dfrac{-\omega_{-1}}{2}\right)\cdot\dfrac{d(-\omega_{-1})}{2}}{\dfrac{2}{\pi}\,\dfrac{Nd(-\omega_{-1})}{2}} \\ &= \frac{\pi(-\omega_{-1})}{2N} = \frac{\pi(\pi - 2\theta)}{2MN}. \end{aligned} \tag{2.14}$$

In virtue of (2.13) and (2.14) we deduce

$$|p_\nu| - p_{-1} \geq \frac{(2\pi^2+8)\theta + (8\nu + 4 - \pi^2)\pi}{2\pi MN} \geq 0, \quad \nu = 1,\ 2,\ \ldots. \tag{2.15}$$

It remains to be shown that $|p_0| \geq p_{-1}$. We notice that (2.15) with $\nu = 0$ still holds for θ satisfying $\theta \geq (\pi^2 - 4)\pi/(2\pi^2 + 8)$. So it suffices to show $|p_0| \geq p_{-1}$ in the case where

$$\theta < \frac{\pi^2 - 4}{2\pi^2 + 8}\,\pi. \tag{2.16}$$

Put $p(x) = 2\sin x \cdot \sin dx/\sin Ndx$. Then $p(x)$ is increasing for $0 \leq x < \pi/Nd$. It is also seen that $|p_0| = p(\omega_0/2) = p\left((\theta + \pi/2)/M\right)$ and $p_{-1} = p(\omega_{-1}/2) = p\left((\pi/2 - \theta)/M\right)$.

(i) If $(\theta + \pi/2)/M \leq \pi/Nd$, it immediately follows from the monotonicity of $p(x)$ described above that

$$|p_0| = p((\theta + \pi/2)/M) > p((\pi/2 - \theta)/M) = p_{-1}.$$

(ii) If $(\theta+\pi/2)/M > \pi/Nd$, we note that $(\theta+\pi/2)/M$ belongs between π/Nd and $2\pi/Nd$. Indeed, by the assumption (Ia) and (2.16), we have

$$\frac{\theta+\pi/2}{M} \le \frac{\pi}{2Nd}\frac{\theta+\pi/2}{\pi/2-\theta} \le \frac{\pi^2}{8}\frac{\pi}{Nd},$$

and therefore

$$\frac{\pi}{Nd} < \frac{\theta+\pi/2}{M} < \frac{2\pi}{Nd}. \tag{2.17}$$

Moreover we obtain

$$\frac{\pi/2-\theta}{M} < \frac{2\pi}{Nd} - \frac{\theta+\pi/2}{M} < \frac{\pi}{Nd}. \tag{2.18}$$

Indeed, we have

$$\begin{aligned}\left(\frac{2\pi}{Nd} - \frac{\theta+\pi/2}{M}\right) - \frac{\pi/2-\theta}{M} &= \frac{2\pi}{Nd} - \frac{\pi}{M} \\ &\ge \frac{2\pi}{Nd} - \frac{\pi}{Nd}\frac{\pi/2}{\pi/2-\theta} \\ &\ge \frac{\pi}{Nd}\frac{12-\pi^2}{8} > 0,\end{aligned}$$

by (Ia) and (2.16). In virtue of the property of $p(x)$ that

$$\begin{aligned}|p(x)| &= \left|\frac{2\sin x \cdot \sin dx}{\sin Nd(2\pi/Nd - x)}\right| \ge \left|\frac{2\sin(2\pi/Nd - x)\cdot \sin d(2\pi/Nd - x)}{\sin Nd(2\pi/Nd - x)}\right| \\ &= |p(2\pi/Nd - x))| \qquad \text{for} \qquad \pi/Nd < x < 2\pi/Nd,\end{aligned}$$

we have

$$\left|p\left(\frac{2\pi}{Nd} - \frac{\theta+\pi/2}{M}\right)\right| \le \left|p\left(\frac{\theta+\pi/2}{M}\right)\right| = |p_0|$$

by (2.17). Consequently, $|p_0| \ge p_{-1}$ follows from the monotonicity of $p(x)$ and (2.18).

Case (Ib): $Nd(\pi/2-\theta)/M > \pi/2$. By (2.13) we have

$$|p_\nu| \ge \frac{2(2\nu+1)}{MN}, \qquad \nu = 0, 1, 2, \dots. \tag{2.19}$$

Since $M = 2k_1 + 1 + (N-1)d > (N-1)d$ and $N \ge 3$, we have

$$\frac{\pi}{2} < \frac{Nd}{M}\left(\frac{\pi}{2} - \theta\right) \le \frac{N}{N-1}\frac{\pi}{2} \le \frac{3\pi}{4},$$

that is,

$$\frac{\pi}{4} \le \pi - \frac{Nd}{M}\left(\frac{\pi}{2} - \theta\right) \le \frac{\pi}{2}.$$

We can evaluate p_{-1} as follows:

$$\begin{aligned} p_{-1} &= p\left(\frac{\pi/2-\theta}{M}\right) = \frac{2\sin(\pi/2-\theta)/M \cdot \sin d(\pi/2-\theta)/M}{\sin\{\pi - Nd(\pi/2-\theta)/M\}} \\ &\le \frac{2\cdot(\pi/2-\theta)/M \cdot d(\pi/2-\theta)/M}{2/\pi\ \{\pi - Nd(\pi/2-\theta)/M\}} \\ &\le \frac{d\pi^2}{2M(2M-Nd)}. \end{aligned} \tag{2.20}$$

It follows from (2.19) and (2.20) that

$$\begin{aligned} \frac{|p_\nu|}{p_{-1}} &\ge \frac{2(2\nu+1)}{MN}\cdot\frac{2M(2M-Nd)}{d\pi^2} \\ &= \frac{4(2\nu+1)}{\pi^2}\left(\frac{2M}{Nd}-1\right) \\ &\ge \frac{4(2\nu+1)}{\pi^2}\left(1-\frac{2}{N}\right). \qquad (\text{by} \quad M > (N-1)d\,) \end{aligned} \tag{2.21}$$

It is clear that $|p_\nu| \ge p_{-1}$ for $\nu \ge 4$, because $N \ge 3$. Now we will prove several claims below in order to verify the inequality $|p_\nu| \ge p_{-1}$ in the remaining cases. (We, however, need adhoc arguments for the case $N = 3$ since the same arguments do not apply for the case of $N \ge 4$.) By (2.21), if $N \ge 4$, then $|p_\nu| \ge p_{-1}$ for $\nu \ge 2$.

Claim 1. $|p_1| \ge p_{-1}$ for $N \ge 4$.

By Lemma 2.1, $|p_1|$ is given by

$$\frac{2\,\sin(2\theta+3\pi)/2M \cdot \sin d(2\theta+3\pi)/2M}{\sin Nd(2\theta+3\pi)/2M}.$$

Observing that

$$\frac{d(2\theta+3\pi)}{2M} \le \frac{2\theta+3\pi}{2(N-1)} \le \frac{2\theta+3\pi}{6} \le \frac{2\pi}{3}$$

and noting the inequality $\sin x \ge 3\sqrt{3}x/4\pi$ for $0 \le x \le 2\pi/3$, we have

$$\sin\frac{d(2\theta+3\pi)}{2M} \ge \frac{3\sqrt{3}}{4\pi}\,\frac{d(2\theta+3\pi)}{2M},$$

so that

$$|p_1| \ge 2\cdot\frac{2}{\pi}\,\frac{2\theta+3\pi}{2M}\cdot\frac{3\sqrt{3}}{4\pi}\,\frac{d(2\theta+3\pi)}{2M} \ge \frac{27\sqrt{3}d}{4M^2}.$$

Combining with (2.20), this means

$$\begin{aligned} \frac{|p_1|}{p_{-1}} &\ge \frac{27\sqrt{3}d}{4M^2}\cdot\frac{2M(2M-Nd)}{d\pi^2} = \frac{27\sqrt{3}}{\pi^2}\left(1-\frac{Nd}{2M}\right) \\ &\ge \frac{27\sqrt{3}}{\pi^2}\left(1-\frac{N}{2(N-1)}\right) \ge \frac{9\sqrt{3}}{\pi^2} > 1, \end{aligned}$$

which proves Claim 1.

Claim 2. $|p_0| \geq p_{-1}$ for $N \geq 3$.

Since

$$\begin{aligned} \frac{2\pi}{Nd} - \frac{\theta + \pi/2}{M} &= \frac{4\pi M - \pi N d - 2Nd\theta}{2MNd} \\ &= \frac{4\pi(2k_1+1) + 2\pi(N-2)d + Nd(\pi - 2\theta)}{2MNd} > 0, \end{aligned}$$

we have $(\pi/2 - \theta)/M < (\theta + \pi/2)/M < 2\pi/Nd$. A similar argument to Case (Ia) yields the claim.

Claim 3. $|p_2|$, $|p_3| \geq p_{-1}$ for $N = 3$.

Setting

$$u_\nu := \frac{d(\pi + \theta_\nu)}{2M} \quad \text{and} \quad \theta_\nu := \frac{2\theta}{2\nu + 1}$$

we can see

$$\begin{aligned} \frac{|p_\nu|}{p_{-1}} &= \left| \frac{\sin{(2\nu+1)u_\nu} \cdot \sin{(2\nu+1)u_\nu/d}}{\sin 3(2\nu+1)u_\nu} \cdot \frac{\sin 3u_{-1}}{\sin u_{-1} \cdot \sin u_{-1}/d} \right| \\ &= \left| \frac{\sin{(2\nu+1)u_\nu} \cdot \sin 3u_{-1}}{\sin 3(2\nu+1)u_\nu \cdot \sin u_{-1}} \right| \cdot \left| \frac{\sin{(2\nu+1)u_\nu/d}}{\sin u_{-1}/d} \right| \\ &\geq \left| \frac{\sin 3u_{-1}}{3\sin u_{-1}} \right| \cdot \left| \frac{\sin{(2\nu+1)u_\nu/d}}{\sin u_{-1}/d} \right|, \end{aligned} \tag{2.22}$$

because $|\sin 3x| \leq 3|\sin x|$ for $-\pi/2 \leq x \leq \pi/2$. Since (Ib) implies $Nd > M$, it follows from $M > (N-1)d$ that $1/2N < d/2M < 1/2(N-1)$. Putting $N = 3$, we obtain $1/6 < d/2M < 1/4$, so that $0 < u_{-1} \leq \pi/4$. Noting that the function $\sin 3x/\sin x$ is decreasing for $0 \leq x \leq \pi/4$, we have

$$\frac{\sin 3u_{-1}}{\sin u_{-1}} \geq \left. \frac{\sin 3x}{\sin x} \right|_{x=\pi/4} = 1. \tag{2.23}$$

Also, by $0 < (2\nu+1)u_\nu/d \leq \pi/2$, the estimate

$$\frac{\sin{(2\nu+1)u_\nu/d}}{\sin u_{-1}/d} \geq \frac{2/\pi \cdot (2\nu+1)u_\nu/d}{u_{-1}/d} = \frac{2(2\nu+1)}{\pi} \cdot \frac{u_\nu}{u_{-1}} > \frac{2(2\nu+1)}{\pi} \tag{2.24}$$

holds for $\nu = 2, 3$. Hence, together with (2.22) and (2.23), we deduce that

$$\frac{|p_\nu|}{p_{-1}} \geq \frac{1}{3} \cdot 1 \cdot \frac{2(2\nu+1)}{\pi} \geq \frac{10}{3\pi} > 1, \quad \text{for} \quad \nu = 2, 3,$$

which shows the claim is valid.

Claim 4. $|p_1| \geq p_{-1}$ for $N = 3$.

In the same way as we derived (2.22) we obtain

$$\begin{aligned}\frac{|p_1|}{p_{-1}} &\geq \left|\frac{\sin 3u_1\cdot\sin 3u_{-1}}{\sin 9u_1\cdot\sin u_{-1}}\right|\cdot\frac{2\,(2\cdot 1+1)}{\pi}\\ &= \frac{6}{\pi}\cdot\left|\frac{\sin 3u_1}{\sin 9u_1}\right|\cdot\left|\frac{\sin 3u_{-1}}{\sin u_{-1}}\right|,\end{aligned}$$

where we used the fact that (2.24) holds with $\nu = 1$. Since $\sin 3x = \sin x\cdot(3-4\sin^2 x)$, this can be written by

$$\frac{|p_1|}{p_{-1}} \geq \frac{6}{\pi}\left|\frac{3-4\sin^2 u_{-1}}{3-4\sin^2 3u_1}\right|. \tag{2.25}$$

Moreover, recall that $1/6 < d/2M < 1/4$ (cf. Claim 3). We can see

$$\frac{\pi}{6}-\frac{\theta}{3} < u_{-1} < \frac{\pi}{4}-\frac{\theta}{2} \tag{2.26}$$

and

$$\frac{\pi}{6}+\frac{\theta}{9} < u_1 < \frac{\pi}{4}+\frac{\theta}{6}.$$

It suffices to show Claim 4 under the assumption that p_1 is positive. So, we assume $p_1 > 0$. Since

$$\frac{\pi}{2}+\frac{\theta}{3} < 3u_1 < \frac{3\pi}{4}+\frac{\theta}{2} \quad\text{and}\quad \frac{3\pi}{2}+\theta < 9u_1 < \frac{9\pi}{4}+\frac{3\theta}{2},$$

p_1 is positive if and only if $\sin 9u_1 < 0$, i.e., $3\pi/2+\theta < 9u_1 < 2\pi$. In particular

$$\frac{\pi}{2}+\frac{\theta}{3} < 3u_1 < \frac{2\pi}{3}. \tag{2.27}$$

Noting that, by (2.26) and (2.27), $3-4\sin^2 u_{-1} > 0$ and $3-4\sin^2 3u_1 \leq 0$ hold respectively, we get

$$\begin{aligned}&|\,3-4\sin^2 u_{-1}\,| - |\,3-4\sin^2 3u_1\,|\\ &\quad= 6-4(\sin^2 u_{-1}+\sin^2 3u_1)\\ &\quad\geq 6-4\Big\{\sin^2\Big(\frac{\pi}{4}-\frac{\theta}{2}\Big)+\sin^2\Big(\frac{\pi}{2}+\frac{\theta}{3}\Big)\Big\}\\ &\quad\geq 6-4\Big(\sin^2\frac{\pi}{4}+\sin^2\frac{\pi}{2}\Big)\\ &\quad= 0.\end{aligned} \tag{2.28}$$

Inequalities (2.25) and (2.28) show that $|p_1|/p_{-1} \geq 6/\pi\cdot 1 > 1$. Thus the claim is proved.

The claims above now completes the proof of the lemma in Case (Ib), and therefore that of Step I.

Step II. $|p_\nu| \geq p_{-1}$ for $\nu \leq -2$.

Case (IIa): $Nd(\pi/2-\theta)/M \le \pi/2$. For $\nu \le -2$, put $\nu = -\mu - 1$ ($\mu = 1, 2, \ldots$). Then $\omega_\nu/2 = \{(\theta - \pi/2) - \mu\pi\}/M$. In a similar way to (2.13), we verify

$$|p_\nu| \ge 2 \cdot \frac{2}{\pi}\left|\frac{\omega_\nu}{2}\right| \cdot \frac{1}{N} = \frac{4(\pi/2 - \theta + \mu\pi)}{\pi M N}.$$

Furthermore, (2.14) is valid also in this case, so we have

$$\begin{aligned}\frac{|p_\nu|}{p_{-1}} &\ge \frac{4(\pi/2-\theta+\mu\pi)}{\pi M N} \cdot \frac{2MN}{\pi(\pi-2\theta)} \ge \frac{8(\pi/2-\theta+\pi)}{\pi^2(\pi-2\theta)} \\ &= \frac{4}{\pi^2}\Big(1 + \frac{\pi}{\pi/2-\theta}\Big) \ge \frac{4\cdot 3}{\pi^2} > 1,\end{aligned}$$

as desired.

Case (IIb): $Nd(\pi/2-\theta)/M > \pi/2$. As in Case (IIa), we have $|p_\nu| \ge 4\mu/MN$ for $\mu = 1, 2, \ldots$. Since (2.20) is valid, we see

$$\begin{aligned}\frac{|p_\nu|}{p_{-1}} &\ge \frac{2M(2M-Nd)}{d\pi^2} \cdot \frac{4\mu}{MN} \\ &\ge \frac{8\mu\{2(N-1)d - Nd\}}{\pi^2 dN} \\ &= \frac{8\mu}{\pi^2}\Big(1 - \frac{2}{N}\Big).\end{aligned}$$

So we find that

- If $N \ge 4$, then $|p_\nu| \ge p_{-1}$ for $\mu \ge 3$;
- If $N = 3$, then $|p_\nu| \ge p_{-1}$ for $\mu \ge 4$.

Consider the remaining cases.

Claim 5. $|p_{-2}|, |p_{-3}| \ge p_{-1}$ for $N \ge 4$. It is easily seen that, for $\nu = -2, -3$ (i.e., for $\mu = 1, 2$),

$$\begin{aligned}|p_\nu| &\ge 2 \cdot \frac{2}{\pi}\,\frac{(2\mu+1)\pi/2 - \theta}{M} \cdot \frac{2}{\pi}\,\frac{d((2\mu+1)\pi/2-\theta)}{M} \\ &\ge \frac{8d}{M^2\pi^2}\Big(\frac{2\mu+1}{2}\pi - \theta\Big)^2 \\ &\ge \frac{8d}{M^2\pi^2}(\mu\pi)^2 \ge \frac{8d}{M^2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{|p_\nu|}{p_{-1}} &\ge \frac{8d}{M^2} \cdot \frac{2M(2M-Nd)}{d\pi^2} = \frac{16}{\pi^2}\Big(2 - \frac{Nd}{M}\Big) \\ &\ge \frac{16}{\pi^2}\Big(2 - \frac{N}{N-1}\Big) \ge \frac{32}{3\pi^2} > 1 \quad \text{for} \quad \nu = -2, -3.\end{aligned}$$

Claim 6. $|p_{-3}|, |p_{-4}| \geq p_{-1}$ for $N = 3$.

Put

$$\tilde{u}_\mu := \frac{d(\pi - \tilde{\theta}_\mu)}{2M} \quad \text{and} \quad \tilde{\theta}_\mu := \frac{2\theta}{2\mu + 1}.$$

It then follows, in a similar fashion to Claim 3, that

$$\frac{|p_\nu|}{p_{-1}} \geq \left| \frac{\sin 3\tilde{u}_0}{3 \sin \tilde{u}_0} \right| \cdot \left| \frac{\sin (2\mu+1)\tilde{u}_\mu/d}{\sin \tilde{u}_0/d} \right|,$$

for $\nu = -3$ and -4 (i.e., for $\mu = 2$ and 3). Since $0 \leq d/2M \leq d/2(N-1)d = 1/4$, that is, $0 \leq \tilde{u}_0 \leq \pi/4$, we obtain (see (2.23)):

$$\left| \frac{\sin 3\tilde{u}_0}{3 \sin \tilde{u}_0} \right| \geq \frac{1}{3}. \tag{2.29}$$

Furthermore, we observe

$$\frac{\sin (2\mu+1)\tilde{u}_\mu/d}{\sin \tilde{u}_0/d} \geq \frac{2/\pi \cdot (2\mu+1)\tilde{u}_\mu/d}{\tilde{u}_0/d} = \frac{2(2\mu+1)}{\pi} \frac{\tilde{u}_\mu}{\tilde{u}_0} \geq \frac{2}{\pi}(2\mu+1),$$

because $\tilde{u}_\mu \geq \tilde{u}_0$. Together with (2.29), we have

$$\frac{|p_\nu|}{p_{-1}} \geq \frac{1}{3} \cdot \frac{2}{\pi}(2\mu+1) \geq \frac{10}{3\pi} > 1 \quad \text{for} \quad \mu = 2 \text{ and } 3.$$

This proves Claim 6.

Claim 7. $|p_{-2}| \geq p_{-1}$ for $N = 3$.

The proof is quite parallel to that of Claim 4. Here we note that $\tilde{u}_0 = u_{-1}$ and therefore, from (2.26), $\pi/6 - \theta/3 < \tilde{u}_0 < \pi/4 - \theta/2$ holds, and that $\tilde{u}_1$ satisfies $\pi/6 - \theta/9 < \tilde{u}_1 < \pi/4 - \theta/6$. Then, we can verify a similar inequality to (2.28) by using

$$\frac{2\pi}{3} < 3\tilde{u}_1 < \frac{3\pi}{4} - \frac{\theta}{2} \qquad \left(\text{with } 0 < \theta < \frac{\pi}{6}\right)$$

instead of (2.27).

Thus we have proved Step II; the proof of Lemma 2.4 is completed. *Q.E.D.*

2.4 Asymptotic Stability of (1.1).

Theorems 2.1 and 2.2 are generalized to the system (1.1) of dimension m. Let $p_j e^{i\theta_j}$ ($j = 1, 2, \ldots, m$) be the eigenvalues, denoted with multiplicity, of the matrix A, where p_j and θ_j (possibly 0) are real numbers and $|\theta_j| \leq \pi/2$. Then we can readily verify that the characteristic equation of (1.1) is given by

$$\prod_{j=1}^{m} \left\{ \lambda^{k_N+1} - \lambda^{k_N} + p_j e^{i\theta_j} T(\lambda) \right\} = 0. \tag{2.30}$$

Consequently, applying our theorems to (2.30), we get the following:

Theorem 2.3 *The system* (1.1) *is asymptotically stable if and only if*

$$0 < p_j < \frac{2\sin\left(\dfrac{\pi/2 - |\theta_j|}{k_N + k_1 + 1}\right)\sin\left(\dfrac{d(\pi/2 - |\theta_j|)}{k_N + k_1 + 1}\right)}{\sin\left(\dfrac{Nd(\pi/2 - |\theta_j|)}{k_N + k_1 + 1}\right)}, \quad j = 1,\, 2,\, \ldots,\, m.$$

3 Asymptotic periodicity.

Let the delays $\{k_j\}$ (i.e., the value of d) be fixed and regard the components of the coefficient matrix B as parameters. Then the results in the previous section say that the stability region of the system (1.2) is, in case (i) (section 1), given by a bounded set in the $(\theta,\, p)$-plane:

$$\Sigma_1 = \{(\theta,\, p) \in \mathbf{R}^2 \,|\, 0 < p < p^*,\, 0 < |\theta| < \pi/2\,\},$$

and is, in case (ii), given as a square in the $(p_1,\, p_2)$-plane:

$$\Sigma_2 = \{(p_1,\, p_2) \in \mathbf{R}^2 \,|\, 0 < p_1,\, p_2 < p_0^*\,\},$$

where p^* and p_0^* are the critical values of Theorems 2.1 and 2.2, respectively.

In this section, we investigate the behavior of solutions of (1.2) when the system corresponds to a point on the boundary of its stability region. More precisely, we are concerned with the subsets of $\partial\Sigma_1$ and $\partial\Sigma_2$

$$\Gamma_1 = \{(\theta,\, p) \in \partial\Sigma_1 \,|\, p = p^*\,\}$$

and

$$\Gamma_2 = \{(p_1,\, p_2) \in \partial\Sigma_2 \,|\, p_1 \text{ or } \ p_2 = p_0^*\,\},$$

according to cases (i) and (ii), respectively. We shall show that (1.2) is asymptotically periodic and furthermore give explicit expressions of those periodic solutions.

3.1 Equivalent system.

We will discuss the structure of solutions of the system (1.2). Let $\{z^n\} \subset \mathbf{R}^\ell$, ℓ being $2(k_N + 1)$, be the sequence defined by

$$z^n = {}^t({}^tz_0^n,\, {}^tz_1^n, \ldots,\, {}^tz_{k_N}^n) := {}^t({}^tx_{n-k_N},\, {}^tx_{n-k_N+1}, \ldots,\, {}^tx_n) \quad \text{for} \quad n \in \mathbf{Z}_+.$$

It follows from (1.2) that

$$z^{n+1} = {}^t({}^t x_{n+1-k_N}, \dots, {}^t x_n, {}^t x_{n+1}) = {}^t\Big({}^t z_1^n, \dots, {}^t z_{k_N}^n, {}^t\Big(z_{k_N}^n - A\sum_{j=0}^{N-1} z_{jd}^n\Big)\Big),$$

so that the system (1.2) is equivalent to the ℓ-dimensional system of the first order:

$$z^{n+1} = \hat{B} z^n, \qquad n \in \mathbf{Z}_+, \tag{3.1}$$

where $\hat{B}$ is the $\ell \times \ell$ matrix of the form

$$\hat{B} = \begin{pmatrix} O & I_2 & O & \dots & \dots & \dots & O \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & O \\ O & \dots & \dots & \dots & \dots & O & I_2 \\ B_1 & B_2 & B_3 & \dots & \dots & B_{k_N} & I_2 \end{pmatrix},$$

and where the components of the last row are 2×2 matrices defined as follows:

$$B_s = \begin{cases} -B & (s = jd+1,\ j = 0, 1, \dots, N-1); \\ O & (\text{otherwise}). \end{cases}$$

The structure of solutions of (3.1) is determined by the eigenvalues of the matrix $\hat{B}$. Let $\sigma(\hat{B})$ be the set of eigenvalues of $\hat{B}$, and σ_-, σ_0 and σ_+ denote the eigenvalues which belong to the interior, the boundary and the exterior of the unit disk respectively. And let $P : \mathbf{C}^\ell \to \bigoplus_{\lambda\in\sigma_0\cup\sigma_+} E(\lambda)$ be the projection, where $E(\lambda)$ is the generalized eigenspace of $\hat{B}$ associated with $\lambda \in \sigma(\hat{B})$. Then the solution $z^n = \hat{B}^n z^0$ of (3.1) with the initial value z^0 is asymptotically equivalent to Pz^n in the sense that

$$\|z^n - Pz^n\| \le C\varepsilon^n, \quad n \in \mathbf{Z}_+,$$

where ε is a constant such that $\max\{|\lambda| \,|\, \lambda \in \sigma_-\} < \varepsilon < 1$ and C is a positive constant depending on ε. Note that Pz^n is also a solution of (3.1) since P commutes with $\hat{B}$.

The solution Pz^n is expressed explicitly in terms of a basis of $\bigoplus_{\lambda\in\sigma_0\cup\sigma_+} E(\lambda)$ and its dual. Indeed, let us denote by $E^*(\lambda)$ the generalized eigenspace of $\hat{B}^* = {}^t\hat{B}$, the adjoint of $\hat{B}$, associated with $\lambda \in \sigma(\hat{B})$, that is, $E^*(\lambda) = \bigcup_{\nu\ge1} \operatorname{Coker}(\lambda I_\ell - \hat{B})^\nu$. We use the notation, for a subset $W \subset \mathbf{C}^\ell$, $W^\perp \subset (\mathbf{C}^\ell)^*$ which means the subspace of covectors that vanish on W, that is, $W^\perp = \{\psi \in (\mathbf{C}^\ell)^* \,|\, \langle\phi, \psi\rangle = 0 \text{ for all } \phi \in W\}$. It is well known that

$$E^*(\lambda) \subset E(\mu)^\perp, \quad \text{for} \quad \lambda \neq \mu, \tag{3.2}$$

and

$$E^*(\lambda) \cap E(\lambda)^\perp = \{\, 0 \,\}. \tag{3.3}$$

Now let $\{\psi_1^\lambda, \ldots, \psi_{n(\lambda)}^\lambda\}$ and $\{\phi_1^\lambda, \ldots, \phi_{n(\lambda)}^\lambda\}$ be bases of $E^*(\lambda)$ and $E(\lambda)$ associated with $\lambda \in \sigma(\hat{B})$, respectively, where $n(\lambda)$ stands for the multiplicity of λ. Then (3.3) yields that the matrix $\Psi^\lambda \Phi^\lambda = (\, \langle \phi_j^\lambda,\, \psi_i^\lambda \rangle \,)$ is nonsingular, where Ψ^λ and Φ^λ are $n(\lambda) \times n(\lambda)$ matrices given by

$$\Psi^\lambda = \begin{pmatrix} \psi_1^\lambda \\ \vdots \\ \psi_{n(\lambda)}^\lambda \end{pmatrix} \quad \text{and} \quad \Phi^\lambda = (\phi_1^\lambda, \ldots, \phi_{n(\lambda)}^\lambda),$$

respectively. The dual basis of $\{\phi_1^\lambda, \ldots, \phi_{n(\lambda)}^\lambda\}$, say $\{\tilde{\psi}_1^\lambda, \ldots, \tilde{\psi}_{n(\lambda)}^\lambda\}$, is obtained in such a way that

$$\tilde{\psi}_j^\lambda = (c_1^j, \ldots, c_{n(\lambda)}^j)\Psi^\lambda = c_1^j \psi_1^\lambda + \cdots + c_{n(\lambda)}^j \psi_{n(\lambda)}^\lambda, \tag{3.4}$$

where $(c_1^j, \ldots, c_{n(\lambda)}^j)$ is the solution of a linear equation

$$(c_1^j, \ldots, c_{n(\lambda)}^j)\Psi^\lambda \Phi^\lambda = (0, \ldots, 0,\, 1,\, 0, \ldots,\, 0),$$

where the right hand side is the element in $(\mathbf{C}^{n(\lambda)})^*$ with the j-th component 1 and the others 0. Combining with (3.2) this implies that the projection onto $E(\lambda)$ is represented, via a basis of $E(\lambda)$ and its dual, as $\Phi^\lambda \Psi^\lambda = \phi_1^\lambda \tilde{\psi}_1^\lambda + \cdots + \phi_{n(\lambda)}^\lambda \tilde{\psi}_{n(\lambda)}^\lambda$, so that

$$P = \sum_{\lambda \in \sigma_0 \cup \sigma_+} \left(\phi_1^\lambda \tilde{\psi}_1^\lambda + \cdots + \phi_{n(\lambda)}^\lambda \tilde{\psi}_{n(\lambda)}^\lambda \right), \tag{3.5}$$

and hence the solution Pz^n is given by

$$Pz^n = P\hat{B}^n z^0 = \sum_{\lambda \in \sigma_0 \cup \sigma_+} \left(\phi_1^\lambda \tilde{\psi}_1^\lambda + \cdots + \phi_{n(\lambda)}^\lambda \tilde{\psi}_{n(\lambda)}^\lambda \right) \hat{B}^n z^0. \tag{3.6}$$

In particular, the solution x_n of the system (1.2) with initial values x_{-k_N}, x_{-k_N+1}, $\ldots, x_0$ is asymptotically equivalent to the solution $\mathrm{pr}(Pz^n)$, more precisely, $\|x_n - \mathrm{pr}(Pz^n)\|$ converges exponentially to 0 as n tends to infinity, where $\mathrm{pr} : \mathbf{C}^{\ell-2} \times \mathbf{C}^2 \to \mathbf{C}^2$ is the projection.

3.2 Distribution of characteristic roots in the critical cases.

In this subsection we consider the characteristic roots of (1.2) in the critical cases mentioned in the top of this section. Here the coefficient matrix B of (1.2) is assumed to have one of forms below:

$$\text{(i)}\ \ p\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \text{(ii t)}\ \ \begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}, \qquad \text{(ii d)}\ \ \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

Recalling analysis of characteristic roots in the previous section, we can summarize the distribution of the characteristic roots in the critical cases. We first consider case (i). When $(\theta,\, p) \in \Gamma_1$, we have the following (see Theorem 2.1):

Lemma 3.1 *Let $p = p^*$. Then the characteristic equation has simple roots $e^{i\omega}$ and $e^{-i\omega}$ on the unit circle and the remainder roots belong to the interior of the unit disk, where $\omega := -\mathrm{sgn}\,(\theta)(\pi - 2|\theta|)/(k_N + k_1 + 1)$.*

Remark 3.1 *The value of ω above is just ω_{-1} given in Lemma 2.1 under the condition $0 < \theta \le \pi/2$.*

In the case of (ii t), the characteristic equation is

$$\left(\lambda^{k_N+1} - \lambda^{k_N} + pT(\lambda)\right)^2 = 0.$$

(see $\Delta(\lambda)$ in the previous section.) So, when $p = p_0^*$, we have the following:

Lemma 3.2 *Let $p = p_0^*$. Then the characteristic equation has double roots $e^{i\omega'}$ and $e^{-i\omega'}$ and the remainder roots belong to the interior of the unit disk, where $\omega' := -\pi/(k_N + k_1 + 1)$.*

In the case of (ii d), the characteristic equation is written in the form

$$\left(\lambda^{k_N+1} - \lambda^{k_N} + p_1 T(\lambda)\right)\left(\lambda^{k_N+1} - \lambda^{k_N} + p_2 T(\lambda)\right) = 0.$$

Particularly, for $(p_1,\, p_2) \in \Gamma_2$, we get:

Lemma 3.3 *Let $(p_1,\, p_2) \in \Gamma_2$. Then followings hold:*
(a) If $p_1 = p_0^*$ and $0 < p_2 < p_0^*$, or $0 < p_1 < p_0^*$ and $p_2 = p_0^*$, the characteristic equation has simple roots $e^{i\omega'}$ and $e^{-i\omega'}$ and the remainder roots belong to the interior of the unit disk.
(b) If $p_1 = p_2 = p_0^*$, the characteristic equation has double roots $e^{i\omega'}$ and $e^{-i\omega'}$ and the remainder roots belong to the interior of the unit disk.

Thus we see that, on the boundaries of stability regions, $\sigma_0 = \{e^{i\omega},\, e^{-i\omega}\}$ and $\sigma_+ = \emptyset$ for case (i), and that $\sigma_0 = \{e^{i\omega'},\, e^{-i\omega'}\}$ and $\sigma_+ = \emptyset$ for both cases of (ii t) and (ii d).

3.3 Explicit expressions of asymptotic periodic solutions.

By using the results in subsections 3.1 and 3.2, we obtain explicit expessions of asymptotic periodic solutions of the system (1.2) in the critical cases. In the case of (i), we have the next theorem.

Theorem 3.1 *Let* $0 < |\theta| < \pi/2$ *and* $p = p^*$. *Then the solution* x_n *of* (1.2) *with initial values* $x_{-k_N}, x_{-k_N+1}, \ldots, x_0$ *in* $\mathbf{R}^2$ *is asymptotically equivalent to the solution* x_n^* *given by*

$$x_n = R(n\omega)\Big(I_2 + {}^tB\sum_{j=1}^{N} k_{N-j+1}R(k_j\omega)\Big)^{-1} \cdot \Big\{x_0 + {}^tB\sum_{j=1}^{N}\sum_{r=1}^{N-j+1} R(k_r\omega)\sum_{s\in S(j)} R(s\omega)\,x_{-s}\Big\},$$

where $\omega = -\mathrm{sgn}\,(\theta)\,(\pi - 2|\theta|)/(k_N + k_1 + 1)$, $R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$, $S(j) = \{k_{N-j} + 1,\, k_{N-j} + 2, \ldots, k_{N-j+1}\}$ *for* $j = 1, \ldots, N-1$ *and* $S(N) = \{1,\, 2, \ldots, k_1\}$.

Proof. By Lemma 3.1, $\sigma_+ = \emptyset$ and the characteristic roots on the unit circle consist of $\lambda := e^{i\omega}$ and its conjugate $\overline{\lambda}$, which are simple. Let ϕ_1 and ψ_1 be an eigenvector and an eigen-covector of $\hat{B}$ associated with λ, respectively, so that they generates $E(\lambda)$ and $E^*(\lambda)$, respectively. Direct calculations show that ϕ_1 and ψ_1 are given by

$$\phi_1 = \begin{pmatrix} \lambda^{-k_N}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ \vdots \\ \lambda^{-1}\begin{pmatrix} 1 \\ -i \end{pmatrix} \\ \begin{pmatrix} 1 \\ -i \end{pmatrix} \end{pmatrix} \tag{3.7}$$

and $\psi_1 = (v_0,\, v_1, \ldots, v_{k_N})$ with

$$v_s = \begin{cases} p^*e^{-i\theta}\lambda^{k_N+k_1-s}(1,\, i), & 0 \le s \le d-1, \\ p^*e^{-i\theta}\lambda^{k_N+k_1-s}(1+\lambda^d)(1,\, i), & d \le s \le 2d-1, \\ p^*e^{-i\theta}\lambda^{k_N+k_1-s}(1+\lambda^d+\lambda^{2d})(1,\, i), & 2d \le s \le 3d-1, \\ \qquad\vdots & \qquad\vdots \\ p^*e^{-i\theta}\lambda^{k_N+k_1-s}T(\lambda)(1,\, i), & (N-1)d \le s \le k_N - 1, \\ (1,\, i), & s = k_N. \end{cases} \tag{3.8}$$

Note that an eigenvector (an eigen-covector resp.) associated with $\overline{\lambda}$ is given by $\overline{\phi_1}$ ($\overline{\psi_1}$ resp.), because $\hat{B}$ is real. So $\{\phi_1, \overline{\phi_1}\}$ is a basis of $E(\lambda) \oplus E(\overline{\lambda})$ and, from (3.4), its dual vector is given by $\{\tilde{\psi}_1, \overline{\tilde{\psi}_1}\}$, where $\tilde{\psi}_1$ is defined by $\tilde{\psi}_1 = \psi_1/\langle\phi_1, \psi_1\rangle$. Here we notice

$$\begin{aligned}
\langle\phi_1, \psi_1\rangle &= \sum_{s=0}^{k_N} \Big\langle \lambda^{-k_N+s} \begin{pmatrix} 1 \\ -i \end{pmatrix}, v_s \Big\rangle \\
&= 2 + 2p^* e^{-i\theta} \lambda^{k_1} \Big(\sum_{j=0}^{N-2} (N-j-1) d\lambda^{jd} + k_1 T(\lambda) \Big) \\
&= 2 + 2p^* e^{-i\theta} \sum_{j=0}^{N-1} (k_1 + (N-j-1)d) \lambda^{k_1+jd} \\
&= 2 + 2p^* e^{-i\theta} \sum_{j=1}^{N} k_{N-j+1} \lambda^{k_j}.
\end{aligned}$$

It follows from (3.5) and (3.6) that the projection $P : \mathbf{C}^\ell \to E(\lambda) \oplus E(\overline{\lambda})$ is written in the form

$$P = \phi_1 \tilde{\psi}_1 + \overline{\phi_1}\, \overline{\tilde{\psi}_1},$$

and that

$$P z^n = (\lambda^n \phi_1 \tilde{\psi}_1 + \overline{\lambda}^n\, \overline{\phi_1}\, \overline{\tilde{\psi}_1}) z^0 = 2 \left\{ \Re\left(\lambda^n \phi_1 \tilde{\psi}_1 \right) \right\} z^0.$$

Hence the solution x_n is asymptotically equivalent to x_n^* given by

$$\begin{aligned}
x_n^* &:= \mathrm{pr}(P z^n) \\
&= \Re\Big[\frac{\lambda^n}{1 + p^* e^{-i\theta} \sum_{j=1}^N k_{N-j+1} \lambda^{k_j}} \sum_{s=0}^{k_N} \begin{pmatrix} 1 \\ -i \end{pmatrix} v_s z_s^0 \Big] \\
&= \Re\Big[\frac{\lambda^n}{\delta} \Big\{ p^* e^{-i\theta} \Big(\sum_{j=1}^{N-1} \sum_{r=1}^{j} \lambda^{(r-1)d} \sum_{s=(j-1)d}^{jd-1} \lambda^{k_N+k_1-s} K z_s^0 \\
&\qquad + \sum_{r=1}^{N} \lambda^{(r-1)d} \sum_{s=(N-1)d}^{k_N-1} \lambda^{k_N+k_1-s} K z_s^0 \Big) + K z_{k_N}^0 \Big\} \Big],
\end{aligned}$$

where $\delta = 1 + p^* e^{-i\theta} \sum_{j=1}^N k_{N-j+1} \lambda^{k_j}$ and $K = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Noting $k_r = k_1 + (r-1)d$ and taking $k_N - s$ as index instead of s (we denote this new index by the same letter s), we have

$$\begin{aligned} x_n^* &= \Re\Big[\frac{\lambda^n}{\delta}\Big\{p^*e^{-i\theta}\Big(\sum_{j=1}^{N-1}\sum_{r=1}^{j}\lambda^{k_r}\sum_{s=k_{N-j}+1}^{k_{N-j+1}}\lambda^s K z^0_{k_N-s} \\ &\qquad + \sum_{r=1}^{N}\lambda^{k_r}\sum_{s=1}^{k_1}\lambda^s K z^0_{k_N-s}\Big) + K z^0_{k_N}\Big\}\Big] \\ &= \Re\Big[\frac{\lambda^n}{\delta}\Big\{p^*e^{-i\theta}\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\lambda^{k_r}\sum_{s\in S(j)}\lambda^s K z^0_{k_N-s} + K z^0_{k_N}\Big\}\Big]. \end{aligned}$$

Now let $z_s^0 = {}^t(\xi_s, \eta_s)$ and $\zeta_s = \xi_s + i\eta_s$ for $s = 0, \ldots, k_N$. Since

$$K z_s^0 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} \xi_s \\ \eta_s \end{pmatrix} = \begin{pmatrix} \zeta_s \\ -i\zeta_s \end{pmatrix},$$

we have

$$\begin{aligned} x_n^* &= \Re\Big[\frac{\lambda^n}{\delta}\Big\{p^*e^{-i\theta}\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\lambda^{k_r}\sum_{s\in S(j)}\lambda^s\begin{pmatrix} \zeta_{k_N-s} \\ -i\zeta_{k_N-s} \end{pmatrix} + \begin{pmatrix} \zeta_{k_N} \\ -i\zeta_{k_N} \end{pmatrix}\Big\}\Big] \\ &= \begin{pmatrix} \Re\,\zeta \\ \Im\,\zeta \end{pmatrix}, \end{aligned}$$

where ζ is the complex number given by

$$\zeta = \frac{\lambda^n}{\delta}\Big\{p^*e^{-i\theta}\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\lambda^{k_r}\sum_{s\in S(j)}\lambda^s\zeta_{k_N-s} + \zeta_{k_N}\Big\}.$$

By virtue of the real representation of $\mathbf{C}$, x_n^* is written by

$$\rho(\lambda)^n\rho(\delta)^{-1}\Big\{\rho(p^*e^{-i\theta})\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\rho(\lambda)^{k_r}\sum_{s\in S(j)}\rho(\lambda)^s z^0_{k_N-s} + z^0_{k_N}\Big\},$$

where $\rho : \mathbf{C}\setminus\{0\} \to GL(2,\mathbf{R})$ maps a complex number $\alpha+i\beta$ into the matrix $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$.
Since $\rho(p^*e^{-i\theta}) = {}^tB$, we deduce

$$\begin{aligned} x_n^* &= R(n\omega)\Big(I_2 + {}^tB\sum_{j=1}^{N}k_{N-j+1}R(k_j\omega)\Big)^{-1} \\ &\qquad \cdot\Big\{{}^tB\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}R(k_r\omega)\sum_{s\in S(j)}R(s\omega)\,z^0_{k_N-s} + z^0_{k_N}\Big\} \\ &= R(n\omega)\Big(I_2 + {}^tB\sum_{j=1}^{N}k_{N-j+1}R(k_j\omega)\Big)^{-1} \\ &\qquad \cdot\Big\{{}^tB\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}R(k_r\omega)\sum_{s\in S(j)}R(s\omega)\,x_{-s} + x_0\Big\}. \end{aligned}$$

This proves Theorem 3.1. *Q.E.D.*

Remark 3.2 *If $\omega = -\mathrm{sgn}\,(\theta)\,(\pi - 2|\theta|)/(k_N + k_1 + 1)$ is rational, i.e., $\theta \in \pi\mathbf{Q}$, then x_n^* is a periodic solution of (1.2). On the other hand, if $\omega \notin \pi\mathbf{Q}$, the ω-limit set of x_n^*, and therefore of x_n, is the circle at the center 0 with radius*

$$\left\| D^{-1}\Big\{ x_0 + {}^tB \sum_{j=1}^{N} \sum_{r=1}^{N-j+1} R\,(k_r\omega) \sum_{s\in S(j)} R\,(s\omega)\, x_{-s} \Big\} \right\|,$$

D being $I_2 + {}^tB \sum_{j=1}^N k_{N-j+1} R(k_j\omega)$. In any case every solution approaches a bounded solution depending linearly on its initial values. In particular the zero solution of (1.2) is stable in the critical case of $p = p^*$.

To the contrary, in the case of (ii t), the zero solution of (1.2) is unstable when p coincides with the critical value p_0^* of Lemma 3.2. More precisely, we have:

Theorem 3.2 *Suppose that the condition $p = p_0^*$ holds in the case of* (ii t). *Then there exists a solution of* (1.2) *which diverges as $n \to \infty$.*

Proof. We know by Lemma 3.2 that $\sigma_+ = \emptyset$ and $\sigma_0 = \{\lambda_0, \overline{\lambda_0}\}$, where $\lambda_0 := e^{i\omega'}$ is a double root. It is easy to see that $\dim \mathrm{Ker}\,(\lambda_0 I_m - \hat{B}) = 1$, so that $E(\lambda_0)$ is generated by an eigenvector ϕ_1' and a generalized eigenvector ϕ_2' such that

$$\hat{B}\phi_2' = \phi_1' + \lambda_0\phi_2'. \tag{3.9}$$

Similarlly, we observe $E^*(\lambda_0) = \mathrm{span}\{\psi_1', \psi_2'\}$, where ψ_1' and ψ_2' are an eigen-covector and a generalized eigen-covector, respectively, satisfying

$$\psi_2'\hat{B} = \psi_1' + \lambda_0\psi_2'. \tag{3.10}$$

Since

$$\begin{aligned} \langle \hat{B}\phi_2', \psi_1' \rangle &= \langle \lambda_0\phi_2' + \phi_1', \psi_1' \rangle \\ &= \lambda_0 \langle \phi_2', \psi_1' \rangle + \langle \phi_1', \psi_1' \rangle, \end{aligned}$$

and also since $\langle \hat{B}\phi_2', \psi_1' \rangle = \langle \phi_2', \psi_1'\hat{B} \rangle = \lambda_0 \langle \phi_2', \psi_1' \rangle$, we see that $\langle \phi_1', \psi_1' \rangle = 0$. A similar argument yields $\langle \phi_2', \psi_1' \rangle = \langle \phi_1', \psi_2' \rangle$. In virtue of the argument in subsection 3.1, we find $\langle \phi_2', \psi_1' \rangle = \langle \phi_1', \psi_2' \rangle \neq 0$. From (3.4), the dual basis $\{\psi_1^*, \psi_2^*\}$ of $\{\phi_1', \phi_2'\}$ is given by

$$\psi_1^* = -\frac{\langle \phi_2', \psi_2' \rangle}{\langle \phi_1', \psi_2' \rangle \langle \phi_2', \psi_1' \rangle}\psi_1' + \frac{1}{\langle \phi_1', \psi_2' \rangle}\psi_2' \quad \text{and} \quad \psi_2^* = \frac{1}{\langle \phi_2', \psi_1' \rangle}\psi_1'. \tag{3.11}$$

Also we note that $E(\overline{\lambda_0}) = \mathrm{span}\{\overline{\phi_1'}, \overline{\phi_2'}\}$ and that $\{\overline{\psi_1^*}, \overline{\psi_2^*}\}$ gives the dual basis of $\{\overline{\phi_1'}, \overline{\phi_2'}\}$, since $\hat{B}$ is real. So, from (3.5) and (3.6)

$$\begin{aligned} Pz^n &= (\phi_1'\psi_1^* + \phi_2'\psi_2^* + \overline{\phi_1'}\,\overline{\psi_1^*} + \overline{\phi_2'}\,\overline{\psi_2^*})\hat{B}^n z^0 \\ &= 2\{\Re\,(\phi_1'\psi_1^* + \phi_2'\psi_2^*)\}\hat{B}^n z^0, \end{aligned}$$

where $P : \mathbf{C}^\ell \to E(\lambda_0) \oplus E(\overline{\lambda_0})$ is the projection. Since $\psi_1'\hat{B}^n = \lambda_0^n\psi_1'$ and $\psi_2'\hat{B}^n = n\lambda_0^{n-1}\psi_1' + \lambda_0^n\psi_2'$, it follows from (3.11) that

$$\begin{aligned} Pz^n = 2\Bigg[\Re\Bigg\{&\frac{-\langle\phi_2', \psi_2'\rangle\lambda_0^n + \langle\phi_2', \psi_1'\rangle n\lambda_0^{n-1}}{\langle\phi_1', \psi_2'\rangle\langle\phi_2', \psi_1'\rangle}\phi_1'\psi_1' \\ &+\frac{\lambda_0^n}{\langle\phi_1', \psi_2'\rangle}\phi_1'\psi_2' + \frac{\lambda_0^n}{\langle\phi_2', \psi_1'\rangle}\phi_2'\psi_1'\Bigg\}\Bigg] z^0. \end{aligned}$$

Consequently, we deduce that if initial value z^0 satisfies $\mathrm{pr}(\phi_1'\psi_1'z^0) \neq 0$ (the components of ϕ_1' and ψ_1' will be given in the proof of the next theorem), $\|\mathrm{pr}\,(Pz^n)\|$, and hence $\|x_n\|$, diverges as $n \to \infty$. The proof is completed. *Q.E.D.*

Finally, we consider the asymptotic behavior of solutions of (1.2) on the boundary of its stability region in the case of (ii d). In the following, the notation E_{ij} means the 2×2 matrix with (i, j)-component 1 and the others 0.

Theorem 3.3 *Suppose that* $(p_1, p_2) \in \Gamma_2$ *in the case of* (iid). *Then the solution* x_n*, starting from initial values* $x_{-k}, x_{-k+1}, \ldots, x_0$*, satisfies the following:*

(a) *If* $p_1 = p_0^*$ *and* $0 < p_2 < p_0^*$*, then the solution* x_n *is asymptotically equivalent to the periodic solution* $x_n^{(1)}$ *expressed by*

$$2E_{11}R(n\omega')\Big(I_2 + p_0^*\sum_{j=1}^{N} k_{N-j+1}R(k_j\omega')\Big)^{-1} \cdot\Big\{E_{11}x_0 + p_0^*\sum_{j=1}^{N}\sum_{r=1}^{N-j+1} R(k_r\omega')\sum_{s\in S(j)} R(s\omega')E_{11}x_{-s}\Big\},$$

where $\omega' = -\pi/(k_N + k_1 + 1)$.

(b) *If* $0 < p_1 < p_0^*$ *and* $p_2 = p_0^*$*, then* x_n *is asymptotically equivalent to the periodic solution* $x_n^{(2)}$ *expressed by*

$$2E_{22}R(n\omega')\Big(I_2 + p_0^*\sum_{j=1}^{N} k_{N-j+1}R(k_j\omega')\Big)^{-1} \cdot\Big\{E_{22}x_0 + p_0^*\sum_{j=1}^{N}\sum_{r=1}^{N-j+1} R(k_r\omega')\sum_{s\in S(j)} R(s\omega')E_{22}x_{-s}\Big\}.$$

(c) *If* $p_1 = p_2 = p_0^*$, *then* x_n *is asymptotically equivalent to the periodic solution* $x_n^{(1)} + x_n^{(2)}$. *In particular, the zero solution of* (1.2) *is stable.*

Proof. (a) By Lemma 3.3(a), $\sigma_0 = \{\lambda_0, \overline{\lambda_0}\}$ (λ_0 : simple) and $\sigma_+ = \emptyset$. It is easy to see that a vector given by

$$\begin{pmatrix} \lambda_0^{-k_N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \vdots \\ \lambda^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \tag{3.12}$$

generates $E(\lambda_0)$. We denote this by ϕ_1' since this vector is an eigenvector associated with λ_0 in the proof of Theorem 3.2 as well. We also note that $\psi_1'' = \left(v_0', v_1', \dots, v_{k_N}'\right)$ with

$$v_s' = \begin{cases} p_0^* \lambda_0^{k_N + k_1 - s}(1, 0), & 0 \le s \le d-1, \\ p_0^* \lambda_0^{k_N + k_1 - s}(1 + \lambda_0^d)(1, 0), & d \le s \le 2d-1, \\ p_0^* \lambda_0^{k_N + k_1 - s}(1 + \lambda_0^d + \lambda_0^{2d})(1, 0), & 2d \le s \le 3d-1, \\ \qquad \vdots & \qquad \vdots \\ p_0^* \lambda_0^{k_N + k_1 - s} T(\lambda_0)(1, 0), & (N-1)d \le s \le k_N - 1, \\ (1, 0), & s = k_N \end{cases} \tag{3.13}$$

generates $E^*(\lambda_0)$. So, the dual of ϕ_1' is given by $\tilde{\psi}_1'' = \psi_1''/\langle \phi_1', \psi_1'' \rangle$, which gives the projection $P : \mathbf{C}^\ell \to E(\lambda_0) \oplus E^*(\overline{\lambda_0})$ as $P = \phi_1' \tilde{\psi}_1'' + \overline{\phi_1'}\, \overline{\tilde{\psi}_1''}$, so that from (3.6)

$$Pz^n = (\phi_1' \tilde{\psi}_1'' + \overline{\phi_1'}\, \overline{\tilde{\psi}_1''}) \hat{B}^n z^0 = 2 \left(\Re \left(\lambda_0^n \phi_1 \tilde{\psi}_1'' z^0 \right) \right).$$

Noting that

$$\delta_0 := \langle \phi_1', \psi_1'' \rangle = 1 + p_0^* \sum_{j=1}^{N} k_{N-j+1} \lambda_0^{k_j},$$

we see

$$\begin{aligned} x_n^* &= \mathrm{pr}(Pz^n) \\ &= 2 \Re \left[\frac{\lambda_0^n}{1 + p_0^* \sum_{j=1}^N k_{N-j+1} \lambda_0^{k_j}} \sum_{s=0}^{k_N} \begin{pmatrix} 1 \\ 0 \end{pmatrix} v_s' z_s^0 \right] \end{aligned}$$

$$
\begin{aligned}
&= 2\,\Re\Big[\frac{\lambda_0^n}{\delta_0}\Big\{p_0^*\Big(\sum_{j=1}^{N-1}\sum_{r=1}^{j}\lambda_0^{k_r}\sum_{s=k_{N-j}+1}^{k_{N-j+1}}\lambda_0^s E_{11} z^0_{k_N-s} \\
&\qquad\qquad + \sum_{r=1}^{N}\lambda_0^{k_r}\sum_{s=1}^{k_1}\lambda_0^s E_{11} z^0_{k_N-s}\Big) + E_{11} z^0_{k_N}\Big\}\Big] \\
&= 2\,\Re\Big[\frac{\lambda_0^n}{\delta_0}\Big(E_{11}x_0 + p_0^*\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}\lambda_0^{k_r}\sum_{s\in S(j)}\lambda_0^s E_{11}x_{-s}\Big)\Big].
\end{aligned}
$$

By using the identity

$$
\Re(\gamma z) = E_{11}\rho(\gamma)E_{11}z + E_{22}\rho(\gamma)E_{22}z, \quad \text{for } \gamma \in \mathbf{C} \text{ and } z \in \mathbf{R}^2,
$$

we find that

$$
\begin{aligned}
x_n^* &= 2E_{11}R(n\omega')\Big(I_2 + p_0^*\sum_{j=1}^{N}k_{N-j+1}R(k_j\omega')\Big)^{-1} \\
&\qquad \cdot\Big\{E_{11}x_0 + p_0^*\sum_{j=1}^{N}\sum_{r=1}^{N-j+1}R(k_r\omega')\sum_{s\in S(j)}R(k_s\omega')E_{11}x_{-s}\Big\},
\end{aligned}
$$

which completes the proof of (a).

(b) In this case it is easily verified that $E(\lambda_0) = \operatorname{span}\{\phi_1''\}$ with

$$
\phi_1'' = \begin{pmatrix} \lambda_0^{-k_N}\begin{pmatrix}0\\1\end{pmatrix} \\ \vdots \\ \lambda_0^{-1}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix},
$$

and that $E^*(\lambda_0)$ is generated by the following covector $\psi_1' = \Big(v_0'',\, v_1'', \ldots, v_{k_N}''\Big)$ with

$$
v_s'' = \begin{cases}
p_0^*\lambda_0^{k_N+k_1-s}(0,\,1), & 0 \le s \le d-1, \\
p_0^*\lambda_0^{k_N+k_1-s}(1+\lambda_0^d)(0,\,1), & d \le s \le 2d-1, \\
p_0^*\lambda_0^{k_N+k_1-s}(1+\lambda_0^d+\lambda_0^{2d})(0,\,1), & 2d \le s \le 3d-1, \\
\qquad\vdots & \qquad\vdots \\
p_0^*\lambda_0^{k_N+k_1-s}T(\lambda_0)(0,\,1), & (N-1)d \le s \le k_N-1, \\
(0,\,1), & s = k_N,
\end{cases}
$$

where we used the notation ψ_1' since it is the same as in Theorem 3.2. The rest of the proof is quite analogous to the preceding one.

(c) If $p_1 = p_2 = p_0^*$, then it is observed that span $\{\phi_1',\, \phi_1''\} \subset \mathrm{Ker}\,(\lambda_0 I_m - \hat{A})$. Hence, Lemma 3.3(b) shows $E(\lambda_0) = \mathrm{span}\,\{\phi_1',\, \phi_1''\}$. Similarlly we deduce that $E^*(\lambda_0) = \mathrm{span}\,\{\psi_1',\, \psi_1''\}$. Thus, the projection $P : \mathbf{C}^\ell \to E(\lambda_0) \oplus E^*(\overline{\lambda_0})$ is expressed as $P = \phi_1'\tilde{\psi}_1'' + \phi_1''\tilde{\psi}_1'$ with $\tilde{\psi}_1' = \psi_1'/\langle \phi_1'',\, \psi_1' \rangle$. Consequently,

$$\begin{aligned} x_n^* &= 2\left[\Re\{\lambda_0^n \mathrm{pr}(\phi_1'\tilde{\psi}_1'' z^0)\}\right] + 2\left[\Re\{\lambda_0^n \mathrm{pr}(\phi_1''\tilde{\psi}_1' z^0)\}\right] \\ &= x_n^{(1)} + x_n^{(2)}. \end{aligned}$$

The proof is completed. *Q.E.D.*

As an application of Theorem 3.3, we can see the asymptotic periodicity of the scalar equation (1.3);

$$u_{n+1} - u_n + p\sum_{j=1}^{N} u_{n-k_j} = 0,$$

and obtain explicit expressions of periodic solutions appearing in the critical case. We know that (1.3) is asymptotically stable if and only if $0 < p < p_0^*$ ([9], or Theorem 2.2). In the case of $p = p_0^*$, we have the following corollary.

Corollary 3.1 *Let $p = p_0^*$ hold in the equation* (1.3). *Then the solution u_n with initial values $u_{-k_N}, u_{-k_N+1}, \ldots, u_0 \in \mathbf{R}$ is asymptotically equivalent to the periodic solution u_n^* expressed by*

$$\begin{aligned} u_n^* = \frac{2}{\delta_1}\Big\{ &(\cos n\omega' + p_0^* \sum_{j=1}^{N} k_{N-j+1}\cos(n-k_j)\omega')u_0 \\ &+ p_0^* {\sum}'\Big(\cos(n+k_r+s)\omega' + p_0^*\sum_{q=1}^{N} k_{N-q+1}\cos(n+k_r-k_q+s)\omega'\Big)u_{-s}\Big\}, \end{aligned}$$

where $\omega' = -\pi/(k_N + k_1 + 1)$,

$$\delta_1 = 1 + 2p_0^*\sum_{j=1}^{N} k_{N-j+1}\cos k_j\omega' + {p_0^*}^2 \sum_{j,\,q=1}^{N} k_{N-j+1}k_{N-q+1}\cos(k_j - k_q)\omega'$$

and the summation $\sum'$ is carried out by the indices j, r and s which run over the ranges $j = 1, \ldots, N$, $r = 1, \ldots, N - j + 1$ and $s \in S(j)$, respectively.

Outline of the proof. It is obvious that u_n approaches the first component of the solution given in, for instance, Theorem 3.3 (c), i.e., that of $x_n^{(1)}$, with initial values $x_{-s} =$

${}^t(u_{-s},\, 0)$, $s = 0, \dots, k_N$. In this setting,

$$\begin{aligned}
&\Big(I_2 + p_0^* \sum_{j=1}^{N} k_{N-j+1} R(k_j\omega')\Big)^{-1} \\
&\quad = \frac{1}{\delta_1} \begin{pmatrix} 1 + p_0^* \sum_{j=1}^{N} k_{N-j+1} \cos k_j\omega' & p_0^* \sum_{j=1}^{N} k_{N-j+1} \sin k_j\omega' \\ -p_0^* \sum_{j=1}^{N} k_{N-j+1} \sin k_j\omega' & 1 + p_0^* \sum_{j=1}^{N} k_{N-j+1} \cos k_j\omega' \end{pmatrix},
\end{aligned}$$

and hence we have

$$\begin{aligned}
&\delta_1 \Big(I_2 + p_0^* \sum_{j=1}^{N} k_{N-j+1} R(k_j\omega')\Big)^{-1} \\
&\qquad \cdot \Big\{E_{11}x_0 + p_0^* \sum_{j=1}^{N} \sum_{r=1}^{N-j+1} R(k_r\omega') \sum_{s\in S(j)} R(s\omega') E_{11} x_{-s}\Big\} \\
&= \begin{pmatrix} \Big(1 + p_0^* \sum_{j=1}^{N} k_{N-j+1} \cos k_j\omega'\Big) u_0 \\ -p_0^* \sum_{j=1}^{N} k_{N-j+1} \sin k_j\omega' \cdot u_0 \end{pmatrix} \\
&\quad + \begin{pmatrix} p_0^* \sum{}' \Big\{\cos(k_r + s)\omega' + p_0^* \sum_{q=1}^{N} k_{N-q+1} \cos(k_r - k_q + s)\omega'\Big\} u_{-s} \\ p_0^* \sum{}' \Big\{\sin(k_r + s)\omega' + p_0^* \sum_{q=1}^{N} k_{N-q+1} \sin(k_r - k_q + s)\omega'\Big\} u_{-s} \end{pmatrix}.
\end{aligned}$$

The expression of $x_n^{(1)}$ in Theorem 3.3 shows the desired asymptotic form of the solution u_n. *Q.E.D.*

References

[1] S. N. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.

[2] V. L. Kocic and G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order with Applications, Kluwer Academic Publishers, Dordrecht, 1993.

[3] S. A. Kurkulis, The asymptotic stability of $x_{n+1} - ax_n + bx_{n-k} = 0$, J. Math. Anal. Appl., 188 (1994), 719–731.

[4] S. A. Levin and R. M. May, A note on difference-delay equations, Theoret. Popul. Biol., 9 (1976), 178–187.

[5] H. Matsunaga and T. Hara, The asymptotic stability of a two-dimensional linear delay difference equation, Dynam. Contin. Discrete Impuls. Systems, 6 (1999), 465–473.

[6] H. Matsunaga, R. Ogita and K. Murakami, Asymptotic behavior of a system of higher order linear difference equations, Nonlinear Anal., 47 (2001), 4667–4677.

[7] Y. Nagabuchi, Asymptotic stability for a linear difference system with two delays, Nonlinear Anal., vol.47 (2001), 4117–4128.

[8] Y. Nagabuchi, Asymptotic periodic solutions for a two-dimensional linear delay difference system, Dynam. Contin. Discrete Impuls. Systems, 9 (2002), 187–200.

[9] R. Ogita, H. Matsunaga and T. Hara, Asymptotic stability condition for a class of linear delay difference equations of higher order, J. Math. Anal. Appl., 248 (2000), 83–96.

[10] V. G. Papanicolaou, On the asymptotic stability of a class of linear difference equations, *Math. Magazine*, **69**, (1996), 34–43.

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