Simultaneous effects of homogenization and vanishing viscosity in fully nonlinear elliptic equations*

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1 Introduction

We consider the partial differential equation

$$(\mathrm{P})_\varepsilon \qquad F(x, x/\varepsilon, u^\varepsilon(x), Du^\varepsilon(x), \delta D^2 u^\varepsilon(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

where ε and $\delta \equiv \delta(\varepsilon)$ are two positive parameters, $F \in C(\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n)$, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices, u^ε is the unknown, and Du^ε and D^2u^ε denote the gradient and Hessian of u^ε, respectively. The parameter δ will be given as a function of ε, that is, $\delta = \delta(\varepsilon)$. A typical example of $\delta(\varepsilon)$ is: $\delta(\varepsilon) = \varepsilon^a$, where $0 \le a < \infty$. We always assume

(A1) F is uniformly elliptic, that is, there are constants $0 < \theta \le \Theta$ for which if $X, Y \in \mathcal{S}^n$ and $Y \ge 0$, then

$$F(x, y, u, p, X) - \Theta \operatorname{tr} Y \le F(x, y, u, p, X + Y) \le F(x, y, u, p, X) - \theta \operatorname{tr} Y;$$

(A2) the function: $y \mapsto F(x, y, u, p, X)$ is periodic with period $\mathbf{Z}^n$, that is,

$$F(x, y + z, u, p, X) = F(x, y, u, p, X) \qquad \text{for all } z \in \mathbf{Z}^n;$$

and

(A3) there is a constant $\lambda > 0$ such that the function: $u \mapsto F(x, y, u, p, X) - \lambda u$ is non-decreasing in $\mathbf{R}$ for any $(x, y, p, X) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{S}^n$.

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* Major part of this paper was prepared by the first author and delivered in a meeting held at Tokyo Metropolitan University on August, 30, 1999.

1) Died on October 2, 1999, when he was a graduate student of Tokyo Metropolitan University.

2) Supported in part by Grant-in-Aid for Scientific Research, No. 12440044, JSPS.

We investigate the asymptotic behavior of the solution u^ε of $(\mathrm{P})_\varepsilon$ as $\varepsilon \to 0$. The parameters ε and δ represent a state in the processes of homogenization and vanishing viscosity in $(\mathrm{P})_\varepsilon$, respectively, and our motivation to studying $(\mathrm{P})_\varepsilon$ is to understand the simultaneous effects of the periodic homogenization and vanishing viscosity in $(\mathrm{P})_\varepsilon$.

Let us describe briefly our results on the effects of the homogenization and vanishing viscosity. For this, we formulate the following three cell problems. Henceforth $C(\mathbf{R}^n/\mathbf{Z}^n)$ denotes the space of periodic functions u on $\mathbf{R}^n$ with period $\mathbf{Z}^n$. Fix $(\hat{x}, \hat{u}, \hat{p}, \hat{X}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$. The first cell problem is to find a pair of $\mu \in \mathbf{R}$ and $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_2 \qquad F(\hat{x}, y, \hat{u}, \hat{p}, \hat{X} + D^2 v(y)) = \mu \qquad \text{in } \mathbf{R}^n.$$

The second cell problem is to find a pair $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_{12} \qquad F(\hat{x}, y, \hat{u}, \hat{p} + Dv(y), D^2 v(y)) = \mu \qquad \text{in } \mathbf{R}^n.$$

The third cell problem is to find a pair $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a (viscosity) solution of

$$(\mathrm{CP})_1 \qquad F(\hat{x}, y, \hat{u}, \hat{p} + Dv(y), 0) = \mu \qquad \text{in } \mathbf{R}^n.$$

In this paper we deal with fully nonlinear PDE which may be degenerate elliptic and which may not have classical solutions, and we adapt the notion of viscosity solution (see [CIL2]). Henceforth we suppress the word "viscosity" and, for instance, we call a viscosity solution simply a solution.

Under appropriate hypotheses each of these problems $(\mathrm{CP})_2$, $(\mathrm{CP})_{12}$, and $(\mathrm{CP})_1$ has a solution (μ, v) and moreover, the value of μ is determined uniquely while the function v is not determined uniquely. The correspondence of $(\hat{x}, \hat{u}, \hat{p}, \hat{X})$ to this value μ is called the homogenized or effective function and denoted by $\bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X})$, $\bar{F}_{12}(\hat{x}, \hat{u}, \hat{p})$, and $\bar{F}_1(\hat{x}, \hat{u}, \hat{p})$, respectively, in problems $(\mathrm{CP})_2$, $(\mathrm{CP})_{12}$, and $(\mathrm{CP})_1$.

Four cases arise in our study of the asymptotics for $(\mathrm{P})_\varepsilon$. Case 1: $\lim_{\varepsilon\to 0} \delta(\varepsilon) \in (0, \infty)$. Case 2: $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = \infty$. Case 3: $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon \in (0, \infty)$. Case 4: $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = 0$. We may assume by a simple normalization that $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 1$ in Case 1 and $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = 1$ in Case 3.

Our main results state that under appropriate hypotheses, the solutions u^ε of $(\mathrm{P})_\varepsilon$ converge uniformly on $\mathbf{R}^n$, to the solution $u \in \mathrm{BUC}(\mathbf{R}^n)$ of

$$\bar{F}_2(x, u(x), Du(x), D^2 u(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

$$\bar{F}_2(x, u(x), Du(x), 0) = 0 \qquad \text{in } \mathbf{R}^n,$$

$$\bar{F}_{12}(x, u(x), Du(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

and

$$\bar{F}_1(x, u(x), Du(x)) = 0 \qquad \text{in } \mathbf{R}^n$$

in Cases 1, 2, 3, and 4, respectively. Indeed, these results in Cases 1 and 3 have already been obtained in [E2].

In Case 4 the vanishing of viscosity is fastest and the result says that in order to find the limit PDE one firstly sets $\delta = 0$ in $(\mathrm{P})_\varepsilon$, i.e., sends the "viscosity" to zero, and secondly homogenizes the resulting PDE. On the other hand, in Case 2 the result says that in order to find the limit PDE one firstly fixes $\delta > 0$ and homogenizes the PDE (as in Case 1 one gets $\bar{F}_2(x, u, Du, \delta D^2 u) = 0$), and secondly sends the "viscosity" to zero, to obtain $\bar{F}_2(x, u, Du, 0) = 0$.

For a general overview of homogenizations of partial differential equations we refer to [BLP].

The paper is organized as follows. Section 2 is devoted to studying cell problems. In Section 3 theorems on the convergence of solutions of $(\mathrm{CP})_\varepsilon$ to that of the corresponding homogenized equations are established. Section 4 provides proofs of technical lemmas which are needed in Sections 2 and 3.

2 Cell problems

We begin this section by stating our assumptions on F.

(A4) For each $R > 0$,

$$F \in \mathrm{BUC}(\mathbf{R}^n \times \mathbf{R}^n \times [-R, R] \times B(0, R) \times B^{n\times n}(0, R)),$$

where $B^{n\times n}(0, R)$ denotes the ball in $\mathcal{S}^n$ of radius R with center at the origin.

We call any continuous function $\omega : [0, \infty) \to [0, \infty)$ a *modulus* if $\omega(0) = 0$ and ω is non-decreasing in $[0, \infty)$.

(A5) For each $R > 0$ there is a modulus ω_R such that for $(u, p, X) \in [-R, R] \times \mathbf{R}^n \times \mathcal{S}^n$ and $x, y, \xi, \eta \in \mathbf{R}^n$,

$$|F(x, \xi, u, p, X) - F(y, \eta, u, p, X)| \le \omega_R\big((|x - y| + |\xi - \eta|)(1 + |p| + \|X\|)\big),$$

where $\|X\| = \sup_{\xi \in B(0,1)} |X\xi| = \max_{i=1,\dots,n} |\lambda_i(X)|$, with $\lambda_i(X)$ denoting the eigenvalues of X.

(A6) For each $R > 0$ there is a constant $C_R > 0$ such that for $(x, u, p) \in \mathbf{R}^n \times [-R, R] \times \mathbf{R}^n$,

$$|F(x, \xi, u, p, 0)| \le C_R(1 + |p|).$$

(A7) For each $R > 0$

$$\lim_{r\to\infty} \inf\{F(x,y,u,p,0) \mid (x,y,u,p) \in \mathbf{R}^{2n} \times [-R,R] \times \mathbf{R}^n,\ |p| \geq r\} = \infty.$$

(A8) For each $(x,u) \in \mathbf{R}^n \times \mathbf{R}$ and $R > 0$ there exists a constant $L \equiv L(R,x,u) > 0$ such that for all $y \in \mathbf{R}^n$, $p,q \in B(0,R)$, and $X \in \mathcal{S}^n$,

$$|F(x,y,u,p,X) - F(x,y,q,X)| \leq L|p-q|.$$

Fix $(\hat{x},\hat{u},\hat{p},\hat{X}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$. Define

$$\hat{F}(y,q,Y) = F(\hat{x},y,\hat{u},\hat{p}+q,\hat{X}+Y) \quad \text{for } (y,q,Y) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{S}^n.$$

Then consider the following cell problem (CP): find a pair $(\mu,v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a solution of

$$\text{(CP)} \qquad \hat{F}(y,Dv(y),D^2v(y)) = \mu \qquad \text{in } \mathbf{R}^n.$$

We call such a pair (μ,v) a solution of (CP).

Theorem 2.1. *Assume that (A1), (A2), (A5), and (A6) hold. Then: (a) There exists a solution $(\mu,v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ of (CP). (b) If $(\mu,v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ and $(\nu,w) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ are solutions of (CP), then $\mu = \nu$. If moreover (A8) holds, then $u(x) = v(x) + C$ for all $x \in \mathbf{R}^n$ and for some constant $C \in \mathbf{R}$. (c) If $(\mu,v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ is a solution of (CP), then v is Lipschitz continuous in $\mathbf{R}^n$.*

To prove this theorem, we need Krylov-Safonov C^α estimates, which we state for the equation of the form

$$\text{(2.1)} \qquad \lambda u(x) + G(x,u(x),Du(x),D^2u(x)) = 0 \qquad \text{in } \mathbf{R}^n,$$

where $\lambda \geq 0$ is a constant and G is a continuous function and satisfies:

(2.2) $G \in \mathrm{BUC}\big(\mathbf{R}^n \times [-R,R] \times B(0,R) \times B^{n\times n}(0,R)\big)$ for all $R > 0$.

(2.3) There are constants $0 < \theta \leq \Theta < \infty$ such that for all $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$, $u \in \mathbf{R}$, and $X,Y \in \mathcal{S}^n$, if $Y \geq 0$, then

$$G(x,u,p,X) - \Theta \operatorname{tr} Y \leq G(x,u,p,X+Y) \leq G(x,u,p,X) - \theta \operatorname{tr} Y.$$

Note that condition (2.3) implies the Lipschitz continuity of $G(x,u,p,X)$ in the variable X. More precisely,

$$|G(x,u,p,X) - G(x,u,p,Y)| \leq n\Theta\|X-Y\|$$

for all $x, p \in \mathbf{R}^n$, $u \in \mathbf{R}$, and $X, Y \in \mathcal{S}^n$.

(2.4) The function $u \mapsto G(x,u,p,X)$ is non-decreasing in $\mathbf{R}$ for each $(x,p,X) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{S}^n$.

(2.5) For each $R > 0$ there is a modulus ω_R such that for $(u,p,X) \in [-R,R] \times \mathbf{R}^n \times \mathcal{S}^n$ and $x, y \in \mathbf{R}^n$,

$$|G(x,u,p,X) - G(y,u,p,X) \leq \omega_R\big(|x-y|(1+|p|+\|X\|)\big).$$

(2.6) For each $R > 0$ there is a constant $C_R > 0$ such that for all $x, p \in \mathbf{R}^n$ and $u \in [-R,R]$,

$$|G(x,u,p,0)| \leq C_R(1+|p|).$$

(2.7) For each $R > 0$ there is a constant $L_R > 0$ such that for all $(x,u) \in \mathbf{R}^n \times [-R,R]$, $p, q \in B(0,R)$, and $X \in \mathcal{S}^n$,

$$|G(x,u,p,X) - G(x,u,q,X)| \leq L_R|p-q|.$$

In the above assumptions, because of the convexity of the domain $\mathbf{R}^n$ for x, y, we may assume that ω_R grows at most linearly. This observation is useful in our proof of Lemma 2.3.

Lemma 2.2. *Assume that (2.3) and (2.6) hold and that $\lambda = 0$. Then for each $R > 0$ there exist constants $\alpha \equiv \alpha(n,\theta,\Theta,C_R) \in (0,1)$ and $C \equiv C(n,\theta,\Theta,C_R) > 0$, where C_R is the constant from (2.6), such that if $u \in C(\mathbf{R}^n)$ is a solution of (2.1) and if $\|u\|_{L^\infty(\mathbf{R}^n)} \leq R$, then*

$$|u(x) - u(y)| \leq C|x-y|^\alpha \quad \textit{if } \ |x-y| \leq 1.$$

We do not give here the proof of this lemma since the result is somehow standard and instead we refer to [CC, CCKS, KT, T] and just give the following remark: for $(x,u,p,X) \in \mathbf{R}^n \times [-R,R] \times \mathbf{R}^n \times \mathcal{S}^n$ we have

$$\begin{aligned} G(x,u,p,X) &= G(x,u,p,X_+ - X_-) \leq G(x,u,p,-X_-) - \theta \operatorname{tr} X_+ \\ &\leq G(x,u,p,0) - \theta \operatorname{tr} X_+ + \Theta \operatorname{tr} X_- \leq C_R(1+|p|) + \mathcal{P}^+(X), \end{aligned}$$

where $X_+ := \frac{1}{2}\left(X + (X^2)^{1/2}\right)$, $X_- := -X + X_+$, and $\mathcal{P}^+(X) := -\theta \operatorname{tr} X_+ + \Theta \operatorname{tr} X_-$. Therefore the solution u of Lemma 2.2 satisfies

$$\mathcal{P}^+(D^2u(x)) + C_R(1+|Du(x)|) \geq 0 \quad \text{in } \mathbf{R}^n.$$

Similarly, we see that u satisfies

$$\mathcal{P}^-(D^2u(x)) - C_R(1+|Du(x)|) \le 0 \quad \text{in } \mathbf{R}^n,$$

where $\mathcal{P}^-(X) := -\Theta \operatorname{tr} X_+ + \theta \operatorname{tr} X_-$.

It may be worth noting here that if the function $G(x,u,p,X)$ is independent of u, then the constants C_R of (2.6) and therefore C of Lemma 2.2 can be chosen independently of R.

Lemma 2.3. *Assume that (2.2)–(2.5) hold. Let* $u \in \mathrm{USC}(\mathbf{R}^n)$ *and* $v \in \mathrm{LSC}(\mathbf{R}^n)$ *be bounded sub- and supersolutions of (2.1), respectively, and let* $R > 0$ *be a constant such that*

$$\max\{\|u\|_{L^\infty(\mathbf{R}^n)}, \|v\|_{L^\infty(\mathbf{R}^n)}\} \le R.$$

Then there is a constant $C \equiv C(n,\theta,\Theta,\omega_R,R) > 0$, *where* ω_R *is the modulus from (2.5), such that*

$$u(x) - v(y) \le \sup_{\mathbf{R}^n}(u-v)_+ + C|x-y| \qquad \text{for all } x,y \in \mathbf{R}^n. \tag{2.8}$$

An assertion close to the above can be found in [IL] (see [IL, (3.19)]). We have chosen condition (2.5) which is much stronger than needed and indeed more restrictive than [IL, (3.2)]. This choice is made for simplicity of the presentation.

Outline of proof. By a careful review of the proof of [IL, (3.19)], we find a constant $C_1 > 0$ depending only on n, θ, Θ, R, ω_R for which we have

$$u(x) - v(y) \le \max_{B(z,1)}(u-v)_+ + C_1|x-y| \quad \text{for all } x,y \in B(z,1) \text{ and } z \in \mathbf{R}^n.$$

This immediately yields (2.8), with an appropriate $C > 0$. QED

A form of the strong maximum principle for (2.1) can be stated as follows:

Lemma 2.4. *Assume that* $\lambda = 0$ *and the function* $G(x,u,p,X)$ *is independent of* u *and that (2.3) and (2.7) hold. If* $u, v \in C(\mathbf{R}^n)$ *be bounded solutions of (2.1), then* $u(x) = v(x) + C$ *for all* $x \in \mathbf{R}^n$ *and for some constant* $C \in \mathbf{R}$.

A proof of Lemma 2.4 can be found in Section 4.

Lemma 2.5. *Assume that* $\lambda > 0$ *and that (2.2)–(2.5) hold. If* $v \in \mathrm{USC}(\mathbf{R}^n)$ *and* $w \in \mathrm{LSC}(\mathbf{R}^n)$ *are bounded sub- and supersolutions of (2.1), respectively, then* $v \le w$ *in* $\mathbf{R}^n$.

A proof of this lemma can be found in Section 4. See [IL, Theorem III.1] and also [T, CCKS] for similar results under "structure condition".

Proof of Theorem 2.1. Fix $\beta > 0$, and we consider the problem

$$(\mathrm{CP})_\beta \qquad \beta v^\beta(y) + \hat{F}(y, Dv^\beta(y), D^2 v^\beta(y)) = 0 \qquad \text{in } \mathbf{R}^n.$$

If we set $M = \max_{\xi \in \mathbf{R}^n} |\hat{F}(\xi, 0, 0)|$, then the constants M/β and $-M/\beta$ are respectively a supersolution and a subsolution of $(\mathrm{CP})_\beta$. In order to build a solution of $(\mathrm{CP})_\beta$, we use Perron's method. Indeed, setting

$$v^\beta(x) = \sup\{w(y) \mid w \ \text{ a subsolution of } (\mathrm{CP})_\beta,\ |w(y)| \le M/\beta \ \ \forall y \in \mathbf{R}^n\},$$

for $x \in \mathbf{R}^n$, we see that the function v^β is a solution of $(\mathrm{CP})_\beta$ in the sense that $(v^\beta)^*$, the upper semicontinuous envelope of v^β, is a subsolution and $(v^\beta)_*$, the lower semicontinuous envelope of v^β, is a supersolution of $(\mathrm{CP})_\beta$.

Applying Lemma 2.5, we see that $(v^\beta)^* \le (v^\beta)_*$ in $\mathbf{R}^n$, i.e., $v^\beta \in C(\mathbf{R}^n)$, and that v^β is a unique solution of $(\mathrm{CP})_\beta$.

It is obvious from the uniqueness of bounded solutions of $(\mathrm{CP})_\beta$, a consequence of Lemma 2.5, that v^β is periodic with period $\mathbf{Z}^n$, i.e., $v^\beta \in C(\mathbf{R}^n/\mathbf{Z}^n)$.

Since $|v^\beta(y)| \le M/\beta$ by comparison, $\{\beta v^\beta(0) \mid \beta > 0\}$ is bounded in $\mathbf{R}$. We can choose a sequence $0 < \beta_j \to 0$ as $j \to \infty$ such that $\beta_j v^{\beta_j}(0) \to -\mu$, as $j \to \infty$, for some $\mu \in [-M, M]$. Now, by virtue of Lemma 2.2, we see that there exist constants $\alpha \in (0,1)$ and $C > 0$ such that

$$|v^\beta(x) - v^\beta(y)| \le C|x - y|^\alpha \quad \text{ for all } x, y \in \mathbf{R}^n,\ \beta > 0.$$

Hence, the family of functions $v_j : y \mapsto v^{\beta_j}(y) - \min_{\mathbf{R}^n} v^{\beta_j}$, with $j \in \mathbf{N}$, is uniformly bounded and equi-continuous on $\mathbf{R}^n$. In view of the periodicity of v_j, we may hence assume that $v_j(y) \to v(y)$ uniformly on $\mathbf{R}^n$, as $j \to \infty$, for some function $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$, and also that as $j \to \infty$,

$$|\beta_j v^{\beta_j}(y) + \mu| \le C\beta_j |y|^\alpha + |\beta_j v^{\beta_j}(0) + \mu| \to 0.$$

Now, sending $j \to \infty$, we see that v is a solution of

$$\hat{F}(y, Dv(y), D^2 v(y)) = \mu \quad \text{ in } \mathbf{R}^n.$$

Thus, (μ, v) has all the properties required for the proof of (a).

Next, we turn to the proof of (b). Let (μ, v), $(\nu, w) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ be two solutions of cell problem (CP). In order to show that $\mu = \nu$, we suppose that $\mu < \nu$. We may assume by adding a constant to v that $v > w$ in $\mathbf{R}^n$. Then, for sufficiently small $\beta > 0$, we observe that v and w are, respectively, a subsolution and a supersolution of

$(\mathrm{CP})_\beta$, with $\hat F(y,p,X)$ replaced by $\hat F(y,p,X) - \frac{\mu+\nu}{2}$. By Lemma 2.5, we see that $v \le w$ in $\mathbf{R}^n$. This contradiction shows that $\mu = \nu$.

Now, we apply the strong maximum principle (Lemma 2.4) to v, w and conclude that $v(y) = w(y) + C$ for some constant $C \in \mathbf{R}$.

Finally, part (c) is an immediate consequence of Lemma 2.3. QED

By Theorem 2.1 problem (CP) has a solution $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ and the constant μ is uniquely determined . In view of the dependence of μ on $\hat x, \hat u, \hat p$ and $\hat X$, we write

$$\mu = \bar F(\hat x, \hat u, \hat p, \hat X).$$

We call the function $\bar F$ on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ the *homogenized* or *effective* function of F.

Let us observe that under the assumptions of Theorem 2.1 the value $\mu = \bar F(\hat x, \hat u, \hat p, \hat X)$ is characterized by the condition that if $\nu \ge \mu$ then there is a subsolution $w \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of

$$\hat F(y, Dw(y), D^2 w(y)) = \nu \quad \text{in } \mathbf{R}^n \tag{2.9}$$

and if $\nu < \mu$ then there is no subsolution $w \in \mathrm{BUC}(\mathbf{R}^n)$ of (2.9). Indeed, by the choice of μ, there is a solution $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of (2.9) with ν replaced by μ and for $\nu \ge \mu$, v is a subsolution of (2.9). On the other hand, if there were a subsolution $w \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of (2.9) with $\nu < \mu$, then we would have

$$\begin{aligned} \varepsilon w(x) + \hat F(x, Dw(x), D^2 w(x)) &\le \frac{\mu+\nu}{2} \quad \text{in } \mathbf{R}^n, \\ \varepsilon v(x) + \hat F(x, Dv(x), D^2 v(x)) &\ge \frac{\mu+\nu}{2} \quad \text{in } \mathbf{R}^n \end{aligned}$$

for sufficiently small $\varepsilon > 0$. Here and henceforth, these inequalities are understood in the viscosity sense in this context. Then by comparison, we get $w \le v$ in $\mathbf{R}^n$, which implies that $w \le v + C$ in $\mathbf{R}^n$ for any constant $C \in \mathbf{R}$. This contradiction verifies our characterization of $\bar F(\hat x, \hat u, \hat p, \hat X)$.

The above observation, of course, can be stated as

$$\bar F(\hat x, \hat u, \hat p, \hat X) = \min\{\nu \in \mathbf{R} \mid (2.9) \text{ has a subsolution } w \in C(\mathbf{R}^n/\mathbf{Z}^n)\}. \tag{2.10}$$

Similarly, under the assumptions of Theorem 2.1 we have

$$\bar F(\hat x, \hat u, \hat p, \hat X) = \max\{\nu \in \mathbf{R} \mid (2.9) \text{ has a supersolution } w \in C(\mathbf{R}^n/\mathbf{Z}^n)\}. \tag{2.11}$$

The effective function $\bar F$ inherits properties (A1), (A3), (A5), (A6), and (A7). That is, we have:

Proposition 2.6. *Assume that (A1), (A2), (A5), and (A6) hold. Then : (a) For all* $(x, u, p, X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$,

$$\min_{y \in \mathbf{R}^n} F(x, y, u, p, X) \leq \bar{F}(x, u, p, X) \leq \max_{y \in \mathbf{R}^n} F(x, y, u, p, X).$$

(b) If (A4) holds, then for each $R > 0$,

$$\bar{F} \in \mathrm{BUC}\big(\mathbf{R}^n \times [-R, R] \times B(0, R) \times B^{n\times n}(0, R)\big).$$

(c) For $(x, u, p) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ *and* $X, Y \in \mathcal{S}^n$, *if* $Y \geq 0$, *then*

$$\bar{F}(x, u, p, X) - \Theta \operatorname{tr} Y \leq \bar{F}(x, u, p, X + Y) \leq \bar{F}(x, u, p, X) - \theta \operatorname{tr} Y.$$

(d) If (A3) holds then the function: $u \mapsto \bar{F}(x, u, p, X) - \lambda u$ *is non-decreasing in* $\mathbf{R}$ *for any* $(x, p, X) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathcal{S}^n$. *(e) For each* $R > 0$ *there is a modulus* ω_R *such that for* $x, y \in \mathbf{R}^n$ *and* $(u, p, X) \in [-R, R] \times \mathbf{R}^n \times \mathcal{S}^n$,

$$|\bar{F}(x, u, p, X) - \bar{F}(y, u, p, X)| \leq \omega_R\big(|x - y|(1 + |p| + \|X\|)\big).$$

(f) For each $R > 0$ *there is a constant* $C_R > 0$ *such that for* $x, p \in \mathbf{R}^n$ *and* $u \in [-R, R]$,

$$|\bar{F}(x, u, p, 0)| \leq C_R(1 + |p|).$$

(g) If (A7) holds, then

$$\lim_{r\to\infty} \inf\{\bar{F}(x, u, p, 0) \mid (x, u, p) \in \mathbf{R}^n \times [-R, R] \times \mathbf{R}^n,\ |p| \geq r\} = \infty \qquad \text{for } R > 0.$$

The next lemma is useful in the following arguments, which is adapted from [J2, CKSS].

Lemma 2.7. *Assume that (2.2)–(2.5) hold. Let* $u \in C(\mathbf{R}^n)$ *be a bounded solution of (2.1), with* $\lambda = 0$. *Then : (a)* u *is Lipschitz continuous in* $\mathbf{R}^n$. *(b) Let* $R > 0$ *be a constant such that* $\|Du\|_{L^\infty(\mathbf{R}^n)} \leq R$ *and* $M_R > 0$ *a constant such that*

$$|G(x, u, p, 0)| \leq M_R \qquad \text{for } (x, u, p) \in \mathbf{R}^n \times [-R, R] \times B(0, R).$$

Then for each $\varepsilon > 0$ *there are functions* $v^\pm \in C(\mathbf{R}^n) \cap W^{2,\infty}(\mathbf{R}^n)$ *and a constant* $C \equiv C(\varepsilon, n, \omega_R, \theta, \Theta, R, M_R) > 0$, *where* ω_R *is the modulus from (2.5), such that*

$$\|u - v^\pm\|_{L^\infty(\mathbf{R}^n)} < \varepsilon, \qquad \|v^\pm\|_{L^\infty(\mathbf{R}^n)} \leq \|u\|_{L^\infty(\mathbf{R}^n)}, \tag{2.12}$$

$$\|Dv^\pm\|_{L^\infty(\mathbf{R}^n)} \leq \|Du\|_{L^\infty(\mathbf{R}^n)}, \qquad \|v^\pm\|_{W^{2,\infty}(\mathbf{R}^n)} \leq C, \tag{2.13}$$

and

$$G(x, v^+(x), Dv^+(x), D^2v^+(x)) \ge -\varepsilon \quad \text{in } \mathbf{R}^n, \tag{2.14}$$

$$G(x, v^-(x), Dv^-(x), D^2v^-(x)) \le \varepsilon \quad \text{in } \mathbf{R}^n. \tag{2.15}$$

A proof of this lemma can be found in Section 4.

Proof of Proposition 2.6. Let $(x, u, p, X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ and $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be a solution of

$$F(x, y, u, p + Dv(y), X + D^2v(y)) = \bar{F}(x, u, p, X) \quad \text{for } y \in \mathbf{R}^n.$$

Let $y^+, y^- \in \mathbf{R}^n$ be points where v attains its maximum and minimum, respectively. We have

$$F(x, y^+, u, p, X) \le \bar{F}(x, u, p, X) \le F(x, y^-, u, p, X),$$

from which follows assertion (a).

Next, let $(x, u, p, X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ and let $Y \in \mathcal{S}^n$ satisfy $Y \ge 0$. There is a solution $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of

$$F(x, y, u, p + Dv(y), X + Y + D^2v(y)) = \bar{F}(x, u, p, X + Y) \quad \text{for } y \in \mathbf{R}^n.$$

Using (A1), we see that v is a subsolution of

$$F(x, y, u, p + Dv(y), X + D^2v(y)) = \bar{F}(x, u, p, X + Y) + \Theta \operatorname{tr} Y \quad \text{for } y \in \mathbf{R}^n,$$

and a supersolution of

$$F(x, y, u, p + Dv(y), X + D^2v(y)) = \bar{F}(x, u, p, X + Y) + \theta \operatorname{tr} Y \quad \text{for } y \in \mathbf{R}^n.$$

Accordingly, by (2.10) and (2.11), we deduce that

$$\bar{F}(x, u, p, X + Y) + \theta \operatorname{tr} Y \le \bar{F}(x, u, p, X) \le \bar{F}(x, u, p, X + Y) + \Theta \operatorname{tr} Y,$$

which completes the proof of (c).

Fix $R > 0$ and $\varepsilon > 0$. Fix $(x, u, p, X) \in \mathbf{R}^n \times [-R, R] \times B(0, R) \times B^{n\times}(0, R)$, and choose a solution $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of

$$F(x, \xi, p + Dv(\xi), X + D^2v(\xi)) = \bar{F}(x, u, p, X) \quad \text{for } \xi \in \mathbf{R}^n.$$

Using (c), (a), and (A6), we see that if we set $G(\xi, q, Y) = F(x, \xi, p + q, X + Y) - \bar{F}(x, u, p, X)$, then

$$\begin{aligned} |G(\xi, q, 0)| &\le |F(x, \xi, u, p + q, 0)| + |\bar{F}(x, u, p, 0)| + 2n\Theta\|X\| \\ &\le 2C_R(1 + |p| + |q|) + 2n\Theta R \le \big(2C_R(1 + R) + 2n\Theta R\big)(1 + |q|). \end{aligned}$$

By Lemmas 2.2 and 2.3 we find a constant $C_1 > 0$ depending only on n, θ, Θ, R such that $\|Dv\|_{L^\infty(\mathbf{R}^n)} \le C_1$. Next by Lemma 2.7 we see that there are a function $w \in \mathrm{BUC}(\mathbf{R}^n)$ and a constant $C_2 > 0$ depending only on $\varepsilon, n, \theta, \Theta, R$ such that w is a subsolution of

$$F(x, \xi, p + Dv(\xi), X + D^2 v(\xi)) = \bar{F}(x, u, p, X) + \varepsilon \quad \text{for } \xi \in \mathbf{R}^n,$$

and such that

$$\|w\|_{L^\infty(\mathbf{R}^n)} \le \|v\|_{L^\infty(\mathbf{R}^n)}, \qquad \|Dw\|_{L^\infty(\mathbf{R}^n)} \le \|Dw\|_{L^\infty(\mathbf{R}^n)}, \qquad \|D^2 w\|_{L^\infty(\mathbf{R}^n)} \le C_2.$$

Let $(y, t, q, Y) \in \mathbf{R}^n \times [-R, R] \times B(0, R) \times B^{n\times n}(0, R)$. By assumption (A4), there is a constant $\delta \in (0, 1)$ depending only on R, C_1, C_2 such that if $|x - y| + |u - t| + |p - q| + \|X - Y\| < \delta$ and $(r, Z) \in B(0, C_1) \times B^{n\times n}(0, C_2)$, then

$$|F(x, \xi, u, p + r, X + Z) - F(y, \xi, t, q + r, Y + Z)| < \varepsilon.$$

Accordingly, we have

$$F(y, \xi, t, q + Dw(\xi), Y + D^2 w(\xi)) \le \bar{F}(x, u, p, X) + 2\varepsilon \quad \text{for } \xi \in \mathbf{R}^n.$$

In view of (2.10) this shows that

$$\bar{F}(y, t, q, Y) \le \bar{F}(x, u, p, X) + 2\varepsilon.$$

By symmetry, we have

$$\bar{F}(x, u, p, X) \le \bar{F}(y, t, q, Y) + 2\varepsilon,$$

and hence, $|\bar{F}(x, u, p, X) - \bar{F}(y, t, q, Y)| < 2\varepsilon$. This proves the required uniform continuity of $\bar{F}$. The boundedness of $\bar{F}$ on $\mathbf{R}^n \times \mathbf{R}^n \times [-R, R] \times B(0, R) \times B^{n\times n}(0, R)$ for each $R > 0$ follows immediately from (a).

Now, we turn to (d) and assume that (A3) holds. Let $(x, u, p, X) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ and $r \ge 0$. Let $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be the solution of

$$F(x, y, u + r, p + Dv(y), X + D^2 v(y)) = \bar{F}(x, u + r, p, X) \quad \text{for } y \in \mathbf{R}^n.$$

Using (A3), we infer that v is a subsolution of

$$F(x, y, u, p + Dv(y), X + D^2 v(y)) + \lambda r \le \bar{F}(x, u + r, p, X) \quad \text{for } y \in \mathbf{R}^n,$$

and moreover that $\bar{F}(x, u, p, X) \le \bar{F}(x, u + r, p, X) - \lambda r$, which was to be shown.

Next we prove (e). Fix $R > 0$ and let ω_R be the modulus from (A5). Let $(u,p,X) \in [-R,R] \times \mathbf{R}^n \times \mathcal{S}^n$ and $x, y \in \mathbf{R}^n$, and let $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be a solution of

$$F(x,\xi,u,p+Dv(\xi),X+D^2v(\xi)) = \bar{F}(x,u,p,X) \quad \text{for } \xi \in \mathbf{R}^n.$$

Fix $\varepsilon \in (0,1)$. Since $v \in W^{1,\infty}(\mathbf{R}^n)$ by (c) of Theorem 2.1, by virtue of Lemma 2.7 there are a function $w \in C(\mathbf{R}^n/\mathbf{Z}^n) \cap W^{2,\infty}(\mathbf{R}^n)$ and a constant $C \equiv C(\varepsilon) > 0$ such that

$$\|v-w\|_{L^\infty(\mathbf{R}^n)} < \varepsilon, \qquad \|Dw\|_{L^\infty(\mathbf{R}^n)} \le \|Dv\|_{L^\infty(\mathbf{R}^n)}, \qquad \|D^2w\|_{L^\infty(\mathbf{R}^n)} \le C,$$

and w satisfies

$$F(x,\xi,u,p+Dw(\xi),X+D^2w(\xi)) \le \bar{F}(x,u,p,X)+\varepsilon \quad \text{for } \xi \in \mathbf{R}^n.$$

(Regarding the periodicity of w, consult the proof of Lemma 2.7.) In view of Lemmas 2.2 and 2.3, there is a constant $C_1 > 0$ which depends only on n, θ, Θ, C_R, ω_R, where C_R is the constant from (A6), such that $\|Dv\|_{L^\infty(\mathbf{R}^n)} \le C_1$.

Hence, by (A5), we see that w satisfies

$$\begin{aligned} &F(y,\xi,u,p+Dw(\xi),X+D^2w(\xi)) \\ &\qquad \le \bar{F}(x,u,p,X)+\varepsilon+\omega_R\big(|x-y|(1+C_1+C+|p|+\|X\|)\big) \quad \text{for } \xi \in \mathbf{R}^n, \end{aligned}$$

which guarantees that

$$\bar{F}(y,u,p,X) \le \bar{F}(x,u,p,X)+\varepsilon+\omega_R\big(|x-y|(1+C_1+C+|p|+\|X\|)\big).$$

By symmetry, we conclude that

$$|\bar{F}(y,u,p,X)-\bar{F}(x,u,p,X)| \le \varepsilon+\omega_R\big(|x-y|(1+C_1+C+|p|+\|X\|)\big).$$

Define the function $\sigma_R : [0,\infty) \to [0,\infty)$ by

$$\sigma_R(r) = \inf\{\varepsilon+\omega_R\big((1+C_1+C(\varepsilon))r\big) \mid \varepsilon \in (0,1)\},$$

which is upper semicontinuous and non-negative in $[0,\infty)$ and satisfies $\sigma_R(0)=0$. We have

$$|\bar{F}(y,u,p,X)-\bar{F}(x,u,p,X)| \le \sigma_R\big(|x-y|(1+|p|+\|X\|)\big).$$

Furthermore we may assume that $\sigma_R \in C([0,\infty))$; otherwise we may replace σ_R by a continuous function in $[0,\infty)$. We finish the proof of (e).

Finally we prove (f) and (g). Fix $R > 0$ and $(x, u, p) \in \mathbf{R}^n \times [-R, R] \times \mathbf{R}^n$. By (a) we have

$$\min_{y \in \mathbf{R}^n} F(x, y, u, p, 0) \leq \bar{F}(x, u, p, 0) \leq \max_{y \in \mathbf{R}^n} F(x, y, u, p, 0).$$

(A6) and (A7), respectively, yield

$$|\bar{F}(x, u, p, 0)| \leq C_R(1 + |p|),$$

and

$$\lim_{r \to \infty} \inf\{\bar{F}(x, u, p, 0) \ |(x, p) \in \mathbf{R}^{2n},\ |p| \geq r\}$$
$$\geq \lim_{r \to \infty} \inf\{F(x, y, u, p, 0) \mid (x, y, p) \in \mathbf{R}^{3n},\ |p| \geq r\} = \infty.$$

Thus the proof is complete. QED

Now, fix $(\hat{x}, \hat{u}, \hat{p}, \hat{X}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ and consider cell problems of finding a pair $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ such that v is a solution, respectively, of

$$\text{(CP)}_2 \qquad F(\hat{x}, y, \hat{u}, \hat{p}, \hat{X} + D^2 v(y)) = \mu \quad \text{for } y \in \mathbf{R}^n,$$

and

$$\text{(CP)}_{12} \qquad F(\hat{x}, y, \hat{u}, \hat{p} + Dv(y), D^2 v(y)) = \mu \quad \text{for } y \in \mathbf{R}^n.$$

These problems are special cases of (CP). (CP)_{12} does not depend on $\hat{X}$, and it is exactly problem (CP) with $\hat{X} = 0$. We denote the homogenized function associated with (CP)_{12} by $\bar{F}_{12}$, which is a function of $(\hat{x}, \hat{u}, \hat{p})$. We obviously have

$$\bar{F}_{12}(\hat{x}, \hat{u}, \hat{p}) = \bar{F}(\hat{x}, \hat{u}, \hat{p}, 0). \tag{2.16}$$

Let us define the function $F(\cdot; \hat{p})$ on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ by

$$F(x, y, u, p, X; \hat{p}) = F(x, y, u, \hat{p}, X),$$

which is a function independent of p. Then (CP) with $F(\cdot; \hat{p})$ in place of F is problem (CP)_2. Let $\bar{F}(\cdot; \hat{p})$ and $\bar{F}_{12}$ denote the homogenized functions associated with (CP) having $F(\cdot; \hat{p})$ and with (CP)_{12}, respectively. It is clear that for all $q \in \mathbf{R}^n$,

$$\bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) = \bar{F}(\hat{x}, \hat{u}, q, \hat{X}; \hat{p}). \tag{2.17}$$

Next, we consider the cell problem

$$\text{(CP)}_1 \qquad F(\hat{x}, y, \hat{u}, \hat{p} + Dv(y), 0) = \mu \quad \text{for } y \in \mathbf{R}^n,$$

where the unknown is a pair $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ and $(\hat{x}, \hat{u}, \hat{p}) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ is an arbitrary point. This is a first-order PDE for the function v and so does not have the uniform ellipticity unlike previous cell problems.

Theorem 2.8. *Assume that (A2)–(A4) and (A7) hold. Then: (a) There exists a solution* $(\mu, v) \in C(\mathbf{R}^n/\mathbf{Z}^n)$ *of* $(CP)_1$. *(b) If* (μ, v), $(\nu, w) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ *are solutions of cell problem* $(CP)_1$, *then* $\mu = \nu$. *(c) If* $(\mu, v) \in \mathbf{R} \times C(\mathbf{R}^n/\mathbf{Z}^n)$ *is a solution of* $(CP)_1$, *then* v *is Lipschitz continuous in* $\mathbf{R}^n$.

It is clear that in Theorem 2.8, the requirement of (A2) to $F(x, y, u, p, X)$ is only needed for $X = 0$.

We do not give the proof of Theorem 2.8 and instead refer to [E2], where a result [E2, Lemma 2.1] similar to ours is proved under a bit stronger assumptions. Consult e.g. [CIL2] for technicalities in the current generality.

In view of Theorem 2.8, we can define the homogenized function $\bar{F}_1$ associated with cell problem $(\mathrm{CP})_1$, which is a function on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. That is, $\bar{F}(\hat{x}, \hat{u}, \hat{p})$ is defined as the value μ for which there is a solution $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ of $(\mathrm{CP})_1$.

Proportion 2.9. *Under the hypotheses of Theorem 2.8, we have: (a) For all* $(x, u, p) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$,

$$\min_{y \in \mathbf{R}^n} F(x, y, u, p) \le \bar{F}_1(x, u, p) \le \max_{y \in \mathbf{R}^n} F(x, y, u, p).$$

(b) For all $R > 0$, $\bar{F}_1 \in C(\mathbf{R}^n \times [-R, R] \times B(0, R))$. *(c) If (A3) holds, then the function* $u \mapsto \bar{F}_1(x, u, p) - \lambda u$ *is non-decreasing in* $\mathbf{R}$, *where* λ *is the constant from (A3). (d) For any* $R > 0$, $\lim_{r \to \infty} \inf\{\bar{F}_1(x, u, p) \mid (x, u, p) \in \mathbf{R}^n \times [-R, R] \times \mathbf{R}^n,\ |p| \ge r\} = \infty$.

Outline of proof. Most of arguments of the proofs of (a), (b), (d), and (g) of Proposition 2.6 apply to show assertion (a), (b), (c), and (d), respectively, with obvious modifications. We do not need (and can not use either) Lemmas 2.2, 2.3, and 2.7 here. In this respect the following observation is useful. Let $R > 0$, $(x, u, p) \in \mathbf{R}^n \times [-R, R] \times B(0, R)$, and $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be a solution of

$$F(x, y, u, p + Dv(y), 0) = \bar{F}_1(x, u, p) \quad \text{for } y \in \mathbf{R}^n. \tag{2.18}$$

Noting that

$$|\bar{F}_1(x, u, p)| \le C_R := \sup\{|F(\xi, y, t, q, 0)| \mid (\xi, y, t, q) \in \mathbf{R}^{2n} \times [-R, R] \times B(0, R)\},$$

in view of (A7) we may choose a constant $L > 0$ so that for all $(\xi, y, t, q) \in \mathbf{R}^n \times \mathbf{R}^n \times [-R, R] \times \mathbf{R}^n$, if $|q| > L$, then

$$F(\xi, y, t, q, 0) > C_R.$$

Now (2.18) implies that v is a subsolution of

$$|Dv(y)| \leq L + R \qquad \text{in } \mathbf{R}^n,$$

which gives the Lipschitz bound $\|Dv\|_{L^\infty(\mathbf{R}^n)} \leq L + R$. The bound L can be chosen so that it depends on x, u, p only through $R > 0$. QED

3 Convergence results

We show in this section that, as $\varepsilon \to 0$, the solutions u^ε of $(\mathrm{P})_\varepsilon$ converge to the solution u of one of the homogenized equations

$$\bar{F}_2(x, u, Du(x), D^2u(x)) = 0 \quad \text{in } \mathbf{R}^n, \tag{3.1}$$

$$\bar{F}_2(x, u, Du(x), 0) = 0 \quad \text{in } \mathbf{R}^n, \tag{3.2}$$

$$\bar{F}_{12}(x, u, Du(x)) = 0 \quad \text{in } \mathbf{R}^n, \tag{3.3}$$

and

$$\bar{F}_1(x, u, Du(x)) = 0 \quad \text{in } \mathbf{R}^n. \tag{3.4}$$

We begin with the existence and uniqueness theorem for $(\mathrm{P})_\varepsilon$.

Theorem 3.1. *Let $\varepsilon > 0$. Assume that (A1) and (A3)–(A5) hold. Then there exists a unique bounded solution u^ε of $(P)_\varepsilon$ and u^ε is Lipschitz continuous in $\mathbf{R}^n$.*

Outline of proof. The first step is to establish the comparison principle between bounded upper semicontinuous subsolutions and bounded lower semicontinuous supersolutions of $(\mathrm{P})_\varepsilon$. To this end, one may follow the proof of [IL, Theorem III.1] with a minor and standard modification which takes care of the non-compactness of the domain $\mathbf{R}^n$. (See also the proof of Lemma 2.5 in Section 4.)

The second step is to use the Perron procedure and to establish the existence of a bounded solution of $(\mathrm{P})_\varepsilon$. By (A3) and (A4) the constants

$$\pm \sup_{x,y \in \mathbf{R}^n} |F(x, y, 0, 0, 0)|/\lambda$$

are respectively super- and subsolutions of $(\mathrm{P})_\varepsilon$. Thus the function

$$u^\varepsilon(x) = \sup\{v(x) \mid v \text{ a subsolution of } (P)_\varepsilon,\ |v(y)| \leq M/\lambda \ \text{ for } y \in \mathbf{R}^n\},$$

where $M = \sup_{x,y\in\mathbf{R}^n} |F(x, y, 0, 0, 0)|$, is a solution of $(\mathrm{P})_\varepsilon$. The comparison principle guarantees that u^ε is continuous in $\mathbf{R}^n$. Lemma 2.3 gives a Lipschitz bound for u^ε. QED

We next state comparison and existence results for (3.1), (3.2), (3.3), and (3.4) in the following theorems.

Theorem 3.2. *Assume that (A1)–(A6). Then: (a) If $u \in \mathrm{USC}(\mathbf{R}^n)$ and $v \in \mathrm{LSC}(\mathbf{R}^n)$ are bounded sub- and supersolutions of (3.1), respectively, then $u \le v$ in $\mathbf{R}^n$. (b) There exists a solution $u \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ of (3.1).*

Remark. Due to Theorem 2.1, under the hypotheses of this or the next theorems the respective homogenized functions $\bar{F}_2$, $\bar{F}_2(\cdot, 0)$, and $\bar{F}_{12}$ are well-defined.

Proof. The function $\bar{F}_2$ has all the properties described in (b)–(f) of Proportion 2.6. The proof of (a) is similar to that of Lemma 2.5, which we leave to the reader. One can prove (b) as in the same way as the proof of Theorem 3.1. Also, (b) is a consequence of the proof of Theorem 3.5 below. QED

Theorem 3.3. *Assume that (A1)–(A7) hold. Then the conclusions of Theorem 3.2 hold both for (3.2) and for (3.3).*

Proof. The function $\bar{F}_2(\cdot, 0)$ and $\bar{F}_{12}$ have all the properties described in (b), (d)–(g) of Proposition 2.6. The comparison and existence assertions are a consequence of classical results (see e.g. [CIL1]). The Lipschitz regularity of the solutions of (3.2) or (3.3) are a classical result as well (see, e.g., the outline of proof of Proposition 2.9). QED

Theorem 3.4. *Assume that (A2)–(A5) and (A7) hold. Then the conclusions of Theorem 3.2 hold for (3.4).*

Remark. By Theorem 2.8, under the hypotheses of the above theorem the homogenized functions $\bar{F}_1$ is well-defined.

Proof. The function $\bar{F}_1$ has all the properties described in (b)–(d) of Proposition 2.9. As before, the comparison and existence assertions and the Lipschitz regularity property are consequences of classical results. QED

We state our results on convergence of solutions of $(\mathrm{P})_\varepsilon$ according to the limit equations.

Theorem 3.5. *Assume that $\lim_{\varepsilon \searrow 0} \delta(\varepsilon) = 1$ and that (A1)-(A6) hold. Let $u^\varepsilon \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ be the solution of $(P)_\varepsilon$ for each $\varepsilon > 0$ and $u \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ the solution of (3.1). Then $u^\varepsilon(x) \to u(x)$ uniformly on $\mathbf{R}^n$ as $\varepsilon \to 0$.*

We remark that, by Theorems 3.1 and 3.2, u^ε and u of the above theorem exist uniquely.

The assertion of Theorem 3.5 is close to and a bit stronger than that of [E2, Theorem 3.3]. Indeed, our result holds under slightly weaker assumptions on F.

We use the following lemma in the proof of Theorem 3.5.

Lemma 3.6. *Let Ω be an open subset of $\mathbf{R}^n$, $u \in \mathrm{USC}(\Omega)$, and $v \in C(\Omega) \cap W^{2,\infty}(\Omega)$. Let $\hat{x} \in \Omega$ and $(p, X) \in J^{2,+}(u - v)(\hat{x})$. Then there is a $Y \in \mathcal{S}^n$ such that*

$$(Dv(\hat{x}), Y) \in \overline{J}^2 v(\hat{x}),$$
$$(p + Dv(\hat{x}), X + Y) \in J^{2,+}u(\hat{x}),$$

where $\overline{J}^2 v(x)$ denotes the set of those points $(q, Y) \in \mathbf{R}^n \times \mathcal{S}^n$ for which there is a sequence $x_j \to x$ such that v is twice differentiable at x_j, i.e., it has the Taylor expansion at x_j up to second order terms, and $(Dv(x_j), D^2 v(x_j)) \to (q, Y)$.

See [CIL2] for the definition of $\overline{J}^{2,+}u$, etc. A proof will be given in Section 4.

Proof of Theorem 3.5. We only prove the local uniform convergence of u^ε to u. We just refer the reader to [HI] for an argument how to improve to the global uniform convergence.

Setting $M = \sup_{x,y \in \mathbf{R}^n} |F(x, y, 0, 0, 0)|$, in view of the construction of u^ε in the proof of Theorem 3.1, we see that $\|u^\varepsilon\|_{L^\infty(\mathbf{R}^n)} \le M/\lambda$ for all $\varepsilon > 0$.

We define functions $\overline{u}$ and $\underline{u}$ on $\mathbf{R}^n$ by

$$\overline{u}(x) = \lim_{r \searrow 0} \sup\{u^\varepsilon(y) \mid y \in \mathbf{R}^n,\ |y - x| \le r\},$$

$$\underline{u}(x) = \lim_{r \searrow 0} \inf\{u^\varepsilon(y) \mid y \in \mathbf{R}^n,\ |y - x| \le r\},$$

and will show that $\overline{u}$ and $\underline{u}$ are a subsolution and a supersolution of (3.1), respectively. Once this is done, we conclude by Theorem 3.2 that $\overline{u} \le u \le \underline{u}$ in $\mathbf{R}^n$ and moreover that $u^\varepsilon(x) \to u(x)$ locally uniformly in $\mathbf{R}^n$ as $\varepsilon \to 0$, which was to be shown.

We show that $\overline{u}$ is a subsolution of (3.1) and omit the proof of the assertion that $\underline{u}$ is a supersolution of (3.1), since the proofs of these facts are symmetric.

Fix $\varphi \in C^2(\mathbf{R}^n)$ and $\hat{x} \in \mathbf{R}^n$ so that $\overline{u} - \varphi$ has a strict maximum at $\hat{x}$. Let $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be a solution of

$$F(\hat{x}, y, \hat{u}, \hat{p}, \hat{X} + D^2 v(y)) = \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) \quad \text{for } y \in \mathbf{R}^n,$$

where $\hat{u} := \overline{u}(\hat{x})$, $\hat{p} := D\varphi(\hat{x})$, and $\hat{X} := D^2 v(\hat{x})$. (Theorem 2.1 guarantees the existence of such a v.)

Fix $\gamma > 0$ and, in view of Lemma 2.7, choose a function $w \in C(\mathbf{R}^n/\mathbf{Z}^n) \cap W^{2,\infty}(\mathbf{R}^n)$ so that w is a supersolution of

$$F(\hat{x}, y, \hat{u}, \hat{p}, \hat{X} + D^2 w(y)) = \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) - \gamma \quad \text{for } y \in \mathbf{R}^n.$$

By the definition of $\overline{u}$, there are sequences $0 < \varepsilon_j \to 0$ and $x_j \to \hat{x}$ such that for each j, the function $u^\varepsilon(x) - \varphi(x) - \varepsilon^2 w(x/\varepsilon)$, with $\varepsilon = \varepsilon_j$, has a local maximum at x_j and $u^\varepsilon(x_j) \to \hat{u}$ as $\varepsilon = \varepsilon_j \to 0$.

Assume for the moment that $w \in C^2(\mathbf{R}^n)$, which is not true in general. Then we have

$$F(x_j, x_j/\varepsilon, u^\varepsilon(x_j), D\varphi(x_j) + \varepsilon Dw(x_j/\varepsilon), \delta D^2\varphi(x_j) + \delta D^2 w(x_j/\varepsilon)) \le 0, \tag{3.5}$$

$$F(\hat{x}, x_j/\varepsilon, \hat{u}, \hat{p}, \hat{X} + D^2 w(x_j/\varepsilon)) \ge \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) - \gamma, \tag{3.6}$$

and, sending $j \to \infty$ and using the periodicity of F and w, we obtain

$$0 \ge F(\hat{x}, \xi, \hat{u}, \hat{p}, \hat{X} + D^2 w(\xi)) \ge \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) - \gamma$$

for some $\xi \in \mathbf{R}^n$. Because of the arbitrariness of $\gamma > 0$, we conclude that

$$\bar{F}_2(\hat{x}, \hat{u}, \hat{p}, \hat{X}) \le 0. \tag{3.7}$$

In the general case we use Lemma 3.6, to find an $X_j \in \mathcal{S}^n$ for each j such that

$$(Dw(x_j/\varepsilon), X_j) \in \overline{J}^2 w(x_j/\varepsilon),$$
$$(D\varphi(x_j) + \varepsilon Dw(x_j/\varepsilon), D^2\varphi(x_j/\varepsilon) + X_j) \in J^{2,+}u^\varepsilon(x_j).$$

Then we obtain inequalities (3.5) and (3.6) with X_j in place of $D^2 w(x_j/\varepsilon)$ and then proceed exactly as above, to conclude (3.7). The proof is now complete. QED

Theorem 3.7. *Assume that* $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ *and* $\lim_{\varepsilon\to 0} \delta(\varepsilon)/\varepsilon = \infty$ *and that (A1)–(A7) hold. Let* u^ε, $u \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ *be the solutions of* $(P)_\varepsilon$ *and of (3.2), respectively. Then* $u^\varepsilon(x) \to u(x)$ *uniformly on* $\mathbf{R}^n$ *as* $\varepsilon \to 0$.

We remark that u^ε and u in the above theorem exist and are unique by Theorems 3.1 and 3.3.

The proof below is similar to the previous one, and so we give just its outline.

Outline of proof. Again we only prove the local uniform convergence of u^ε to u.

We define $\overline{u}$ and $\underline{u}$ as in the previous proof with current $\{u^\varepsilon\}_{\varepsilon>0}$. It is enough to show that $\overline{u}$ and $\underline{u}$ are sub- and supersolutions of (3.2), respectively.

Again, we only prove that $\overline{u}$ is a subsolution of (3.2). Fix $\varphi \in C^2(\mathbf{R}^n)$ and $\hat{x} \in \mathbf{R}^n$ so that $\overline{u} - \varphi$ attains a strict maximum at $\hat{x}$. Let $v \in C(\mathbf{R}^n/\mathbf{Z}^n)$ be a solution of

$$F(\hat{x}, y, \hat{u}, \hat{p}, D^2 v(y)) = \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, 0) \quad \text{for } y \in \mathbf{R}^n,$$

where $\hat{u} := \overline{u}(\hat{x})$ and $\hat{p} := D\varphi(\hat{x})$. Let $\gamma > 0$ be an arbitrary number and choose $w \in W^{2,\infty}(\mathbf{R}^n) \cap C(\mathbf{R}^n/\mathbf{Z}^n)$ so that w is a supersolution of

$$F(\hat{x}, y, \hat{u}, \hat{p}, D^2 w(y)) = \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, 0) - \gamma \quad \text{for } y \in \mathbf{R}^n.$$

We can find sequences $0 < \varepsilon_j \to 0$ and $x_j \to \hat{x}$, as $j \to \infty$, such that for each $\varepsilon = \varepsilon_j$ the function $u^\varepsilon - \varphi - \varepsilon^2\delta^{-1}w(x/\varepsilon)$ attains a maximum at x_j and $u^\varepsilon(x_j) \to \hat{u}$, with $\varepsilon = \varepsilon_j$, as $j \to \infty$. Proceeding as in the previous proof, we first get

$$F(x_j, x_j/\varepsilon, u^\varepsilon(x_j), D\varphi(x_j) + \varepsilon\delta^{-1}p_j, \delta D^2\varphi(x_j) + X_j) \leq 0,$$
$$F(\hat{x}, x_j/\varepsilon, \hat{u}, \hat{p} + \varepsilon p_j, X_j) \geq \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, 0) - \gamma$$

for some $(p_j, X_j) \in \overline{J}^2 w(x_j/\varepsilon)$ and for $\varepsilon = \varepsilon_j$ and all j, and then, passing to the limit as $j \to \infty$,

$$0 \geq F(\hat{x}, \xi, \hat{u}, \hat{p}, X) \geq \bar{F}_2(\hat{x}, \hat{u}, \hat{p}, 0) - \gamma$$

for some $(\xi, X) \in \mathbf{R}^n \times \mathcal{S}^n$. This shows that $\bar{F}_2(\hat{x}, \hat{u}, \hat{p}, 0) \leq 0$, which completes the proof. QED

Theorem 3.8. *Assume that (A1)–(A7) hold and that* $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = 1$. *Then the solution* $u^\varepsilon \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ *converges to the solution* $u \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ *of (3.3) uniformly on* $\mathbf{R}^n$ *as* $\varepsilon \to 0$.

Again this theorem is close to [E2, Theorem 4.4]. Our assumptions are slightly weaker than those of [E2, Theorem 4.4].

The proof below is similar to those for Theorems 3.5 and 3.7.

Outline of proof. We follow the proof of the previous two theorems. To see that the function $\overline{u}$, which is defined as in the previous proof, is a subsolution of (3.3), we fix $\varphi \in C^2(\mathbf{R}^n)$ and $\hat{x} \in \mathbf{R}^n$ so that $\overline{u} - \varphi$ attains a strict maximum at $\hat{x}$. Then we choose a function $w \in C(\mathbf{R}^n) \cap W^{2,\infty}(\mathbf{R}^n)$ so that

$$F(\hat{x}, y, \hat{u}, \hat{p} + Dw(y), D^2w(y)) = \bar{F}_{12}(\hat{x}, \hat{u}, \hat{p}) - \gamma \quad \text{for } y \in \mathbf{R}^n,$$

where $\gamma > 0$ is an arbitrarily fixed number and $\hat{u}, \hat{p}$ are defined as in the same fashion as before.

For $\varepsilon > 0$ we consider the function $u^\varepsilon(x) - \varphi(x) - \varepsilon^2\delta^{-1}w(x/\varepsilon)$ and its maximum point x_ε. We then get

$$F(x_\varepsilon, x_\varepsilon/\varepsilon, u^\varepsilon(x_\varepsilon), D\varphi(x_\varepsilon) + \varepsilon\delta^{-1}p_\varepsilon, \delta D^2\varphi(x_\varepsilon) + X_\varepsilon) \leq 0,$$
$$F(\hat{x}, x_\varepsilon/\varepsilon, \hat{u}, \hat{p} + p_\varepsilon, X_\varepsilon) \geq \bar{F}_{12}(\hat{x}, \hat{u}, \hat{p}) - \gamma,$$

where $(p_\varepsilon, X_\varepsilon) \in \overline{J}^2 w(x_\varepsilon/\varepsilon)$. The rest of the arguments are the same as before. QED

Theorem 3.9. *Assume that (A1)–(A5) and (A7) hold and that* $\lim_{\varepsilon\to 0}\delta(\varepsilon)/\varepsilon = 0$. *Then the solution* $u^\varepsilon \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ *of* $(P)_\varepsilon$ *converges to the solution* $u \in C(\mathbf{R}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ *of (3.4) uniformly on* $\mathbf{R}^n$ *as* $\varepsilon \to 0$.

Remark that the existence of u^ε and u in the above theorem is assured by Theorems 3.1 and 3.4.

Outline of proof. We argue as before and then come up to the situation that we have a function $\varphi \in C^2(\mathbf{R}^n)$ and a point $\hat{x} \in \mathbf{R}^n$ such that $\overline{u} - \varphi$ attains a strict maximum at $\hat{x}$, where $\overline{u}$ is defined as usual.

The next step is to fix a solution $v \in C(\mathbf{R}^n/\mathbf{Z}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ of

$$F(\hat{x}, y, \hat{u}, \hat{p} + Dv(y), 0) \geq \bar{F}_{12}(\hat{x}, \hat{u}, \hat{p}) \quad \text{for } y \in \mathbf{R}^n,$$

where $\hat{u} := \overline{u}(\hat{x}),\ \hat{p} := D\varphi(\hat{x})$.

Since this equation is not uniformly elliptic, we cannot use Lemmas 2.7 and 3.6 in this proof. Instead, we utilize inf-convolutions of v to find a function $w \in C(\mathbf{R}^n/\mathbf{Z}^n) \cap W^{1,\infty}(\mathbf{R}^n)$ which is a supersolution of

$$F(\hat{x}, y, \hat{u}, \hat{p} + Dw(y), 0) \geq \bar{F}_{12}(\hat{x}, \hat{u}, \hat{p}) - \gamma \quad \text{for } y \in \mathbf{R}^n,$$

where $\gamma > 0$ is an arbitrarily fixed number, and which is semiconcave in $\mathbf{R}^n$. Choose a constant $L > 0$ so that the function $w(x) - (L/2)|x|^2$ is concave in $\mathbf{R}^n$. Note that if $p \in D^+w(x)$, then $(p, LI) \in J^{2,+}w(x)$ and that $D^+w(x) \neq \emptyset$ for all $x \in \mathbf{R}^n$. Note as well that, since w is almost everywhere differentiable, $\overline{D}w(x) \neq \emptyset$, where $\overline{D}w(x)$ denotes the set of $p \in \mathbf{R}^n$ for which there is a sequence $y_j \to x$ such that w is differentiable at y_j and $Dw(y_j) \to p$ as $j \to \infty$, and that $\overline{D}w(x) \subset D^+w(x)$ for all $x \in \mathbf{R}^n$.

Fix sequences $0 < \varepsilon_j \to 0$ and $\mathbf{R}^n \ni x_j \to \hat{x}$ so that for each j, the function $u^{\varepsilon_j}(x) - \varphi(x) - \varepsilon_j w(x/\varepsilon_j)$ has a local maximum at x_j and $u^{\varepsilon_j}(x_j) \to \hat{u}$ as $j \to \infty$. Fix j and write $\varepsilon = \varepsilon_j$. Fix $p_j \in \overline{D}w(x_j/\varepsilon)$ and observe that

$$(D\varphi(x_j) + p_j, D^2\varphi(x_j) + \varepsilon^{-1}LI) \in J^{2,+}u^\varepsilon(x_j).$$

We then get

$$F\big(x_j, x_j/\varepsilon, u^\varepsilon(x_j), D\varphi(x_j) + p_j, \delta(D^2\varphi(x_j) + \varepsilon^{-1}LI)\big) \leq 0,$$
$$F(\hat{x}, x_j/\varepsilon, \hat{u}, \hat{p} + p_j, 0) \geq \bar{F}_1(\hat{x}, \hat{u}, \hat{p}) - \gamma.$$

Sending $j \to \infty$, we get

$$0 \geq F\big(\hat{x}, y, \hat{u}, \hat{p} + p, 0\big) \geq \bar{F}_1(\hat{x}, \hat{u}, \hat{p}) - \gamma$$

for some $y, p \in \mathbf{R}^n$, and conclude that $\overline{u}$ is a subsolution of (3.4). QED

4 Proof of lemmas

Proof of Lemma 2.5. Let $\varepsilon \in (0,1)$ and $\alpha > 0$, and set

$$\Phi(x,y) = u(y) - v(y) - \frac{\alpha}{2}|x-y|^2 - \frac{\varepsilon}{2}|x|^2 \quad \text{for } x, y \in \mathbf{R}^n.$$

This function clearly attains a maximum at a point $(\bar{x}, \bar{y}) \in \mathbf{R}^{2n}$.

We argue by contradiction and, in order to get a contradiction, we suppose that $M := \sup_{\mathbf{R}^n}(u-v) > 0$.

We choose a sequence $\{x_j\} \subset \mathbf{R}^n$ so that as $j \to \infty$,

$$(u-v)(x_j) \to M.$$

We may assume by replacing $\{x_j\}$ by a subsequence that the sequence of functions $G(x+x_j, u, p, X)$ on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}^n$ is locally uniformly convergent to some function $\bar{G}$. Define $\overline{u}$ and $\underline{u}$ by

$$\begin{aligned} \overline{u}(x) &= \lim_{r \searrow 0} \sup\{u(y+x_j) \mid y \in B(x_j, r),\ j > 1/r\}, \\ \underline{v}(x) &= \lim_{r \searrow 0} \inf\{v(y+x_j) \mid y \in B(x_j, r),\ j > 1/r\}. \end{aligned}$$

Replacing G, u, and v by $\bar{G}$, $\overline{u}$, and $\underline{v}$, respectively, we may assume that $u - v$ attains its maximum at the origin.

Then we follow the proof of [IL, Theorem III.1] with minor modifications. The first step is to observe that there are $X_{\alpha,\varepsilon}, Y_{\alpha,\varepsilon} \in \mathcal{S}^n$ such that

$$\begin{gathered} \lambda u(\bar{x}) + G(\bar{x}, u(\bar{x}), \alpha(\bar{x}-\bar{y}) + \varepsilon \bar{x}, X_{\alpha,\varepsilon} + \varepsilon I) \le 0, \\ \lambda v(\bar{y}) + G(\bar{y}, v(\bar{y}), \alpha(\bar{x}-\bar{y}), -Y_{\alpha,\varepsilon}) \ge 0, \\ -3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_{\alpha,\varepsilon} & 0 \\ 0 & Y_{\alpha,\varepsilon} \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \end{gathered}$$

Subtracting the second from the first of the above inequalities and sending $\varepsilon \to 0$, we get

$$\lambda M + \theta |\operatorname{tr}(X_\alpha + Y_\alpha)| \le \omega_R(r_\alpha(1 + R + \|X_\alpha\|)) \tag{4.1}$$

for some $X_\alpha, Y_\alpha \in \mathcal{S}^n$ satisfying

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X_\alpha & 0 \\ 0 & Y_\alpha \end{pmatrix} \le 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4.2}$$

where $r_\alpha := \liminf_{\varepsilon \to 0} |\bar{x} - \bar{y}|$ and for some $R > 0$. Here we have used the following observations. By Lemma 2.3, we have

$$u(x) - v(y) \le M + C_1|x-y| \quad \text{for } x, y \in \mathbf{R}^n$$

for some constant $C_1 > 0$, and hence,

$$M \leq u(\bar{x}) - v(\bar{y}) - \frac{\alpha}{2}|\bar{x}-\bar{y}|^2 - \frac{\varepsilon}{2}|\bar{x}|^2 \leq M + C|\bar{x}-\bar{y}| - \frac{\alpha}{2}|\bar{x}-\bar{y}|^2 - \frac{\varepsilon}{2}|\bar{x}|^2.$$

Therefore,

$$\alpha|\bar{x}-\bar{y}| \leq 2C_1.$$

Similarly we have

$$\varepsilon|\bar{x}|^2 \leq 2(u(\bar{x}) - v(\bar{y}) - M),$$

and

$$\varepsilon|\bar{x}| \leq C_2$$

for some constant $C_2 > 0$. Hence, we have

$$\alpha|\bar{x}-\bar{y}| + \varepsilon|\bar{x}| \leq 2C_1 + C_2.$$

¿From (4.1) and (4.2), we argue as in the proof of [IL, Theorem III.1], to obtain a contradiction after letting $\alpha \to \infty$. QED

Proof of Lemma 2.7. The assertion (a) follows immediately from Lemma 2.3.

To show (b), we use the sup-inf and inf-sup convolutions ([LL]) and the techniques of their use as adapted in [J2, CKSS]. We only prove the existence of v^- having the required properties. It is left to reader to show the existence of v^+ fulfilling the required properties.

Fix any $\varepsilon > 0$ and we first observe that the sup-convolution of u^ε gives a nice approximation as a subsolution to (2.1). This is basically well-known, but the point is to check this fact in our current situation. We only deal with the subsolution case, and it is left to the reader to examine the other case .

Thus let $\varphi \in C^2(\mathbf{R}^n)$ and $\hat{y} \in \mathbf{R}^n$ be such that $u - \varphi$ attains a maximum at $\hat{y}$, and we will show that

$$G(\hat{y}, u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2\varphi(\hat{y})) \leq \sigma(\varepsilon)$$

for some function $\sigma : (0,\infty) \to [0,\infty]$ satisfying $\sigma(+0) = 0$ which depends only on n, θ, ω_R, and R.

Indeed, according to the definition

$$u^\varepsilon(y) = \max_{x \in \mathbf{R}^n} \Big(u(x) - \frac{1}{2\varepsilon}|x-y|^2 \Big),$$

we find a point $\hat{x} \in \mathbf{R}^n$ such that the function

$$u(x) - \varphi(y) - \frac{1}{2\varepsilon}|x-y|^2$$

attains a maximum at $(\hat{x}, \hat{y})$.

Now, by the maximum principle [CIL2, Theorem 3.2], we see that for some $X, Y \in \mathcal{S}^n$, we have

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \tag{4.3}$$

$$G\Big(\hat{x}, u(\hat{x}), \frac{1}{\varepsilon}(\hat{x}-\hat{y}), X\Big) \leq 0, \tag{4.4}$$

$$-D^2\varphi(\hat{y}) \leq Y. \tag{4.5}$$

Note that (4.3) and (4.5) together yield

$$\begin{pmatrix} X & 0 \\ 0 & -D^2\varphi(\hat{y}) \end{pmatrix} \leq \frac{3}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{4.6}$$

Since $D\varphi(\hat{y}) + \varepsilon^{-1}(\hat{y}-\hat{x}) = 0$ and $u(\hat{x}) \geq u^\varepsilon(\hat{y})$, from (4.4) we get

$$G(\hat{y} + \varepsilon D\varphi(\hat{y}), u^\varepsilon(\hat{y}), D\varphi(\hat{y}), X) \leq 0.$$

It is known (see [IL, Lemma III.1]) that (4.6) implies

$$\|X\| \leq C_1(\varepsilon^{-1/2}|\operatorname{tr}(X - D^2\varphi(\hat{y}))|^{1/2} + |\operatorname{tr}(X - D^2\varphi(\hat{y}))|) \tag{4.7}$$

for some constant $C_1 > 0$ which depends only on n. Note that, since $D\varphi(\hat{y}) = (1/\varepsilon)(\hat{x} - \hat{y}) \in D^+u(\hat{x})$, we have $|D\varphi(\hat{y})| \leq \|Du\|_{L^\infty(\mathbf{R}^n)} \leq R$ and that, since $X + Y \leq 0$, $X - D\varphi(\hat{y}) \leq 0$. Using (4.6), (4.7), and this observation, we calculate that

$$\begin{aligned} 0 &\geq G\big(\hat{y} + \varepsilon D\varphi(\hat{y}), u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2\varphi(\hat{y}) + (X - D^2\varphi(\hat{y})\big) \\ &\geq G\big(\hat{y} + \varepsilon D\varphi(\hat{y}), u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2\varphi(\hat{y})\big) + \theta|\operatorname{tr}(X - D^2\varphi(\hat{y})| \\ &\geq G\big(\hat{y}, u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2\varphi(\hat{y})\big) - \sup_{t\geq 0}\Big(-\theta t + \omega_R\big(\varepsilon R[1 + R + C_1(\varepsilon^{-\frac{1}{2}}t^{\frac{1}{2}} + t)]\big)\Big) \end{aligned}$$

Setting

$$\sigma(\varepsilon) = \sup_{t\geq 0}\Big(-\theta t + \omega_R\big(\varepsilon R(1 + R + C_1[\varepsilon^{-\frac{1}{2}}t^{\frac{1}{2}} + t])\big)\Big),$$

we observe that $\sigma : (0,\infty) \to [0,\infty]$ satisfies $\sigma(+0) = 0$ and that

$$G(\hat{y}, u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2\varphi(\hat{y})) \leq \sigma(\varepsilon).$$

Thus, as far as $\sigma(\varepsilon) < \infty$, u^ε is a subsolution of

$$G(y, u^\varepsilon(\hat{y}), D\varphi(\hat{y}), D^2u^\varepsilon(y)) \leq \sigma(\varepsilon) \quad \text{in } \mathbf{R}^n. \tag{4.8}$$

Note that σ depends only on n, θ, ω_R, and R.

Fix such a function σ, which may be assumed to be non-decreasing, and choose $r > 0$ so that $\sigma(r) < \infty$.

Let $\varepsilon > 0$ and $\delta > 0$ satisfy $\varepsilon + \delta \leq r$. As is observed in [CKSS, S], we have

(4.10) $\left(u^{\varepsilon+\delta}\right)_\delta \geq u^\varepsilon \geq u$ in $\mathbf{R}^n$,

(4.11) $\left(u^{\varepsilon+\delta}\right)_\delta(y) - \frac{1}{2\delta}|y|^2$ is concave in $\mathbf{R}^n$,

(4.12) $\left(u^{\varepsilon+\delta}\right)_\delta(y) + \frac{1}{2\varepsilon}|y|^2$ is convex in $\mathbf{R}^n$,

and moreover,

(4.13) if $\left(u^{\varepsilon+\delta}\right)_\delta$ is twice differentiable and $\left(u^{\varepsilon+\delta}\right)_\delta(\hat{y}) > u^\varepsilon(\hat{y})$ at a point $\hat{y} \in \mathbf{R}^n$, then the matrix $D^2\left(u^{\varepsilon+\delta}\right)_\delta(\hat{y})$ has $1/\delta$ as an eigenvalue.

In particular, the function $\left(u^{\varepsilon+\delta}\right)_\delta$ has the $W^{2,\infty}$ regularity.

We write v for $\left(u^{\varepsilon+\delta}\right)_\delta$. Let $\hat{y} \in \mathbf{R}^n$ be a point such that $\left(u^{\varepsilon+\delta}\right)_\delta$ is twice differentiable at $\hat{y}$ and $\left(u^{\varepsilon+\delta}\right)_\delta(\hat{y}) > u^\varepsilon(\hat{y})$. Since $D^2v(\hat{y}) + \frac{1}{\varepsilon}I \geq 0$, using (4.13), we get

$$\operatorname{tr}\left(D^2v(\hat{y}) + \frac{1}{\varepsilon}I\right) \geq \frac{1}{\delta} - \frac{1}{\varepsilon}.$$

Furthermore, recalling that $\|v\|_{L^\infty(\mathbf{R}^n)} \leq \|u\|_{L^\infty(\mathbf{R}^n)} \leq R$, if $\delta \leq \varepsilon$ then we have

$$\begin{aligned}
G(\hat{y}, v(\hat{y}), Dv(\hat{y}), D^2v(\hat{y})) &= G\Big(\hat{y}, v(\hat{y}), Dv(\hat{y}), -\frac{1}{\varepsilon}I + D^2v(\hat{y}) + \frac{1}{\varepsilon}I\Big) \\
&\leq G\Big(\hat{y}, v(\hat{y}), Dv(\hat{y}), -\frac{1}{\varepsilon}I\Big) - \theta \operatorname{tr}\left(D^2v(\hat{y}) + \frac{1}{\varepsilon}I\right) \\
&\leq G(\hat{y}, v(\hat{y}), Dv(\hat{y}), 0) + \Theta\frac{n}{\varepsilon} - \theta\Big(\frac{1}{\delta} - \frac{1}{\varepsilon}\Big) \\
&\leq M_R + \frac{n\Theta + \theta}{\varepsilon} - \frac{\theta}{\delta}.
\end{aligned}$$

Fix $\varepsilon \in (0, r)$ and choose $\delta \equiv \delta(\varepsilon) \in (0, \varepsilon)$ so that $M_R + \frac{n\Theta+\theta}{\varepsilon} - \frac{\theta}{\delta} \leq 0$. Then it is easy to check that $v = (u^{\varepsilon+\delta})_\delta$ is a subsolution of

$$G(y, v(y), Dv(y), D^2v(y)) \leq \sigma(\varepsilon) \quad \text{in } \mathbf{R}^n.$$

Note here that the function $\varepsilon \mapsto \delta(\varepsilon)$ depends only on n, θ, Θ, and M_R.

Finally, noting the well-known facts that there is a modulus γ, which depends only on R and the function δ of ε, such that

$$\|(u^{\varepsilon+\delta})_\delta - u\|_{L^\infty(\mathbf{R}^n)} \leq \gamma(\varepsilon)$$

and that $\|v\|_{L^\infty(\mathbf{R}^n)} \le \|u\|_{L^\infty(\mathbf{R}^n)}$ and $\|Dv\|_{L^\infty(\mathbf{R}^n)} \le \|Du\|_{L^\infty(\mathbf{R}^n)}$, we finish the proof. QED

Lemma 4.1. *Let Ω be an open subset of $\mathbf{R}^n$ and $f : \Omega \to \mathbf{R}^n$ a semiconvex function. Then*

$$J^{2,+}f(x) \subset \overline{J}^2 f(x) \qquad \textit{for } x \in \Omega.$$

This is an observation due to Jensen [J1].

Proof. Fix $x \in \Omega$ and $(p, X) \in J^{2,+}f(x)$, and choose a function $\varphi \in C^2(\Omega)$ so that $(f-\varphi)(y) = 0 < (f-\varphi)(x)$ for $y \in \Omega \setminus \{x\}$. As a simple consequence of [CIL2, Theorem A.2 and Lemma A.3], we find sequences $\Omega \ni x_j \to x$ and $\mathbf{R}^n \times \mathcal{S}^n \ni (p_j, X_j) \to (0,0)$ such that $(p_j, X_j) \in J^2(f-\varphi)(x_j)$ for all j. ¿From this, we conclude that $(p, X) \in \overline{J}^2 f(x)$. QED

Proof of Lemma 2.4. Set $w(x) = u(x) - v(x)$ for $x \in \mathbf{R}^n$. Choose $R > 0$ so that

$$\max\{\|u\|_{L^\infty(\mathbf{R}^n)}, \|v\|_{L^\infty(\mathbf{R}^n)}, \|Du\|_{L^\infty(\mathbf{R}^n)}, \|Dv\|_{L^\infty(\mathbf{R}^n)}\} \le R.$$

We begin by showing that

$$\mathcal{P}^-(D^2 w) - L_R|Dw| \le 0 \quad \text{in } \mathbf{R}^n. \tag{4.14}$$

To this aim, we first apply Lemma 2.7 to u and v to find for each $\varepsilon > 0$ $u_\varepsilon, v_\varepsilon \in W^{2,\infty}(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ such that

$$\begin{aligned} &\|u - u_\varepsilon\|_{L^\infty(\mathbf{R}^n)} \le \varepsilon, \qquad \|v - v_\varepsilon\|_{L^\infty(\mathbf{R}^n)} \le \varepsilon, \\ &G(x, Du_\varepsilon(x), D^2 u_\varepsilon(x)) \le \varepsilon, \\ &G(x, Dv_\varepsilon(x), D^2 v_\varepsilon(x)) \ge \varepsilon. \end{aligned}$$

Set $w_\varepsilon(x) = u_\varepsilon(x) - v_\varepsilon(x)$ for $x \in \mathbf{R}^n$. Let $x \in \mathbf{R}^n$ and $(p, X) \in J^{2,+}w_\varepsilon(x)$. By Lemma 3.6, we find a $Y \in \mathcal{S}^n$ such that $(Dv_\varepsilon(x), Y) \in \overline{J}^2 v_\varepsilon(x)$ and $(p + Dv_\varepsilon(x), X + Y) \in J^{2,+}u(x)$. Now, setting $(r, Z) = (p + Dv_\varepsilon(x), X + Y)$, we have

$$\begin{aligned} 2\varepsilon &\ge G(x, r, Z) - G(x, Dv_\varepsilon(x), Y) \\ &\ge G(x, r, Y) - \Theta\,\mathrm{tr}\,(Z - Y)_+ + \theta\,\mathrm{tr}\,(Z - Y)_- - G(x, Dv_\varepsilon(x), Y) \\ &\ge L_R|r - Dv_\varepsilon(x)| + \mathcal{P}^-(Z - Y) \ge \mathcal{P}^-(X) - L_R|p|. \end{aligned}$$

Therefore,

$$\mathcal{P}^-(D^2 w_\varepsilon(x)) - L_R|Dw_\varepsilon(x)| \le 2\varepsilon \quad \text{in } \mathbf{R}^n.$$

According to the stability property of viscosity solutions, we conclude by sending $\varepsilon \to 0$ that (4.14) holds.

Using a limiting argument as in the proof of Lemma 2.7, we see that we may assume that $u - v$ attains a maximum at some point $z \in \mathbf{R}^n$. The rest of the proof is somehow standard, for which we refer to [BD]. QED

The second author learned from Mariko Arisawa that w satisfies (4.14) in the above proof. He is grateful to her for this.

Proof of Lemma 3.6. We choose a function $\varphi \in C^2(\Omega)$ so that $v - \varphi$ attains a maximum at $\hat{x}$. Since v is semiconcave, we can find a $Y \in \mathcal{S}^n$ such that $(Dv(\hat{x}), Y) \in J^{2,+}v(\hat{x})$. Since v is semiconvex, by Lemma 4.1, we have $J^{2,+}v(\hat{x}) \subset \overline{J}^2 v(\hat{x})$. Now, by the assumption that $u - v - \varphi$ has a maximum at $\hat{x}$, we see that $J^{2,+}(v + \varphi)(\hat{x}) \subset J^{2,+}u(\hat{x})$. Combining these we get

$$(p + Dv(\hat{x}), X + Y) \in (p, X) + J^{2,+}v(\hat{x}) \subset J^{2,+}u(\hat{x}).$$

This concludes the proof. QED

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Funkcialaj Ekvacioj (http://www.math.kobe-u.ac.jp/˜fe) **46** (2003), 63–88