STABILITY OF CONSTANT EQUILIBRIUM FOR THE MAXWELL-HIGGS EQUATIONS

Yoshio TSUTSUMI†

Mathematical Institute, Tohoku University
Sendai 980-8578, JAPAN

Dedicated to Professor T. Nishida
on his 60th birthday

§1. Introduction and theorem

We consider the Maxwell-Higgs equations in space time dimensions 2 + 1 and 3 + 1:

$$\partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = j^\mu, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.1}$$

$$D_\mu D^\mu \varphi = \frac{1}{2}\big(m^2\varphi - \frac{2m^2}{M^2}|\varphi|^2\varphi\big), \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.2}$$

where $n = 2$ or 3, M and m are positive constants, A^μ are real-valued functions, φ is a complex-valued function and

$$j^\mu = -i\{\varphi\overline{(D^\mu\varphi)} - (D^\mu\varphi)\overline{\varphi}\},$$
$$D^\mu = \partial^\mu + iA^\mu.$$

Here and hereafter, we follow the convention that Greek indices take values in $\{0, 1, \cdots, n\}$ while Latin indices are valued in $\{1, \cdots, n\}$. Indices repeated lower and upper are summed. The space $\mathbf{R}^{n+1}$ is the $n+1$ dimensional Euclidean space equipped with the flat Minkowski metric

$$(g_{\alpha\beta}) = diag(1, -1, \cdots, -1).$$

Indices are raised and lowered using the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. We put $x^0 = t$ and $\partial_\alpha = \partial/\partial x^\alpha$.

The potential $V(|\varphi|) = \frac{m^2}{4M^2}(M^2/2 - |\varphi|^2)^2$ associated with the right hand side of (1.2) is called the Higgs potential and it has equilibria 0 and $z = Me^{i\theta}/\sqrt{2}$, $\theta \in \mathbf{R}$. The latter equilibria correspond to the degenerate ground state, which is

†Partially supported by Grant-In-Aid for Scientific Research (B) (no. 12440033), JSPS

the vacuum with vacuum expectation value $M/\sqrt{2}$. All equilibria $z = Me^{i\theta}/\sqrt{2}$, $\theta \in \mathbf{R}$ are equivalent from a physical point of view, because this system has $U(1)$ symmetry. If one of them is spontaneously chosen, for example, if $z = M/\sqrt{2}$ is chosen as a vacuum equilibrium, then the so-called spontaneous breakdown of symmetry will happen and the photon described by A^μ will have mass M. This is called the Higgs mechanism.

We look at this mechanism in more details (see [1, Section 13.3]). We first note that equations (1.1)-(1.2) are invariant under the following gauge transformation:

$$\tilde{A}^\mu = A^\mu - \partial^\mu \chi,$$
$$\tilde{\varphi} = e^{-i\chi}\varphi,$$

where $\chi(t,x)$ is an arbitrary smooth real-valued function. If φ is close to the vacuum equilibrium $M/\sqrt{2}$, then we can take $\chi = \arg\varphi$, $-\pi < \arg\varphi \leq \pi$. In that case, both $\tilde{A}^\mu$ and $\tilde{\varphi}$ become real-valued functions. Such choice of gauge is called the unitary gauge. Again we denote $\tilde{A}^\mu$ and $\tilde{\varphi}$ by A^μ and φ, respectively. We put $\varphi = \phi + M/\sqrt{2}$ and suppose that the fluctuation (A^μ, ϕ) from the vacuum $(0,0)$ is small in the new dynamical variables. Then, the Cauchy problem for (1.1)-(1.2) is reduced to the following:

$$(\Box + M^2)A^\mu - \partial^\mu\partial_\nu A^\nu = -2\sqrt{2}MA^\mu\phi - 2A^\mu\phi^2, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.3}$$

$$(\Box + m^2)\phi = \frac{M}{\sqrt{2}}A_\nu A^\nu + A_\nu A^\nu\phi - \frac{3m^2}{\sqrt{2}M}\phi^2 - \frac{m^2}{M^2}\phi^3, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.4}$$

$$(A^\mu(0), \partial_t A^\mu(0)) = (\alpha^\mu, \beta^\mu), \quad (\phi(0), \partial_t\phi(0)) = (\phi_0, \phi_1), \tag{1.5}$$

$$\phi\partial_\mu A^\mu + 2A_\mu\partial^\mu\phi + \frac{M}{\sqrt{2}}\partial_\mu A^\mu = 0, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.6}$$

where $\Box = \partial_\mu\partial^\mu$. The constraint (1.6) is a gauge condition associated with the unitary gauge.

If $\phi > -M/\sqrt{2}$, we can rewrite (1.3) by using (1.6) as follows.

$$\begin{aligned}(\Box + M^2)A^\mu &= 2\partial^\mu\left((\phi + M/\sqrt{2})^{-1}A_\nu\partial^\nu\phi\right) \\ &\quad - 2\sqrt{2}MA^\mu\phi - 2A^\mu\phi^2 \\ &= \frac{2\sqrt{2}}{M}\partial^\mu(A_\nu\partial^\nu\phi) - 2\sqrt{2}MA^\mu\phi \\ &\quad -2A^\mu\phi^2 + f(\phi, A_\nu, \partial^\nu\phi, \partial^\mu A_\nu, \partial^\mu\partial^\nu\phi), \quad (t,x) \in \mathbf{R}^{1+n},\end{aligned} \tag{1.7}$$

where f is a nonlinear function belonging to

$$C^\infty\big((-M/\sqrt{2},\infty)\times\mathbf{R}^{1+n}\times\mathbf{R}^{1+n}\times\mathbf{R}^{1+n}\times\mathbf{R}^{1+n}\big)$$

and f is cubic around zero.

Remark 1.1. In (1.7) the nonlinearity includes the second derivatives of ϕ. Nevertheless, the system (1.4) and (1.7) can actually be handled as a semilinear system of Klein-Gordon equations with mass terms M and m (see the proof of Lemma 2.4 in Section 2).

Remark 1.2. If the solution (A^ν,ϕ) of (1.3)-(1.5) satisfies the unitary gauge condition (1.6), the following compatibility conditions on initial data must hold.

$$\phi_0\beta^0+\phi_0\partial_j\alpha^j+2(\alpha_0\phi_1+\alpha_j\partial^j\phi_0)+\frac{M}{\sqrt{2}}(\beta^0+\partial_j\alpha^j)=0, \tag{1.8}$$

$$(\Delta-M^2)\alpha^0-\partial_j\beta^j-2\sqrt{2}M\alpha^0\phi_0-2\alpha^0\phi_0^2=0. \tag{1.9}$$

The relations (1.8) and (1.9) follow from (1.6) and (1.3) with $\mu=0$, respectively. Furthermore, if $\phi>-M/\sqrt{2}$, the solution of (1.3)-(1.6) satisfies equation (1.7). Conversely, if the initial data satisfy (1.8) and (1.9), then the solution of (1.7) and (1.4) must automatically satisfy the unitary gauge condition (1.6) and consequently satisfy (1.3), as long as $\phi>-M/\sqrt{2}$ (for the details, see the proof of Lemma 2.4 (ii) in Section 2). Therefore, in most cases, we consider the problem (1.7), (1.4) and (1.5) under assumptions (1.8) and (1.9) instead of (1.3)-(1.6).

The unique global solvability of the Cauchy problem of (1.1)-(1.2) was proved by Eardley and Moncrief [9], Burzlaff and Moncrief [4] and Schwarz [34] for the classical solution and by Klainerman and Machedon [27] and Keel [23] for the finite energy solution without restriction on size of initial data. In this paper, we study the stability of constant vacuum equilibria for (1.1)-(1.2), which is reduced to the stability of zero solution for (1.3)-(1.6). The stability problem of constant equilibria seems to be important, because if they are not stable, the unitary gauge can not be defined for all times. This problem apears in the abelian Higgs model for $n=3$ and in the relativistic superconductivity theory for $n=2$ (for the physical background, see [1], [22] and [40]).

Remark 1.3. The existence of topologically non-trivial stationary solutions is known (see, e.g., [3], [22] and [40]). These solutions are called vortices for $n=2$ and monopoles for $n=3$. The stability and instability of these solutions are studied, for example, by [19] and [20], but many problems still remain open to be solved.

Before we state our theorem, we list several notations. For k, $s \in \mathbf{N} \cup \{0\}$, we define the weighted Sobolev space $H^{k,s}$ as follows.

$$H^{k,s} = \{v \in L^2;\ \|v\|_{H^{k,s}} < \infty\}$$

with the norm

$$\|v\|_{H^{k,s}} = \Big(\int_{\mathbf{R}^n} |(1+|x|^2)^{s/2}(1-\Delta)^{k/2}v(x)|^2\ dx\Big)^{1/2}.$$

For simplicity, we write $\|u(t)\|_{H^k} = \|u(t)\|_{H^{k,0}}$. We put

$$\Omega_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \qquad \mu \neq \nu.$$

We denote the generators of the Poincaré group by

$$\Gamma = (\Gamma_j;\ j = 1, \cdots, \frac{n^2+3n+2}{2}) = (\partial_\mu, \Omega_{\mu j};\ \mu = 0, 1, \cdots, n,\ j = 1, \cdots, n,\ \mu < j).$$

For $k \in \mathbf{N}$, $l \in \mathbf{R}$ and $u \in C([0,\infty); \mathcal{S}(\mathbf{R}^n))$, we put

$$\begin{aligned}
|u(t,x)|_k &= \sum_{|\alpha|\le k} |\Gamma^\alpha u(t,x)|,\\
\|u(t)\|_{k,l} &= \sum_{|\alpha|\le k} \|(1+t+|x|)^l \Gamma^\alpha u(t)\|_{L^2},\\
\|u(t)\|_k &= \|u(t)\|_{k,0}.
\end{aligned}$$

For $s \in \mathbf{R}$, let $[s]$ denote the largest integer that does not exceed s. We put $\partial = (\partial_0, \partial_1, \cdots, \partial_n)$.

We have the following theorem.

Theorem 1.1. *Let $n = 2$ or 3, let k be an integer with $k \ge 24$ and let $0 < \delta \le 1/4$. We assume that $2M \neq m$ for $n = 2$ and $(\alpha^\mu, \beta^\mu, \phi_0, \phi_1) \in H^{k-1,k-1} \oplus H^{k-2,k-1} \oplus H^{k,k-1} \oplus H^{k-1,k-1}$. Then, there exists an $\varepsilon_0 > 0$ such that if the initial data $(\alpha^\mu, \beta^\mu, \phi_0, \phi_1)$ satisfy*

$$\sum_{\mu=0}^{n} \big(\|\alpha^\mu\|_{H^{k-1,k-1}} + \|\beta^\mu\|_{H^{k-2,k-1}}\big) + \|\phi_0\|_{H^{k,k-1}} + \|\phi_1\|_{H^{k-1,k-1}} \le \varepsilon$$

for some ε with $0 < \varepsilon < \varepsilon_0$ and the compatibility conditions (1.8) and (1.9) for the unitary gauge are satisfied, then the Cauchy problem (1.3)-(1.6) has a unique global solution (A^μ, ϕ) satisfying

$$\partial^\alpha\Gamma^\beta A^\mu(t), \quad \partial^\alpha\Gamma^\gamma\phi(t) \in C(\mathbf{R}; L^2(\mathbf{R}^n)), \tag{1.10}$$

$$|\alpha| \le 1, |\beta| \le k-2, |\gamma| \le k-1,$$

$$\sum_{|\alpha|\le 2} \left(\|\partial^\alpha A^\mu(t)\|_{k-6} + \|\partial^\alpha \phi(t)\|_{k-5} \right) \le C\varepsilon, \quad t \in \mathbf{R}, \tag{1.11}$$

$$\sum_{|\alpha|\le 1} (1+|t|)^{-\delta} \left(\|\partial^\alpha A^\mu(t)\|_{k-2} + \|\partial^\alpha \phi(t)\|_{k-1} \right) \le C\varepsilon, \quad t \in \mathbf{R}, \tag{1.12}$$

$$\|\phi(t)\|_{L^\infty} \le \frac{3M}{4\sqrt{2}}, \quad t \in \mathbf{R}, \tag{1.13}$$

$$\sum_{|\alpha|\le 2} \left(|\partial^\alpha A^\mu(t,x)|_{k-12} + |\partial^\alpha \phi(t,x)|_{k-11} \right) \le C(1+|t|+|x|)^{-n/2}, \quad (t,x) \in \mathbf{R}^{1+n}, \tag{1.14}$$

where C does not depend on ε. Furthermore, A^μ and ϕ have free profiles $(\alpha_\pm^\mu, \beta_\pm^\mu, \phi_{0\pm}, \phi_{1\pm}) \in H^{k-5} \oplus H^{k-6} \oplus H^{k-4} \oplus H^{k-5}$ such that

$$\|A^\mu(t) - a_\pm^\mu(t)\|_{H^{k-5}} + \|\partial_t A^\mu(t) - \partial_t a_\pm^\mu(t)\|_{H^{k-6}} \longrightarrow 0 \quad (t \to \pm\infty), \tag{1.15}$$

$$\|\phi(t) - \phi_\pm(t)\|_{H^{k-4}} + \|\partial_t \phi(t) - \partial_t \phi_\pm(t)\|_{H^{k-5}} \longrightarrow 0 \quad (t \to \pm\infty), \tag{1.16}$$

where $a_\pm^\mu$ and $\phi_\pm$ are the solutions of the following free Klein-Gordon equations:

$$\Box a_\pm^\mu + M^2 a_\pm^\mu = 0, \qquad (a_\pm^\mu(0), \partial_t a_\pm^\mu(0)) = (\alpha_\pm^\mu, \beta_\pm^\mu), \quad \partial_\mu a_\pm^\mu = 0,$$
$$\Box \phi_\pm + m^2 \phi_\pm = 0, \qquad (\phi_\pm(0), \partial_t \phi_\pm(0)) = (\phi_{0\pm}, \phi_{1\pm}).$$

Remark 1.4. (i) The time decay property (1.14) in Theorem 1.1 implies the asymptotical stability of zero solution for (1.3)-(1.4), which corresponds to the asymptotical stability of the constant vacuum equilibrium $(0, M/\sqrt{2})$ for (1.1)-(1.2) in the unitary gauge. By the $U(1)$ symmetry of the original system (1.1)-(1.2), we can conclude that other constant vacuum equilibria $(0, Me^{i\theta}/\sqrt{2})$, $\theta \in \mathbf{R}$ are asymptotically stable in the same sense.

(ii) By the argument of Glassey [18] and Matsumura [30], we can prove that if $n = 2$ and $2M = m$, the relations (1.14), (1.15) and (1.16) fail in a certain sense. More precisely, the wave operators can not be well-defined for $n = 2$ and $2M = m$. In that case, the long range phenomenon should take place.

(iii) When we consider the vortex solutions and the monopole solutions, the case of $2M = m$ is of special interest (see Plohr [33] and Jaffe and Taubes [22]).

The standard proof of global existence results for small initial data consists of the energy estimate and the time decay estimate. Theorem 1.1 for $n = 3$ follows immediately from the $L^\infty - L^2$ decay estimate by Klainerman [25] (see also Hörmander [21], Georgiev [12]-[15]). In the two dimensional case, this decay estimate is not sufficient for the proof of Theorem 1.1 and we need to use the technique of normal form by Shatah [34] in addtion to the $L^\infty - L^2$ decay estimate (see Ozawa, Tsutaya and Tsutsumi [31] and [32]). However, Shatah's argument does not always work if $M \neq m$. For example, in our problem the restriction $2M > m$ is required to apply Shatah's argument of normal form. Here we improve the proof by Kosecki [28] to overcome this difficulty, which gives an alternative proof of the result in [31] or [32] as well as an extension to the system of nonlinear Klein-Gordon equations with different mass terms.

We illustrate our proof of Theorem 1.1 for $n = 2$. The proof of Theorem 1.1 is based on the null condition technique, which compensates for the insufficiency of decay rate in two spatial dimensions. In [26], Klainerman introduced the notion of the null condition to prove the global existence of solutions for massless nonlinear wave equations with small initial data (see also Christodoulou [6]). Here, we give not a precise definition of the null condition but several typical examples of quadratic forms satisfying the null condition. For smooth functions $u(t,x)$ and $v(t,x)$, we define the quadratic forms $Q_0(u,v)$ and $Q_{\mu\nu}(u,v)$ as follows.

$$Q_0(u,v) = \partial_0 u \partial_0 v - \nabla u \cdot \nabla v, \tag{1.18}$$

$$Q_{\mu\nu}(u,v) = \partial_\mu u \partial_\nu v - \partial_\nu u \partial_\mu v, \quad \mu \neq \nu. \tag{1.19}$$

It is known that $Q_0(u,v)$ and $Q_{\mu\nu}(u,v)$ satisfy the null condition (see [26]). Let $S = x^\mu \partial_\mu$. These quadratic forms can be rewritten by using $\Omega_{\mu j}$ and S in the following forms.

$$Q_0(u,v) = t^{-1}\big(Su\partial_0 v - \partial_j u \Omega_{0j} v\big), \tag{1.20}$$

$$Q_{jk}(u,v) = t^{-1}\big(\partial_j u \Omega_{0k} v - \partial_k u \Omega_{0j} v + \Omega_{jk} u \partial_0 v\big), \tag{1.21}$$

$$Q_{0j}(u,v) = -Q_{j0}(u,v) = t^{-1}\big(\partial_0 u \Omega_{0j} v - \Omega_{0j} u \partial_0 v\big). \tag{1.22}$$

These relations show that the quadratic forms Q_0 and $Q_{\mu\nu}$ have better decay estimates than other quadratic nonlinearity. On the other hand, we have the following commutation relations.

$$[\Omega_{\mu j}, \Box] = 0, \qquad [S, \Box] = -2\Box.$$

Because of the second commutation relation, the radial vector field S is incompatible with the Klein-Gordon operator $\Box + M^2$, while it is useful for the D'Alembertian

$\Box$. Therefore, we need avoid to use Q_0 when we consider the nonlinear Klein-Gordon equation. However, because of the first commutation relation, the $\Omega_{\mu j}$'s are compatible with both the Klein-Gordon and the wave equations. So, $Q_{\mu\nu}$ are often called strong null forms (see Georgiev [12]). We introduce new dynamical variables to rewrite equations (1.3) and (1.4) so that all the quadratic nonlinear terms of the resulting equations can be expressed in terms of the strong null forms $Q_{\mu\nu}$. This is inspired by Bachelot [2] and Kosecki [28]. Our proof is a kind of combination of the $L^\infty - L^2$ decay estimate and the normal form like the proofs by Bachelot [2, Part V] and Kosecki [28].

The plan of this paper is the following. In Section 2, we summarize several lemmas needed for the proof of Theorem 1.1. Especially, Lemma 2.2 in Section 2 is a key to our proof of Theorem 1.1, which enables us to control the solution globally in time. At the end of Section 2, we prove the unique local existence of solution for (1.3)-(1.6) (see Lemma 2.4). In Section 3, we prove Theorem 1.1 by using the lemmas in Section 2.

§2. Lemmas

We first state the lemma concerning the time decay estimate for the inhomogeneous linear Klein-Gordon equation. We consider the following Klein-Gordon equation.

$$\Box u + M^2 u = f(t,x), \quad t \in \mathbf{R}, \quad x \in \mathbf{R}^n, \tag{2.1}$$

$$u(0,x) = u_0(x), \quad \partial_0 u(0,x) = u_1(x), \quad x \in \mathbf{R}^n. \tag{2.2}$$

We have the following lemma.

Lemma 2.1. *Assume that $n = 2$ or 3. Let k be an arbitrary nonnegative integer and let u be a solution of (2.1)-(2.2). We put $l(n) = [(n+5)/2]$. Then, we have the following estimate:*

$$\begin{aligned}(1+|t|+|x|)^{n/2}|u(t,x)|_k \le C\big\{&\|u_0\|_{H^{k+l(n)+1,k+l(n)+3}} + \|u_1\|_{H^{k+l(n),k+l(n)+2}} \\ &+ \sum_{j=0}^{\infty} 2^j \sup_{s\in I_j\cap[0,t]} \|f(s)\|_{k+l(n)+2,1}\big\}, \quad t > 0, \quad x \in \mathbf{R}^n,\end{aligned}$$

where $I_0 = [0,1]$, $I_j = [2^{j-1}, 2^{j+1}]$ $(j \ge 1)$ and C is a positive constant depending only on k. For $t < 0$, the same estimate as above holds.

This kind of decay estimate for the inhomogeneous linear Klein-Gordon equation was first proved by Klainerman [25] and was improved by Bachelot[2], Hörmander [21] and Georgiev [13]-[16]. For the proof of Lemma 2.1, see [14, Theorem 1].

In the following Lemmas 2.2 and 2.3, we do not use the convention that indices repeated are summed. We consider a system of Klein-Gordon equations:

$$(2.3)\qquad (\Box + M^2)u = au^2 + \sum_{1\le j\le l_1} b_j^{(1)} u v_j + \sum_{1\le j\le l_2} b_j^{(2)} u w_j + \sum_{1\le j\le k\le l_1} c_{jk}^{(1)} v_j v_k + \sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}} c_{jk}^{(2)} v_j w_k + \sum_{1\le j\le k\le l_2} c_{jk}^{(3)} w_j w_k + f \equiv F, \qquad (t,x)\in \mathbf{R}^{1+n}$$

$$(2.4)\qquad (\Box + M^2)v_j = g_j, \quad (t,x)\in \mathbf{R}^{1+n}, \quad 1\le j\le l_1,$$

$$(2.5)\qquad (\Box + m^2)w_j = h_j, \quad (t,x)\in \mathbf{R}^{1+n}, \quad 1\le j\le l_2,$$

where a, $b_j^{(q)}$ and $c_{jk}^{(q)}$ are constants, l_1 and l_2 are positive integers and f, g_j and h_j are smooth functions. We now state the key lemma to our proof of Theorem 1.1, which reveals that quadratic nonlinear Klein-Gordon equations generically have the null condition structure.

Lemma 2.2. *Let u, v_j and w_j be solutions of (2.3), (2.4) and (2.5), respectively. Assume that $(2M-m)(M-2m)\neq 0$. Then, there exist constants p, $q_j^{(1)}, \tilde{q}_j^{(1)} (1\le j\le l_1)$, $q_j^{(2)}, \tilde{q}_j^{(2)} (1\le j\le l_2)$, $r_{jk}^{(1)} (1\le j,k\le l_1)$, $r_{jk}^{(2)}, \tilde{r}_{jk}^{(2)} (1\le j\le l_1, 1\le k\le l_2)$, and $r_{jk}^{(3)}, \tilde{r}_{jk}^{(3)} (1\le j,k\le l_2)$ such that if we put*

$$(2.6)\qquad \begin{aligned} z_\nu = {} & \partial_\nu u - 2pu\partial_\nu u - \sum_{1\le j\le l_1} (q_j^{(1)} \partial_\nu u v_j + \tilde{q}_j^{(1)} u \partial_\nu v_j) \\ & - \sum_{1\le j\le l_2} (q_j^{(2)} \partial_\nu u w_j + \tilde{q}_j^{(2)} u \partial_\nu w_j) - \sum_{1\le j,k\le l_1} (r_{jk}^{(1)} \partial_\nu v_j v_k + \tilde{r}_{jk}^{(1)} v_j \partial_\nu v_k) \\ & - \sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}} (r_{jk}^{(2)} \partial_\nu v_j w_k + \tilde{r}_{jk}^{(2)} v_j \partial_\nu w_k) - \sum_{1\le j,k\le l_2} (r_{jk}^{(3)} \partial_\nu w_j w_k + \tilde{r}_{jk}^{(3)} w_j \partial_\nu w_k), \end{aligned}$$

then z_ν satisfies the following equation.

$$(2.7)\qquad \begin{aligned} (\Box + M^2) z_\nu = {} & -4p \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(u, \partial^\mu u) \\ & - 2 \sum_{1\le j\le l_1} \Big(q_j^{(1)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(v_j, \partial^\mu u) + \tilde{q}_j^{(1)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(u, \partial^\mu v_j) \Big) \\ & - 2 \sum_{1\le j\le l_2} \Big(q_j^{(2)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(w_j, \partial^\mu u) + \tilde{q}_j^{(2)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(u, \partial^\mu w_j) \Big) \\ & - 2 \sum_{1\le j,k\le l_1} \Big(r_{jk}^{(1)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(v_k, \partial^\mu v_j) + \tilde{r}_{jk}^{(1)} \sum_{\substack{0\le \mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(v_j, \partial^\mu v_k) \Big) \end{aligned}$$

$$
\begin{aligned}
&- 2 \sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}} \Big(r_{jk}^{(2)} \sum_{\substack{0\le\mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(w_k, \partial^\mu v_j) + \tilde r_{jk}^{(2)} \sum_{\substack{0\le\mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(v_j, \partial^\mu w_k)\Big)\\
&- 2 \sum_{1\le j,k\le l_2} \Big(r_{jk}^{(3)} \sum_{\substack{0\le\mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(w_k, \partial^\mu w_j) + \tilde r_{jk}^{(3)} \sum_{\substack{0\le\mu\le n\\ \mu\neq\nu}} Q_{\mu\nu}(w_j, \partial^\mu w_k)\Big)\\
&+ H,
\end{aligned}
$$

where

$$
\begin{aligned}
H = {} & -2p(3F\partial_\nu u + u\partial_\nu F)\\
& - \sum_{1\le j\le l_1} \big(q_j^{(1)}\partial_\nu F v_j + (q_j^{(1)} + 2\tilde q_j^{(1)})\partial_\nu u g_j + (2q_j^{(1)} + \tilde q_j^{(1)})F\partial_\nu v_j + \tilde q_j^{(1)} u\partial_\nu g_j\big)\\
& - \sum_{1\le j\le l_2} \big(q_j^{(2)}\partial_\nu F w_j + (q_j^{(2)} + 2\tilde q_j^{(2)})\partial_\nu u h_j + (2q_j^{(2)} + \tilde q_j^{(2)})F\partial_\nu w_j + \tilde q_j^{(2)} u\partial_\nu h_j\big)\\
& - \sum_{1\le j,k\le l_1} \big(r_{jk}^{(1)}\partial_\nu g_j v_k + (r_{jk}^{(1)} + 2\tilde r_{jk}^{(1)})\partial_\nu v_j g_k + (2r_{jk}^{(1)} + \tilde r_{jk}^{(1)})g_j\partial_\nu v_k + \tilde r_{jk}^{(1)} v_j\partial_\nu g_k\big)\\
& - \sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}} \big(r_{jk}^{(2)}\partial_\nu g_j w_k + (r_{jk}^{(2)} + 2\tilde r_{jk}^{(2)})\partial_\nu v_j h_k + (2r_{jk}^{(2)} + \tilde r_{jk}^{(2)})g_j\partial_\nu w_k + \tilde r_{jk}^{(2)} v_j\partial_\nu h_k\big)\\
& - \sum_{1\le j,k\le l_2} \big(r_{jk}^{(3)}\partial_\nu h_j w_k + (r_{jk}^{(3)} + 2\tilde r_{jk}^{(3)})\partial_\nu w_j h_k + (2r_{jk}^{(3)} + \tilde r_{jk}^{(3)})h_j\partial_\nu w_k + \tilde r_{jk}^{(3)} w_j\partial_\nu h_k\big).
\end{aligned}
\tag{2.8}
$$

Especially, if $c_{jk}^{(3)} = 0$, *we do not have to assume* $M \neq 2m$ *and if* $b_j^{(2)} = c_{jk}^{(2)} = 0$, *we do not have to assume* $2M \neq m$.

Remark 2.1. When we apply Lemma 2.2 to equation (1.7), we choose the v_j's and the w_j's as follows.

$$
\begin{aligned}
&(v_j;\ 1 \le j \le 2(1+n)) = (A_\nu, \partial^\mu A_\nu;\ 0 \le \nu \le n),\\
&(w_j;\ 1 \le j \le 2(1+n)) = (\partial^\nu\phi, \partial^\mu\partial^\nu\phi;\ 1 \le \nu \le n).
\end{aligned}
$$

In the above choice, we regard A_μ on the right hand side of (1.7) as one of the v_j's and so $a = b_j^{(s)} = 0$. We can employ another choice of the v_j's so that A^μ itself is excluded from the v_j's. When we apply Lemma 2.2 to equation (1.4), we choose the v_j's and the w_j's as follows.

$$
\begin{aligned}
&(v_j;\ 1 \le j \le 1+n) = (A^\nu;\ 0 \le \nu \le n),\\
&w_j = 0.
\end{aligned}
$$

In this case, we do not need to consider equation (2.4). At the end of Section 3, we state a remark on the application of Lemma 2.2 to the general quasilinear case (see Concluding Remark in Section 3).

For the proof of Lemma 2.2, we start with the following observation due to Kosecki [28].

Lemma 2.3. *Let u be a smooth function on $\mathbf{R}^{1+n}$ and let v be a solution of the following inhomogeneous linear Klein-Gordon equation.*

$$\Box v + M^2 v = h, \quad (t,x) \in \mathbf{R}^{1+n},$$

where h is a smooth function on $\mathbf{R}^{1+n}$. Then, the following identity holds.

$$Q_0(u, \partial_\nu v) = \sum_{\substack{0 \le \mu \le n \\ \mu \ne \nu}} Q_{\mu\nu}(u, \partial^\mu v) + (\partial_\nu u)(-M^2 v + h),$$
$$0 \le \nu \le n.$$

Lemma 2.3 follows from a direct calculation (see [28, Lemma 2.1]). Now we state the proof of Lemma 2.2.

Proof of Lemma 2.2. Let z_ν be difined as in (2.6). We let the Klein-Gordon operator $(\Box + M^2)$ act on z_ν to have

$$(\Box + M^2) z_\nu = G_1 + G_2 + G_3,$$

where

$$\begin{aligned}
(2.9) \qquad G_1 = & -4pQ_0(u, \partial_\nu u) \\
& - 2 \sum_{1 \le j \le l_1} \left(q_j^{(1)} Q_0(\partial_\nu u, v_j) + \tilde{q}_j^{(1)} Q_0(u, \partial_\nu v_j) \right) \\
& - 2 \sum_{1 \le j \le l_2} \left(q_j^{(2)} Q_0(\partial_\nu u, w_j) + \tilde{q}_j^{(2)} Q_0(u, \partial_\nu w_j) \right) \\
& - 2 \sum_{1 \le j,k \le l_1} \left(r_{jk}^{(1)} Q_0(\partial_\nu v_j, v_k) + \tilde{r}_{jk}^{(1)} Q_0(v_j, \partial_\nu v_k) \right) \\
& - 2 \sum_{\substack{1 \le j \le l_1 \\ 1 \le k \le l_2}} \left(r_{jk}^{(2)} Q_0(\partial_\nu v_j, w_k) + \tilde{r}_{jk}^{(2)} Q_0(v_j, \partial_\nu w_k) \right) \\
& - 2 \sum_{1 \le j,k \le l_2} \left(r_{jk}^{(3)} Q_0(\partial_\nu w_j, w_k) + \tilde{r}_{jk}^{(3)} Q_0(w_j, \partial_\nu w_k) \right),
\end{aligned}$$

$$\begin{aligned}
(2.10) \qquad G_2 = & -2p(F\partial_\nu u + u\partial_\nu F) \\
& -2 \sum_{1 \le j \le l_1} \left(q_j^{(1)} (\partial_\nu F v_j + \partial_\nu u g_j) + \tilde{q}_j^{(1)} (F\partial_\nu v_j + u\partial_\nu g_j) \right) \\
& -2 \sum_{1 \le j \le l_2} \left(q_j^{(2)} (\partial_\nu F w_j + \partial_\nu u h_j) + \tilde{q}_j^{(2)} (F\partial_\nu w_j + u\partial_\nu h_j) \right) \\
& -2 \sum_{1 \le j,k \le l_1} \left(r_{jk}^{(1)} (\partial_\nu g_j v_k + \partial_\nu v_j g_k) + \tilde{r}_{jk}^{(1)} (g_j \partial_\nu v_k + v_j \partial_\nu g_k) \right)
\end{aligned}$$

$$-2\sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}}\left(r_{jk}^{(2)}(\partial_\nu g_j w_k+\partial_\nu h_k)+\tilde r_{jk}^{(2)}(g_j\partial_\nu w_k+v_j\partial_\nu h_k)\right)$$
$$-2\sum_{1\le j,k\le l_2}\left(r_{jk}^{(3)}(\partial_\nu h_j w_k+\partial_\nu w_j h_k)+\tilde r_{jk}^{(3)}(h_j\partial_\nu w_k+w_j\partial_\nu h_k)\right),$$

(2.11)
$$\begin{aligned}G_3 &= 2(a+M^2p)u\partial_\nu u\\ &+\sum_{1\le j\le l_1}\{(b_j^{(1)}+M^2q_j^{(1)})\partial_\nu u v_j+(b_j^{(1)}+M^2\tilde q_j^{(1)})u\partial_\nu v_j\}\\ &+\sum_{1\le j\le l_2}\{(b_j^{(2)}+m^2q_j^{(2)})\partial_\nu u w_j+(b_j^{(2)}+m^2\tilde q_j^{(2)})u\partial_\nu w_j\}\\ &+\sum_{1\le j,k\le l_1}\{(c_{jk}^{(1)}+M^2r_{jk}^{(1)})\partial_\nu v_j v_k+(c_{jk}^{(1)}+M^2\tilde r_{jk}^{(1)})v_j\partial_\nu v_k\}\\ &+\sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}}\{(c_{jk}^{(2)}+m^2r_{jk}^{(2)})\partial_\nu v_j w_k+(c_{jk}^{(2)}+m^2\tilde r_{jk}^{(2)})v_j\partial_\nu w_k\}\\ &+\sum_{1\le j,k\le l_2}\{(c_{jk}^{(3)}-(M^2-2m^2)r_{jk}^{(3)})\partial_\nu w_j w_k\\ &\qquad+(c_{jk}^{(3)}-(M^2-2m^2)\tilde r_{jk}^{(3)})w_j\partial_\nu w_k\}.\end{aligned}$$

Here, we apply Lemma 2.3 to G_1 and so we have (2.7) with the following extra terms in the right hand side.

(2.12)
$$\begin{aligned}&2(a+3M^2p)u\partial_\nu u\\ &+\sum_{1\le j\le l_1}\{(b_j^{(1)}+M^2q_j^{(1)}+2M^2\tilde q_j^{(1)})\partial_\nu u v_j+(b_j^{(1)}+2M^2q_j^{(1)}+M^2\tilde q_j^{(1)})u\partial_\nu v_j\}\\ &+\sum_{1\le j\le l_2}\{(b_j^{(2)}+m^2q_j^{(2)}+2m^2\tilde q_j^{(2)})\partial_\nu u w_j+(b_j^{(2)}+2M^2q_j^{(2)}+m^2\tilde q_j^{(2)})u\partial_\nu w_j\}\\ &+\sum_{1\le j,k\le l_1}\{(c_{jk}^{(1)}+M^2r_{jk}^{(1)}+2M^2\tilde r_{jk}^{(1)})\partial_\nu v_j v_k+(c_{jk}^{(1)}+2M^2r_{jk}^{(1)}+M^2\tilde r_{jk}^{(1)})v_j\partial_\nu v_k\}\\ &+\sum_{\substack{1\le j\le l_1\\ 1\le k\le l_2}}\{(c_{jk}^{(2)}+m^2r_{jk}^{(2)}+2m^2\tilde r_{jk}^{(2)})\partial_\nu v_j w_k+(c_{jk}^{(2)}+2M^2r_{jk}^{(2)}+m^2\tilde r_{jk}^{(2)})v_j\partial_\nu w_k\}\\ &+\sum_{1\le j,k\le l_2}\{(c_{jk}^{(3)}-(M^2-2m^2)r_{jk}^{(3)}+2m^2\tilde r_{jk}^{(3)})\partial_\nu w_j w_k\\ &\qquad+(c_{jk}^{(3)}+2m^2r_{jk}^{(3)}-(M^2-2m^2)\tilde r_{jk}^{(3)})w_j\partial_\nu w_k\}.\end{aligned}$$

If all terms vanish in (2.12), we have the desired equation (2.7). Therefore, we have only to solve the following algebraic equations.

(2.13)
$$a+3M^2p=0,$$

(2.14)
$$b_j^{(1)}+M^2 q_j^{(1)} + 2M^2\tilde{q}_j^{(1)} = 0, \quad b_j^{(1)} + +2M^2 q_j^{(1)} + M^2\tilde{q}_j^{(1)} = 0,$$
(2.15)
$$b_j^{(2)}+m^2 q_j^{(2)} + 2m^2\tilde{q}_j^{(2)} = 0, \quad b_j^{(2)} + 2M^2 q_j^{(2)} + m^2\tilde{q}_j^{(2)} = 0,$$
(2.16)
$$c_{jk}^{(1)}+M^2 r_{jk}^{(1)} + 2M^2\tilde{r}_{jk}^{(1)} = 0, \quad c_{jk}^{(1)} + 2M^2 r_{jk}^{(1)} + M^2\tilde{r}_{jk}^{(1)} = 0,$$
(2.17)
$$c_{jk}^{(2)}+m^2 r_{jk}^{(2)} + 2m^2\tilde{r}_{jk}^{(2)} = 0, \quad c_{jk}^{(2)} + 2M^2 r_{jk}^{(2)} + m^2\tilde{r}_{jk}^{(2)} = 0,$$
(2.18)
$$c_{jk}^{(3)}-(M^2 - 2m^2)r_{jk}^{(3)} + 2m^2\tilde{r}_{jk}^{(3)} = 0,$$
$$c_{jk}^{(3)}+2m^2 r_{jk}^{(3)} - (M^2 - 2m^2)\tilde{r}_{jk}^{(3)} = 0.$$

A necessary and sufficient condition for all equations (2.13)-(2.18) to have solutions a, $q_j^{(s)}$, $\tilde{q}_j^{(s)}$, $r_{jk}^{(s)}$ and $\tilde{r}_{jk}^{(s)}$ is the following:

$$(2M - m)(M - 2m) \neq 0.$$

The restriction $2M \neq m$ comes from (2.15) and (2.17), and the restriction $M \neq 2m$ comes from (2.18). Accordingly, if $b_j^{(2)} = c_{jk}^{(2)} = 0$, we do not need the restriction $2M \neq m$ and if $c_{jk}^{(3)} = 0$, we do not need the restriction $M \neq 2m$. This completes the proof of Lemma 2.2. $\square$

We now mention several remarks on Lemma 2.2.

Remark 2.2. (i) Even if we consider complex-valued solutions, Lemma 2.2 also holds.

(ii) If either M or m vanishes, Lemma 2.2 still holds for a certain class of quadratic nonlinearity. Because (2.18) in the proof of Lemma 2.2 is solvable, when either M or m vanishes. Therefore, for example, we can apply Lemma 2.2 to the first equation in the following system:

$$(\Box + M^2)u = v^2,$$
$$\Box v = f(u, v, \partial u, \partial v).$$

This system is closely related to the Dirac-Proca system in the intermediate vector boson model for the weak interaction of elementary particles (see Aitchison and Hey [1, Section 10.1]). The argument analogous to Lemma 2.2 was applied to the Dirac-Proca equations in [39].

(iii) Suppose that f is cubic nonlinearity in (2.3) and g_j and h_j are quadratic nonlinearity in (2.4) and (2.5). Then, Lemma 2.2 implies that if we consider new

dynamical variables z_ν instead of u itself, equation (2.3) can be transformed into a new system of Klein-Gordon equations with strong null forms and cubic nonlinearity only.

We conclud this section by giving the local existence theorem for the Cauchy problem (1.3)-(1.6). Because the unique local solvability of the problem (1.3)-(1.6) seems to be interesting itself. We consider the problem (1.4)-(1.7) instead of (1.3)-(1.6). The proof of the following Lmma 2.4 reveals a semilinear feature of the system (1.4) and (1.7), that is, we can regard all nonlinearity in (1.4) and (1.7) as lower order perturbation with respect to the d'Alembertian $\Box$. From now on, we again use the convention that indices repeated are summed.

Lemma 2.4. *Let n be spatial dimensions with $n \geq 2$ and let k be an integer with $k \geq [n/2] + 2$.*

(i) We assume that

$$(\alpha^\mu, \beta^\mu, \phi_0, \phi_1) \in H^{k-1,k-1} \oplus H^{k-2,k-1} \oplus H^{k,k-1} \oplus H^{k-1,k-1}.$$

Then, there exist $\kappa_0 > 0$ and $T > 0$ such that if the initial data $(\alpha^\mu, \beta^\mu, \phi_0, \phi_1)$ satisfy

$$\|\alpha^\mu\|_{H^{[n/2]+1}} + \|\beta^\mu\|_{H^{[n/2]}} + \|\phi_0\|_{H^{[n/2]+2}} + \|\phi_1\|_{H^{[n/2]+1}} \leq \kappa$$

for some κ with $0 < \kappa \leq \kappa_0$ and the compatibility conditions (1.8) and (1.9) for the unitary gauge are satisfied, then the Cauchy problem (1.4)-(1.7) has a unique local solution (A^μ, ϕ) on $[0,T]$ satisfying

$$\begin{aligned}
&\partial^\alpha A^\mu(t), \quad \partial^\beta \phi(t) \in C([0,T]; H^{0,k-1}), \qquad |\alpha| \leq k-1, \quad |\beta| \leq k, \\
&\sum_{j=0}^{1} \big(\|\partial_0^j A^\mu(t)\|_{H^{[n/2]+1-j}} + \|\partial_0^j \phi(t)\|_{H^{[n/2]+2-j}} \big) \leq C\kappa, \quad t \in [0,T], \\
&\|\phi(t)\|_{L^\infty} \leq \frac{3M}{4\sqrt{2}}, \quad t \in [0,T],
\end{aligned}$$

where T is determined only by n and κ_0, and C does not depend on κ.

(ii) Let I be an arbitray interval in $\mathbf{R}$. We put

$$\begin{aligned}
X_I^k = \big\{ (A^\mu, \phi);\ &\partial^\alpha A^\mu(t), \partial^\beta \phi(t) \in C(I; H^{0,k-1}), \quad |\alpha| \leq k-1, |\beta| \leq k, \\
&\|\phi(t)\|_{L^\infty} \leq \frac{3M}{4\sqrt{2}}, \quad t \in I \big\}.
\end{aligned}$$

Then, the solution in X_I^k of (1.7), (1.4) and (1.5) with supplementary conditions (1.8) and (1.9) is a solution in X_I^k of (1.3)-(1.6) and the converse is also true.

Proof. We first show part (ii). For that purpose, we observe that if the initial data satisfy (1.8)-(1.9) and $\phi > -M/\sqrt{2}$, the problem (1.3)-(1.6) is equivalent to (1.7),

(1.4) and (1.5) in the following sense. We put $\varphi = \phi + M/\sqrt{2}$ and rewrite (1.3)-(1.4) and (1.6) as follows.

$$\Box A^\mu - \partial^\mu \partial_\nu A^\nu = -2A^\mu \varphi^2, \tag{2.19}$$

$$\Box \varphi = A_\nu A^\nu \varphi + \frac{1}{2}\big(m^2\varphi - \frac{2m^2}{M^2}|\varphi|^2\varphi\big), \tag{2.20}$$

$$(\partial_\nu A^\nu)\varphi + 2A_\nu \partial^\nu \varphi = 0. \tag{2.21}$$

We note that when we take the derivatives in x^μ of equations (2.19) and sum up the resulting equations over μ, we have equation (2.21), that is, the unitary gauge condition. Using the unitary gauge conditon (2.21), we replace equations (2.19) by the following.

$$\Box A^\mu + \partial^\mu(2A_\nu(\partial^\nu\varphi)\varphi^{-1}) = -2A^\mu\varphi^2. \tag{2.22}$$

Equation (2.22) makes sense, since φ is expected to remain close to $M/\sqrt{2}$. We now regard equations (2.22) and (2.20) as a system with which we start. If we have a regular solution (A^μ, φ) of system (2.22) and (2.20) with $\varphi(t,x) > 0$ and initial data at $t = 0$ satisfy (1.8)-(1.9), we can easily restore the unitary gauge condition (2.21). Indeed, we take the derivatives in x^μ of equations (2.22) and sum up the resulting equations over μ to obtain

$$(\Box + 2\varphi^2)(\partial_\nu A^\nu + 2A_\nu(\partial^\nu\varphi)\varphi^{-1}) = 0. \tag{2.23}$$

By assumptions (1.8)-(1.9) and equation (2.22) with $\mu = 0$, we also have

$$\partial_0^j(\partial_\nu A^\nu + 2A_\nu(\partial^\nu\varphi)\varphi^{-1})(0) = 0, \quad j = 0, 1. \tag{2.24}$$

Therefore, if $\varphi(t,x) > 0$, we can derive (2.21) from (2.23) and (2.24). Thus, we conclude that the Cauchy problem of system (2.19)-(2.21) is equivalent to that of (2.22) and (2.20) under assumptions (1.8) and (1.9), as long as $\varphi > 0$. This implies that if the initial data satisfy (1.8)-(1.9) and $\phi > -M/\sqrt{2}$, the problem (1.3)-(1.6) is equivalent to (1.7), (1.4) and (1.5), which shows (ii) of Lemma 2.4.

We next show part (i), that is, the unique local existence of solution for (1.4)-(1.7). For that purpose, we consider the following nonlinear mapping.

$$N[(A^\mu, \phi)] = (a^\mu, b) + (N_1, N_2),$$

where $\omega = (-\Delta)^{1/2}$ and

$$
\begin{aligned}
a^\mu(t) &= \cos\omega t\ \alpha^\mu + \omega^{-1}\sin\omega t\ \beta^\mu, \\
b(t) &= \cos\omega t\ \phi_0 + \omega^{-1}\sin\omega t\ \phi_1, \\
N_1[(A^\mu,\phi)] &= \int_0^t \omega^{-1}\sin\omega(t-s)\ \big[-\partial^\mu\{2A_\nu(\partial^\nu\phi)(\phi+M/\sqrt{2})^{-1}\} \\
&\qquad\qquad - 2A^\mu(\phi+M/\sqrt{2})^2\big]\ ds, \\
N_2[(A^\mu,\phi)] &= \int_0^t \omega^{-1}\sin\omega(t-s)\ \big(-m^2\phi + \frac{M}{\sqrt{2}}A_\nu A^\nu \\
&\qquad\qquad + A_\nu A^\nu\phi - \frac{3m^2}{\sqrt{2}M}\phi^2 - \frac{m^2}{M^2}\phi^3\big)\ ds.
\end{aligned}
$$

Let T be a constant with $0 < T \le 1$ to be determined later. We put

$$
\mathcal{H} = \big(\bigcap_{j=0}^{1} C^j([0,T];H^{[n/2]+1-j})\big)^{1+n} \oplus \big(\bigcap_{j=0}^{1} C^j([0,T];H^{[n/2]+2-j})\big),
$$

$$
\|(A^\mu,\phi)\|_{\mathcal{H}} = \sum_{\mu=0}^{n}\sum_{j=0}^{1}\big(\sup_{0\le t\le T}\|\partial_t^j A^\mu(t)\|_{H^{[n/2]+1-j}} + \sup_{0\le t\le T}\|\partial_t^j\phi(t)\|_{H^{[n/2]+2-j}}\big),
$$

$$
(A^\mu,\phi)\in\mathcal{H}.
$$

We consider A^μ as an element in $C([0,T];H^{[n/2]+1})$, while ϕ is considered as an element in $C([0,T];H^{[n/2]+2})$. Because the nonlinear function N_1 includes the second derivatives of ϕ and we only have by the Sobolev embbeding theorem

$$
N_1[(A^\mu,\phi)] \in C([0,T];H^{[n/2]+1})
$$

for $(A^\mu,\phi)\in\mathcal{H}$. On the other hand, we note that $N_2[(A^\mu,\phi)]$ belongs to $C([0,T];H^{[n/2]+2})$ for $(A^\mu,\phi)\in\mathcal{H}$. By the assumption on initial data, we have

$$
\|(a^\mu,b)\|_{\mathcal{H}} \le C_0\kappa,
$$

where C_0 is a positive constant independent of κ and T with $0\le T\le 1$. We define the complete metric space X_T as follows.

$$
X_T = \{(A^\mu,\phi)\in\mathcal{H};\ \|(A^\mu,\phi)\|_{\mathcal{H}} \le 4C_0\kappa\}
$$

with the metric

$$
d((A^\mu,\phi),(B^\mu,\psi)) = \|(A^\mu-B^\mu,\phi-\psi)\|_{\mathcal{H}}.
$$

Now, we choose $\kappa > 0$ so small that for any second component ϕ of element in X_T, we can have by the Sobolev embedding theorem

$$
\|\phi(t)\|_{L^\infty} \le \frac{3M}{4\sqrt{2}}, \quad t\in[0,T],
$$

which implies

$$\varphi(t,x) = \phi(t,x) + \frac{M}{\sqrt{2}} \geq \frac{M}{4\sqrt{2}}, \quad (t,x) \in [0,T] \times \mathbf{R}^n.$$

Then, if we choose $T > 0$ small, the standard contraction argument yields the unique fixed point $(A^\mu, \phi) \in X_T$ of the nonlinear mapping N (for the details, see, e.g., Ginibre and Velo [17]). This fixed point $(A^\mu, \phi) \in X_T$ is a unique local solution on $[0,T]$ of the Cauchy problem (1.7), (1.4) and (1.5) with supplementary conditions (1.8) and (1.9). The rest of Lemma 2.4 (i) is easily proved. $\square$

§3. Proof of Theorem 1.1

In this section, we give an outline of the proof of estimates (1.11), (1.12) and (1.13) in Theorem 1.1 for $n = 2$, which is the main part of the proof for Theorem 1.1. If we have showed (1.11), (1.12) and (1.13), it is easy to prove the rest of Theorem 1.1.

The difficulty to treat quadratic nonlinearity for $n = 2$ consists in the insufficiency of decay rate. For example, We take the $L^2(\mathbf{R}^2)$ norm of the first term on the right hand side of (1.7) to have by Hölder's inequality

$$\|\sqrt{2}MA^\mu\phi\|_{L^2} \leq C\|A^\mu(t)\|_{L^\infty}\|\phi(t)\|_{L^2}. \tag{3.1}$$

Since we may expect that $\|\phi(t)\|_{L^2}$ is bounded for all t and

$$\|A^\mu(t)\|_{L^\infty} \sim |t|^{-1} \quad (|t| \to \infty),$$

the right hand side of (3.1) is not integrable on $\mathbf{R}$. So, it is difficult to control the quadratic nonlinear terms on the right hand side of (1.7) and (1.4) as they are. Therefore, we introduce new dynamical variables$((B^\mu)_\nu, \psi_\nu)$ as follows:

$$\begin{aligned}(B^\mu)_\nu = {} & \partial_\nu A^\mu - C_1\partial_\nu\partial^\mu A_\gamma\partial^\gamma\phi - C_2\partial^\mu A_\gamma\partial^\eta\partial_\nu\phi \\ & - C_3\partial_\nu A_\gamma\partial^\gamma\partial^\mu\phi - C_4 A_\gamma\partial^\gamma\partial^\mu\partial_\nu\phi \\ & - C_5\partial_\nu A^\mu\phi - C_6 A^\mu\partial_\nu\phi,\end{aligned} \tag{3.2}$$

$$\psi_\nu = \partial_\nu\phi - 2C_7\partial_\nu A_\eta A^\eta - 2C_8\phi\partial_\nu\phi, \tag{3.3}$$

where C_j, $1 \leq j \leq 8$ are real constants to be determined by Lemma 2.2. We apply Lemma 2.2 to equations (1.7) and (1.4), by choosing $a = b_j^{(s)} = c_{jk}^{(1)} = c_{jk}^{(3)} = 0$ and $l_1 = l_2 = 2(1+n)$ for (1.7) and $b_j^{(s)} = c_{jk}^{(1)} = c_{jk}^{(2)} = 0$ and $l_2 = 1$ with the roles of M and m exchanged for (1.4) (see Remark 2.1 in Section 2). Here we note that when we apply Lemma 2.2 to (1.7), we regard A^μ on the right hand side of (1.7)

as one of the v_j's and so $a = b_j^{(s)} = 0$. If we choose C_j, $1 \le j \le 8$ by Lemma 2.2, we have the following new system with respect to dynamical variables $((B^\mu)_\nu, \psi_\nu)$.

$$
\begin{aligned}
(3.4)\qquad (\Box+M^2)(B^\mu)_\nu &= \sum_{\gamma=0}^{2} \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{1\eta}^{(\nu)} Q_{\eta\nu}(\partial^\mu A_\gamma, \partial^\eta\partial^\gamma\phi) \\
&+ \sum_{\gamma=0}^{2} \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{2\eta}^{(\nu)} Q_{\eta\nu}(\partial^\gamma\phi, \partial^\eta\partial^\mu A_\gamma) + \sum_{\gamma=0}^{2} \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{3\eta}^{(\nu)} Q_{\eta\nu}(A_\gamma, \partial^\eta\partial^\gamma\partial^\mu\phi) \\
&\sum_{\gamma=0}^{2} \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{4\eta}^{(\nu)} Q_{\eta\nu}(\partial^\gamma\partial^\mu\phi, \partial^\eta A_\gamma) + \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{5\eta}^{(\nu)} Q_{\eta\nu}(A^\mu, \partial^\eta\phi) \\
&+ \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{6\eta}^{(\nu)} Q_{\eta\nu}(\phi, \partial^\eta A^\mu) \\
&+ N_1^{(\nu)}(\partial^\alpha A^\gamma, \partial^\beta\phi;\ |\alpha|\le 3, |\beta|\le 4), \quad x\in\mathbf{R}^2, \\
(3.5)\qquad (\Box+m^2)\psi_\nu &= \sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{7\eta}^{(\nu)} \sum_{0\le\mu\le 2} Q_{\eta\nu}(A_\mu, \partial^\eta A^\mu) \\
&\sum_{\substack{0\le\eta\le 2\\ \eta\neq\nu}} D_{8\eta}^{(\nu)} Q_{\eta\nu}(\phi, \partial^\eta\phi) \\
&+ N_2^{(\nu)}(\partial^\alpha A^\gamma, \partial^\beta\phi;\ |\alpha|\le 2, |\beta|\le 3), \quad x\in\mathbf{R}^2,
\end{aligned}
$$

where $D_{j\eta}^{(\nu)}$ are constants and $N_1^{(\nu)}$ and $N_2^{(\nu)}$ are cubic nonlinearity with respect to $(\partial^\alpha A^\gamma, \partial^\beta\phi;\ |\alpha|\le 3, |\beta|\le 4)$ and $(\partial^\alpha A^\gamma, \partial^\beta\phi;\ |\alpha|\le 2, |\beta|\le 3)$, respectively.

All terms on the right hand side of (3.4) and (3.5) are strong null forms or cubic nonlinearity, which have the sufficient decay property.

For simplicity, we consider the case of $t \ge 0$ only, because the proof for the case $t \le 0$ is the same. We put

$$
\begin{aligned}
(3.6)\qquad |||(A^\mu,\phi)(t)||| &= \sum_{0\le\mu\le 2} \sum_{|\alpha|\le 2} \sup_{0\le s\le t} \big(\|\partial^\alpha A^\mu(s)\|_{k-6} + \|\partial^\alpha\phi(s)\|_{k-5}\big) \\
&+ \sum_{0\le\mu\le 2} \sum_{|\alpha|\le 1} \sup_{0\le s\le t} (1+s)^{-\delta}\big(\|\partial^\alpha A^\mu(s)\|_{k-2} + \|\partial^\alpha\phi(s)\|_{k-1}\big) \\
&+ \sum_{0\le\mu\le 2} \sum_{|\alpha|\le 2} \sup_{\substack{s\in[0,t)\\ x\in\mathbf{R}^2}} (1+s+|x|)^{-1}\big(|\partial^\alpha A^\mu(s,x)|_{k-12} + |\partial^\alpha\phi(s,x)|_{k-11}\big).
\end{aligned}
$$

By using the standard argument, Lemma 2.1 and (1.21)-(1.22), we can derive the energy estimates of order $\le k-4$ and of order $\le k-5$ and the decay estimates of order $\le k-9$ and of order $\le k-8$ from (3.4) and (3.5) for $\partial^\alpha(B^\mu)_\nu$ and $\partial^\alpha\psi_\nu$),

$|\alpha| \le 1$, respectively. From these estimates and (3.2)-(3.3), we easily obtain the energy estimates of order $\le k-6$ and of order $\le k-5$ and the decay estimates of order $\le k-11$ and of order $\le k-10$ for $\partial^\alpha A^\mu$ and $\partial^\alpha \phi$, $1 \le |\alpha| \le 2$, respectively. Once we have the estimates of derivatives of the solution, we recover the estimates of the solution itself by using the fact that the Klein-Gordon equation has a mass term. Thus, we have

(3.7)
$$\begin{aligned}
&\sum_{0\le\mu\le 2}\sum_{|\alpha|\le 2}\left(\|\partial^\alpha A^\mu(t)\|_{k-6} + \|\partial^\alpha\phi(t)\|_{k-5}\right)\\
&+\sum_{0\le\mu\le 2}\sum_{|\alpha|\le 2}(1+t+|x|)^{-1}\left(|\partial^\alpha A^\mu(t,x)|_{k-11} + |\partial^\alpha\phi(t,x)|_{k-10}\right)\\
&\le C\left(\varepsilon + |||(A^\mu,\phi)(t)|||^2 + |||(A^\mu,\phi)(t)|||^4\right), \quad t\ge 0, \quad x\in\mathbf{R}^2.
\end{aligned}$$

But the energy estimates derived from (3.4) and (3.5) have an undesirable feature of derivative loss. Hence, we need to return to the original system of (1.3) and (1.4), when we derive the estimates of higher order. Equations (1.3) and (1.4) yield the energy estimates without derivative loss, but we lose the boundedness for $\|\partial^\alpha A^\mu(t)\|_{k-2}$, $\|\partial^\alpha\phi(t)\|_{k-1}$, $|\alpha|\le 1$ by $|t|^\delta$. Accordingly, we have

(3.8)
$$\begin{aligned}
&\sum_{0\le\mu\le 2}\sum_{|\alpha|\le 1}\sup_{0\le s\le t}(1+s)^{-\delta}\left(\|\partial^\alpha A^\mu(s)\|_{k-2} + \|\partial^\alpha\phi(s)\|_{k-1}\right)\\
&\le C\left(\varepsilon + |||(A^\mu,\phi)(t)|||^2 + |||(A^\mu,\phi)(t)|||^4\right), \quad t\ge 0.
\end{aligned}$$

If we choose $\varepsilon > 0$ sufficiently small for any positive constant η, then estimates (3.7) and (3.8) implies that

$$|||(A^\mu,\phi)(t)||| \le \eta, \quad t\ge 0,$$

which shows (1.11), (1.12) and (1.14). For the details, see, e.g., [26], [28] and [31].

Concluding Remark. (i) The proof in this paper is applicable to the quadratic quasi-linear case, that is, the case that the quadratic nonlinearity depends on derivatives of unknown functions up to second order except for the second derivative in t and depends linearly on the second derivatives. In that case, we have only to choose $(v_j) = (\partial_\mu u, \partial_{\mu j} u)$ to apply Lemma 2.2 to the equation of u.

(ii) The problem whether the constant equilibria are stable for $2M = m$ or not seems very interesting. There are several papers treating the global solution of system with long range effect and the small data blowup (see, e.g., [7], [8], [10] and [11] for the long range problem and [24] for the small data blowup problem). The

author does not know whether the proofs in those papers will be applicable to our case $2M = m$.

(iii) Recently, Sunagawa [38] has studied a system of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension by using an argument analogous to Lemma 2.2 in Section 2. The difference between the arguments in this paper and in [38] is how to eliminate undesirable terms Q_0 in the process of transforming the original equation into a new system with strong null forms and cubic nonlinearity only. In this paper, we consider derivatives of unknown functions to eliminate terms Q_0, while the definition of new dynamical variables includes terms Q_0 in [38].

Acknowledgments. The author would like to express his deep gratitude to Professor S. Alinhac for pointing out a mistake in the first version of this paper about the formulation of the unitary gauge condition.

References

[1] I. J. R. Aitchison and A. J. G. Hey, *Gauge Theories in Particle Physics, A Practical Introduction*, Second Edition, Adam Hilger, Bristol and Philadelphia, 1989.

[2] A. Bachelot, *Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon*, Ann. Inst. Henri Poincaré, Phys. Théor. **48** (1988), 387–422.

[3] M. S. Berger and Y. Y. Chen, *Symmetric vortices for the Ginzburg-Landau equations of superconductivity and the nonlinear desingularization phenomenon*, J. Funct. Anal. **82** (1989), 259–295.

[4] J. Burzlaff and V. Moncrief, *The global existence of time-dependent vortex solutions*, J. Math. Phys. **26** (1985), 1368–1372.

[5] Y. Choquet-Bruhat and D. Christodoulou, *Existence of global solutions of the Yang-Mills, Higgs and spinor field equations in 3 + 1 dimensions*, Ann. Scient. Ec. Norm. Sup., 4^{e} Série **14** (1981), 481–506.

[6] D. Christodoulou, *Global solutions of nonlinear hyperbolic equations for small initial data*, Comm. Pure Appl. Math. **39** (1986), 267–282.

[7] D. Chiristodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space*, Princeton University Press, 1993.

[8] J.-M. Delort, *Existence globale et comportement asymptotique pour l'équation de Klein-Gordon quasilinéaire à données petites en dimension 1*, Ann. Scient. Ec. Norm. Sup., 4^{e} Série **34** (2001), 1–61.

[9] D. Eardley and V. Moncrief, *The global existence of Yang-Mills-Higgs fields in 4-dimensional Minkowski space. I. Local existence and smoothness properties, II. Completion of proof*, Comm. Math. Phys. **83** (1982), 171–191, 193–212.

[10] M. Flato, J.C.H. Simon and E. Taflin, *On global solutions of the Maxwell-Dirac equations*, Comm. Math. Phys. **112** (1987), 21–49.

[11] M. Flato, J.C.H. Simon and E. Taflin, *Asymptotic Completeness, Global Existence and the Infrared Problem for the Maxwell-Dirac equations*, Memoirs Amer. Math. Soc., No. 606, 1997.

[12] V. Georgiev, *Global solution of the system of wave and Klein-Gordon equations*, Math. Z. **203** (1990), 683–698.

[13] V. Georgiev, *Small amplitude solutions of the Maxwell-Dirac equations*, Indiana Univ. Math. J. **40** (1991), 845–883.

[14] V. Georgiev, *Decay estimates for the Klein-Gordon equation*, Comm. Part. Diff. Eqs. **17** (1992), 1111-1139.

[15] V. Georgiev and S. Lucente, *Weighted Sobolev spaces applied to nonlinear Klein-Gordon equation*, C. R. Acad. Sci. Paris, Série I **329** (1999), 21–26.

[16] V. Georgiev and P. Popivanov, *Global solution to the two-dimensional Klein-Gordon equation*, Comm. Part. Diff. Eqs. **16** (1991), 941–995.
[17] J. Ginibre and G. Velo, *The Cauchy problem for coupled Yang-Mills and scalar fields in the Lorentz gauge*, Ann. Inst. Henri Poincaré, Phys. Théor. **36** (1982), 59–78.
[18] R. T. Glassey, *On the asymptotic behavior of nonlinear wave equation*, Trans. Amer. Math. Soc. **182** (1973), 187–200.
[19] Y. Guo, *Instability of symmetric vortices with large charge and coupling constant*, Comm. Pure Appl. Math. **49** (1996), 1051–1080.
[20] S. Gustafson and I. M. Sigal, *The stability of magnetic vortices*, Comm. Math. Phys. **212** (2000), 257–275.
[21] L. Hörmander, *Lecture on Nonlinear Hyperbolic Differential Equations*, Springer-Verlag, Berlin-New York-Heidelberg, 1997.
[22] A. Jaffe and C. Taubes, *Vortices and Monopoles*, Birkhäuser, 1980.
[23] M. Keel, *Global existence for critical power Yang-Mills-Higgs equations in* $\mathbb{R}^{3+1}$, Comm. Part. Diff. Eqs. **22** (1997), 1161–1225.
[24] M. Keel and T. Tao, *Small data blow-up for semilinear Klein-Gordon equations*, Amer. J. Math. **121** (1999), 629–669.
[25] S. Klainerman, *Global existence of small amplitude solutions to nonlinear Klein-Gordon equations in four space time dimensions*, Comm. Pure Appl. Math. **38** (1985), 631–641.
[26] S. Klainerman, *The null condition and global existence to nonlinear wave equations*, Lect. Appl. Math. **23** (1986), 293–326.
[27] S. Klainerman and M. Machedon, *Finite energy solutions for the Yang-Mills equations*, Ann. Math. **14** (1995), 39–119.
[28] R. Kosecki, *The unit condition and global existence for a class of nonlinear Klein-Gordon equations*, J. Diff. Eqs. **100** (1992), 257–268.
[29] H. Lindblad and C. D. Sogge, *Restriction theorems and semilinear Klein-Gordon equations in* $(1+3)$*- dimensions*, Duke Math. J. **85** (1996), 227–252.
[30] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. RIMS, Kyoto Univ. **12** (1976), 169–189.
[31] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Global existence and asymptotic behavior of solutions for the Klein-Gordon equations with quadratic nonlinearity in two space dimensions*, Math. Z. **222** (1996), 341–362.
[32] T. Ozawa, K. Tsutaya and Y. Tsutsumi, *Remarks on the Klein-Gordon equation with quadratic nonlinearity in two space dimensions*, in the Proceedings of "Nonlinear Waves", pp. 383–392, GAKUTO International Series, Mathematical Sciences and Applications, Vol. 10, 1997.
[33] B. Plohr, *The behavior at infinity of isotropic vortices and monopoles*, J. Math. Phys. **22** (1981), 2184–2190.
[34] M. Schwarz, Jr., *Global solutions of Maxwell-Higgs on Minkowski space*, J. Math. Anal. Appl. **229** (1999), 426–440.
[35] J. Shatah, *Normal forms and quadratic nonlinear Klein-Gordon equations*, Comm. Pure Appl. Math. **38** (1985), 685–696.
[36] J. C. H. Simon, *A wave operator for a non-linear Klein-Gordon equation*, Lett. Math. Phys. **7** (1983), 387–398.
[37] J. C. H. Simon and E. Taflin, *The Cauchy problem for non-linear Klein-Gordon equations*, Comm. Math. Phys. **152** (1993), 433–478.
[38] H. Sunagawa, *On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimensions*, preprint, 2001.
[39] Y. Tsutsumi, *Global solutions for the Dirac-Proca equations with small initial data in 3 + 1 space time dimensions*, preprint, 2001.
[40] C. H. Taubes, *A brief survey of the Yang-Mills-Higgs equations on* $\mathbb{R}^3$, in "Nonlinear partial differential equations and their applications, Collège de France Seminar, Vol. VII (Paris, 1983–1984)", edited by H. Brézis and J.-L. Lions, pp. 257–271, Res. Notes in Math., **122**, Pitman, Boston, MA, 1985.