Approximations of the Relativistic Euler-Poisson-Darboux Equation

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1 Introduction and conclusions

In [2] we investigated the Cauchy problem to the relativistic Euler equation

$$(rE) \qquad \frac{\partial}{\partial t}\frac{\rho + Pu^2/c^4}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{(\rho + P/c^2)u}{1-u^2/c^2} = 0,$$

$$\frac{\partial}{\partial t}\frac{(\rho + P/c^2)u}{1-u^2/c^2} + \frac{\partial}{\partial x}\frac{P+\rho u^2}{1-u^2/c^2} = 0.$$

Here c is a positive constant, the speed of light, and P is a given smooth function of ρ satisfying the assumption

(A): $P(\rho) > 0, 0 < P' = dP/d\rho < c^2, 0 < P'' = d^2P/d\rho^2$ for $\rho > 0$, and

$$(1) \qquad P = A_0\rho^{5/3}(1 + \sum_{j=1}^{\infty} A_j(\rho^{2/3}/c^2)^j)$$

as $\rho \to 0$. Here A_0 is a positive constant and $\sum A_j z^j$ is a power series with positive radius of convergence.

In order to prove the existence of global weak solutions to the Cauchy problem by the theory of compensated compactness, we have to solve *the relativistic Euler-Poisson-Darboux equation*

$$(rEPD) \qquad \eta_{xx} - \eta_{yy} + A(x,y)\eta_y + B(x,y)\eta_x = 0,$$

where the independent variables are

$$(2) \qquad x = \frac{c}{2}\log\frac{c+u}{c-u}, \qquad y = \int_0^\rho \frac{\sqrt{P'}}{\rho + P/c^2}d\rho$$

and the unknown function $\eta(x,y)$ is an entropy to (rE). The coefficients of $(rEPD)$ are given by

$$A(x,y) = \frac{1}{\sqrt{P'}}(1 - \frac{P'}{c^2} - \frac{\rho + P/c^2}{2P'}P'')\frac{1+P'u^2/c^4}{1-P'u^2/c^4},$$

$$B(x,y) = -\frac{2u/c^2}{1-P'u^2/c^4}(1 - \frac{P'}{c^2} - \frac{\rho+P/c^2}{2P'}P'').$$

By the definition (2) and the assumption (1), we see that $A(x, y)$ and $B(x, y)$ are of the form

$$\begin{aligned} A(x,y) &= \frac{2}{y} + \epsilon y a(\epsilon x^2, \epsilon y^2) = \frac{2}{y} + \epsilon y \sum_{j+k\geq 0} a_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\ a_{00} &= -\frac{4}{9}(1 + \frac{7}{20}\frac{A_1}{A_0}), \\ B(x,y) &= -\frac{4}{3}\epsilon x b(\epsilon x^2, \epsilon y^2) = -\frac{4}{3}\epsilon x \sum_{j+k\geq 0} b_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\ b_{00} &= 1. \end{aligned}$$

Here and hereafter we denote

$$\epsilon = 1/c^2. \tag{3}$$

Introducing the new unknown V by

$$\frac{\partial \eta}{\partial y} = yV, \qquad \eta(x,y) = IV(x,y) = \int_0^y YV(x,Y)dY, \tag{4}$$

the singularity of $A(x, y)$ in $(rEPD)$ can be eliminated and $(rEPD)$ is reduced to the equation

$$V_{yy} - V_{xx} = \epsilon(yaV_y - \frac{4}{3}xbV_x + 2(a + \epsilon y^2 D_2 a)V - \frac{8}{3}\epsilon x D_2 b I V_x).$$

The problem

$$V_{yy} - V_{xx} = \epsilon(yaV_y - \frac{4}{3}xbV_x + 2(a + \epsilon y^2 D_2 a)V - \frac{8}{3}\epsilon x D_2 b I V_x), \tag{Q}$$

$$V|_{y=0} = 0, \qquad V_y|_{y=0} = 4\phi(x)$$

admits a unique solution V for any smooth ϕ given by a formula

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi - x,\epsilon)\phi(\xi)d\xi. \tag{5}$$

For the proof see [2], Section 5. $G(x, y, z, \epsilon)$ is a smooth function of $|x| < \infty, y \geq 0, |z| \leq y$. Therefore by defining

$$K(x,y,z,\epsilon) = JG(x,y,z,\epsilon) = \int_{|z|}^y YG(x,Y,z,\epsilon)dY, \tag{6}$$

we have a formula

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi - x,\epsilon)\phi(\xi)d\xi \tag{7}$$

for solutions of $(rEPD)$. We call this formula *the relativistic Darboux formula* and K *the relativistic Darboux kernel.* We know

$$K = (y^2 - z^2)(1 + O(\epsilon y)). \tag{8}$$

The purpose of this article is to study the properties of this relativistic Darboux kernel. The motivations are as follows.

First we see that $K(x,y,z,\epsilon)$ tends to the Darboux kernel $K(y,z)=y^2-z^2$ as ϵ tends to 0. Therefore the solution η of $(rEPD)$ tends to those of the Euler-Poisson-Darboux equation

$$\eta_{uu}-\eta_{yy}+\frac{2}{y}\eta_y=0$$

as ϵ tends to 0. But η is an entropy and there is a gap between the convergence of entropies and the convergence of solutions (ρ,u) of the Euler equation. We conjecture that weak solutions to (rE) contain a subsequence which converges to a weak solution of the non-relativistic Euler equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0,$$
$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho u^2+P)=0$$

as $\epsilon=1/c^2$ tends to 0. Actually this is the case if we assume $P=A\rho, A$ being a constant $<c^2$, since the total variations of the solutions obtained by Glimm's scheme in [5] can be estimated uniformly with respect to ϵ. See [4] for a proof. But this conjecture is not yet proved if we assume a more realistic equation of states (A). In order to approach this problem it is necessary to give a particuler account of the dependence of K upon ϵ. We want to find the first order term of the expansion of K with respect to ϵ. Using such detailed informations, we might discuss the properties of weak solutions to the equation with the parameter ϵ which are obtained by the compensated compactness method. Maybe a general theory cannot be expected and we must consider according to the situations. Second although we can guarantee the existence of weak solutions for any bounded initial data $\rho|_{t=0}=\rho^0(x), u|_{t=0}=u^0(x)$ such that

$$0\le\rho^0(x)\le M,\qquad |\frac{c}{2}\log\frac{c+u^0(x)}{c-u^0(x)}|\le M$$

provided that $\epsilon=1/c^2\le\epsilon_0(M)$, $\epsilon_0(M)$ being a positive number depending upon M, but what happens if ϵ is not so small and M is large? Observing the proof of [2], we find that we use the fact that $K(x,y,z,\epsilon)>0$ on the considered region in which $|z|<y$. Thus we wonder whether $K>0$ if ϵ is not so small and x,y are large. In other words the global behavior of the kernel K is of interest. In order to consider this problem, we also should give a particular account of the dependence of K upon ϵ. These are the motivations of this study.

The conclusions of the present study are

Theorem 1 *The relativistic Darboux kernel $K(x,y,z,\epsilon)$ is analytic in ϵ and*

$$K(x,y,z,\epsilon)=(y^2-z^2)(1+\sum_{\nu=1}^{\infty}K_\nu(x,y,z)\epsilon^\nu),$$

where $K_\nu(x,y,z)$ is a homogeneous polynomial of x,y,z of order 2ν of the form

$$K_\nu=\sum_{i+2j+k=2\nu}K_{ijk}x^iy^{2j}z^k.$$

The power series is convergent for $|\epsilon| \leq \delta/M^2$, where $|x|+|y| \leq M, |z| \leq |y|$ and δ is a positive constant. Particularly

$$K_1 = (\frac{5a_{00}}{16} - \frac{1}{4})y^2 - (\frac{a_{00}}{16} + \frac{1}{12})z^2 - \frac{2}{3}xz.$$

Theorem 2 *The relativistic Darboux kernel $K(x,y,z,\epsilon)$ is approximated by $K^a(x,y,z,\epsilon)$ given in the following manner. If $A_1 < 0$, then*

$$K^a(x,y,z,\epsilon) = 2\int_{|z|}^{y} I_0(\sqrt{\kappa(Y^2-z^2)})\exp(\epsilon(\frac{1}{3}x^2 + \alpha Y^2))Y dY;$$

If $A_1 = 0$, then

$$K^a(x,y,z,\epsilon) = -\frac{9}{\epsilon}(\exp(-\frac{\epsilon}{9}y^2) - \exp(-\frac{\epsilon}{9}z^2))\exp(\epsilon(\frac{1}{3}x^2));$$

If $A_1 > 0$, then

$$K^a(x,y,z,\epsilon) = 2\int_{|z|}^{y} J_0(\sqrt{-\kappa(Y^2-z^2)})\exp(\epsilon(\frac{1}{3}x^2 + \alpha Y^2))Y dY.$$

Here I_0 is the modified Bessel function of order 0, J_0 is the Bessel function of order 0, $\alpha = -(1 + 7A_1/20A_0)/9$, $\kappa = -(7A_1/30A_0)\epsilon$ and for any smooth ϕ the function

$$\eta(x,y) = \int_{x-y}^{x+y} K^a(x,y,\xi - x,\epsilon)\phi(\xi)d\xi$$

satisfies the equation

$$(\clubsuit) \qquad \eta_{xx} - \eta_{yy} + (\frac{2}{y} + \epsilon a_{00}y)\eta_y - \frac{4}{3}\epsilon x\eta_x = -\epsilon^2(\frac{4}{9}x^2 - \frac{a_{00}^2}{4}y^2)\eta - \epsilon^2\frac{a_{00}^2}{2}I\eta,$$

which is congruent with (rEPD) modulo $O(\epsilon^2)$.

2 Power series expansion in ϵ

The equation of (Q) can be rewritten as

$$V_{yy} - V_{xx} = \epsilon((yaV)_y - \frac{4}{3}(xbV)_x + cV - \frac{8}{3}\epsilon(xfIV)_x + \frac{8}{3}\epsilon hIV).$$

Here

$$\begin{aligned}
c &= a + \frac{4}{3}b + \frac{8}{3}\epsilon x^2 D_1 b = \sum_{j+k\geq 0} c_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\
c_{00} &= a_{00} + \frac{4}{3}, \\
f &= D_2 b = \sum_{j+k\geq 0} f_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\
h &= D_2 b + 2\epsilon x^2 D_1 D_2 b = \sum_{j+k\geq 0} h_{jk}(\epsilon x^2)^j(\epsilon y^2)^k.
\end{aligned}$$

Therefore the kernel G can be constructed by the iteration

$$\begin{aligned} G^{(0)} &= 2, \\ G^{(n+1)} &= 2+\frac{\epsilon}{2}(T_1G^{(n)}-\frac{4}{3}T_2G^{(n)}+T_3G^{(n)})+\frac{4}{3}\epsilon^2(-T_4JG^{(n)}+T_5JG^{(n)}), \end{aligned}$$

where

$$\begin{aligned} T_1G(x,y,z) &= \int_{\frac{y+z}{2}}^{y} Ya(\epsilon(x-y+Y)^2,\epsilon Y^2)G(x-y+Y,Y,z+y-Y)dY + \\ &+ \int_{\frac{y-z}{2}}^{y} Ya(\epsilon(x+y-Y)^2,\epsilon Y^2)G(x+y-Y,Y,z-y+Y)dY, \\ T_2G(x,y,z) &= \int_{\frac{y-z}{2}}^{y} (x+y-Y)b(\epsilon(x+y-Y)^2,\epsilon Y^2)G(x+y-Y,Y,z-y+Y)dY + \\ &- \int_{\frac{y+z}{2}}^{y} (x-y+Y)b(\epsilon(x-y+Y)^2,\epsilon Y^2)G(x-y+Y,Y,z+y-Y)dY, \\ T_3G(x,y,z) &= \int\int_{D(x,y,z)} c(\epsilon X^2,\epsilon Y^2)G(X,Y,z+x-X)dXdY, \end{aligned}$$

where

$$\begin{aligned} D(x,y,z) &= \{(X,Y): z+x-X\le Y, -z-x+X\le Y, \\ &\quad Y\le -x+y+X, Y\le x+y-X\} \\ T_4JG(x,y,z) &= \int_{\frac{y-z}{2}}^{y} (x+y-Y)f(\epsilon(x+y-Y)^2,\epsilon Y^2)JG(x+y-Y,Y,z-y+Y)dY + \\ &- \int_{\frac{y+z}{2}}^{y} (x-y+Y)f(\epsilon(x-y+Y)^2,\epsilon Y^2)JG(x-y+Y,Y,z+y-Y)dY, \\ T_5JG(x,y,z) &= \int\int_{D(x,y,z)} h(\epsilon X^2,\epsilon Y^2)JG(X,Y,z+x-X)dXdY. \end{aligned}$$

We fix $\delta>0$ such that the power series $a(X,Y)=\sum a_{jk}X^jY^k$ and $b(X,Y)=\sum_{jk}X^jY^k$ are convergent for $|X|\vee|Y|\le\delta$. Suppose that (x,y,z) is confined to a compact subset of R^3 such that $|x|+|y|\le M, |z|\le|y|$. Keeping in mind that the arguements of a,b,c,f,h in the integrands are estimated by

$$|x-y+Y|, |x+y-Y|, |Y|, |X|\le M$$

for $|z|\le|y|$, we see inductively that $G^{(n)}(x,y,z,\epsilon)$ is analytic function of ϵ provided that $M^2|\epsilon|\le\delta$. Here we consider, for $y<0$,

$$D(x,y,z)=\{(X,Y):(X,-Y)\in D(x,-y,z)\}$$

and

$$\begin{aligned} &\int\int_{D(x,y,z)} F(X,Y,z+x-Y)dXdY = \\ &= \frac{1}{2}\int_{x-y}^{x+z}\int_{x+z}^{x+y} F(\frac{X+Y}{2},\frac{X-Y}{2},z+x-\frac{X+Y}{2})dXdY \end{aligned}$$

for any function F. On the other hand it is easy to see inductively that the iteration satisfy

$$|G^{(n+1)}(x,y,z,\epsilon) - G^{(n)}(x,y,z,\epsilon)| \leq \frac{M_1^{n+1}|y|^{n+1}}{(n+1)!}$$

with a large constant $M_1 = M_1(M,\delta)$ independent of ϵ. Thus $G^{(n)}$ converges to the limit G uniformly on $|x|+|y| \leq M, |z| \leq |y|, |\epsilon| \leq \delta/M^2$. This implies that G is analytic in $\epsilon(\leq \delta/M^2)$ provided that $|x|+|y| \leq M, |z| \leq |y|$. (See [1], Chap.V. Th.1.) In other words, $G(x,y,z,\epsilon)$ admits a power series expansion

$$G(x,y,z,\epsilon) = 2 + \sum_{\nu=1}^{\infty} G_\nu(x,y,z)\epsilon^\nu, \tag{9}$$

which is convergent for $|\epsilon| \leq \delta/M^2$ provided that $|x|+|y| \leq M, |z| \leq |y|$, and satisfies the integral equation

$$G = 2 + \frac{\epsilon}{2}(T_1G - \frac{4}{3}T_2G) + \frac{4}{3}\epsilon^2(-T_4JG + T_5JG). \tag{10}$$

Proposition 1 *We have*

$$G_1(x,y,z) = \frac{a_{00}}{4}(5y^2 - 3z^2) - \frac{1}{3}(4xz + 3y^2 - z^2). \tag{11}$$

Proof. Inserting the expansion (9) to the integral equation (10), and comparing the terms of order ϵ, we see

$$\begin{aligned} G_1 &= a_{00}(\int_{\frac{y+z}{2}}^{y} YdY + \int_{\frac{y-z}{2}}^{y} YdY) \\ &- \frac{4}{3}(\int_{\frac{y-z}{2}}^{y}(x+y-Y)dY - \int_{\frac{y+z}{2}}^{y}(x-y+Y)dY) + c_{00}\int\int_{D(x,y,z)} dXdY. \end{aligned}$$

Computing the right hand side, we have (11). QED.

Of course $G_\nu, \nu \geq 2$ can be explicitely computed successively. But we prove the following general property.

Proposition 2 *$G_\nu(x,y,z)$ is a homogeneous polynomial in x,y,z of order 2ν of the form*

$$G_\nu(x,y,z) = \sum_{i+2j+k=2\nu} G_{ijk}x^i y^{2j} z^k. \tag{12}$$

Proof. The assertion is equivalent to the following set of properties:

(P1) $G_\nu(x,y,z)$ is a polynomial in x,y,z of order $\leq 2\nu$;

(P2) $G_\nu(\lambda x, \lambda y, \lambda z) = \lambda^{2\nu}G_\nu(x,y,z)$;

(P3) $G_\nu(x,-y,z) = G_\nu(x,y,z)$.

Suppose that $G_1, ..., G_\nu$ satisfy (12). Then $G_{\nu+1}$ is given by comparing the terms from the substituion of the expansion $G = \sum G_\nu \epsilon^\nu$ to the integral equation (10). The result is

$$G_{\nu+1} = \frac{1}{2}((T_1G)_\nu - \frac{4}{3}(T_2G)_\nu + (T_3G)_\nu) + \frac{4}{3}(-(T_4JG)_{\nu-1} + (T_5JG)_{\nu-1}),$$

$$
\begin{aligned}
(T_1G)_\nu &= \sum_{2l+2m+i+2j+k=2\nu} a_{lm}G_{ijk}L^1_{lmijk},\\
L^1_{lmijk}(x,y,z) &= \int_{\frac{y+z}{2}}^{y} Y^{2m+2j+1}(x-y+Y)^{2l+i}(z+y-Y)^k dY + \\
&+ \int_{\frac{y-z}{2}}^{y} Y^{2m+2j+1}(x+y-Y)^{2l+i}(z-y+Y)^k dY,\\
(T_2G)_\nu &= \sum_{2l+2m+i+2j+k=2\nu} b_{lm}G_{ijk}L^2_{lmijk},\\
L^2_{lmijk}(x,y,z) &= \int_{\frac{y-z}{2}}^{y} (x+y-Y)^{2l+i+1}Y^{2m+2j}(z-y+Y)^k dY + \\
&- \int_{\frac{y+z}{2}}^{y} (x-y+Y)^{2l+i+1}Y^{2m+2j}(z+y-Y)^k dY,\\
(T_3G)_\nu &= \frac{1}{2}\sum_{2l+2m+i+2j+k=2\nu} c_{lm}G_{ijk}L^3_{lmijk},\\
L^3_{lmijk}(x,y,z) &= \int_{x-y}^{x+z}\int_{x+z}^{x+y} (\frac{X+Y}{2})^{2l+i}(\frac{X-Y}{2})^{2m+2j}(z+x-\frac{X+Y}{2})^k dXdY,\\
(T_4JG)_{\nu-1} &= \sum_{2l+2m+i+2j+k=2\nu} f_{lm}G^*_{ijk}L^4_{lmijk},\\
L^4_{lmijk}(x,y,z) &= \int_{\frac{y-z}{2}}^{y} (x+y-Y)^{2l+i+1}Y^{2m+2j}(z-y+Y)^k dY + \\
&- \int_{\frac{y+z}{2}}^{y} (x-y+Y)^{2l+i+1}Y^{2m+2j}(z+y-Y)^k dY.
\end{aligned}
$$

Here

$$
JG = \sum_\nu (\sum_{i+2j+k=2\nu+2} G^*_{ijk}x^i y^{2j} z^k)\epsilon^\nu.
$$

We note

$$
\begin{aligned}
JG_\nu &= \int_{|z|}^{y} YG_\nu(x,Y,z)dY = \sum_{i+2j+k=2\nu} G_{ijk}\int_{|z|}^{y} x^i Y^{2j+1}z^k dY\\
&= \sum_{i+2j+k=2\nu} G_{ijk}\frac{y^{2j+2}-z^{2j+2}}{2(j+1)}x^i z^k = \sum_{i+2j+k=2\nu+2} G^*_{ijk}x^i y^{2j} z^k.\\
(T_5JG)_{\nu-1} &= \frac{1}{2}\sum_{2l+2m+i+2j+k=2\nu} h_{lm}G^*_{ijk}L^5_{lmijk},\\
L^5_{lmijk} &= \int_{x-y}^{x+z}\int_{x+z}^{x+y} (\frac{X+Y}{2})^{2l+i}(\frac{X-Y}{2})^{2m+2j}(z+x-\frac{X+Y}{2})^k dXdY.
\end{aligned}
$$

Therefore (P1) and (P2) are obviously verified for $G_{\nu+1}$, since

$$
L^1_{lmijk}(\lambda x, \lambda y, \lambda z) = \lambda^{2\nu+2}L^1_{lmijk}(x,y,z)
$$

and so on. As for (P3), we observe

$$
L^1_{lmijk}(x,-y,z) = \int_{\frac{-y+z}{2}}^{-y} Y^{2m+2j+1}(x+y+Y)^{2l+i}(z-y-Y)^k dY +
$$

$$\begin{aligned}
&+ \int_{\frac{-y-z}{2}}^{-y} Y^{2m+2j+1}(x-y-Y)^{2l+i}(z+y+Y)^k dY \\
&= \int_{\frac{y-z}{2}}^{y} Y^{2m+2j+1}(x+y-Y)^{2l+i}(z-y+Y)^k dY + \\
&+ \int_{\frac{y+z}{2}}^{y} Y^{2m+2j+1}(x-y+Y)^{2l+i}(z+y-Y)^k dY \\
&= L^1_{lmijk}(x,y,z).
\end{aligned}$$

This implies $(T_1 G)_\nu(x,-y,z) = (T_1 G)_\nu(x,y,z)$. We see

$$\begin{aligned}
L^2_{lmijk}(x,-y,z) &= \int_{\frac{-y-z}{2}}^{-y} (x-y-Y)^{2l+i+1} Y^{2m+2j}(z+y+Y)^k dY + \\
&- \int_{\frac{-y+z}{2}}^{-y} (x+y+Y)^{2l+i+1} Y^{2m+2j}(z-y-Y)^k dY + \\
&= -\int_{\frac{y+z}{2}}^{y} (x-y+Y)^{2l+i+1} Y^{2m+2j}(z+y-Y)^k dY + \\
&+ \int_{\frac{y-z}{2}}^{y} (x+y-Y)^{2l+i+1} Y^{2m+2j}(z-y+Y)^k dY \\
&= L^2_{lmijk}(x,y,z).
\end{aligned}$$

This implies $(T_2 G)_\nu(x,-y,z) = (T_2 G)_\nu(x,y,z)$. We see

$$\begin{aligned}
L^3_{lmijk}(x,-y,z) &= \int_{x+y}^{x+z}\int_{x+z}^{x-y} (\frac{X+Y}{2})^{2l+i}(\frac{X-Y}{2})^{2m+2j}(z+x-\frac{X+Y}{2})^k dXdY \\
&= \int_{z+x}^{x-y}\int_{x+y}^{x+z} (\frac{Y+X}{2})^{2l+i}(\frac{Y-X}{2})^{2m+2j}(z+x-\frac{Y+X}{2})^k dYdX \\
&= \int_{x-y}^{x+z}\int_{x+z}^{x+y} ...dYdX \\
&= L^3_{lmijk}(x,y,z).
\end{aligned}$$

This implies $(T_3 G)_\nu(x,-y,z) = (T_3 G)_\nu(x,y,z)$. $T_4 JG$ can be treated as $T_2 G$, and $T_5 JG$ as $T_3 G$. Thus $G_{\nu+1}$ enjoys (P3). QED.

Combining Propositions 1 and 2, it is easy to prove Theorem 1. Here we note that

$$\int_{|z|}^{y} YY^{2j} dY = \frac{1}{2(j+1)}(y^2-z^2)\sum_{\mu=0}^{j} y^{2\mu} z^{2(j-\mu)}.$$

3 Approximation by the telegraph equation

It is difficult to derive informations on the behavior of the relativistic Darboux kernel for large $\epsilon x^2, \epsilon y^2$ from the expansion obtained in the previous section, in which ϵM^2 is restricted to be small for $|x|+|y| \le M$. So, in this section we propose another approximation.

Neglecting the terms of order ϵ^2, we approximate the relativistic Euler-Poisson-Darboux equation ($rEPD$) as

$$\eta_{xx} - \eta_{yy} + (\frac{2}{y} + \epsilon a_{00} y)\eta_y - \frac{4}{3}\epsilon x \eta_x = 0 \tag{13}$$

Then the approximate equation for $V = \eta_y/y$ is

$$V_{yy} - V_{xx} = \epsilon(a_{00} y V_y - \frac{4}{3} x V_x + 2a_{00} V). \tag{14}$$

If we introduce the variable W by

$$V = e^{\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}y^2)} W,$$

in order to eliminate the first order derivatives, then the equation (14) is reduced to

$$W_{yy} - W_{xx} = (\epsilon(\frac{3}{2}a_{00} + \frac{2}{3}) + \epsilon^2(\frac{a_{00}^2}{4}y^2 - \frac{4}{9}x^2))W.$$

Let us neglect the terms of order ϵ^2 again. Then the problem (Q) is approximated by the problem

$$(Q') \qquad W_{yy} - W_{xx} = \kappa W,$$

$$W|_{y=0} = 0, \qquad W_y|_{y=0} = 4\phi(x).$$

Here

$$\kappa = \epsilon(\frac{3}{2}a_{00} + \frac{2}{3}) = -\frac{7}{30}\epsilon\frac{A_1}{A_0}. \tag{15}$$

Inversely if $W(x, y)$ is a solution of the problem (Q'), then the function $V(x, y) = e^{\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}y^2)} W(x, y)$ is an exact solution of the equation

$$V_{yy} - V_{xx} = \epsilon(a_{00} y V_y - \frac{4}{3} x V_x + 2a_{00} V) + \epsilon^2(\frac{4}{9}x^2 - \frac{a_{00}^2}{4}y^2)V$$

and the function $\eta(x, y) = IV(x, y) = \int_0^y YV(x, Y)dY$ is an exact solution of the equation

$$(\clubsuit) \qquad \eta_{xx} - \eta_{yy} + (\frac{2}{y} + \epsilon a_{00} y)\eta_y - \frac{4}{3}\epsilon x \eta_x = -\epsilon^2(\frac{4}{9}x^2 - \frac{a_{00}^2}{4}y^2)\eta - \epsilon^2\frac{a_{00}^2}{2} I\eta,$$

which is congruent with $(rEPD)$ modulo $O(\epsilon^2)$.

Now the equation of (Q') is nothing but *the telegraph equation (the Klein-Goldon equation)*, of which the solution formula by the Riemann function is well known. See, e.g., [3], Part I, Chap. 5, Sec. 3.

Case I: Suppose $\kappa > 0 (\Leftrightarrow a_{00} > -4/9 \Leftrightarrow A_1 < 0)$. In this case the solution of (Q') is given by

$$W(x, y) = 2\int_{x-y}^{x+y} I_0(\sqrt{\kappa(y^2 - (\xi - x)^2)})\phi(\xi)d\xi.$$

Here I_0 is the modified Bessel function of order 0:

$$I_0(z) = J_0(\sqrt{-1}z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2^k k!)^2}.$$

Hence an approximation of G is given by

$$G^a(x, y, z, \epsilon) = 2I_0(\sqrt{\kappa(y^2 - z^2)}) \exp(\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}y^2)).$$

The function $I_0(z)$ is monotone increasing as $z \to +\infty$ and it is known that

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}(1 + O(\frac{1}{z}))$$

as $z \to +\infty$. ([6], p.203, (2).) Therefore $G^a > 0$ for $|z| < y$ and we know the asymptotic behavior of G^a for large y. Hence the approximation of the relativistic Darboux kernel is

$$K^a(x, y, z, \epsilon) = 2\int_{|z|}^{y} I_0(\sqrt{\kappa(Y^2 - z^2)}) \exp(\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}Y^2))Y dY.$$

Of course

$$\eta(x, y) = \int_{x-y}^{x+y} K^a(x, y, \xi - x)\phi(\xi)d\xi$$

satisfies (♣) for any smooth ϕ. We see $K^a > 0$ for $|z| < y$ in this case.

Case II: Suppose $\kappa < 0 (\Leftrightarrow a_{00} < -4/9 \Leftrightarrow A_1 > 0)$. In this case the solution of (Q') is given by

$$W(x, y) = 2\int_{x-y}^{x+y} J_0(\sqrt{-\kappa(y^2 - (\xi - x)^2)})\phi(\xi)d\xi.$$

Here J_0 is the Bessel function of order 0:

$$J_0(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2^k k!)^2}.$$

It is known that J_0 is oscillating and

$$J_0(z) = \sqrt{\frac{2}{\pi z}}(\cos(z - \frac{\pi}{4}) + O(\frac{1}{z}))$$

as $z \to +\infty$. ([6], p.199. (1).) Hence

$$G^a(x, y, z, \epsilon) = 2J_0(\sqrt{-\kappa(y^2 - z^2)}) \exp(\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}y^2))$$

has infinitely many zero as y varies for fixed x, z. Therefore

$$K^a(x, y, z, \epsilon) = 2\int_{|z|}^{y} J_0(\sqrt{-\kappa(Y^2 - z^2)}) \exp(\epsilon(\frac{1}{3}x^2 + \frac{a_{00}}{4}Y^2))Y dY$$

is oscillating as y increases, but *we do not know whether K^a has zero or not.* But it is known that $K^a(0, y, 0)$ has the positive limit

$$\frac{4}{-\epsilon a_{00}} \exp(-\frac{\kappa}{\epsilon a_{00}})$$

as $y \to +\infty$. This is the Weber's integral. See [6], p.393,(1).

Summing up we have Theorem 2.

Remark. If we develop K^a of Section 3 as the power series in ϵ, the first order term does not coincide with K_1 of Section 2, since K^a is a function of x^2, y^2, z^2 and does not contain the interaction term xz. This seems little bit curious. Maybe this is a counter example to show we cannot expect that an approximation of the solution formula would derive the same result with an approximation of the equation itself.

4 Generalization

In this section we extend the above conclusions to a more general pressure law. Instead of (1) we assume

$$P = A_0\rho^\gamma(1 + \sum_{j=1}^{\infty} A_j(\rho^{\gamma-1}/c^2)^j), \qquad \gamma = 1 + \frac{2}{2N+1}, \tag{16}$$

where N is a positive integer. If $N = 1$, then $\gamma = 5/3$ and we have discussed this particular case. The case $N \geq 2$ can be treated in a similar manner.

The coefficients of $(rEPD)$ are of the form

$$\begin{aligned}
A(x,y) &= \frac{2N}{y} + \epsilon y a(\epsilon x^2, \epsilon y^2) = \frac{2N}{y} + \epsilon y \sum_{j+k\geq 0} a_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\
a_{00} &= -\frac{4}{3(2N+1)}(1 + \frac{2N+5}{4(2N+3)}\frac{A_1}{A_0}), \\
B(x,y) &= -\frac{4N}{2N+1}\epsilon x b(\epsilon x^2, \epsilon y^2) = -\frac{4N}{2N+1}\epsilon x \sum_{j+k\geq 0} b_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\
b_{00} &= 1.
\end{aligned}$$

We introduce the sequence of variables $\eta_0 = \eta, \eta_1, ..., \eta_N = V$ by

$$\frac{\partial \eta_j}{\partial y} = y\eta_{j+1}, \qquad \eta_j(x,y) = I\eta_{j+1} = \int_0^y Y\eta_{j+1}(x,Y)dY.$$

The equation for $v = \eta_j$ is

$$\begin{aligned}
v_{yy} - v_{xx} &= (\frac{2(N-j)}{y} + \epsilon y a)v_y - \frac{4N}{2N+1}\epsilon x b v_x + 2j\epsilon(a + \epsilon y^2 D_2 a)v + \\
&+ \sum_{\mu=1}^{j} \epsilon^{\mu+1} x f_{j\mu} I^\mu v_x + \sum_{\mu=1}^{j} \epsilon^{\mu+1} h_{j\mu} I^\mu v,
\end{aligned}$$

where $f_{j\mu}(\epsilon x^2, \epsilon y^2)$ and $h_{j\mu}(\epsilon x^2, \epsilon y^2)$ are determind inductively by

$$\begin{aligned}
f_{j+1,1} &= f_{j1} - \frac{8}{2N+1}D_2 b, \\
f_{j+1,\mu} &= f_{j\mu} + 2D_2 f_{j,\mu-1} \quad (\mu \geq 2), \\
h_{j+1,1} &= h_{j1} + 4j(2D_2 a + \epsilon y^2 D_2 D_2 a), \\
h_{j+1,\mu} &= h_{j\mu} + 2D_2 h_{j,\mu-1} \qquad (\mu \geq 2).
\end{aligned}$$

Therefore the problem to be solved is

$$(Q) \qquad V_{yy} - V_{xx} = \epsilon(ayV_y - \frac{4N}{2N+1}bxV_x + 2N(a + \epsilon y^2 D_2 a)V) +$$

$$\begin{aligned}
&+ \sum_{\mu=1}^{N} \epsilon^{\mu+1} x f_\mu I^\mu V_x + \sum_{\mu=1}^{N} \epsilon^{\mu+1} h_\mu I^\mu V, \\
V|_{y=0} &= 0, \qquad V_y|_{y=0} = 2^{N+1} N! \phi(x),
\end{aligned}$$

where $f_\mu = f_{N\mu}, h_\mu = h_{N\mu}$. For any smooth ϕ the problem (Q) admits a unique solution

$$V(x,y) = \int_{x-y}^{x+y} G(x,y,\xi - x,\epsilon)\phi(\xi)d\xi.$$

We know

$$G = 2^N N! + O(\epsilon y).$$

Defining the relativistic Darboux kernel K by $K = J^N G$, we have the relativistic Darboux formula

$$\eta(x,y) = \int_{x-y}^{x+y} K(x,y,\xi - x,\epsilon)\phi(\xi)d\xi$$

for solutions of $(rEPD)$.

We rewrite the equation of (Q) as

$$\begin{aligned} V_{yy} - V_{xx} &= \epsilon((ayV)_y - \frac{4N}{2N+1}(bxV)_x + cV) + \\ &+ \sum_{\mu=1}^{N} \epsilon^{\mu+1}(xf_\mu I^\mu V)_x + \sum_{\mu=1}^{N} \epsilon^{\mu+1} g_\mu I^\mu V, \end{aligned}$$

where

$$\begin{aligned} c &= (2N-1)a + 2(N-1)\epsilon y^2 D_2 a + \frac{4N}{2N+1}b + \frac{8N}{2N+1}\epsilon x^2 D_2 b \\ &= \sum_{j+k\geq 0} c_{jk}(\epsilon x^2)^j(\epsilon y^2)^k, \\ c_{00} &= (2N-1)a_{00} + \frac{4N}{2N+1}, \\ g_\mu &= h_\mu - f_\mu - 2\epsilon x^2 D_1 f_\mu = \sum_{j+k\geq 0} g_{\mu,jk}(\epsilon x^2)^j(\epsilon y^2)^k. \end{aligned}$$

The kernel G is constructed by the iteration

$$\begin{aligned} &G^{(0)} = 2^N N!, \\ &G^{(n+1)} = 2^N N! + \frac{\epsilon}{2}(T_1 G^{(n)} - \frac{4N}{2N+1}T_2 G^{(n)} + T_3 G^{(n)}) + \\ + &\sum_{\mu=1}^{N} \epsilon^{\mu+1} T_{4\mu} J^\mu G^{(n)} + \sum_{\mu=1}^{N} \epsilon^{\mu+1} T_{5\mu} J^\mu G^{(n)}. \end{aligned}$$

Here $T_1, T_2, ...$ are defined in a similar manner to Section 2. Of course G satisfies the integral equation

$$\begin{aligned} &G = 2^N N! + \frac{\epsilon}{2}(T_1 G - \frac{4N}{2N+1}T_2 G + T_3 G) + \\ + &\sum_{mu=1}^{N} \epsilon^{\mu+1} T_{4\mu} J^\mu G + \sum_{\mu=1}^{N} \epsilon^{\mu+1} T_{5\mu} J^\mu G. \end{aligned}$$

Then it can be proved that $G(x,y,z,\epsilon)$ is analytic in ϵ,

$$G(x,y,z,\epsilon) = 2^N N! + \sum_{\nu=1}^{\infty} G_\nu(x,y,z)\epsilon^\nu,$$

and $G_\nu(x,y,z)$ is a polynomial in x,y,z of the form

$$G_\nu(x,y,z) = \sum_{i+2j+k=2\nu} G_{ijk} x^i y^{2j} z^k.$$

We note that it can be proved inductively that

$$J^\mu G_\nu = (y^2 - z^2)^\mu \sum_{i+2j+k=2\nu} G^{*\mu}_{ijk} x^1 y^{2j} z^k,$$

keeping in mind that

$$\begin{aligned}\int_{|z|}^{y} Y(Y^2 - z^2)^l x^i Y^{2j} z^k dY &= \frac{1}{2}\int_0^{y^2-z^2} t^l x^i (t+z^2)^j z^k dt \\ &= \frac{1}{2}(y^2-z^2)^{l+1} \sum_{m=0}^{j} \binom{j}{m} \frac{1}{l+m+1} x^i (y^2-z^2)^m z^{k+2(j-m)}.\end{aligned}$$

Therefore, putting $K = J^N G$, we have

Theorem 3 *The relativistic Darboux kernel K is of the form*

$$K(x,y,z,\epsilon) = (y^2-z^2)^N (1 + \sum_{\nu=1}^{\infty} K_\nu(x,y,z)\epsilon^\nu),$$

in which K_ν is a homogeneous polynomial in x,y,z of the form

$$K_\nu(x,y,z) = \sum_{i+2j+k=2\nu} K_{ijk} x^i y^{2j} z^k.$$

The power series is convergent for $|\epsilon| \le \delta/M^2$, where $|x|+|y| \le M, |z| \le |y|$ and δ is a positive constant.

Neglecting the terms of order ϵ^2, we approximate the equation of (Q) as

$$V_{yy} - V_{xx} = \epsilon(a_{00} y V_y - \frac{4N}{2N+1} x V_x + 2N a_{00} V).$$

$W = \exp(-\epsilon(\frac{N}{2N+1}x^2 + \frac{a_{00}}{4}y^2)V$ satisfies

$$W_{yy} - W_{xx} = (\kappa + \epsilon^2(\frac{4N(N-2)}{(2N+1)^2}x^2 + \frac{a_{00}}{4}y^2))W,$$

where

$$\kappa = \epsilon(\frac{4N-1}{2}a_{00} + \frac{2N}{2N+1}) = -\epsilon(\frac{2(N-1)}{3(2N+1)} + \frac{(4N-1)(2N+5)}{6(2N+1)(2N+3)}\frac{A_1}{A_0}).$$

Neglecting the terms of order ϵ^2 again, we get the problem

$$(Q') \qquad W_{yy} - W_{xx} = \kappa W$$

$$W|_{y=0} = 0, \qquad W_y|_{y=0} = 2^{N+1}N!\phi(x).$$

Inversely if W is a solution of (Q') then $v = \eta_{N-k} = I^k V, V = \exp(\epsilon(\frac{N}{2N+1}x^2 + \frac{a_{00}}{4}y^2))W$, satisfies

$$\begin{aligned} v_{yy} - v_{xx} &= (\frac{2k}{y} + \epsilon a_{00} y)v_y - \frac{4N}{2N+1}\epsilon x v_x + 2\epsilon(N-k)a_{00}v + \\ &+ 4\epsilon^2((\frac{N}{2N+1}x)^2 - (\frac{a_{00}}{4}y)^2)v + \frac{\epsilon^2 a_{00}^2 k}{2} Iv. \end{aligned}$$

Hence $\eta = I^N V$ is an exact solution of the equation

$$(\clubsuit) \qquad \begin{aligned} &\eta_{xx} - \eta_{yy} + (\frac{2N}{y} + \epsilon a_{00})\eta_y - \frac{4N}{2N+1}\epsilon x \eta_x = \\ &= -4\epsilon^2((\frac{N}{2N+1}x)^2 - (\frac{a_{00}}{4}y)^2)\eta - \epsilon^2\frac{a_{00}^2 N}{2} I\eta. \end{aligned}$$

If $\kappa > 0 (\Leftrightarrow a_{00} > -\frac{4N}{(2N+1)(4N+1)} \Leftrightarrow A_1 < -\frac{4(N-1)(2N+3)}{(4N-1)(2N+5)}A_0)$, then the solution formula for (Q') is

$$W(x,y) = 2^N N! \int_{x-y}^{x+y} I_0(\sqrt{\kappa(y^2 - (\xi - x)^2)})\phi(\xi)d\xi$$

and the approximate kernel is

$$G^a(x,y,z,\epsilon) = 2^N N! I_0(\sqrt{\kappa(y^2 - z^2)})\exp(\epsilon(\frac{N}{2N+1}x^2 + \frac{a_{00}}{4}y^2)).$$

If $\kappa < 0$, we replace I_0 by J_0. Thus we have

Theorem 4 *The relativistic Darboux kernel is approximated by K^a given in the following manner. If $A_1 < -\frac{4(N-1)(2N+3)}{(4N-1)(2N+5)}A_0$, then*

$$\begin{aligned} K^a(x,y,z,\epsilon) &= 2^N N! \int_{|z|}^{y} Y_{N-1}dY_{N-1} \dots \int_{|z|}^{Y_1} Y dY I_0(\sqrt{\kappa(Y^2 - z^2)}) \times \\ &\times \exp(\epsilon(\frac{N}{2N+1}x^2 + \alpha Y^2)); \end{aligned}$$

If $A_1 = -\frac{4(N-1)(2N+3)}{(4N-1)(2N+5)}A_0$, then

$$K^a(x,y,z,\epsilon) = (y^2 - z^2)^N \sum_{k=0}^{\infty} (\epsilon\alpha)^k \frac{N!}{(k+N)!}(y^2 - z^2)^k e^{\epsilon(\frac{N}{2N+1}x^2 + \alpha z^2)};$$

If $A_1 > -\frac{4(N-1)(2N+3)}{(4N-1)(2N+5)}A_0$, then I_0 is replaced by J_0. Here $\alpha = -\frac{1}{3(2N+1)}(1 + \frac{2N+5}{4(2N+3)}\frac{A_1}{A_0}), \kappa = -\epsilon(\frac{2(N-1)}{3(2N+1)} + \frac{(4N-1)(2N+5)}{6(2N+1)(2N+3)}\frac{A_1}{A_0})$ and for any smooth ϕ the function

$$\eta(x,y) = \int_{x-y}^{x+y} K^a(x,y,\xi - x,\epsilon)\phi(\xi)d\xi$$

satisfies the equation $(\clubsuit)$, which is congruent with (rEPD) modulo $O(\epsilon^2)$.

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References

[1] H. Cartan, Théorie Élémentaire des Fonctions Analytiques d'une ou Plusieures Variables Complexes, Herman, Paris, 1962.

[2] C.-H. Hsu, S.-S. Lin and T. Makino, On the relativistic Euler equation, Methods and Applications of Analysis, 8(2001), 159-208.

[3] N. S. Košljakov, E. B. Gliner, M. M. Smirnov, Differencial'nye Uravnenija Matematičeskoĭ Fiziki, Gos. Izd. Fiz.-Mat. Lit., Moskva, 1962.

[4] M. Lu and S. Ukai, Non-relativistic global limits of weak solutions of the relativistic Euler equation, J. Math. Kyoto Univ., 38-3(1998), 525-537.

[5] J. Smoller and B. Temple, Global solutions of the relativistic Euler equations, Comm. Math. Phys., 156(1993), 67-99.

[6] G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge UP, 1922/1944.

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