Rational solutions of the fourth Painlevé equation in two variables

By

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1 Introduction

The present article concerns rational solutions of the fourth Painlevé equation in two variables

$$\begin{cases} \dfrac{\partial u_k}{\partial z_j} = \dfrac{\partial L_j}{\partial v_k}, \\ \dfrac{\partial v_k}{\partial z_j} = -\dfrac{\partial L_j}{\partial u_k}, \quad (j, k = 1, 2), \end{cases} \tag{P}$$

where L_1, L_2 are

$$\begin{aligned} L_1 &= {v_1}^2 - u_2{v_2}^2 + v_1(u_2 - {u_1}^2 - z_1) \\ &\quad + v_2(-u_1u_2 + u_2z_2 + \kappa_0) + \kappa_\infty u_1, \\ L_2 &= -2u_2v_1v_2 - u_2(u_1 + z_2){v_2}^2 + v_1(-u_2u_1 + u_2z_2 + \kappa_0) \\ &\quad + v_2(-{u_2}^2 + u_2{z_2}^2 + u_2z_1 + u_1\kappa_0 + z_2\kappa_0) + \kappa_\infty u_2. \end{aligned}$$

In [3], we derived a non-linear differential equation in g variables from a holonomic deformation of a certain linear differential equation. When $g = 1$, this non-linear differential equation is equivalent to the fourth Painlevé equation. The equation (P) corresponds to this non-linear differential equation in $g = 2$. By the general theorey of Miwa [7], it is known that (P) has the Painlevé property. The equation (P) is expected to have many interesting properties as the Painlevé equations in one variable.

In this paper, we find all rational solutions of (P). These rational solutions suggest a symmetry of the transformation group of solutions.

2 Statement of Theorems

Theorem 1 *If the equation* (P) *has a rational solution*

$$u_j = \frac{Q_{1,j}(z_1, z_2)}{P_{1,j}(z_1, z_2)}, \ v_j = \frac{Q_{2,j}(z_1, z_2)}{P_{2,j}(z_1, z_2)}, \quad (P_{i,j}, Q_{i,j} \in \mathbb{C}[z_1, z_2]),$$

then κ_0 and κ_∞ satisfy the following condition

$$\begin{cases} \cdot\ \kappa_0, \kappa_\infty \ \textit{are integers}, \\ \cdot\ \kappa_\infty \geq 0, \ \kappa_\infty \leq \kappa_0 - 1 \ \textit{or} \ \kappa_\infty \leq -1, \ \kappa_\infty \geq \kappa_0. \end{cases} \tag{$*$}$$

Theorem 2 *If the parameters* κ_0 *and* κ_∞ *satisfy the condition* $(*)$, *then* (P) *has a unique rational solution.*

Theorem 3 *If* $\kappa_\infty = 0$ *and* κ_0 *is a positive integer, then the unique rational solution is written in the form*

$$u_1 = \frac{\partial}{\partial z_1}\log\tau, \quad u_2 = \frac{\partial}{\partial z_2}\log\tau, \quad v_1 = v_2 = 0,$$

where

$$\tau = \operatorname*{Res}_{s=z_2}(z_2 - s)^{-\kappa_0}\exp\left(\frac{1}{3}s^3 + z_1 s\right)ds.$$

3 Verifiation of the theorems

3.1 Review of the fourth Painlevé equation in two variables

For the equation (P), following facts hold. (See [3],[4],[5].)

Theorem A *If we define new variables* $\zeta_1, \zeta_2, \eta_1, \eta_2$, $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *as*

$$\begin{aligned}
\zeta_1 &= -u_1 + \frac{-u_2 v_2 + \kappa_\infty}{v_1},\\
\zeta_2 &= \frac{v_2}{v_1}(-v_2 u_2 + \kappa_0),\\
\eta_1 &= -v_1 + z_1 + {\zeta_1}^2 - \zeta_2,\\
\eta_2 &= \frac{v_1}{v_2} + z_2 - \zeta_1,\\
\mathcal{L}_j &= L_j + u_j, \quad (j = 1, 2),
\end{aligned}$$

then $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are written in the form*

$$\begin{aligned}
\mathcal{L}_1 &= {\eta_1}^2 - \zeta_2{\eta_2}^2 + \eta_1(\zeta_2 - {\zeta_1}^2 - z_1)\\
&\quad + \eta_2(-\zeta_1\zeta_2 + \zeta_2 z_2 + \kappa_0) + \kappa_\infty\zeta_1,\\
\mathcal{L}_2 &= -2\zeta_2\eta_1\eta_2 - \zeta_2(\zeta_1 + z_2){\eta_2}^2 + \eta_1(-\zeta_2\zeta_1 + \zeta_2 z_2 + \kappa_0)\\
&\quad + \eta_2(-{\zeta_2}^2 + \zeta_2{z_2}^2 + \zeta_2 z_1 + \zeta_1\kappa_0 + z_2\kappa_0) + \kappa_\infty\zeta_2,
\end{aligned}$$

i.e., $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \zeta,\\
v &\longrightarrow \eta,\\
\kappa_\infty &\longrightarrow \kappa_\infty - 1
\end{aligned}$$

in L_j. *Moreover, the transformation* $(u, v, L, z) \to (\zeta, \eta, \mathcal{L}, z)$ *is canonical transformation.*

Theorem B *If we define new variables* $\bar{\zeta}_1, \bar{\zeta}_2, \bar{\eta}_1, \bar{\eta}_2$, $\bar{\mathcal{L}}_1$ *and* $\bar{\mathcal{L}}_2$ *as*

$$\begin{aligned}
\bar{\zeta}_1 &= v_2 + (u_2 + v_1 - u_1 z_2 + v_2 u_1 - z_1)\frac{1}{u_1 + v_2 - z_2},\\
\bar{\zeta}_2 &= \left\{ {v_2}^2(u_1 + z_2) + v_2(u_2 + 2v_1 - z_1 - {z_2}^2) - u_1 v_1 - v_1 z_2 + \kappa_0 - \kappa_\infty - 1 \right\} \frac{1}{u_1 + v_2 - z_2},\\
\bar{\eta}_1 &= v_2(u_1 + v_2 - z_2),\\
\bar{\eta}_2 &= -u_1 - v_2 + z_2 + \frac{\kappa_0 - 1}{\bar{\zeta}_2},\\
\bar{\mathcal{L}}_1 &= L_1 - v_2,\\
\bar{\mathcal{L}}_2 &= L_2 - (u_1 v_2 + v_1 + v_2 z_2),
\end{aligned}$$

then the explicit forms of $\bar{\mathcal{L}}_1$ *and* $\bar{\mathcal{L}}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \bar{\zeta},\\
v &\longrightarrow \bar{\eta},\\
\kappa_0 &\longrightarrow \kappa_0 - 1
\end{aligned}$$

in L_j. *Moreover, the transformation* $(u, v, L, z) \to (\bar{\zeta}, \bar{\eta}, \bar{\mathcal{L}}, z)$ *is canonical transformation.*

Theorem C *If we define new variables* $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\eta}_1, \hat{\eta}_2$, $\hat{\mathcal{L}}_1$ *and* $\hat{\mathcal{L}}_2$ *as*

$$\begin{aligned}
\hat{\zeta}_1 &= u_1,\\
\hat{\zeta}_2 &= u_2,\\
\hat{\eta}_1 &= v_1,\\
\hat{\eta}_2 &= v_2 - \frac{\kappa_0}{u_2},\\
\hat{\mathcal{L}}_1 &= L_1 - \kappa_0 z_2,\\
\hat{\mathcal{L}}_2 &= L_2 - \kappa_0(z_1 + {z_2}^2),
\end{aligned}$$

then the explicit forms of $\hat{\mathcal{L}}_1$ *and* $\hat{\mathcal{L}}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \hat{\zeta},\\
v &\longrightarrow \hat{\eta},\\
\kappa_0 &\longrightarrow -\kappa_0,\\
\kappa_\infty &\longrightarrow \kappa_\infty - \kappa_0
\end{aligned}$$

in L_j. *Moreover, the transformation* $(u, v, L, z) \to (\hat{\zeta}, \hat{\eta}, \hat{\mathcal{L}}, z)$ *is canonical transformation.*

Theorem D *If* $\kappa_\infty = 0$, *the equation* (P) *has a special solution*

$$\begin{aligned}
u_j &= \frac{\partial}{\partial z_j} \log \tau,\\
v_j &= 0, \quad (j = 1, 2),
\end{aligned}$$

where

$$\tau = \int_c (z_2 - s)^{-\kappa_0} \exp\left(\frac{1}{3}s^3 + z_1 s\right) ds.$$