Rational solutions of the fourth Painlevé equation in two variables

By

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1 Introduction

The present article concerns rational solutions of the fourth Painlevé equation in two variables

$$\begin{cases} \dfrac{\partial u_k}{\partial z_j} = \dfrac{\partial L_j}{\partial v_k}, \\ \dfrac{\partial v_k}{\partial z_j} = -\dfrac{\partial L_j}{\partial u_k}, \quad (j,k=1,2), \end{cases} \tag{P}$$

where L_1, L_2 are

$$\begin{aligned} L_1 &= {v_1}^2 - u_2{v_2}^2 + v_1(u_2 - {u_1}^2 - z_1) \\ &\quad + v_2(-u_1u_2 + u_2z_2 + \kappa_0) + \kappa_\infty u_1, \\ L_2 &= -2u_2v_1v_2 - u_2(u_1+z_2){v_2}^2 + v_1(-u_2u_1 + u_2z_2 + \kappa_0) \\ &\quad + v_2(-{u_2}^2 + u_2{z_2}^2 + u_2z_1 + u_1\kappa_0 + z_2\kappa_0) + \kappa_\infty u_2. \end{aligned}$$

In [3], we derived a non-linear differential equation in g variables from a holonomic deformation of a certain linear differential equation. When $g = 1$, this non-linear differential equation is equivalent to the fourth Painlevé equation. The equation (P) corresponds to this non-linear differential equation in $g = 2$. By the general theory of Miwa [7], it is known that (P) has the Painlevé property. The equation (P) is expected to have many interesting properties as the Painlevé equations in one variable.

In this paper, we find all rational solutions of (P). These rational solutions suggest a symmetry of the transformation group of solutions.

2 Statement of Theorems

Theorem 1 *If the equation* (P) *has a rational solution*

$$u_j = \frac{Q_{1,j}(z_1,z_2)}{P_{1,j}(z_1,z_2)},\ v_j = \frac{Q_{2,j}(z_1,z_2)}{P_{2,j}(z_1,z_2)}, \quad (P_{i,j}, Q_{i,j} \in \mathbb{C}[z_1,z_2]),$$

then κ_0 and κ_∞ satisfy the following condition

$$\begin{cases} \cdot\ \kappa_0, \kappa_\infty \text{ are integers,} \\ \cdot\ \kappa_\infty \geq 0,\ \kappa_\infty \leq \kappa_0 - 1 \text{ or } \kappa_\infty \leq -1,\ \kappa_\infty \geq \kappa_0. \end{cases} \tag{$*$}$$

Theorem 2 *If the parameters* κ_0 *and* κ_∞ *satisfy the condition* $(*)$*, then* (P) *has a unique rational solution.*

Theorem 3 *If* $\kappa_\infty = 0$ *and* κ_0 *is a positive integer, then the unique rational solution is written in the form*

$$u_1 = \frac{\partial}{\partial z_1} \log \tau, \quad u_2 = \frac{\partial}{\partial z_2} \log \tau, \quad v_1 = v_2 = 0,$$

where

$$\tau = \operatorname*{Res}_{s=z_2} (z_2 - s)^{-\kappa_0} \exp\left(\frac{1}{3}s^3 + z_1 s\right) ds.$$

3 Verifiation of the theorems

3.1 Review of the fourth Painlevé equation in two variables

For the equation (P), following facts hold. (See [3],[4],[5].)

Theorem A *If we define new variables* $\zeta_1, \zeta_2, \eta_1, \eta_2$*,* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *as*

$$\begin{aligned}
\zeta_1 &= -u_1 + \frac{-u_2 v_2 + \kappa_\infty}{v_1}, \\
\zeta_2 &= \frac{v_2}{v_1}(-v_2 u_2 + \kappa_0), \\
\eta_1 &= -v_1 + z_1 + {\zeta_1}^2 - \zeta_2, \\
\eta_2 &= \frac{v_1}{v_2} + z_2 - \zeta_1, \\
\mathcal{L}_j &= L_j + u_j, \quad (j = 1, 2),
\end{aligned}$$

then $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are written in the form*

$$\begin{aligned}
\mathcal{L}_1 &= {\eta_1}^2 - \zeta_2 {\eta_2}^2 + \eta_1(\zeta_2 - {\zeta_1}^2 - z_1) \\
&\quad + \eta_2(-\zeta_1\zeta_2 + \zeta_2 z_2 + \kappa_0) + \kappa_\infty \zeta_1, \\
\mathcal{L}_2 &= -2\zeta_2\eta_1\eta_2 - \zeta_2(\zeta_1 + z_2){\eta_2}^2 + \eta_1(-\zeta_2\zeta_1 + \zeta_2 z_2 + \kappa_0) \\
&\quad + \eta_2(-{\zeta_2}^2 + \zeta_2 {z_2}^2 + \zeta_2 z_1 + \zeta_1\kappa_0 + z_2\kappa_0) + \kappa_\infty \zeta_2,
\end{aligned}$$

i.e., $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \zeta, \\
v &\longrightarrow \eta, \\
\kappa_\infty &\longrightarrow \kappa_\infty - 1
\end{aligned}$$

in L_j*. Moreover, the transformation* $(u, v, L, z) \to (\zeta, \eta, \mathcal{L}, z)$ *is canonical transformation.*

Theorem B *If we define new variables* $\bar{\zeta}_1, \bar{\zeta}_2, \bar{\eta}_1, \bar{\eta}_2$, $\bar{\mathcal{L}}_1$ *and* $\bar{\mathcal{L}}_2$ *as*

$$\begin{aligned}
\bar{\zeta}_1 &= v_2 + (u_2 + v_1 - u_1 z_2 + v_2 u_1 - z_1)\frac{1}{u_1 + v_2 - z_2},\\
\bar{\zeta}_2 &= \left\{ {v_2}^2(u_1 + z_2) + v_2(u_2 + 2v_1 - z_1 - {z_2}^2) - u_1 v_1 - v_1 z_2 + \kappa_0 - \kappa_\infty - 1 \right\} \frac{1}{u_1 + v_2 - z_2},\\
\bar{\eta}_1 &= v_2(u_1 + v_2 - z_2),\\
\bar{\eta}_2 &= -u_1 - v_2 + z_2 + \frac{\kappa_0 - 1}{\bar{\zeta}_2},\\
\bar{\mathcal{L}}_1 &= L_1 - v_2,\\
\bar{\mathcal{L}}_2 &= L_2 - (u_1 v_2 + v_1 + v_2 z_2),
\end{aligned}$$

then the explicit forms of $\bar{\mathcal{L}}_1$ *and* $\bar{\mathcal{L}}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \bar{\zeta},\\
v &\longrightarrow \bar{\eta},\\
\kappa_0 &\longrightarrow \kappa_0 - 1
\end{aligned}$$

in L_j*. Moreover, the transformation* $(u, v, L, z) \to (\bar{\zeta}, \bar{\eta}, \bar{\mathcal{L}}, z)$ *is canonical transformation.*

Theorem C *If we define new variables* $\hat{\zeta}_1, \hat{\zeta}_2, \hat{\eta}_1, \hat{\eta}_2$, $\hat{\mathcal{L}}_1$ *and* $\hat{\mathcal{L}}_2$ *as*

$$\begin{aligned}
\hat{\zeta}_1 &= u_1,\\
\hat{\zeta}_2 &= u_2,\\
\hat{\eta}_1 &= v_1,\\
\hat{\eta}_2 &= v_2 - \frac{\kappa_0}{u_2},\\
\hat{\mathcal{L}}_1 &= L_1 - \kappa_0 z_2,\\
\hat{\mathcal{L}}_2 &= L_2 - \kappa_0(z_1 + {z_2}^2),
\end{aligned}$$

then the explicit forms of $\hat{\mathcal{L}}_1$ *and* $\hat{\mathcal{L}}_2$ *are given by the replacement of*

$$\begin{aligned}
u &\longrightarrow \hat{\zeta},\\
v &\longrightarrow \hat{\eta},\\
\kappa_0 &\longrightarrow -\kappa_0,\\
\kappa_\infty &\longrightarrow \kappa_\infty - \kappa_0
\end{aligned}$$

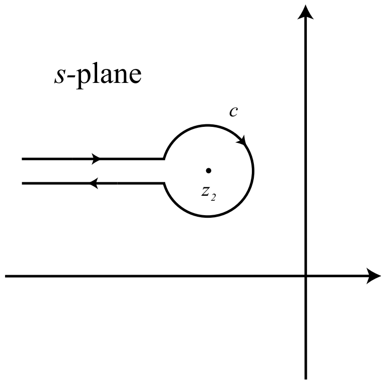
in L_j*. Moreover, the transformation* $(u, v, L, z) \to (\hat{\zeta}, \hat{\eta}, \hat{\mathcal{L}}, z)$ *is canonical transformation.*

Theorem D *If* $\kappa_\infty = 0$*, the equation* (P) *has a special solution*

$$\begin{aligned}
u_j &= \frac{\partial}{\partial z_j} \log \tau,\\
v_j &= 0, \quad (j = 1, 2),
\end{aligned}$$

where

$$\tau = \int_c (z_2 - s)^{-\kappa_0} \exp\left(\frac{1}{3}s^3 + z_1 s\right) ds.$$



By virtue of Theorem D and Theorem 2, Theorem 3 is obtained immediately. Hence we will prove Theorem 1 and Theorem 2.

3.2 Strategy of the proof

Suppose that

$$u_j = \frac{Q_{1,j}(z_1,z_2)}{P_{1,j}(z_1,z_2)},\ v_j = \frac{Q_{2,j}(z_1,z_2)}{P_{2,j}(z_1,z_2)}, \quad (P_{i,j}, Q_{i,j} \in \mathbb{C}[z_1,z_2])$$

is a rational solution of (P). Let a, b be complex numbers such that

$$P_{i,j}(a,b) \neq 0 \quad (i,j = 1,2).$$

Let

$$\begin{cases} u_j = f_j(z_1) + \displaystyle\sum_{k=1}^{\infty} f_j^{(k)}(z_1)\cdot(z_2-b)^k, \\ v_j = g_j(z_1) + \displaystyle\sum_{k=1}^{\infty} g_j^{(k)}(z_1)\cdot(z_2-b)^k \end{cases} \tag{1}$$

be the expansion around $(z_1,z_2)=(a,b)$, where $f_j(z_1), f_j^{(k)}(z_1), g_j(z_1), g_j^{(k)}(z_1)$ are rational functions in z_1. By substituting the expansion (1) into (P), we find that $(f_1(z_1), f_2(z_1), g_1(z_1), g_2(z_1))$ is a rational solution of the differential equation

$$\begin{cases} \dfrac{df_1}{dz_1} = -{f_1}^2 + f_2 + 2g_1 - z_1, \\ \dfrac{df_2}{dz_1} = -f_1\cdot f_2 - 2f_2\cdot g_2 + b\cdot f_2 + \kappa_0, \\ \dfrac{dg_1}{dz_1} = 2f_1\cdot g_1 + f_2\cdot g_2 - \kappa_\infty, \\ \dfrac{dg_2}{dz_1} = {g_2}^2 + f_1\cdot g_2 - g_1 - b\cdot g_2. \end{cases} \tag{E}$$

Hence, if there exists a rational solution of (P) for a parameter $(\kappa_0, \kappa_\infty)$, there exists a rational solution of (E) for the same parameter $(\kappa_0, \kappa_\infty)$. Based on this consideration, we prove Theorem 1 and Theorem 2 by following strategy:

1) find all rational solutions of (E),

2) construct rational solutions of (P) from rational solutions of (E).

3.3 Properties of rational solutions of (E)

From the equation (E), we find that f_2 and g_1 are written in the form

$$f_2 = {f_1}' + {f_1}^2 - 2g_1 + z_1, \tag{2}$$
$$g_1 = -{g_2}' + f_1 \cdot g_2 + {g_2}^2 - b \cdot g_2, \tag{3}$$

where $'$ denotes the differentiation with respect to z_1. Eliminating f_2, g_1 from (E), we get

$$\begin{aligned} &{f_1}'' + 3(f_1 - \frac{b}{3}) \cdot {f_1}' + 6(f_1 - \frac{b}{3}) \cdot {g_2}' + {f_1}^3 - {f_1}^2 \cdot (6g_2 + b) \\ &\quad + f_1 \cdot (-6{g_2}^2 + 8g_2 \cdot b + z_1) + 2{g_2}^2 \cdot b - 2g_2 \cdot b^2 - b \cdot z_1 - \kappa_0 + 2\kappa_\infty + 1 = 0, \\ &{g_2}'' - 3(f_1 - \frac{b}{3}) \cdot {g_2}' - 2{g_2}^3 + 2{g_2}^2 \cdot b + g_2 \cdot (3{f_1}^2 - 2f_1 \cdot b + z_1) - \kappa_\infty = 0. \end{aligned}$$

We put

$$b = 3B, \quad f_1 = f + B,$$

then the above equations are written in the form

$$\begin{cases} f'' + 3f' \cdot f + f^3 - 6f^2 \cdot g_2 + f \cdot (6{g_2}' - 6{g_2}^2 + 12B \cdot g_2 + z_1 - 3B^2) \\ \qquad\qquad\qquad\qquad - 2B \cdot z_1 - 2B^3 - \kappa_0 + 2\kappa_\infty + 1 = 0, \\ {g_2}'' - 3f \cdot {g_2}' + 3f^2 \cdot g_2 - 2{g_2}^3 + 6B \cdot {g_2}^2 + (z_1 - 3B^2) \cdot g_2 - \kappa_\infty = 0. \end{cases} \tag{E$'$}$$

Notice that finding all rational solutions of (E) is equivalent to finding all rational solutions of (E$'$).

Lemma 1 *If* (E$'$) *has rational solutions, then the parameters* κ_0 *and* κ_∞ *must be integers.*

To prove the Lemma 1, we prepair sublemmas.

Sublemma 1 *Let* p *be a finite complex number and* $(f, g_2) = (\varphi, \psi)$ *be a rational solution of* (E$'$). *If* φ *or* ψ *has a pole at* $z_1 = p$, *the order of pole is 1.*

Proof. We consider the proof in the following three cases:

1) $\varphi = 0$,

2) $\psi = 0$,

3) $\varphi \neq 0$ and $\psi \neq 0$.

Case 1). From the first equation of (E$'$), we get $B = 0$ and $\kappa_0 - 2\kappa_\infty - 1 = 0$. Hence, the second equation of (E$'$) is reduced to the second Painlevé equation

$${g_2}'' - 2{g_2}^3 + z_1 \cdot g_2 - \kappa_\infty = 0.$$

Substituting the power series

$$g_2 = (z_1 - p)^{-m} \sum_{k\geq 0} b_k (z_1 - p)^k, \quad (b_0 \neq 0, m > 0),$$

we find $m = 1$. Therefore the sublemma holds.

Case 2). From the second equation, we get $\kappa_\infty = 0$. The first equation is reduced to

$$f'' + 3f'f + f^3 + f \cdot (z_1 - 3B^2) - 2Bz_1 - 2B^3 - \kappa_0 + 1 = 0.$$

Substituting the power series

$$f = (z_1 - p)^{-n} \sum_{k\geq 0} a_k (z_1 - p)^k, \quad (a_0 \neq 0, n > 0),$$

we find $n = 1$. Therefore the sublemma holds.

Case 3). Let

$$\varphi = (z_1 - p)^{-n} \sum_{k\geq 0} a_k (z_1 - p)^k, \tag{4}$$

$$\psi = (z_1 - p)^{-m} \sum_{k\geq 0} b_k (z_1 - p)^k, \quad (a_0 \neq 0, b_0 \neq 0; m, n \in \mathbb{Z}) \tag{5}$$

be the expansion of φ and ψ. We will show that $n \leq 1$ and $m \leq 1$. Firstly, we assume that $n \geq 2$. Substituting (4) and (5) into (E′), we find m is equal to n. Putting $m = n$, we get

$$\begin{cases} {a_0}^3 - 6{a_0}^2 b_0 - 6a_0{b_0}^2 = 0, \\ 3{a_0}^2 b_0 - 2{b_0}^3 = 0. \end{cases} \tag{6}$$

The solution of (6) is $(a_0, b_0) = (0, 0)$. This is contradiction. Secondly, we assume that $m \geq 2$. In this case, we can show that $m = n$ by the same method. From above discussion, this is contradiction. Therefore n and m must be less equal than one. □

Sublemma 2 *Let p be a finite complex number and $(f, g_2) = (\varphi, \psi)$ be a rational solution of* (E′). *We put*

$$\alpha = \operatorname*{Res}_{z_1=p} \varphi \, dz_1, \quad \beta = \operatorname*{Res}_{z_1=p} \psi \, dz_1.$$

Then (α, β) is one of the following pairs:

$$(0,0), (0,1), (0,-1), (1,0), (2,0), (1,-2), (-1,1), (-1,-1), (-2,2).$$

Proof. By Sublemma 1, φ and ψ are written in the form

$$\varphi = \frac{\alpha}{z_1 - p} + \sum_{k\geq 0} \alpha_k (z_1 - p)^k,$$

$$\psi = \frac{\beta}{z_1 - p} + \sum_{k\geq 0} \beta_k (z_1 - p)^k,$$

where α and β are not necessarily non-zero complex numbers. Substituting these expansions into (E$'$), we get

$$\alpha\{\alpha^2 - 3\alpha(2\beta+1) - 2(3\beta^2+3\beta-1)\} = 0,$$
$$\beta\{2\beta^2 - (3\alpha^2+3\alpha+2)\} = 0.$$

The solutions of this equation are

$$(0,0), (0,1), (0,-1), (1,0), (2,0), (1,-2), (-1,1), (-1,-1), (-2,2).$$

Hence the sublemma holds. □

Sublemma 3 *Let* $(f, g_2) = (\varphi, \psi)$ *be a rational solution of* (E$'$)*, then* φ *and* ψ *are holomorphic at* $z_1 = \infty$*.*

Proof. Putting $f = 2B + F$, $z_1 = 1/\zeta$, we get

$$\begin{cases} \zeta^4 \dfrac{d^2}{d\zeta^2}F + \zeta^2(-3F + 2\zeta - 6B)\dfrac{d}{d\zeta}F + F^3 - 6F^2(g_2 - B) + F(-6\zeta^2\dfrac{d}{d\zeta}g_2 \\ \qquad - 6{g_2}^2 - 12Bg_2 + \dfrac{1}{\zeta} + 9B^2) - 12B\zeta^2\dfrac{d}{d\zeta}g_2 - 12B{g_2}^2 - \kappa_0 + 2\kappa_\infty + 1 = 0, \\ \zeta^4\dfrac{d^2}{d\zeta^2}g_2 + \zeta^2(3F + 2\zeta + 6B)\dfrac{d}{d\zeta}g_2 - 2{g_2}^3 + 6B{g_2}^2 \\ \qquad + (3F^2 + 12BF + \dfrac{1}{\zeta} + 9B^2)g_2 - \kappa_\infty = 0. \end{cases} \tag{E$'_\infty$}$$

Let $(F, g_2) = (\tilde\varphi, \tilde\psi)$ be a rational solution of (E$'_\infty$). We show that $\tilde\varphi$ and $\tilde\psi$ are holomorphic at $\zeta = 0$. We consider the proof in the following three cases:

1) $\tilde\varphi = 0$,

2) $\tilde\psi = 0$,

3) $\tilde\varphi \neq 0$ and $\tilde\psi \neq 0$.

Case 1). When $\tilde\varphi = 0$, (E$'_\infty$) is reduced to

$$\begin{cases} -12B\zeta^2\dfrac{d}{d\zeta}\tilde\psi - 12B\tilde\psi^2 - \kappa_0 + 2\kappa_\infty + 1 = 0, \\ \zeta^4\dfrac{d^2}{d\zeta^2}\tilde\psi + 2\zeta^2(\zeta + 3B)\dfrac{d}{d\zeta}\tilde\psi - 2\tilde\psi^3 + 6B\tilde\psi^2 + (\dfrac{1}{\zeta} + 9B^2)\tilde\psi - \kappa_\infty = 0. \end{cases} \tag{7}$$

If $\zeta = 0$ is a pole of $\tilde\psi$, $\tilde\psi$ is expanded in the form

$$\tilde\psi = \frac{1}{\zeta^n}(b_0 + b_1\zeta + b_2\zeta^2 + \cdots), \quad (n \in \mathbb{N}, b_0 \neq 0).$$

Substituting the expansion into (7), we get $b_0 = 0$. This is contradiction. Hence $\tilde\psi$ is holomorphic at $\zeta = 0$.

Case 2). When $\tilde{\psi}=0$, (E'_{∞}) is reduced to

$$\begin{cases} \zeta^4\dfrac{d^2}{d\zeta^2}\tilde{\varphi}+\zeta^2(-3\tilde{\varphi}+2\zeta-6B)\dfrac{d}{d\zeta}\tilde{\varphi}+\tilde{\varphi}^3+6B\tilde{\varphi}^2+\tilde{\varphi}(\dfrac{1}{\zeta}+9B^2)-\kappa_0+2\kappa_\infty+1=0, \\ \kappa_\infty=0 \end{cases} \tag{8}$$

If $\zeta=0$ is a pole of $\tilde{\varphi}$, $\tilde{\varphi}$ is expanded in the form

$$\tilde{\varphi}=\frac{1}{\zeta^m}(a_0+a_1\zeta+a_2\zeta^2+\cdots), \quad (m\in\mathbb{N}, a_0\neq 0).$$

Substituting the expansion into (8), we get $a_0=0$. This is contradiction. Hence $\tilde{\varphi}$ is holomorphic at $\zeta=0$.

Case 3). Let

$$\begin{aligned} &\tilde{\varphi}=\zeta^m(a_0+a_1\zeta+\cdots), \\ &\tilde{\psi}=\zeta^n(b_0+b_1\zeta+\cdots), \quad (m,n\in\mathbb{Z}, a_0\neq 0, b_0\neq 0) \end{aligned}$$

be the expansion of $\tilde{\varphi}$ and $\tilde{\psi}$. Substituting above expansions into (E'_{∞}), we find that $m,n<0$ or $m,n\geq 0$. If $m,n<0$, we can easily show that n is equal to m. Assuming that $m=n<0$, we get

$$\begin{cases} {a_0}^3-6{a_0}^2b_0-6a_0{b_0}^2=0, \\ -2{b_0}^3+3{a_0}^2b_0=0. \end{cases} \tag{9}$$

The solution of (9) is $(a_0,b_0)=(0,0)$. This is contradiction. Hence m and n must be nonnegative integers.

By the consideration from case 1) to 3), we find that $\tilde{\varphi}$ and $\tilde{\psi}$ are holomorphic at $\zeta=0$. Therefore Sublemma 3 holds. □

Sublemma 4 *Let* $(f,g_2)=(\varphi,\psi)$ *be a rational solution of* (E′). *Then we have*

$$\operatorname*{Res}_{z_1=\infty}\varphi\,dz_1=-\kappa_0+2\kappa_\infty+1, \quad \operatorname*{Res}_{z_1=\infty}\psi\,dz_1=-\kappa_\infty.$$

Sublemma 4 is proved in the same way as the proof of Sublemma 2. We omit the proof.

Proof of the Lemma 1. By the residue theorem, the sum of all residues of φ is zero. On the other hund, the sum of all residues of φ is "$-\kappa_0+2\kappa_\infty+$ some integer" from Sublemma 2 and 4. Therefore $-\kappa_0+2\kappa_\infty$ is an integer. By the sum of all residues of ψ, we conclude that κ_∞ is an integer in the same way. Hence Lemma 1 holds. □

3.4 Bäcklund transformations of (E)

We put

$$F^{(1)}_{\infty+} \;=\; -f_1-f_2\cdot\frac{-f_1-g_2+b}{{f_1}^2-f_2-g_1+z_1}+\frac{-\kappa_0+\kappa_\infty+1}{{f_1}^2-f_2-g_1+z_1},$$

$$
\begin{aligned}
F^{(2)}_{\infty+} &= (-f_1 - g_2 + b) \cdot \frac{f_1 \cdot f_2 + f_2 \cdot g_2 - b \cdot f_2 - \kappa_0}{{f_1}^2 - f_2 - g_1 + z_1}, \\
G^{(1)}_{\infty+} &= {f_1}^2 - f_2 - g_1 + z_1, \\
G^{(2)}_{\infty+} &= -\frac{{f_1}^2 - f_2 - g_1 + z_1}{-f_1 - g_2 + b},
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{\infty-} &= \frac{-f_1 \cdot g_1 - f_2 \cdot g_2 + \kappa_\infty}{g_1}, \\
F^{(2)}_{\infty-} &= -g_2 \cdot \frac{f_2 \cdot g_2 - \kappa_0}{g_1}, \\
G^{(1)}_{\infty-} &= -g_1 + z_1 - g_2 \cdot \frac{f_2 \cdot g_2 - \kappa_0}{g_1} + \left(\frac{-f_1 \cdot g_1 - f_2 \cdot g_2 + \kappa_\infty}{g_1} \right)^2, \\
G^{(2)}_{\infty-} &= \frac{b \cdot g_2 + g_1}{g_2} + \frac{f_1 \cdot g_1 + f_2 \cdot g_2 - \kappa_\infty}{g_1},
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{0+} &= b - g_2 + \frac{\kappa_0}{f_2} - \frac{f_2 \cdot g_2}{\kappa_0 - f_2 \cdot g_2}, \\
F^{(2)}_{0+} &= z_1 + b^2 - 2g_1 + (f_1 + b) \cdot \frac{\kappa_0 - f_2 \cdot g_2}{f_2} + f_2 \cdot \frac{f_1 \cdot g_1 + f_2 \cdot g_2 - b \cdot g_1 - \kappa_\infty}{\kappa_0 - f_2 \cdot g_2}, \\
G^{(1)}_{0+} &= \left(\frac{f_2 \cdot g_1}{\kappa_0 - f_2 \cdot g_2} \right)^2 + f_2 \cdot \frac{-f_1 \cdot g_1 - b \cdot g_1 - \kappa_0 + \kappa_\infty}{\kappa_0 - f_2 \cdot g_2} + f_2, \\
G^{(2)}_{0+} &= \frac{f_2 \cdot g_1}{\kappa_0 - f_2 \cdot g_2},
\end{aligned}
$$

$$
\begin{aligned}
F^{(1)}_{0-} &= f_1 + g_2 + \frac{{f_1}^2 - f_2 - g_1 + z_1}{-f_1 - g_2 + b}, \\
F^{(2)}_{0-} &= b \cdot g_2 + f_1 \cdot g_2 + g_1 + g_2 \cdot \frac{{f_1}^2 - f_2 - g_1 + z_1}{-f_1 - g_2 + b} + \frac{-\kappa_0 + \kappa_\infty + 1}{-f_1 - g_2 + b}, \\
G^{(1)}_{0-} &= g_2 \cdot (f_1 + g_2 - b), \\
\frac{G^{(2)}_{0-}}{f_1 + g_2 - b} &= \frac{\kappa_0 - 1}{{g_2}^2 \cdot (f_1 + b) - g_2 \cdot (-f_2 - 2g_1 + z_1 + b^2) + g_1 \cdot (f_1 - b) + \kappa_0 - \kappa_\infty - 1} - 1.
\end{aligned}
$$

If (f_1, f_2, g_1, g_2) is a solution of (E), $(F^{(1)}_{*\sharp}, F^{(2)}_{*\sharp}, G^{(1)}_{*\sharp}, G^{(2)}_{*\sharp})$ $(* = 0 \text{ or } \infty, \sharp = + \text{ or } -)$ is a solution of the equation which is obtained by the replacement of

$$
\kappa_* \longrightarrow \begin{cases} \kappa_* + 1 & (\sharp = +), \\ \kappa_* - 1 & (\sharp = -) \end{cases}
$$

in (E). This transformation $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{*\sharp}, F^{(2)}_{*\sharp}, G^{(1)}_{*\sharp}, G^{(2)}_{*\sharp})$ is obtained from Theorem A and Theorem B. We ommit the calculation.

When $g_1 = 0$ or $g_2 = 0$, the transformation $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{\infty-}, F^{(2)}_{\infty-}, G^{(1)}_{\infty-}, G^{(2)}_{\infty-})$ is not defined. Substituting $g_2 = 0$ into (E), we get

$$\begin{aligned}
&{f_1}' = -{f_1}^2 + f_2 + 2g_1 - z_1,\\
&{f_2}' = -f_1 f_2 + b f_2 + \kappa_0,\\
&{g_1}' = 2f_1 g_1 - \kappa_\infty,\\
&0 = g_1.
\end{aligned}$$

Since $g_1 = 0$, κ_∞ must be zero. Similarly, substituting $g_1 = 0$ into (E), we get $\kappa_\infty = 0$ or $\kappa_0 = \kappa_\infty$. This implies that the transformation $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{\infty-}, F^{(2)}_{\infty-}, G^{(1)}_{\infty-}, G^{(2)}_{\infty-})$ is defined when $\kappa_0 \neq \kappa_\infty$ and $\kappa_\infty \neq 0$. We get the following Sublemma and Lemma in this manner.

Sublemma 5 *1)* $g_1 = 0 \Rightarrow \kappa_\infty = 0$ *or* $\kappa_0 = \kappa_\infty$.

2) $g_2 = 0 \Rightarrow \kappa_\infty = 0$.

3) ${f_1}^2 - f_2 - g_1 + z_1 = 0 \Rightarrow \kappa_\infty = -1$ *or* $\kappa_0 - \kappa_\infty - 1 = 0$.

4) $b - f_1 - g_2 = 0 \Rightarrow \kappa_0 - \kappa_\infty - 1 = 0$.

5) $f_2 = 0 \Rightarrow \kappa_0 = 0$.

6) $f_2 \cdot g_2 - \kappa_0 = 0 \Rightarrow \kappa_0 = 0$ *or* $\kappa_0 = \kappa_\infty$.

7) $g_2 \cdot z_1 - {g_2}^2(b + f_1) + g_2(b^2 - 2g_1 - f_2) + g_1(b - f_1) - \kappa_0 + \kappa_\infty + 1 = 0 \Rightarrow \kappa_0 = 1$ *or* $\kappa_0 - \kappa_\infty - 1 = 0$.

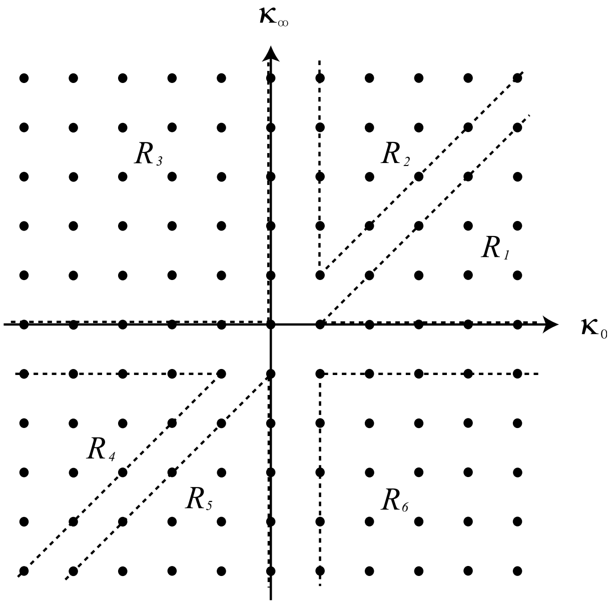
Lemma 2 *1) When* $\kappa_\infty \neq -1, \kappa_\infty \neq \kappa_0 - 1$, *the transformation* $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{\infty+}, F^{(2)}_{\infty+}, G^{(1)}_{\infty+}, G^{(2)}_{\infty+})$ *is defined.*

2) When $\kappa_\infty \neq 0, \kappa_\infty \neq \kappa_0$, *the transformation* $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{\infty-}, F^{(2)}_{\infty-}, G^{(1)}_{\infty-}, G^{(2)}_{\infty-})$ *is defined.*

3) When $\kappa_0 \neq 0, \kappa_\infty \neq \kappa_0$, *the transformation* $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{0+}, F^{(2)}_{0+}, G^{(1)}_{0+}, G^{(2)}_{0+})$ *is defined.*

4) When $\kappa_0 \neq 1, \kappa_\infty \neq \kappa_0 - 1$, *the transformation* $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{0-}, F^{(2)}_{0-}, G^{(1)}_{0-}, G^{(2)}_{0-})$ *is defined.*

Based on Lemma 2, we divide $\mathbb{Z}^2$ into six regions $R_1, R_2, \cdots, R_6$ as follows.



Notice that if $(\kappa_0, \kappa_\infty), (\kappa'_0, \kappa'_\infty) \in R_j$ and (f_1, f_2, g_1, g_2) is a solution of (E) with parameter $(\kappa_0, \kappa_\infty)$, we can always construct a solution of (E) with parameter $(\kappa'_0, \kappa'_\infty)$ from (f_1, f_2, g_1, g_2). Hence, examining the equation (E) with parameter $(\kappa_0, \kappa_\infty) \in \{(1,0),(1,1),(0,0),(-1,-1),(0,-1),(1,-1)\}$, we can find all rational solutions. From now on , $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(\alpha,\beta)}$ (or $(\mathrm{E}')|_{(\kappa_0,\kappa_\infty)=(\alpha,\beta)}$) denotes the equation (E) (or (E′)) with parameter $(\kappa_0, \kappa_\infty) = (\alpha, \beta)$.

3.5 Rational solutions of (E)

In this subsection, we show the following proposition:

Proposition 1 *1) $(E)|_{(\kappa_0,\kappa_\infty)=(\alpha,\beta)}$ has a rational solution $\Leftrightarrow (\alpha,\beta) \in R_1 \cup R_4$.*

2) When $(\alpha,\beta) \in R_1 \cup R_4$, $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(\alpha,\beta)}$ has a unique rational solution.

To show the proposition, we prove following lemmas.

Lemma 3 $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(1,0)}$ *has no rational solutions except for* $(f_1, f_2, g_1, g_2) = (b, z_1 + b^2, 0, 0)$.

Lemma 4 $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(-1,-1)}$ *has no rational solutions except for* $(f_1, f_2, g_1, g_2) = (b, z_1+b^2, 0, -1/(z_1+b^2))$.

Lemma 5 *When* $(\kappa_0, \kappa_\infty) \in \{(1,1),(0,0),(0,-1),(1,-1)\}$, (E) *has no rational solutions.*

Using Lemma 3, Lemma 4, Lemma 5 and the transformation $(f_1, f_2, g_1, g_2) \to (F^{(1)}_{*\sharp}, F^{(2)}_{*\sharp}, G^{(1)}_{*\sharp}, G^{(2)}_{*\sharp})$ stated in subsection 3.4, we obtain Proposition 1.

Sublemma 6 *Let* $(F, g_2) = (\varphi, \psi)$ *be a rational solution of* (E'_∞). *If* $\kappa_\infty = 0$, ψ *must be zero.*

Proof. We consider the proof in the following three cases:

1) $\varphi = 0$ and $B \neq 0$,

2) $\varphi = 0$ and $B = 0$,

3) $\varphi \neq 0$.

Case 1). Substituting $\varphi = 0$, (E'_∞) is reduced to

$$\begin{cases} -12B\zeta^2 \dfrac{d}{d\zeta}\psi - 12B\psi^2 - \kappa_0 + 2\kappa_\infty + 1 = 0, \\ \zeta^4 \dfrac{d^2}{d\zeta^2}\psi + 2\zeta^2(\zeta + 3B)\dfrac{d}{d\zeta}\psi - 2\psi^3 + 6B\psi^2 + (\dfrac{1}{\zeta} + 9B^2)\psi - \kappa_\infty = 0. \end{cases} \tag{10}$$

Eliminating $d\psi/d\zeta$, we get

$$\psi = \frac{3 \cdot B \cdot (\kappa_0 - 1)\zeta}{(54B^3 + \kappa_0 - 2\kappa_\infty - 1) \cdot \zeta + 6B}. \tag{11}$$

Substituting $\kappa_\infty = 0$ and (11) into the first equation in (10), we find $\kappa_0 = 1$. Therefore $\psi = 0$.

Case 2). When $B = 0$, (10) is

$$\begin{cases} -\kappa_0 + 2\kappa_\infty + 1 = 0, \\ \zeta^4 \dfrac{d^2}{d\zeta^2}\psi + 2\zeta^3 \dfrac{d}{d\zeta}\psi - 2\psi^3 + \dfrac{1}{\zeta}\psi - \kappa_\infty = 0. \end{cases}$$

Putting $\zeta = 1/z_1$, the second equation is

$$\frac{d^2}{dz_1^2}\psi - 2\psi^3 + z_1\psi - \kappa_\infty = 0. \tag{12}$$

This is the second Painlevé equation. It is known that (12) has the unique rational solution $\psi = 0$ when $\kappa_\infty = 0$. Hence Sublemma holds.

Case 3). By Sublemma 3, φ and ψ are written in the form

$$\begin{cases} \varphi = \zeta^m(a_0 + a_1\zeta + a_2\zeta^2 + \cdots) \quad (a_0 \neq 0; m \in \mathbb{Z}_{\geq 0}), \\ \psi = \zeta^n(b_0 + b_1\zeta + b_2\zeta^2 + \cdots) \quad (n \in \mathbb{Z}_{\geq 0}). \end{cases} \tag{13}$$

If we assume $\psi \neq 0$, we can assume $b_0 \neq 0$. Substituting $\kappa_\infty = 0$ and (13) into the second equation in (E'_∞), we find $m = -1$. This is contradiction. Hence ψ must be zero. □

Proof of Lemma 3. The Lemma 3 is equivalent to the following claim.

Claim (E')$|_{(\kappa_0,\kappa_\infty)=(1,0)}$ has no rational solutions except for $(f, g_2) = (2B, 0)$, where $B = b/3$.

We will show the claim. Putting $f = F + 2B$ and $z_1 = 1/\zeta$, we get (E'_∞). Let $(F, g_2) = (\varphi, \psi)$ be a rational solution of (E'_∞)$|_{(\kappa_0,\kappa_\infty)=(1,0)}$. From Sublemma 6 and Sublemma 3, φ is holomorphic at $\zeta = 0$ and $\psi = 0$.

We assume that φ is not equal to zero. Let

$$\varphi = \zeta^m(\alpha_0 + \alpha_1\zeta + \alpha_2\zeta^2 + \cdots), \quad (m \geq 0,\ \alpha_0 \neq 0)$$

be the expansion of φ around $\zeta = 0$. Substituting $\psi = 0,\ \kappa_0 = 1,\ \kappa_\infty = 0$ and this expansion into (E'_∞), we find α_0 must be zero. This is contradiction. Hence $\varphi = 0$. Therefore $(\mathrm{E}'_\infty)|_{(\kappa_0,\kappa_\infty)=(1,0)}$ has no rational solutions except for $(F, g_2) = (0, 0)$. Hence the claim holds. □

Sublemma 7 *Let* (f_1, f_2, g_1, g_2) *be a solution of* (E). *Putting*

$$\begin{aligned} \bar{f}_1 &= f_1, \\ \bar{f}_2 &= f_2, \\ \bar{g}_1 &= g_1, \\ \bar{g}_2 &= g_2 - \frac{\kappa_0}{f_2}, \end{aligned}$$

$(\bar{f}_1, \bar{f}_2, \bar{g}_1, \bar{g}_2)$ *is a solution of the equation which is obtained by the replacement of*

$$\kappa_0 \longrightarrow -\kappa_0, \quad \kappa_\infty \longrightarrow \kappa_\infty - \kappa_0$$

in (E).

This transformation is obtained from Theorem C. We ommit the proof.
Using this Sublemma, we get Lemma 4 from Lemma 3.

Sublemma 8 $(\mathrm{E}')|_{(\kappa_0,\kappa_\infty)=(0,0)}$ *has no rational solutions.*

Proof. Putting

$$\begin{aligned} f &\longrightarrow F + 2B, \\ z_1 &\longrightarrow 1/\zeta, \end{aligned}$$

the equation $(\mathrm{E}')|_{(\kappa_0,\kappa_\infty)=(0,0)}$ changes into $(\mathrm{E}'_\infty)|_{(\kappa_0,\kappa_\infty)=(0,0)}$. We will show that $(\mathrm{E}'_\infty)|_{(\kappa_0,\kappa_\infty)=(0,0)}$ has no rational solutions. Let $(F, g_2) = (\varphi, \psi)$ be a rational solution of (E'_∞). By Sublemma 6, ψ must be zero. When $\psi = 0$, $(\mathrm{E}'_\infty)|_{(\kappa_0,\kappa_\infty)=(0,0)}$ is reduced to

$$\zeta^4 \frac{d^2}{d\zeta^2}F + \zeta^2(-3F + 2\zeta - 6B)\frac{d}{d\zeta}F + F^3 + 6BF^2 + F\left(\frac{1}{\zeta} + 9B^2\right) + 1 = 0.$$

Substituting the expansion

$$F = \zeta^m(\alpha_0 + \alpha_1\zeta + \alpha_2\zeta^2 + \cdots), \quad (\alpha_0 \neq 0),$$

we get $m = 1$. This implies that $z_1 = \infty$ is a pole of φ. This contradicts Sublemma 3. □

Sublemma 9 *Let* (f_1, f_2, g_1, g_2) *be a solution of* (E). *Putting*

$$\begin{aligned}\bar{f}_1 &= -f_1,\\ \bar{f}_2 &= f_2,\\ \bar{g}_1 &= -g_1 + {f_1}^2 - f_2 + z_1,\\ \bar{g}_2 &= g_2 + f_1 - b,\end{aligned}$$

$(\bar{f}_1, \bar{f}_2, \bar{g}_1, \bar{g}_2)$ *is a solution of the equation which is obtained by the replacement of*

$$\kappa_\infty \longrightarrow \kappa_0 - \kappa_\infty - 1, \quad b \longrightarrow -b$$

in (E).

Proof. One can check the sublemma by direct calculation. □

Sublemma 10 *When* $(\kappa_0, \kappa_\infty) \in \{(1,1),(0,-1),(1,-1)\}$, (E) *has no rational solutions.*

Proof. By Sublemma 8 and Sublemma 9, we find that $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(0,-1)}$ has no rational solutions. Since $(0,-1),(-1,-2) \in R_5$, we can transform the equation $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(0,-1)}$ into the equation $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(-1,-2)}$ by an appropriate birational transformation. Hence $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(-1,-2)}$ has no rational solutions. Using Sublemma 7, we find $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(1,-1)}$ has no rational solutions. Similarly, by Sublemma 9, we find $(\mathrm{E})|_{(\kappa_0,\kappa_\infty)=(1,1)}$ has no rational solutions. □

By Sublemma 8 and Sublemma 10, we get Lemma 5.

3.6 Construction of rational solutions of (P)

Let $(f_1(z_1), f_2(z_1), g_1(z_1), g_2(z_1))$ be a rational solution of (E). We put

$$\begin{aligned}u_j &= f_j + \sum_{k=1}^{\infty} {f_j}^{(k)}(z_1)\cdot(z_2-b)^k,\\ v_j &= g_j + \sum_{k=1}^{\infty} {g_j}^{(k)}(z_1)\cdot(z_2-b)^k, \quad (j=1,2).\end{aligned}$$

Substituting (u_1, u_2, v_1, v_2) into (P), we get

$$(h+1)\cdot {f_1}^{(h+1)} = -\sum_{\substack{i+j=h\\ i,j\geq 0}} {f_2}^{(i)}(2{g_2}^{(j)} + {f_1}^{(j)}) + b{f_2}^{(h)} + {f_2}^{(h-1)} + \kappa_0\delta_{h,0}, \tag{14}$$

$$(h+1)\cdot {g_1}^{(h+1)} = \sum_{\substack{i+j+k=h\\ i,j,k\geq 0}} {f_2}^{(i)}{g_2}^{(j)}{g_2}^{(k)} + \sum_{\substack{i+j=h\\ i,j\geq 0}} {f_2}^{(i)}{g_1}^{(j)} - \kappa_0{g_2}^{(h)}, \tag{15}$$

$$\begin{aligned}(h+1)\cdot {f_2}^{(h+1)} = &\sum_{\substack{i+j=h\\ i,j\geq 0}} {f_2}^{(i)}(-2{g_1}^{(j)} - 2b{g_2}^{(j)} - {f_2}^{(j)}) - 2\sum_{\substack{i+j+k=h\\ i,j,k\geq 0}} {f_1}^{(i)}{f_2}^{(j)}{g_2}^{(k)}\\ &- 2\sum_{\substack{i+j=h-1\\ i,j\geq 0}} {f_2}^{(i)}{g_2}^{(j)} + (b^2+z_1){f_2}^{(h)} + 2b{f_2}^{(h-1)} + {f_2}^{(h-2)}\end{aligned} \tag{16}$$

$$
\begin{aligned}
&\quad + \kappa_0 {f_1}^{(h)} + \kappa_0(b\delta_{h,0} + \delta_{h,1}),\\
(h+1)\cdot {g_2}^{(h+1)} =& \sum_{\substack{i+j=h\\ i,j\geq 0}} {g_2}^{(i)}(2{f_2}^{(j)} + 2{g_1}^{(j)} + b{g_2}^{(h)}) + \sum_{\substack{i+j=h\\ i,j\geq 0}} {f_1}^{(i)}{g_1}^{(j)} \qquad (17)\\
&+ \sum_{\substack{i+j+k=h\\ i,j,k\geq 0}} {f_1}^{(i)}{g_2}^{(j)}{g_2}^{(k)} + \sum_{\substack{i+j=h-1\\ i,j\geq 0}} {g_2}^{(i)}{g_2}^{(j)} - b{g_1}^{(h)} - (b^2+z_1){g_2}^{(h)}\\
&- {g_1}^{(h-1)} - 2b{g_2}^{(h-1)} - {g_2}^{(h-2)} - \kappa_\infty\delta_{k,0}, \quad (h = 0,1,2,3,\cdots),
\end{aligned}
$$

where

$$
{f_j}^{(k)} = \begin{cases} f_j & (k=0) \\ 0 & (k<0) \end{cases}, \quad {g_j}^{(k)} = \begin{cases} g_j & (k=0) \\ 0 & (k<0) \end{cases}, \quad (j=1,2).
$$

By Lemma 3, $(f_1, f_2, g_1, g_2) = (b, z_1 + b^2, 0, 0)$ is the unique rational solution of (E) for $(\kappa_0, \kappa_\infty) = (1, 0)$. Substituting $g_1 = g_2 = 0$ into (15),(17), we find ${g_j}^{(k)} = 0$ $(k = 1,2,3,\cdots)$. Putting ${g_j}^{(k)} = 0$ $(k = 0,1,2,\cdots)$ in the recurrence relations (14) and (16), we get

$$
(h+1)\cdot {f_1}^{(h+1)} = -\sum_{\substack{i+j=h\\ i,j\geq 0}} {f_2}^{(i)}{f_1}^{(j)} + b{f_2}^{(h)} + {f_2}^{(h-1)} + \kappa_0\delta_{h,0}, \qquad (18)
$$

$$
\begin{aligned}
(h+1)\cdot {f_2}^{(h+1)} =& -\sum_{\substack{i+j=h\\ i,j\geq 0}} {f_2}^{(i)}{f_2}^{(j)} + (b^2+z_1){f_2}^{(h)} + 2b{f_2}^{(h-1)} + {f_2}^{(h-2)} \qquad (19)\\
&+ \kappa_0 {f_1}^{(h)} + \kappa_0(b\delta_{h,0} + \delta_{h,1}).
\end{aligned}
$$

Substituting ${f_1}^{(0)} = b,\ {f_2}^{(0)} = z_1 + b^2$ into (18) and (19), we find

$$
\begin{aligned}
{f_1}^{(1)} &= 1, & {f_2}^{(1)} &= 2b,\\
{f_1}^{(2)} &= 0, & {f_2}^{(2)} &= 1,\\
{f_1}^{(3)} &= 0, & {f_2}^{(3)} &= 0,\\
{f_1}^{(4)} &= 0, & {f_2}^{(4)} &= 0.
\end{aligned}
$$

Using this fact and induction, we can show ${f_j}^{(k)} = 0$ $(k \geq 4)$. Hence

$$
\begin{cases}
u_1 = z_2,\\
u_2 = z_1 + {z_2}^2,\\
v_1 = 0,\\
v_2 = 0.
\end{cases} \qquad (20)
$$

By the method of composition, (u_1, u_2, v_1, v_2) satisfies the equation (P). From above consideration, we get the following Lemma.

Lemma 6 *When* $(\kappa_0, \kappa_\infty) = (1, 0)$, (P) *has a unique rational solution. The unique rational solution of* (P) *is given by* (20).

Remark The unique rational solution (20) is obtained from Theorem D. In fact, putting $\kappa_0 = 1$, the function τ is

$$
\tau = \int_c (z_2 - s)^{-\kappa_0} \exp\left(\frac{1}{3}s^3 + z_1 s\right) ds = -\exp\left(\frac{1}{3}{z_2}^3 + z_1 z_2\right).
$$

Hence

$$u_j = \frac{\partial}{\partial z_j}\log\tau,$$
$$v_j = 0, \quad (j = 1, 2)$$

coincide with (20). Notice that if κ_0 is a positive integer and $\kappa_\infty = 0$, τ is written in the form

$$\tau = (\text{polynomial of } z_1, z_2)\cdot\exp\left(\frac{1}{3}{z_2}^3 + z_1 z_2\right).$$

This implies that u_1, u_2 are rational functions of z_1, z_2. By Proposition 1 and the method of construction of solution (P) stated in (14) - (17), we find this $(u_1, u_2, 0, 0)$ is a unique rational solution of (P) for $\kappa_0 \in \mathbb{Z}_{>0}$ and $\kappa_\infty = 0$.

3.7 Proof of Theorem 1 and Theorem 2

From 1) of Proposition 1, we find (P) has no rational solutions, if κ_0 and κ_∞ do not satisfy the condition $(*)$ (i.e., $(\kappa_0, \kappa_\infty) \notin R_1 \cup R_4$). Therefore Theorem 1 holds. We will prove (P) has a unique rational solution, if $(\kappa_0, \kappa_\infty) \in R_1 \cup R_4$.

Sublemma 11 *When $\kappa_0 \in \mathbb{Z}_{>0}$ and $\kappa_\infty \in \mathbb{Z}_{\geq 0}$, u_2 and $\kappa_0 - u_1\cdot u_2 - u_2\cdot v_2 + u_2\cdot z_2$ are not zero identically.*

Proof. The explicit form of (P) is

$$\frac{\partial u_1}{\partial z_1} = 2v_1 + u_2 - {u_1}^2 - z_1, \tag{21}$$
$$\frac{\partial u_1}{\partial z_2} = -2u_2\cdot v_2 - u_1\cdot u_2 + u_2\cdot z_2 + \kappa_0,$$
$$\frac{\partial v_1}{\partial z_1} = 2u_1\cdot v_1 + u_2\cdot v_2 - \kappa_\infty, \tag{22}$$
$$\frac{\partial v_1}{\partial z_2} = u_2\cdot {v_2}^2 + u_2\cdot v_1 - \kappa_0\cdot v_2,$$
$$\frac{\partial u_2}{\partial z_1} = -2u_2\cdot v_2 - u_1\cdot u_2 + u_2\cdot z_2 + \kappa_0, \tag{23}$$
$$\frac{\partial u_2}{\partial z_2} = -2u_2\cdot v_1 - 2u_2\cdot v_2\cdot(u_1 + z_2) - {u_2}^2 + u_2\cdot {z_2}^2 + u_2\cdot z_1 + \kappa_0\cdot(u_1 + z_2),$$
$$\frac{\partial v_2}{\partial z_1} = {v_2}^2 - v_1 + u_1\cdot v_2 - z_2\cdot v_2, \tag{24}$$
$$\frac{\partial v_2}{\partial z_2} = 2v_1\cdot v_2 + (u_1 + z_2)\cdot {v_2}^2 - v_1\cdot(-u_1 + z_2) - v_2\cdot(-2u_2 + {z_2}^2 + z_1) - \kappa_\infty.$$

Hence, if $u_2 = 0$, κ_0 must be zero from (23). This is contradiction. Therefore u_2 is not zero identically.

If $\kappa_0 - u_1\cdot u_2 - u_2\cdot v_2 + u_2\cdot z_2 = 0$, u_1 is written in the form

$$u_1 = \frac{\kappa_0 - u_2\cdot v_2 + u_2\cdot z_2}{u_2}.$$

Substituting this into (21),(23) and (24), we find

$$v_1 = -u_2 + {v_2}^2 - 2v_2z_2 + z_1 + {z_2}^2 + \frac{2\kappa_0}{u_2} \cdot (-v_2 + z_2) + \left(\frac{\kappa_0}{u_2}\right)^2.$$

From the equation (22), we get $\kappa_\infty = -1$. Hence $\kappa_0 - u_1 \cdot u_2 - u_2 \cdot v_2 + u_2 \cdot z_2$ is not zero identically, too. □

Proposition 2 *When* $(\kappa_0, \kappa_\infty) \in R_1 \cup R_4$, (P) *has a unique rational solution.*

Proof. For a solution (u_1, u_2, v_1, v_2) of (P) with parameter $(\kappa_0, \kappa_\infty) \in R_1$, $(u_1, u_2, v_1, v_2 - \kappa_0/u_2)$ is a solution of (P) with parameter $(-\kappa_0, \kappa_\infty - \kappa_0) \in R_4$. (See Theorem C.) Hence, it is sufficient to show that (P) has a unique rational solution when $(\kappa_0, \kappa_\infty) \in R_1$.

Let (u_1, u_2, v_1, v_2) be a solution of (P) for $(\kappa_0, \kappa_\infty) \in R_1$. Putting

$$\begin{aligned}
\zeta_1 &= -\frac{\kappa_0}{u_2} + u_1 + v_2 + \frac{u_2 \cdot ({u_1}^2 - u_2 - v_1 + z_1)}{\kappa_0 - u_1 \cdot u_2 - u_2 \cdot v_2 + u_2 \cdot z_2}, \\
\zeta_2 &= -{u_1}^2 + u_1 \cdot v_2 + 2v_1 + v_2 \cdot z_2 + u_2 - z_1 - \frac{\kappa_0}{u_2} \cdot (u_1 + z_2) \\
&\quad + u_2 \cdot \frac{(u_1 - z_2) \cdot (-{u_1}^2 + u_2 + v_1 - z_1 + \kappa_\infty + 1)}{\kappa_0 - u_1 \cdot u_2 - u_2 \cdot v_2 + u_2 \cdot z_2}, \\
\eta_1 &= v_2 \cdot (u_1 + v_2 - z_2) + \frac{\kappa_0 \cdot (-u_1 - 2v_2 + z_2)}{u_2} + \left(\frac{\kappa_0}{u_2}\right)^2, \\
\eta_2 &= -u_1 - v_2 + z_2 + \frac{\kappa_0}{u_2},
\end{aligned}$$

$(\zeta_1, \zeta_2, \eta_1, \eta_2)$ is a solution of (P) for $(\kappa_0+1, \kappa_\infty+1)$ by Theorem A and B. From the Remark in subsection 3.6, (P) has a unique rational solution, if $\kappa_0 \in \mathbb{Z}_{>0}$ and $\kappa_\infty = 0$. Using above transformation, we get a rational solution of (P) for any $(\kappa_0, \kappa_\infty) \in R_1$. (Notice that u_2 and $\kappa_0 - u_1 \cdot u_2 - u_2 \cdot v_2 + u_2 \cdot z_2$ are not zero identically from Sublemma 11. Hence above transformation is always defined.) This implies (P) has a rational solution for any $(\kappa_0, \kappa_\infty) \in R_1$. This rational solution is unique because of Proposition 1. □

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References

[1] M.Jimbo, T.Miwa, K.Ueno : *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I*, Physica,**2D**, 306 – 352 (1981).

[2] M.Jimbo, T.Miwa : *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II*, Physica,**2D**, 407 – 448 (1981).

[3] H.Kawamuko : *Studies on the fourth Painlevé equation in several variables*, Ph. D. dissertation, The University of Tokyo, (1997).

[4] H.Kawamuko : *On the holonomic deformation of linear differential equations*, Proc. Japan Acad. Ser. A.,**73**, 152 – 154 (1997).

[5] H.Kawamuko : *On the polynomial Hamiltonian structure associated to the $L(1, g+2; g)$ type*, Proc. Japan Acad. Ser. A.,**73**, 155 – 157 (1997).

[6] H.Kimura : *The degeneration of the two dimensional Garnier system and the polynomial Hamiltonian structure*, Ann.Math.Pura.Appl.,**155**, 25 – 74 (1989).

[7] T.Miwa : *Painlevé property of monodromy preserving deformation equations and the analyticity of τ functions*, Publ. RIMS, Kyoto Univ.,**17**, 703–721 (1981).

[8] Y.Murata : *Rational solutions of the second and fourth Painlevé equations*, Funk.Ekva.,**28**, 1 – 32 (1985).

[9] K.Okamoto : *Studies on the Painlevé equations III*, Math.Ann.,**275**, 221 – 255 (1986).

[10] K.Okamoto : *The Hamiltonians associated to the Painlevé equations*, Pysics. The Painlevé Property, ed. R. Conte, CRM Series in Mathematical Pysics (Springer, 1995), pages 735 – 787.

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