On uniqueness and existence of slowly decaying positive radial solutions for some semilinear elliptic equations

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1 Introduction

In this paper, we consider the following semilinear elliptic equation

$$\Delta u + f(|x|, u) = 0 \;\text{ in }\; \mathbb{R}^n \; (n \geq 3). \tag{1}$$

Since we are only interested in positive radial solutions, we study the following problem :

$$u'' + \frac{n-1}{r}u' + f(r,u) = 0, \tag{2}$$

$$u > 0 \; (r > 0) \text{ and } u \to 0 \text{ as } r \to \infty, \tag{3}$$

where $r > 0$, $r = |x|$, $x \in \mathbb{R}^n$. If $f(r,u) > 0$ for small $u > 0$, then $r^{n-2}u(r)$ is increasing for large $r > 0$ (see the proof of Lemma 2.1), and therefore the solutions of (2)-(3) can be classified into two types:

(R) if $r^{n-2}u(r) \to C < \infty (r \to \infty)$ for some $C > 0$, then $u(r)$ is called a rapidly decaying solution,

(S) if $r^{n-2}u(r) \to \infty (r \to \infty)$, then $u(r)$ is called a slowly decaying solution.

Many authors studied the non-existence or existence and uniqueness of positive radial solutions of (2)-(3) under suitable structure conditions on $f(r,u)$ (see [Ni],[NS],[Pan1],[LN],[SZ],[Li],[Pan2] and the references therein). The case $f(r,u) = u^p$ has been extensively studied. Among them, it is known that if $n/(n-2) < p < (n+2)/(n-2)$, there exists only one slowly decaying positive solution u near infinity, namely $u(r) = \lambda r^{-\alpha}$ with $\alpha = 2/(p-1), \lambda = \{\alpha(n-2-\alpha)\}^{1/(p-1)}$ (see e.g. [SZ]). For the case $f(r,u) = u^p + u^q, n/(n-2) <$

$p < (n+2)/(n-2), p < q$, several authors studied existence and uniqueness of slowly decaying solutions of (2)-(3). In particular, Qi and Lu[QL] proved the existence and uniqueness of slowly decaying solution of (2)-(3) near infinity under the assumptions $n/(n-2) < p < (n+2)/(n-2), p < q$. Furthermore, they showed that if we impose the additional condition $q \leq (n+2)/(n-2)$, then the slowly decaying solution can be extended on $(0, \infty)$ as a singular solution, i.e. $\lim_{r \to 0} u(r) = +\infty$. See also [Pan1],[LN],[SZ] for some previous results on this problem. For the case $q > (n+2)/(n-2)$, the classification of the slowly decaying solution u as $r \to 0$ is difficult and has been an open problem. Very recently, partial results in this direction was obtained by R.Bamón, I.Flores and M.del Pino (see [BFP] for the details).

The purpose of this paper is to show uniqueness and existence of slowly decaying solutions near infinity for general nonlinearity $f(r,u)$ which satisfies certain structure conditions, including the typical one $f(r,u) = u^p + K(r)u^q$, but $K(r)$ is not necessarily bounded. Furthermore, we investigate the sufficient condition on $f(r,u)$ to make a slowly decaying solution singular at $r = 0$.

Throughout this paper, we assume that the constants p and q satisfy the following relations:

$$\frac{n}{n-2} < p < \frac{n+2}{n-2},\ p < q.$$

We also use the notation :

$$\alpha = \frac{2}{p-1},\ \sigma = (q-p)\alpha,\ \lambda = \{\alpha(n-2-\alpha)\}^{\frac{1}{p-1}}.$$

We assume the following conditions for $f(r,u)$.

(A-1) $f(r,u) = 0$ for $u \leq 0$ and $f(r,u)$ is continuous on $(0,\infty) \times (0,\infty)$ and locally Lipschitz continuous with respect to u.

(A-2) There exist positive constants C_0 and δ such that $f(r,u) \geq C_0 u^p$ holds for $u \in (0,\delta)$.

(A-3) There exists a function $K(r)$ such that $f(r,u) = u^p + f_1(r,u)$, $|f_1(r,u)| \leq K(r)u^q$ for $u \in (0,\delta)$ and $K(r) = O(r^l)$ $(r \to \infty)$ for some $l < \sigma$.

(A-4) There exists a function $\widetilde{K}(r)$ such that $|\{f_1(r,u)\}_u| \leq \widetilde{K}(r)u^{q-1}$ for $u \in (0,\delta)$ and $\widetilde{K}(r) = O(r^l)$ $(r \to \infty)$ for some $l < \sigma$.

First, we state the main result on the uniqueness of slowly decaying positive solutions near infinity.

Theorem 1.1. *Suppose that* $(A-1) \sim (A-4)$ *hold. Then any slowly decaying solution of* $(2)-(3)$*, if it exists, satisfies that for any* $\epsilon > 0$

$$r^\alpha u - \lambda = o(r^{-(\sigma - l - \epsilon)}) \text{ as } r \to \infty \tag{4}$$

and there exists at most one slowly decaying solution of $(2)-(3)$ *on* $[r_0, \infty)$ *for any* $r_0 > 0$.

Remark 1.1. *Theorem 1.1 is an extension of [QL], [SZ]. Actually, in [QL] they asserts existence and uniqueness for more general nonlinearity* $f(r,u) = f(u)$ *with* $\lim_{u \to 0} \frac{f(u)-u^p}{u^q} = 1$ *or* $f(r,u) = u^p + K(r)u^q$ *with a bounded function* $K(r)$*. However, their proof, even for the uniqueness, is based on the higher order asymptotic expansion of slowly decaying solutions. We suspect that such higher order asymptotic expansion can not be obtained if there is an oscillation in the higher order terms of* $f(u)$ *or* $K(r)$ *(cf.Theorem 1.2 and 1.3). The method of [QL] cannot be applied for* $f(r,u)$ *satisfying (A-1)∼(A-4) in general.*

We show the estimate (4) by the argument in [QL] and prove Theorem 1.1 by the method in [SZ] by using the estimate (4).
To show existence of slowly decaying solutions, we need the higher order asymptotic expansion for the solution u, since we use the contraction mapping principle as in [QL].

Theorem 1.2. *Let* $f(r,u) = u^p + r^l u^q$ $(l < \sigma)$, $n/(n-2) < p < (n+2)/(n-2)$, $q > p$, $\sigma = (q-p)\alpha$, $\alpha = 2/(p-1)$*. Then there exists a unique slowly decaying solution on* (r_0, ∞) *for sufficiently large* $r_0 > 0$.

Theorem 1.2 is an extension of Theorem 1 in [QL]. Moreover, although we follow the strategy of [QL], we simplify the proof slightly (see Lemma 4.1 and Remark 4.1). For more general nonlinearity, we have the following theorem.

Theorem 1.3. *Let* $f(r,u) = u^p + K(r)g(u)$*. Suppose* $g(u) = u^q(\sum_{n=0}^{\infty} b_n u^n)$ $(u \to 0)$ *and* $K(r) = r^l \sum_{n=0}^{\infty} \alpha_n / r^n$ $(r \to \infty)$, $l < \sigma$*. Furthermore, we assume* $\sigma - l$ *and* $\alpha \in \mathbb{Q}$*. Then there exists a unique slowly decaying solution on* (r_0, ∞) *for sufficiently large* $r_0 > 0$.

In these two cases, the precise asymptotic expansion of the slowly decaying solution will be given in details in section 4 (see Theorem 4.1 and 4.2). See also Remark 4.2 for more general statements. Although basically we follow the strategy of [QL], we simplify the procedure slightly and give the proof in details.
Next, we consider the following problem; when we have a slowly decaying positive solution u on $[r_0, \infty)$, to what extent we can extend the solution u backward into the region $(0, r_0)$. The behavior can be classified as follows. Instead of (A-1),(A-2), we impose a slightly stronger condition:

(A-1)' $f(r,u) = 0$ for $u \leq 0$ and $f(r,u)$ is continuous on $[0,\infty) \times [0,\infty)$ and locally Lipschitz continuous with respect to u.

(A-2)' $f(r,u) \geq 0$ on $(0,\infty) \times (0,\infty)$ and there exist positive constants C_0 and δ such that $f(r,u) \geq C_0 u^p$ holds for any $u \in (0,\delta)$.

Proposition 1.1. *Suppose that (A-1)' and (A-2)' hold. Then the behavior of the solution of (2)-(3) on $(0,\infty)$ is classified into three types:*

1. *singular solution which satisfies $u(r) \to +\infty$ as $r \to +0$.*
2. *regular solution which satisfies $u(r) \to c'$ as $r \to +0$ for some $c' > 0$.*
3. *0-hit solution which satisfies $u(r_1) = 0$ for some $r_1 > 0$.*

Furthermore if $f(r,u)$ satisfies the certain additional structure condition, by using Pohozaev identity (see Section 4), we can extend the slowly decaying solutions of (2)-(3) backward into $(0,+\infty)$ as a singular solution. To state the result precisely, instead of (A-3), we must impose a stronger condition.

(A-3)' There exist a function $L(r)$ and a constant δ_2 such that $f(r,u) = u^p + f_1(r,u)$, $|f_1(r,u)| \leq L(r)u^q$ for $u \in (0,\delta_2)$, $|r\partial f_1/\partial r| \leq ML(r)u^q$ for some $M > 0$, and $L(r) = O(r^l)$ $(r \to \infty)$ for some $l < \sigma$.

Theorem 1.4. *Suppose that (A-1)',(A-2)',(A-3)' hold and $f(r,u)$ satisfies*

$$(5) \quad nF(r,u) - \frac{n-2}{2}uf(r,u) + rF_r(r,u) \geq 0 \text{ and } \not\equiv 0, \ (r,u) \in (0,\infty) \times (0,\delta)$$

for any $\delta > 0$, where $F(r,u) = \int_0^u f(r,t)dt$. Then any positive solution of (2)$-$(3), if it exists on $[r_0,\infty)$ for some $r_0 > 0$, can be extended to $(0,\infty)$ and satisfies

$$(6) \qquad u(r) \to \infty \ (r \to +0).$$

Theorem 1.4 is an extension of Theorem 3 in [QL]. The classification problem for slowly decaying solutions on $(0,\infty)$ is widely open (see [BFP]).

2 Preliminaries

In this section, we prepare some lemmas and give their proofs. First, we give the basic estimates for any solution of (2)-(3). Since $u(r) \to 0$ *as* $r \to \infty$, there exists a constant r_0 such that $0 < u(r) < \delta$ for $r \geq r_0$.
The following Lemma 2.1 and 2.2 are essentially well-known (see [Ni],[NS]). However, we present the proofs under the somewhat general assumptions $(A-1) \sim (A-3)$ to make the paper self-contained.

Lemma 2.1. *Suppose that* $(A-1) \sim (A-3)$ *hold. Then any solution* $u(r)$ *of (2)-(3) satisfies the following estimates*

$$d_1 r^{2-n} \le u(r) \le d_2 r^{-\alpha} \ (r \ge r_0), \tag{7}$$

$$|u'(r)| \le d_3 r^{-\alpha-1} \ (r \ge r_0), \tag{8}$$

where d_1,d_2,d_3 *are positive constants.*

Proof. First, we prove the estimate (7). Let $v(s) = su(r)$, $s = r^{n-2}$. Since

$$v_s(s) = u(r) + \frac{1}{n-2} u'(r) s^{\frac{1}{n-2}} \tag{9}$$

and by (A-2)

$$v_{ss}(s) = \frac{-s^{\frac{4-n}{n-2}} f(r,u)}{(n-2)^2} \le 0, \tag{10}$$

$v_s(s)$ is decreasing on (s_0, ∞), where $s_0 = r_0^{n-2}$. If there exists a positive number $s'_0 \ge s_0$ such that $v_s(s'_0) = -a_0 < 0$, then $v_s(s) \le -a_0$ for any $s(\ge s'_0)$. By the basic formula, we have

$$v(s) - v(s'_0) = \int_{s'_0}^{s} v_s(t)dt \le -a_0(s - s'_0). \tag{11}$$

Letting $s \to \infty$, we get $v(s) \to -\infty$. This contradicts to $v(s) > 0$. Therefore $v_s(s) \ge 0$ for any $s \ge s_0$. This implies that $r^{n-2}u(r)$ is increasing on (r_0, ∞) and

$$r^{n-2}u(r) \ge r_0^{n-2} u(r_0) = d_1, \ \ r \ge r_0.$$

Next we shall prove $u(r) \le d_2 r^{-\alpha}$ on (r_0, ∞) for some positive constant d_2. By (2),

$$(r^{n-1}u')' = -r^{n-1} f(r,u). \tag{12}$$

For any $t > \tau \ge r_0$, integrating the both sides of (12) on $[\tau, t]$, we have

$$t^{n-1}u'(t) - \tau^{n-1}u'(\tau) = -\int_{\tau}^{t} r^{n-1} f(r,u)\ dr \le 0. \tag{13}$$

Therefore $t^{n-1}u'(t) \le \tau^{n-1}u'(\tau)$ for any $t > \tau \ge r_0$. Since $u(r) > 0$ and $u(r) \to 0$ as $r \to \infty$, there exists a sufficiently large number $r'_0 \ge r_0$ such that

$$t^{n-1}u'(t) \le {r'_0}^{n-1} u'(r'_0) < 0 \text{ for any } t > r'_0. \tag{14}$$

Consequently we get $u'(t) < 0$ for any $t > r_0'$. Therefore by (A-2), for $t > \tau > r_0'$,

$$\begin{aligned} -t^{n-1}u'(t) &\geq -t^{n-1}u'(t) + \tau^{n-1}u'(\tau) \\ &= \int_\tau^t r^{n-1} f(r,u)\, dr \\ &\geq C_0 \int_\tau^t r^{n-1} u^p\, dr \\ &= \frac{C_0 u^p(t)(t^n - \tau^n)}{n}. \end{aligned}$$

For $t > 2^{1/n} r_0' = R_0$, choose τ such that $\tau^n = t^n/2$. Then $t > \tau > r_0'$ holds and hence we have

$$\frac{-u'(t)}{u^p(t)} \geq \frac{C_0 t}{2n}, \ t > R_0. \tag{15}$$

Integrating the both sides of (15) on $[R_{0,}t]$,

$$\frac{u^{1-p}(t) - u^{1-p}(R_0)}{p-1} \geq \frac{C_0(t^2 - R_0^2)}{4n}. \tag{16}$$

Choosing t such that $t^2 - R_0^2 \geq t^2/2$ namely $t \geq \sqrt{2}R_0$, we have

$$u^{1-p}(t) \geq \frac{C_0(p-1)t^2}{8n}. \tag{17}$$

By (17), we get (7) as $d_2 = \max\{(8n/[C_0(p-1)])^{1/(p-1)},\ \max_{r_0 \leq r \leq \sqrt{2}R_0}(r^\alpha u(r))\}$. Next we show the estimate (8). By (A-3), there exist positive constants M and $r_1 \geq \max(1, r_0)$ such that $f(r,u) \leq u^p + Mr^l u^q$ for $r \geq r_1$.For $\overline{r} > r > r_1$, integrating (12) on $[r, \overline{r}]$, we have

$$-\overline{r}^{n-1}u'(\overline{r}) + r^{n-1}u'(r) = \int_r^{\overline{r}} s^{n-1} f(s,u)\, ds. \tag{18}$$

Since $u(r) \leq d_2 r^{-\alpha}$ and $n-1+l-\alpha q < n-1+\sigma-\alpha q = n-1-\alpha p$, it follows that

$$\begin{aligned} -\overline{r}^{n-1}u'(\overline{r}) &\leq -r^{n-1}u'(r) + \int_r^{\overline{r}} s^{n-1}\left(u^p + Ms^l u^q\right) ds \\ &\leq -r^{n-1}u'(r) + 2\max(d_2^p, d_2^q M) \int_r^{\overline{r}} s^{n-1-\alpha p}\, ds \\ &= -r^{n-1}u'(r) + C(\overline{r}^{n-\alpha p} - r^{n-\alpha p}) \\ &\leq -r^{n-1}u'(r) + C\overline{r}^{n-\alpha p} \end{aligned}$$

where $C = 2\max(d_2^p, d_2^q M)/(n-\alpha p)$. Here we have used $n - \alpha p > 0$, since $p > n/(n-2)$.
Hence we get

$$-u'(\overline{r}) \leq -\left(\frac{r}{\overline{r}}\right)^{n-1} u'(r) + C\overline{r}^{1-\alpha p}. \tag{19}$$

Let r be fixed. Since $1 - n < 1 - \alpha p$, we get (8). □

The next lemma is an another characterization of slowly decaying solutions.

Lemma 2.2. *Suppose that* $(A-1) \sim (A-3)$ *hold. If* $u(r)$ *is a positive solution of* $(2)-(3)$ *and satisfies* $r^\alpha u(r) \to 0$ *as* $r \to \infty$, $u(r)$ *is a rapidly decaying solution. Therefore, a slowly decaying solution* u *does not satisfy* $r^\alpha u(r) \to 0$.

Proof. Let $v = r^\alpha u$, $r = e^s$ $(r \geq r_0)$.Then $v(s)$ satisfies

$$v'' - av' - bv + v^p + e^{-\sigma s} H(v) = 0 \tag{20}$$

where $a = 2\alpha + 2 - n > 0$, $b = \alpha(n-2-\alpha) > 0$, and $H(v) = v^q\{f(r,u) - u^p\}/u^q$. First we note that $r^\alpha u(r) \to 0$ implies $\{r^\alpha u(r)\}' < 0$ near ∞. Indeed, if there exists a sequence $\{x_k\} \to \infty$ such that $v(x_k)$ is a local maximum, then $v''(x_k) \leq 0, v'(x_k) = 0$, and $v(x_k) \to 0$. Then the left hand side of (20) is negative near ∞. This is a contradiction. Hence

$$-ru' > \alpha u \quad \text{near } \infty. \tag{21}$$

By L'hospital's rule and (21), we have

$$\begin{aligned} 0 &\leq \lim_{r\to\infty} \frac{\int_r^\infty sf(s,u)\,ds}{u} \\ &= \lim_{r\to\infty} \frac{-rf(r,u)}{u'} \\ &= \lim_{r\to\infty} \frac{r^2 f(r,u)}{-ru'} \\ &\leq \lim_{r\to\infty} \frac{r^2(u^p + Mr^\sigma u^q)}{\alpha u} \\ &= \frac{1}{\alpha}\lim_{r\to\infty}(r^\alpha u)^{p-1} + \frac{M}{\alpha}\lim_{r\to\infty}(r^\alpha u)^{q-1} \\ &= 0 \end{aligned}$$

for some $M > 0$. Here we have used the relation $2 + \sigma = \alpha(q-1)$. Hence

$$\int_r^\infty sf(s,u)\,ds < \epsilon u \quad \text{for any } \epsilon > 0. \tag{22}$$

Notice

$$ru' + (n-2)u = \int_r^\infty sf(s,u)\, ds. \tag{23}$$

By (22) and (23), we have

$$u < cr^{-m} \quad \text{for any } m < n-2. \tag{24}$$

Therefore

$$s^{n-1}f(s,u) < s^{n-1}(C^p s^{-mp} + C^q s^{l-mq}). \tag{25}$$

Since $n/(n-2) < p < (n+2)/(n-2)$, note that there exists a positive number $m < n-2$ such that $n + l - mq \le n - mp < 0$. Then, by (25), we have

$$\int_R^\infty s^{n-1} f(s,u)\, ds < \infty \quad \text{for some } R > 0,$$

$$r^{n-2}\int_r^\infty sf(s,u)ds \le \frac{c}{mp-2} r^{n-mp}.$$

Hence by (12) and (23), we get

$$\begin{aligned} \lim_{r\to\infty} r^{n-2}u &= \frac{1}{n-2}\lim_{r\to\infty}\left\{r^{n-2}\int_r^\infty sf(s,u)\, ds - r^{n-1}u'\right\} \\ &= \frac{1}{n-2}\lim_{r\to\infty}(-r^{n-1}u') \\ &= \frac{1}{n-2}\left\{\int_R^\infty s^{n-1}f(s,u)\, ds - R^{n-1}u'(R)\right\} \\ &= c_1 \end{aligned}$$

for some $c_1 > 0$ and the proof of Lemma 2.2 is finished. □

The following lemmas are elementary, but useful in the proof of Lemma 4.1 later (see Section 4).

Lemma 2.3. *Let λ_0, $\mu_0 > 0$ and $G(s) \in L^1(s_0, \infty)$ for some $s_0 > 0$. Suppose that w is a solution of the following ODE,*

$$\begin{cases} w'' - \lambda_0 w' + \mu_0 w - G(s) = 0, \;\; s \in (s_0, \infty) \\ \qquad\qquad w(s) \to 0 \;\; (s \to \infty). \end{cases} \tag{26}$$

Then $w'(s) \to 0$ $(s \to \infty)$ holds.

Proof. Let α, β be two solutions of the characteristic equation $t^2 - \lambda_0 t + \mu_0 = 0$. Then

$$\alpha, \beta = \frac{\lambda_0}{2} \pm \sqrt{\left(\frac{\lambda_0}{2}\right)^2 - \mu_0}.$$

Case (i). If $\left(\frac{\lambda_0}{2}\right)^2 - \mu_0 > 0$, $e^{\alpha s}$ and $e^{\beta s}$ are the fundamental solutions.

Case (ii). If $\left(\frac{\lambda_0}{2}\right)^2 - \mu_0 = 0$, $e^{\alpha s}$ and $se^{\alpha s}$ are the fundamental solutions.

Case (iii). If $\left(\frac{\lambda_0}{2}\right)^2 - \mu_0 < 0$, $e^{\frac{\lambda_0 s}{2}} \cos \gamma s$ and $e^{\frac{\lambda_0 s}{2}} \sin \gamma s$ are the fundamental solutions where $\alpha, \beta = \frac{\lambda_0}{2} + i\gamma$.

In Case (i), we set

$$\phi(s) = \frac{1}{\alpha - \beta} \left\{ -\int_s^\infty e^{\alpha(s-\tau)} G(\tau) d\tau + \int_s^\infty e^{\beta(s-\tau)} G(\tau) d\tau \right\}.$$

Since it is easy to see that $\phi(s)$ satisfies

$$\phi'' - \lambda_0 \phi' + \mu_0 \phi = G,$$

we can write

$$w(s) = C_1 e^{\alpha s} + C_2 e^{\beta s} + \phi(s).$$

Since

$$\begin{aligned} \left| \int_s^\infty e^{\alpha(s-\tau)} G(\tau) d\tau \right| &\le \int_s^\infty e^{\alpha(s-\tau)} \left| G(\tau) \right| d\tau \\ &\le \int_s^\infty \left| G(\tau) \right| d\tau, \end{aligned}$$

$$\phi' = -\frac{\alpha}{\alpha - \beta} \int_s^\infty e^{\alpha(s-\tau)} G(\tau) d\tau + \frac{\beta}{\alpha - \beta} \int_s^\infty e^{\beta(s-\tau)} G(\tau) d\tau$$

and $G \in L^1(s_0, \infty)$, we have

$$|\phi(s)| \to 0, \;\; |\phi'(s)| \to 0.$$

Since $w(s) \to 0$, it follows that

$$C_1 e^{\alpha s} + C_2 e^{\beta s} \;\; (\equiv g(s)) \to 0.$$

If $(C_1, C_2) \neq (0, 0)$, we may assume $\alpha < \beta$ and $C_2 > 0$.Then we get

$$g(s) = C_2 e^{\beta s} \left(1 + \frac{C_1}{C_2} e^{(\alpha - \beta)s} \right) \to \infty.$$

This is a contradiction.Therefore $C_1 = C_2 = 0$ and hence $w(s) = \phi(s)$, which implies $w'(s) \to 0$. The case (ii) and (iii) can be treated in a similar way.
We just mention that in Case (iii), we can take a special solution ϕ as the same one as in Case (i). In Case (ii), we can take

$$\phi(s) = -\int_s^\infty e^{\alpha(s-\tau)}(s-\tau)G(\tau)d\tau.$$

□

The following lemma is stated in the proof of Lemma 2.4 in [QL,1] without its proof. We give the proof for the sake of completeness.

Lemma 2.4. *Let $\sigma_1 > 0, Re\ p_0 < 0$, and g be bounded. Suppose that U is a solution of the following ODE,*

$$\begin{cases} U' + p_0 U + e^{-\sigma_1 s} g(s) = 0, \ s \in (s_0, \infty), \\ U(s) \to 0 \ (s \to \infty). \end{cases}$$

Then for any $s \geq s_0$, we have

$$|U(s)| \leq \frac{M_g e^{-\sigma_1 s}}{\sigma_1 - Re\ p_0}, \ M_g \stackrel{def}{\equiv} \sup_{s \geq s_0} |g(s)| < \infty.$$

Proof. Multiplying the both sides of $U' + p_0 U = -e^{-\sigma_1 s} g(s)$ by $e^{p_0 s}$,

$$(e^{p_0 s} U)' = -e^{p_0 s} e^{-\sigma_1 s} g(s).$$

Hence

$$\phi(s) = e^{-p_0 s} \int_s^\infty e^{(p_0 - \sigma_1)\tau} g(\tau) d\tau$$

satisfies

$$\phi' + p_0 \phi + e^{-\sigma_1 s} g(s) = 0$$

since $e^{-\sigma_1 s} g(s)$ is integrable. So we can write $U(s) = C_1 e^{-p_0 s} + \phi(s)$. Since $e^{p_0 s}\phi(s) \to 0$, $e^{p_0 s} U \to 0$ $(s \to \infty)$, we have $C_1 = 0$ and

$$\begin{aligned} |U(s)| &= |\phi(s)| \\ &\leq \int_s^\infty e^{Re p_0 \cdot (\tau - s)} e^{-\sigma_1 \tau} M_g d\tau \\ &= \frac{M_g e^{-\sigma_1 s}}{\sigma_1 - Re\ p_0}. \end{aligned}$$

□

3 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. First we show the estimate (4) for slowly decaying solutions u.

Proof of the estimate (4) : Let u be a slowly decaying solution near infinity. We define the energy $E(s)$ as follows:

$$E(s) \equiv \frac{1}{2}v'^2 - \frac{b}{2}v^2 + \frac{1}{p+1}v^{p+1} - \int_s^\infty e^{-\sigma t}H(v)v'\,dt, \tag{27}$$

where $v = r^\alpha u, r = e^s$. Since Lemma 2.1, v, v' are bounded and hence $E(s)$ is well-defined and uniformly bounded. Here we have used $|H(v)| \leq Me^{ls}v^q$ with $l < \sigma$ for some $M > 0$ by (A-3). By (20) and (27), we have

$$E'(s) = av'(s)^2 \geq 0, \tag{28}$$

and hence

$$E(t) - E(s) = a\int_s^t v'(\tau)^2 d\tau \geq 0. \tag{29}$$

and

$$\lim_{s\to\infty} E(s) = E_0 < \infty. \tag{30}$$

Letting s_0 be fixed and $t \to \infty$, we have

$$a\int_{s_0}^\infty v'(\tau)^2 d\tau < \infty \tag{31}$$

By (20),(27) and Lemma 2.1, v' and v'' are bounded and $\{(v'(s))^2\}' \leq M$.

Claim 1 : $v'(s) \to 0$ as $s \to \infty$, that is $rd(r^\alpha u)/dr \to 0$ as $r \to \infty$.

Proof. If not, there exist an $\varepsilon_0 > 0$ and a sequence $\{t_n\} \to \infty$ such that $|v'(t_n)|^2 \geq \varepsilon_0$ with $t_n > t_{n-1} + \epsilon_0/2M (M > 0)$. By the boundedness of $\{(v'(s))^2\}'$, we have

$$\begin{aligned} \{v'(s)\}^2 - \{v'(t)\}^2 &= \int_t^s \{(v'(y))^2\}' dy \\ &\geq -M|s-t| \geq -\frac{\epsilon_0}{2} \end{aligned}$$

for any $s > s_0$ such that $|s-t| < \epsilon_0/2M$. Hence if $\{v'(t)\}^2 \geq \epsilon_0$ for some $t > s_0$, then $\{v'(s)\}^2 \geq \epsilon_0/2$ for any $s > s_0$ such that $|s-t| \leq \epsilon_0/4M$. Define $I_n \equiv (t_n - \epsilon_0/4M, t_n + \epsilon_0/4M)$. Then we have

$$\{v'(s)\}^2 \geq \frac{\epsilon_0}{2} \text{ for any } s \in I_n. \tag{32}$$

By the choice of $\{t_n\}$, $\{I_n\}$ is disjoint and then

$$\begin{aligned}\int_{s_0}^{\infty}(v')^2 ds &\geq \sum_{n=1}^{\infty}\int_{I_n}(v')^2 dx\\ &\geq \sum_{n=1}^{\infty}\frac{\epsilon_0^2}{4M}=\infty\end{aligned}$$

This contradicts to (31). Hence we get $v'(s)\to 0$ as $s\to\infty$. $\square$

Next, we shall show the following two claims.

Claim 2 : We have

$$a_n=\inf_{t\in(n,n+1)} v''(t)\to 0 \text{ as } n\to\infty. \tag{33}$$

Therefore there exists a sequence $\{u_n\}$ such that $v''(u_n)\to 0$ as $n\to\infty$.

Proof. Assume that $\lim_{n\to\infty} a_n\neq 0$. Then there exist an $\epsilon_0>0$ and a sequence $\{n_k\}\to\infty$ such that $|a_{n_k}|\geq\epsilon_0$. Thus, we have

$$v'(n_k+1)-v'(n_k)=\int_{n_k}^{n_k+1} v''(s)ds\geq\epsilon_0$$

and the left hand side converges to 0. This is a contradiction. $\square$

Claim 3 : For any slowly decaying solution u of (2)-(3) and $v(s)=r^\alpha u(r), r=e^s$, we have

$$\lim_{s\to\infty} v(s)=b^{\frac{1}{p-1}}=\lambda. \tag{34}$$

Proof. Since $v'(s)\to 0$, there exists $A>0$ such that

$$|v'(\theta)|<\frac{\epsilon}{3}\ (\theta\geq A). \tag{35}$$

Fix $N\in\mathbb{N}$. Then, for any $a'\geq\max(A,u_N)$, we have

$$a'\in[u_{N+j},u_{N+j+1}] \text{ for some } j\geq 0,$$

and

$$|v(a')-\lambda|\leq|v(a')-v(u_{N+j+1})|+|v(u_{N+j+1})-\lambda|. \tag{36}$$

By the mean value theorem and the choice of $\{u_n\}$, we have

$$
\begin{aligned}
(37) \qquad |v(a') - v(u_{N+j+1})| &\leq |v'(\theta)|\,|u_{N+j+1} - a'| \quad (\theta \in [a', u_{N+j+1}]) \\
&\leq 2\,|v'(\theta)| \\
&< \frac{2}{3}\epsilon.
\end{aligned}
$$

By letting $n \to \infty$ in (20), using the facts $v'(u_n), v''(u_n) \to 0$ as $n \to \infty$, and Lemma 2.2, we obtain that for any $\epsilon > 0$, there exists N_0 such that if $n \geq N_0$,

$$
(38) \qquad |v(u_n) - \lambda| < \frac{\epsilon}{3}.
$$

Note that the estimate (37) implies that $\{v(u_n)\}$ does not oscillate between 0 and λ. Hence by (36),(37) and (38), we can conclude $|v(a') - \lambda| < \epsilon$ for $a' \geq \max(A, u_{N_0})$. □

Now, we have

$$
(39) \qquad E_0 = \frac{\lambda^{p+1}}{p+1} - \frac{b\lambda^2}{2}, \; E(s) \leq E_0
$$

and

$$
(40) \qquad \frac{1}{2}(v')^2 + \frac{b}{2}\lambda^2 - \frac{\lambda^{p+1}}{p+1} - \left(\frac{b}{2}v^2 - \frac{v^{p+1}}{p+1}\right) - \int_s^\infty e^{-\sigma t} H(v) v' dt \leq 0.
$$

Here $H(v) \leq M r^l v^q$ for sufficiently large r by (A-3). By L'hospital's rule,

$$
(41) \qquad \lim_{v \to \lambda} \frac{\frac{b}{2}\lambda^2 - \frac{\lambda^{p+1}}{p+1} - \frac{b}{2}v^2 + \frac{v^{p+1}}{p+1}}{(v-\lambda)^2} = \frac{(p-1)b}{2}.
$$

Hence we have

$$
(42) \qquad \frac{b}{2}\lambda^2 - \frac{\lambda^{p+1}}{p+1} - \frac{b}{2}v^2 + \frac{v^{p+1}}{p+1} = \frac{(p-1)b}{2}(v-\lambda)^2 + o[(v-\lambda)^2].
$$

By (40) and (42), we get

$$
(43) \qquad \frac{1}{8}(p-1)b(v-\lambda)^2 \leq M \int_s^\infty e^{-(\sigma - l)t} v^q |v'| dt
$$

and

$$
(44) \qquad \frac{1}{4}(v')^2 \leq M \int_s^\infty e^{-(\sigma - l)t} v^q |v'| dt
$$

for any $s > s_0$, where s_0 is sufficiently large. Taking s_0 sufficiently large such that $v^q|v'| < (\sigma - l)/4M$ for any $s > s_0$, by (44),

$$|v'| \leq e^{-\frac{(\sigma-l)s}{2}}.$$

Taking s_0 sufficiently large furthermore such that $v^q|v'| < 3(\sigma - l)/8M$ for any $s > s_0$, by (44),

$$|v'| \leq e^{-(\frac{1}{2}+\frac{1}{4})(\sigma-l)s}.$$

Iterating this computation, we have

$$|v'| \leq e^{-\left(1-\frac{1}{2^{k+1}}\right)(\sigma-l)s} \text{ for any } k \in \mathbb{N}. \tag{45}$$

Therefore by (43),

$$|v - \lambda| \leq M' e^{-\left(1-\frac{1}{2^{k+1}}\right)(\sigma-l)s} \text{ for some } M' > 0. \tag{46}$$

Taking k sufficiently large such that $\epsilon > (\sigma - l)/2^{k+1}$, we have

$$|v - \lambda| e^{(\sigma-l-\epsilon)s} \leq M' e^{\left(\frac{\sigma-l}{2^{k+1}}-\epsilon\right)s} \to 0 \text{ as } s \to \infty. \tag{47}$$

Hence we get (4). Next we shall show uniqueness of slowly decaying solutions.

Proof of uniqueness : Let u_1 and u_2 be slowly decaying solutions.
Let $G(r,u) = f(r,u) - u^p$, $w = \overline{v_1} - \overline{v_2} = r^\alpha(u_1 - u_2)$, where $\overline{v_i}(r) = r^\alpha u_i(r) - \lambda$ $(i = 1, 2)$.
Then w satisfies

$$w'' + \frac{n-1-2\alpha}{r} w' - \frac{\alpha(n-2-\alpha)}{r^2} w \tag{48}$$

$$+ r^\alpha \left\{ G\left(r, \frac{\lambda + \overline{v_1}}{r^\alpha}\right) - G\left(r, \frac{\lambda + \overline{v_2}}{r^\alpha}\right) \right\} + \frac{(\lambda + \overline{v_1})^p - (\lambda + \overline{v_2})^p}{r^2} = 0$$

Using the mean value theorem and Taylor's expansion, we have

$$w'' + \frac{n-1-2\alpha}{r} w' + \frac{2(n-2-\alpha)}{r^2} w + G_u\left(r, \frac{\lambda + \theta\overline{v_1} + (1-\theta)\overline{v_2}}{r^\alpha}\right) w \tag{49}$$

$$+ \frac{1}{r^2} \left\{ \sum_{k=2}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!} \lambda^{p-k} (\overline{v_1}^k - \overline{v_2}^k) \right\} = 0 \text{ for some } 0 < \theta < 1.$$

Multiplying (49) by $r^2 w'$ and integrating on $[t, T]$, for $T > t \geq t_0$ and sufficiently large t_0,

$$\left[\frac{r^2 w'^2}{2}\right]_t^T + (n-2-2\alpha)\int_t^T r w'^2 dr + \left[(n-2-\alpha)w^2\right]_t^T \tag{50}$$
$$+\int_t^T [g_1(r)-g_2(r)]w' dr + \int_t^T r^2 w w' G_u\left(r, \frac{\lambda+\theta\overline{v_1}+(1-\theta)\overline{v_2}}{r^\alpha}\right) dr = 0,$$

where $g_1(r) - g_2(r) = \lambda^p \sum_{k=2}^{\infty} \frac{p(p-1)\cdots(p-k+1)}{k!}\left\{(\frac{\overline{v_1}}{\lambda})^k - (\frac{\overline{v_2}}{\lambda})^k\right\}$. About the 4th term, by the mean value theorem and (4),

$$\begin{aligned} |\overline{v_1}(r)^k - \overline{v_2}(r)^k| &\leq k\ |\theta\overline{v_1} + (1-\theta)\overline{v_2}|^{k-1}|\overline{v_1}-\overline{v_2}| \\ &\leq k(|\overline{v_1}|+|\overline{v_2}|)^{k-2}|\overline{v_1}-\overline{v_2}|(|\overline{v_1}|+|\overline{v_2}|) \\ &\leq k(M'r^{-\epsilon})^{k-2}|w|M_1 r^{-\epsilon} \text{ for some } M', M_1 > 0 \end{aligned}$$

for any $\epsilon \in (0, \sigma - l)$. Hence we have

$$\int_t^T [g_1(r)-g_2(r)]w' dr \leq M\int_t^T r^{-\epsilon-1}(w^2 + w'^2 r^2)dr, \tag{51}$$

where $M = M_1 \sum_{k=2}^{\infty}\left|\frac{p(p-1)\cdots(p-k+1)}{k!}\right| \lambda^{p-k} k(M't^{-\epsilon})^{k-2}$ and t is fixed. On the other hand, about the 5th term, by (A-4) and (7),

$$\left|\int_t^T r^2 w w' G_u\left(r, \frac{\lambda+\theta\overline{v_1}+(1-\theta)\overline{v_2}}{r^\alpha}\right) dr\right| \leq M_2\int_t^T r^{l-\sigma-1}(r^2 w'^2 + w^2)dr \tag{52}$$

for some $M_2 > 0$. Paying attention to $\epsilon < \sigma - l$ and letting $T \to \infty$ in (50), we get

$$t^2 w'^2 + 2(n-2-\alpha)w^2 \leq c\int_t^\infty r^{-\epsilon-1}\{2(n-2-\alpha)w^2 + r^2 w'^2\}dr \tag{53}$$

Here we have used $\tau^2 w'(\tau)^2 \to 0$ as $\tau \to \infty$ (see Claim 1 in the proof of (4)). Set $F(r) = 2(n-2-\alpha)w(r)^2 + r^2 w'(r)^2$. We have only to show that $F(r) \equiv 0$ for $r \gg 1$. Set $G(t) \equiv c\int_t^\infty r^{-1-\epsilon}F(r)dr$. Then

$$G'(t) = -ct^{-1-\epsilon}F(t) \geq -ct^{-1-\epsilon}G(t). \tag{54}$$

Integrating the both sides of (54) on $[t, R]$, we have

$$\frac{G(R)}{G(t)} \geq e^{\frac{c}{\epsilon}(R^{-\epsilon}-t^{-\epsilon})}. \tag{55}$$

Since $e^{-\frac{c}{\epsilon}R^{-\epsilon}}G(R) \to 0$ as $R \to \infty$,

$$G(t) \equiv 0 \text{ for } t \geq t_0\ ,$$

which implies that $F(t) \equiv 0$ for $t \gg t_0$. Hence $w(r) \equiv 0$ for any $r > r_0$ by the uniqueness of the initial value problem for ordinary differential equations. □

4 Proof of Theorem 1.2 and 1.3

In this section, we give the proof of Theorem 1.2 and 1.3. To show existence, we use the contraction mapping principle as in [QL] and need accurate asymptotic behavior. Using the following lemma, we can get the accurate asymptotic behavior at ∞ for slowly decaying solutions of (2)-(3).

Lemma 4.1. *Let $\lambda_1, \mu_1, \sigma_1 > 0$ and consider the ordinary differential equation*

$$\begin{cases} w'' - \lambda_1 w' + \mu_1 w + e^{-\sigma_1 s} g(s) = 0, \\ w(s) \to 0 \text{ as } s \to \infty \text{ and } g \text{ is bounded.} \end{cases} \tag{56}$$

Then

(i) $w' \to 0$ *as* $s \to \infty$.

(ii) $w = O(e^{-\sigma_1 s})$.

(iii) If there exists a limit $g_\infty \in \mathbb{R}$, $g(s) - g_\infty = O(e^{-\sigma_2 s})$ *for some* $\sigma_2 > 0$, *then*

$$\lim_{s\to\infty} w(s) e^{\sigma_1 s} = -\frac{g_\infty}{\sigma_1(\sigma_1 + \lambda_1) + \mu_1}. \tag{57}$$

Remark 4.1. *Lemma 4.1 is closely related to Lemma 2.4 in [QL]. However we emphasize that the property (i) was assumed in Lemma 2.4 of [QL]. By Lemma 4.1, we can simplify the iteration process in the proof of the accurate asymptotic behavior, namely we do not need to check the property (i) in each iteration process.*

Proof. The property (i) is a consequence of Lemma 2.3. To show (ii), let p_1, p_2 be solutions of the quadratic equation $x^2 + \lambda_1 x + \mu_1 = 0$. Since

$$p_1, p_2 = -\frac{\lambda_1}{2} \pm \sqrt{\left(\frac{\lambda_1}{2}\right)^2 - \mu_1}\ ,$$

we have $Re(p_i) < 0$ $(i = 1, 2)$ and

$$p_1 + p_2 = -\lambda_1,\ p_1 p_2 = \mu_1.$$

Set $U = w' + p_1 w$. Since $U \to 0$ (by (i)) and U satisfies

$$U' + p_2 U + e^{-\sigma_1 s} g(s) = 0,$$

we have by Lemma 2.4

$$|U(s)| \le M e^{-\sigma_1 s} \ \text{ for some } M > 0.$$

Hence

$$w' + p_1 w = U(s) \equiv e^{-\sigma_1 s}\widetilde{U}(s),\ \widetilde{U}(s) \text{ is bounded.} \tag{58}$$

Applying Lemma 2.4 to w, we get

$$|w| \le M' e^{-\sigma_1 s} \text{ for some } M' > 0 \tag{59}$$

and the proof of (ii) is finished. Furthermore, by (58) and (59), we have $|w'| \le ce^{-\sigma_1 s}$ for some $c > 0$. Before starting the proof of (iii), we need another lemma.

Lemma 4.2. *Let $p_1 > 0$, $q_1, r_1 \in \mathbb{R}$ and v be a solution of the following ODE*

$$\begin{cases} v'' - p_1 v' + q_1 v + r_1 + K(s) = 0, \\ K \in L^1(s_0, \infty) \cap L^\infty(s_0, \infty), \\ v, v' \text{ are bounded.} \end{cases}$$

Then

$$\lim_{s\to\infty} v'(s) = 0.$$

The proof of Lemma 4.2 is the same as the proof of $v'(s) \to 0$ in section 3. Therefore we omit the proof. Set $V(s) = e^{\sigma_1 s} w(s)$. By (ii) and $|w'| \le ce^{-\sigma_1 s}$, V, V' are bounded and V satisfies

$$V'' - (2\sigma_1 + \lambda)V' + \left(\sigma_1^2 + \lambda\sigma_1 + \mu_1\right)V + g = 0.$$

Setting $p_1 = 2\sigma_1 + \lambda_1$, $q_1 = \sigma_1^2 + \lambda_1\sigma_1 + \mu_1$, $r_1 = g_\infty$, $K(s) = g(s) - g_\infty$ and applying lemma 4.2 to V, we have $V' \to 0$ $(s \to \infty)$. Then we can prove (iii) in the same way as in Claims in section 3. ☐

Based on Lemma 4.1, we can get the accurate asymptotic expansion. We will give iteration process in details.

Theorem 4.1. *Let u be a slowly decaying solution of (2)-(3) with $f(r,u) = u^p + r^l u^q$ $(l < \sigma), n/(n-2) < p < (n+2)/(n-2)$, $q > p$, $\sigma = (q-p)\alpha$, $\alpha = 2/(p-1)$. Then for any $k \in \mathbb{N}$,*

$$u(r) = r^{-\alpha}\left\{\lambda + \sum_{j=1}^{k-1} C_j r^{-j(\sigma-l)} + O(r^{-k(\sigma-l)})\right\} \text{ at } \infty, \tag{60}$$

where C_j are constants which depend only on l, p, q.

Proof. By (20), for $f(r,u) = u^p + r^l u^q$, we have

$$v'' - av' - bv + v^p + e^{-\sigma_1 s} v^q = 0 \tag{61}$$

where $\sigma_1 \equiv \sigma - l$. Set $v_1 := v - \lambda$. Then $v_1 \to 0$ and $v_1' \to 0$. Since

$$v^p - bv = \sum_{k=1}^{\infty} a_k (v-\lambda)^k \tag{62}$$

$$v^q = \sum_{k=0}^{\infty} b_k (v-\lambda)^k \tag{63}$$

by Taylor's expansion, it follows from (61),(62) and (63) that

$$v_1'' - \lambda_1 v_1' + \mu_1 v_1 + \sum_{k=2}^{\infty} a_k v_1^k + e^{-\sigma_1 s}\left(g_\infty^{(1)} + \sum_{k=1}^{\infty} b_k v_1^k\right) = 0, \tag{64}$$

where $\lambda_1 = a > 0, \mu_1 = a_1 = (p-1)b > 0, g_\infty^{(1)} = b_0 = \lambda^q$. By (4), we notice that $v_1 = O(e^{-\epsilon s})$ for any $\epsilon < \sigma_1$. By Lemma 4.1,

$$\lim_{s\to\infty} v_1 e^{\sigma_1 s} = -\frac{g_\infty^{(1)}}{\sigma_1(\sigma_1+\lambda_1)+\mu_1} = C_1. \tag{65}$$

Set $v_2 = v_1 e^{\sigma_1 s} - C_1$. Then since

$$\begin{aligned} v_1^k &= e^{-k\sigma_1 s}(C_1+v_2)^k \\ &= e^{-k\sigma_1 s}\sum_{m_1=0}^{k}\binom{k}{m_1} C_1^{k-m_1} v_2^{m_1} \end{aligned}$$

$$v_1' = \{v_2' - \sigma_1(C_1+v_2)\}e^{-\sigma_1 s}$$

$$v_1'' = \{v_2'' - 2\sigma_1 v_2' + \sigma_1^2(C_1+v_2)\}e^{-\sigma_1 s},$$

and $(\sigma_1^2 + \lambda_1\sigma_1 + \mu_1)C_1 = -g_\infty^{(1)}$, we have the following equation

$$\begin{aligned} v_2'' - \lambda_2 v_2' + \mu_2 v_2 + e^{-\sigma_1 s}\Bigg\{&\sum_{k=2}^{\infty} a_k e^{-(k-2)\sigma_1 s}\left(\sum_{m_1=0}^{k}\binom{k}{m_1}C_1^{k-m_1}v_2^{m_1}\right) \\ &+\sum_{k=1}^{\infty} b_k e^{-(k-1)\sigma_1 s}\left(\sum_{m_1=0}^{k}\binom{k}{m_1}C_1^{k-m_1}v_2^{m_1}\right)\Bigg\} = 0,\ v_2 \to 0, \end{aligned} \tag{66}$$

where $\lambda_2 = 2\sigma_1 + \lambda_1, \mu_2 = \sigma_1^2 + \lambda_1\sigma_1 + \mu_1$. We can rewrite (66) as follows,

$$v_2'' - \lambda_2 v_2' + \mu_2 v_2 + e^{-\sigma_1 s}\left\{g_\infty^{(2)} + \widetilde{P}^{(2)}(v_2) + \sum_{l_1=1}^{\infty} e^{-l_1\sigma_1 s} P_{l_1}^{(2)}(v_2)\right\} = 0 \tag{67}$$

where $\widetilde{P}^{(2)}(v_2)$ is the polynomial of degree more than one about v_2 and $P^{(2)}_{l_1}(v_2)$ is the polynomial of v_2. By Lemma 4.1, $v_2 = O(e^{-\sigma_1 s})$ and therefore by Lemma 4.1 again, we have

$$\lim_{s\to\infty} v_2 e^{\sigma_1 s} = -\frac{g^{(2)}_\infty}{\sigma_1(\sigma_1+\lambda_2)+\mu_2} := C_2. \tag{68}$$

Set $v_3 := v_2 e^{\sigma_1 s} - C_2$. Then v_3 satisfies

$$v_3'' - \lambda_3 v_3' + \mu_3 v_3 + e^{-\sigma_1 s}\left\{ g^{(3)}_\infty + \widetilde{P}^{(3)}(v_3) + \sum_{l_1=1}^{\infty} e^{-l_1\sigma_1 s} P^{(3)}_{l_1}(v_3)\right\} = 0. \tag{69}$$

By Lemma 4.1 again , we get

$$\lim_{s\to\infty} v_3 e^{\sigma_1 s} = -\frac{g^{(3)}_\infty}{\sigma_1(\sigma_1+\lambda_3)+\mu_3} = C_3. \tag{70}$$

We iterate this computation. For each $j \in \mathbb{N}$, we denote by $\tilde{P}^{(j)}(v_j)$ and $P^{(j)}_{l_1}(v_j)$ ($l_1 \in \mathbb{N}$) the polynomial of degree more than one about v_j and the polynomial of v_j, respectively. Assume that we have

$$v_j'' - \lambda_j v_j' + \mu_j v_j + e^{-\sigma_1 s}\left\{ g^{(j)}_\infty + \widetilde{P}^{(j)}(v_j) + \sum_{l_1=1}^{\infty} e^{-l_1\sigma_1 s} P^{(j)}_{l_1}(v_j)\right\} = 0$$

and

$$\lim_{s\to\infty} v_j e^{\sigma_1 s} = -\frac{g^{(j)}_\infty}{\sigma_1(\sigma_1+\lambda_j)+\mu_j} = C_j.$$

Set $v_{j+1} = v_j e^{\sigma_1 s} - C_j$. Using the same procedure as before, we have the equation

$$v_{j+1}'' - \lambda_{j+1} v_{j+1} + \mu_{j+1} v_{j+1} + e^{-\sigma_1 s}\Bigg\{ g^{(j+1)}_\infty + \widetilde{P}^{(j+1)}(v_{j+1})$$

$$+ \sum_{l_1=1}^{\infty} e^{-l_1\sigma_1 s} P^{(j+1)}_{l_1}(v_{j+1})\Bigg\} = 0.$$

By Lemma 4.1, we get

$$\lim_{s\to\infty} v_{j+1} e^{\sigma_1 s} = -\frac{g^{(j+1)}_\infty}{\sigma_1(\sigma_1+\lambda_{j+1})+\mu_{j+1}} = C_{j+1}.$$

This computation implies that $v_k = O(e^{-\sigma_1 s})$ for any $k \in \mathbb{N}$ and

$$
\begin{aligned}
v_1 &= C_1 e^{-\sigma_1 s} + e^{-\sigma_1 s} v_2 \\
&= C_1 e^{-\sigma_1 s} + C_2 e^{-2\sigma_1 s} + e^{-3\sigma_1 s}(C_3 + v_4) \\
&= \cdots \\
&= C_1 e^{-\sigma_1 s} + C_2 e^{-2\sigma_1 s} + \cdots + e^{-(k-1)\sigma_1 s}(C_{k-1} + v_k) \\
&= C_1 e^{-\sigma_1 s} + C_2 e^{-2\sigma_1 s} + \cdots + C_{k-1} e^{-(k-1)\sigma_1 s} + O(e^{-k\sigma_1 s}).
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

Proof of Theorem 1.2 : Since uniqueness follows from Theorem 1.1, it suffices to show existence. We prove existence of a slowly decaying solution by using the contraction mapping principle. In the iteration process above to obtain $v_k = O(e^{-\sigma_1 s})$, since

$$
g_\infty^{(k)} + \widetilde{P}^{(k)}(v_k) + \sum_{l_1=1}^{\infty} e^{-l_1 \sigma_1 s} P_{l_1}^{(k)}(v_k) = g_\infty^{(k)} + O(e^{-\sigma_1 s}),
$$

we notice that

$$
\begin{aligned}
-bv + v^p + e^{-\sigma_1 s} v^q - \mu_1 v_1 &= \sum_{k=2}^{\infty} a_k v_1^k + e^{-\sigma_1 s} \sum_{k=0}^{\infty} b_k v_1^k \\
&= e^{-\sigma_1 s} g_\infty^{(1)} + e^{-2\sigma_1 s} g_\infty^{(2)} + \cdots + e^{-k\sigma_1 s} g_\infty^{(k)} \\
&\quad + e^{-k\sigma_1 s} \left\{ \widetilde{P}^{(k)}(v_k) + \sum_{l_1=1}^{\infty} e^{-l_1 \sigma_1 s} P_{l_1}^{(k)}(v_k) \right\}.
\end{aligned}
$$

Namely, if v_1 is represented as

$$
v_1 = C_1 e^{-\sigma_1 s} + C_2 e^{-2\sigma_1 s} + \cdots + C_{k-1} e^{-(k-1)\sigma_1 s} + C(s) e^{-k\sigma_1 s}, \ |C(s)| \le C_0,
$$

then it follows that

$$
\sum_{k=2}^{\infty} a_k v_1^k + e^{-\sigma_1 s} \sum_{k=0}^{\infty} b_k v_1^k = e^{-\sigma_1 s} g_\infty^{(1)} + e^{-2\sigma_1 s} g_\infty^{(2)} + \cdots
$$

$$
+ e^{-k\sigma_1 s} g_\infty^{(k)} + O(e^{-(k+1)\sigma_1 s}). \tag{71}
$$

Combining (71) with

$$
\mu_1 v_1 = \mu_1 C_1 e^{-\sigma_1 s} + \mu_1 C_2 e^{-2\sigma_1 s} + \cdots + \mu_1 C_{k-1} e^{-(k-1)\sigma_1 s} + \mu_1 C(s) e^{-k\sigma_1 s},
$$

we have

$$-bv + v^p + e^{-\sigma_1 s} v^q = \sum_{j=1}^{k-1} (\mu_1 C_j + g_\infty^{(j)}) e^{-j\sigma_1 s} \tag{72}$$

$$+(\mu_1 C(s) + g_\infty^{(k)}) e^{-k\sigma_1 s} + O(e^{-(k+1)\sigma_1 s}),$$

where $g_\infty^{(j)} = -C_j[(\sigma_1 + \lambda_j)\sigma_1 + \mu_j], \lambda_j = 2\sigma_1 + \lambda_{j-1}, \mu_j = \sigma_1(\sigma_1 + \lambda_{j-1}) + \mu_{j-1}$. Here, note that

$$\sigma_1(\sigma_1 + \lambda_n) + \mu_n = n\sigma_1(n\sigma_1 + \lambda_1) + \mu_1$$

i.e. for any $n \in \mathbb{N}$,

$$\frac{g_\infty^{(n)} + \mu_1 C_n}{n\sigma_1(n\sigma_1 + \lambda_1)} = -C_n. \tag{73}$$

We define an operator T as follows. For $s \geq s_0$ with some $s_0 > 0$,

$$[Tv](s) = \lambda - \int_s^\infty \int_t^\infty e^{a(t-\tau)}(-bv + v^p + e^{-\sigma_1 \tau} v^q) d\tau dt. \tag{74}$$

Since $\sigma_1 > 0$, there exists $k \in \mathbb{N}$ such that

$$k\sigma_1(k\sigma_1 + \lambda_1) > 2\mu_1. \tag{75}$$

For this k, we take C_0 such that

$$\frac{\mu_1 C_0}{k\sigma_1(k\sigma_1 + \lambda_1)} + 2|C_k| < C_0. \tag{76}$$

Now we define

$$E = \left\{ v \in C[s_0, \infty) \ ; \ v = \lambda + \sum_{j=1}^{k-1} C_j e^{-j\sigma_1 s} + C(s) e^{-k\sigma_1 s}, \right. \tag{77}$$

$$\left. \|v\|_E = \|C(\cdot)\| = \sup_{s \geq s_0} |C(s)| \leq C_0 \right\}.$$

If $v \in E$ is a fixed point of T, noticing that u is a slowly decaying solution when we define $v = r^\alpha u(r)$, we show that T has a unique fixed point. Since E is a bounded, closed subset of $C[s_0, \infty)$, we have only to prove T maps E into itself and T is a contraction map. By (72) $\sim$ (77), we have

$$\begin{aligned} [Tv](s) - \lambda &= -\int_s^\infty \int_t^\infty e^{a(t-\tau)}(-bv + v^p + e^{-\sigma_1 s} v^q) d\tau dt \\ &= -\sum_{j=1}^{k-1} \frac{\mu_1 C_j + g_\infty^{(j)}}{j\sigma_1(j\sigma_1 + \lambda_1)} e^{-j\sigma_1 s} + D(s) e^{-k\sigma_1 s}, \end{aligned}$$

$$
\begin{aligned}
|D(s)| &\le \frac{1}{k\sigma_1(k\sigma_1+\lambda_1)}(\mu_1 C_0 + |g_\infty^{(k)}|) + O(e^{-\sigma_1 s}) \\
&\le \frac{\mu_1 C_0}{k\sigma_1(k\sigma_1+\lambda_1)} + 2|C_k| + O(e^{-\sigma_1 s}) \\
&\le C_0
\end{aligned}
$$

for any $s \ge s_0$, which implies $Tv \in E$. We define

$$
[Tv_i](s) = \lambda + \sum_{j=1}^{k-1} C_j e^{-j\sigma_1 s} + D_i(s) e^{-k\sigma_1 s} \quad (i = 1, 2). \tag{78}
$$

Then it follows from (74),(75),(77),(78) that

$$
\begin{aligned}
\|Tv_1 - Tv_2\|_E &= \sup_{s\ge s_0} |D_1 - D_2| \\
&= \sup_{s\ge s_0} |\{[Tv_1](s) - [Tv_2](s)\} e^{k\sigma_1 s}| \\
&= \sup_{s\ge s_0} \{e^{k\sigma_1 s}| - \int_s^\infty \int_t^\infty e^{a(t-\tau)}[a_1(C_1(\tau) - C_2(\tau))e^{-k\sigma_1\tau} \\
&\quad + a_2(C_1(\tau) - C_2(\tau))e^{-k\sigma_1\tau}\{2\sum_{j=1}^{k-1} C_j e^{-j\sigma_1\tau} \\
&\quad + (C_1(\tau) + C_2(\tau))e^{-k\sigma_1\tau}\} + O(e^{-(k+1)\sigma_1\tau})]d\tau dt|\} \\
&\le \frac{\mu_1 + O(e^{-\sigma_1 s})}{k\sigma_1(k\sigma_1+\lambda_1)} \|v_1 - v_2\|_E .
\end{aligned}
$$

Therefore T is a contraction map in E, which completes the proof of Theorem 1.2. □

To show Theorem 1.3 we establish the following accurate asymptotic expansion.

Theorem 4.2. *Let $f(r,u) = u^p + K(r)g(u)$, where $K(r), g(u)$ are the same as in Theorem 1.3 and $\sigma_1(= \sigma - l), \alpha \in \mathbb{Q}$. Then if $u(r)$ is a slowly decaying solution of (2)-(3), for any $k \in \mathbb{N}$, there exists $\sigma_j > 0$ $(j = 0, 1, 2, \cdots, k)$ such that*

$$
\begin{aligned}
u(r) = r^{-\alpha}\{\lambda + C_1' r^{-\sigma_1} + C_2' r^{-(\sigma_1+\sigma_2)} \\
+ \cdots + C_{k-1}' r^{-(\sigma_1+\cdots+\sigma_{k-1})} + O(r^{-(\sigma_1+\cdots+\sigma_k)})\},\ \sigma_2 \ge \sigma_0,\ \textit{and} \\
\sigma_j = \sigma_0 \quad \textit{for any } j \ge 3,
\end{aligned} \tag{79}
$$

where $C_1', \cdots, C_{k-1}'$ and $\sigma_j (j = 0, 1, 2, \cdots, k)$ are constants which do not depend on u.

Remark 4.2. *Actually, without the restriction $\sigma_1, \alpha \in \mathbb{Q}$, we can show for any k, there exists $\sigma_j > 0$ $(j = 1, 2, \cdots, k)$ such that*

$$
\begin{aligned}
u(r) = r^{-\alpha}\{\lambda + C_1' r^{-\sigma_1} + C_2' r^{-(\sigma_1+\sigma_2)} + \cdots \\
+ C_{k-1}' r^{-(\sigma_1+\cdots+\sigma_{k-1})} + O(r^{-(\sigma_1+\cdots+\sigma_k)})\}.
\end{aligned}
$$

However, when σ_1 or $\alpha \notin \mathbb{Q}$, we do not know whether $\sum_{i=1}^{\infty} \sigma_i = \infty$ holds or not. Once we have $\sum_{i=1}^{\infty} \sigma_i = \infty$, we can show existence as in the proof of Theorem 1.3 even if σ_1 or $\alpha \notin \mathbb{Q}$.

Proof. By (20), $v(= r^{\alpha} u)$ satisfies

$$v'' - av' - bv + v^p + e^{-\sigma_1 s} \left(\sum_{n=0}^{\infty} \alpha_n e^{-ns} \right) v^q \sum_{n=0}^{\infty} b_n e^{-\alpha n s} v^n = 0. \tag{80}$$

Notice that

$$v^q = \sum_{k=0}^{\infty} b_k^{(0)} v_1^k, \ v^n = \sum_{k=0}^{n} b_k^{(n)} v_1^k \tag{81}$$

and $f(r, u)$ satisfies the assumptions (A-1) $\sim$ (A-4), where $b_n, b_k^{(0)}, b_k^{(n)}$ are constants which depend on l, n, p, and q. Then $v_1 (= v - \lambda)$ satisfies

$$\begin{aligned} & v_1'' - \lambda_1 v_1' + \mu_1 v_1 + \sum_{k=2}^{\infty} a_k v_1^k \\ & + e^{-\sigma_1 s} \left(\sum_{n=0}^{\infty} b_k^{(0)} v_1^k \right) \left(\sum_{n=0}^{\infty} \alpha_n e^{-ns} \right) \left\{ \sum_{n=0}^{\infty} b_n e^{-\alpha n s} \left(\sum_{k=0}^{n} b_k^{(n)} v_1^k \right) \right\} \\ & = 0, \end{aligned}$$

where λ_k, μ_k $(k = 1, 2, \cdots)$ are positive constants and a_k is the same as in Theorem 4.1. Hence by Lemma 4.1,

$$\lim_{s \to \infty} e^{\sigma_1 s} v_1 = C_1'.$$

Set $v_2 = e^{\sigma_1 s} v_1 - C_1'$. Then v_2 satisfies

$$\begin{aligned} & v_2'' - \lambda_2 v_2' + \mu_2 v_2 + \sum_{k=2}^{\infty} a_k e^{-(k-1)\sigma_1 s} (C_1' + v_2)^k \\ & + \left\{ \sum_{k=0}^{\infty} b_k^{(0)} e^{-k\sigma_1 s} (C_1' + v_2)^k \right\} \left(\sum_{n=0}^{\infty} \alpha_n e^{-ns} \right) \\ & \times \left\{ \sum_{n=0}^{\infty} b_n e^{-\alpha n s} \left(\sum_{k=0}^{n} b_k^{(n)} e^{-k\sigma_1 s} (C_1' + v_2)^k \right) \right\} \\ & = 0. \end{aligned} \tag{82}$$

Since $\sigma_1, \alpha \in \mathbb{Q}$, we can write

$$\sigma_1 = \frac{\eta_1}{\xi_1}, \alpha = \frac{\eta_2}{\xi_2} \text{ for } \xi_j, \eta_j \in \mathbb{N}.$$

Set

$$\sigma_0 = \frac{1}{\xi_1 \xi_2} \quad (\xi_1\xi_2 = n_0 \in \mathbb{N}).$$

Then we have

$$e^{-s} = e^{-n_0\sigma_0 s}, e^{-\sigma_1 s} = e^{-n_1\sigma_0 s}, e^{-\alpha s} = e^{-n_2\sigma_0 s},$$

where $n_1 = \eta_1\xi_2, n_2 = \eta_2\xi_1$. Therefore we can rewrite (82) as follows.

$$v_2'' - \lambda_1 v_1' + \mu_1 v_2 + e^{-\sigma_2 s}\left(g_\infty^{(2)} + \widetilde{P}^{(2)}(v_2) + \sum_{l_1=1}^{\infty} e^{-l_1\sigma_0 s} P_{l_1}^{(2)}(v_2)\right) = 0,\ \ \sigma_2 \geq \sigma_0,$$

where $\tilde{P}^{(2)}(v_2)$ is the polynomial of degree more than one about v_2 and $P_{l_1}^{(2)}(v_2)$ is the polynomial v_2. By Lemma 4.1 again, we have

$$\lim_{s\to\infty} e^{\sigma_2 s} v_2 = C_2'$$

Iterating this computation as in the proof of Theorem 4.1, we get

$$v_j'' - \lambda_j v_j' + \mu_j v_j + e^{-\sigma_0 s}\left\{g_\infty^{(j)} + \widetilde{P}^{(j)}(v_j) + \sum_{l_1=1}^{\infty} e^{-l_1\sigma_0 s} P_{l_1}^{(j)}(v_j)\right\} = 0$$

and

$$\lim_{s\to\infty} v_j e^{\sigma_0 s} = -\frac{g_\infty^{(j)}}{\sigma_0(\sigma_0+\lambda_j)+\mu_j} = C_j'.$$

This implies (79). □

Proof of Theorem 1.3 : Define

(83)

$$E = \left\{ v \in C[s_0,\infty)\ ;\ v = \lambda + \sum_{j=1}^{k-1} C_j' e^{-(\sigma_1+\sigma_2+\cdots+\sigma_j)s} + C(s) e^{-(\sigma_1+\sigma_2+\cdots+\sigma_k)s},\right.$$
$$\left. \|v\|_E = \|C(\cdot)\| = \sup_{s\geq s_0} |C(s)| \leq C_0 \right\},$$

where $\{C_j'\}_{j=1}^{k-1}$ is the constants appeared in Theorem 4.2. In the same way as in the proof of Theorem 1.2, the nonlinear term in (80) can be written as follows,

$$-bv + v^p + e^{-\sigma_1 s}\left(\sum_{n=0}^{\infty}\alpha_n e^{-ns}\right) v^q \sum_{n=0}^{\infty} b_n e^{-\alpha n s} v^n$$

$$= \sum_{j=1}^{k-1} (\mu_1 C'_j + g_\infty^{(j)}) e^{-(\sigma_1 + \cdots + \sigma_j)s} + (\mu_1 C(s) + g_\infty^{(k)}) e^{-(\sigma_1 + \cdots + \sigma_k)s}$$
$$+ O(e^{-(\sigma_1 + \cdots + \sigma_{k+1})s}),$$

where $g_\infty^{(j)} = -C'_j[(\sigma_j + \lambda_j)\sigma_j + \mu_j]$, $\lambda_j = 2\sigma_{j-1} + \lambda_{j-1}$, $\mu_j = \sigma_{j-1}(\sigma_{j-1} + \lambda_{j-1}) + \mu_{j-1}$. Here notice that

$$\sigma_j(\sigma_j + \lambda_j) + \mu_j = (\sigma_1 + \cdots + \sigma_j)(\sigma_1 + \cdots + \sigma_j + \lambda_1) + \mu_1$$

i.e. for any $j \in \mathbb{N}$,

$$\frac{g_\infty^{(j)} + \mu_1 C'_j}{(\sigma_1 + \cdots + \sigma_j)(\sigma_1 + \cdots + \sigma_j + \lambda_1)} = -C'_j.$$

For $s \geq s_0$ with some $s_0 > 0$ we define

$$[Tv](s) = \lambda - \int_s^\infty \int_t^\infty e^{a(t-\tau)} \Bigg[-bv + v^p$$
$$+ e^{-\sigma_1 \tau} \left(\sum_{n=0}^\infty \alpha_n e^{-n\tau} \right) v^q \sum_{n=0}^\infty b_n e^{-\alpha n \tau} v^n \Bigg] d\tau dt.$$

Since $\sigma_1, \cdots, \sigma_k > 0$, there exists $k \in \mathbb{N}$ such that

$$(\sigma_1 + \cdots + \sigma_k)(\sigma_1 + \cdots + \sigma_k + \lambda_1) \geq 2\mu_1.$$

For this k, we take C_0 such that

$$\frac{\mu_1 C_0}{(\sigma_1 + \cdots + \sigma_k)(\sigma_1 + \cdots + \sigma_k + \lambda_1)} + 2|C'_k| < C_0.$$

The rest of this proof is the same as the one of Theorem 1.2. $\square$

Example (0) If $f(r,u) = u^p + \frac{1}{(1+r)^l}\frac{u^q}{1+u^\nu}$, $\nu \in \mathbb{N}$, $n/(n-2) < p < (n+2)/(n-2)$, $p < q$, $-l < \sigma$, $\sigma + l \in \mathbb{Q}$, $\alpha \in \mathbb{Q}$, there exists a unique slowly decaying solution near infinity.
(1) If $f(r,u) = u^p + \frac{1}{(1+r)^l} u^q$, $p < q$, $-l < \sigma$, $\sigma + l \in \mathbb{Q}$, $\alpha \in \mathbb{Q}$, there exists a unique slowly decaying solution near infinity. In this case, the concrete asymptotic expansion is illustrated as follows:

$$\begin{cases} u(r) = r^{-\alpha}\{\lambda + C_1' r^{-(\sigma+l)} + C_2' r^{-(\sigma+l+1)} + O(r^{-(\sigma+l+2)}) & (\sigma + l \geq 2) \\ u(r) = r^{-\alpha}\{\lambda + C_1' r^{-(\sigma+l)} + C_2' r^{-(\sigma+l+1)} + O(r^{-2(\sigma+l)}) & (1 < \sigma + l < 2) \\ u(r) = r^{-\alpha}\{\lambda + C_1' r^{-1} + C_3' r^{-2} + O(r^{-3})\} & (\sigma + l = 1) \\ u(r) = r^{-\alpha}\{\lambda + C_1' r^{-(\sigma+l)} + C_4' r^{-2(\sigma+l)} + O(r^{-(\sigma+l+1)}) & \left(\frac{1}{2} \leq \sigma + l < 1\right) \\ u(r) = r^{-\alpha}\{\lambda + C_1' r^{-(\sigma+l)} + C_4' r^{-2(\sigma+l)} + O(r^{-3(\sigma+l)}) & \left(0 < \sigma + l < \frac{1}{2}\right) \end{cases},$$

where α_k and $C_1^{'}, \cdots, C_4^{'}$ are constants which depend on l, n, p, and q.
(2) If $f(r,u) = u^p + \frac{1}{1+r^m} u^q$ $(m > 0)$, $\sigma + m \in \mathbb{Q}$, $\alpha \in \mathbb{Q}$, there exists a unique slowly decaying solution near infinity. In this case, we get the following asymptotic expansion,

$$\begin{cases} u(r) = r^{-\alpha} \left\{ \lambda + C_1 r^{-(\sigma+m)} + C_2 r^{-(\sigma+2m)} + O(r^{-2(\sigma+m)}) \right\} \text{ for } m \geq \sigma, \text{ at } \infty \\ u(r) = r^{-\alpha} \left\{ \lambda + C_1 r^{-(\sigma+m)} + C_2 r^{-(\sigma+2m)} + O(r^{-(\sigma+3m)}) \right\} \text{ for } m < \sigma, \text{ at } \infty \end{cases}$$

where $C_1 = -\lambda^q / \{(\sigma+m)(\sigma+m+a) + (p-1)b\}$, $C_2 = \lambda^q / \{(\sigma+2m)^2 + a(\sigma + 2m) + (p-1)b\}$.

5 Proof of Theorem 1.4

In the final section, we shall prove Theorem 1.4. Before starting the proof of Theorem 1.4, we give the proof of Proposition 1.1 concisely.

Proof of Proposition 1.1 : By (12),(A-1)' and (A-2)', notice that $r^{n-1}u'(r)$ is decreasing. If $u'(r) < 0$ for any $r > 0$ and $\lim_{r \to +0} u'(r) \leq 0$, $u(r)$ must be singular or $\lim_{r \to +0} u(r) = C \in (0, +\infty)$. If $\lim_{r \to +0} u(r) = C \in (0, +\infty)$, then it is easy to see that u becomes a distributional solution of (1) on $\mathbb{R}^n$ and hence u should be a regular solution by the elliptic regularity theorem. If $u'(r_2) = 0$ for some $r_2 > 0$, $u(r)$ is increasing on $(0, r_2)$ since $r^{n-1}u'(r)$ is decreasing. Then $u(r)$ must be a 0-hit solution. □

To prove Theorem 1.4, we need the following Pohozaev identity.

Lemma 5.1. *Suppose that (A-1)',(A-2)',(A-3)' hold. Let u be a solution to (2)-(3) on (r_0, ∞). Then, for $\tau > r_0$, the following Pohozaev identity holds.*

$$-\frac{\tau^n}{2} u'(\tau)^2 - \frac{n-2}{2} \tau^{n-1} u(\tau) u'(\tau) - \tau^n F(\tau, u(\tau)) \tag{84}$$

$$= \int_\tau^\infty r^{n-1} \left\{ nF(r, u(r)) - \frac{n-2}{2} u(r) f(r, u(r)) + r F_r(r, u(r)) \right\} dr.$$

Proof. Multiplying (2) by $r^n u'$ and integrating on $[\tau, \infty)$, we have

$$-\frac{\tau^n}{2} u'(\tau)^2 + \frac{n-2}{2} \int_\tau^\infty r^{n-1} u'(r)^2 dr$$

$$= \int_\tau^\infty \left\{ r^n F_r(r, u(r)) + n r^{n-1} F(r, u(r)) \right\} dr + \tau^n F(\tau, u(\tau)),$$

since $u(r) \leq d_2 r^{-\alpha}$, $|u'(r)| \leq d_3 r^{-\alpha-1}$ and $n - 2 - 2\alpha < 0$. Note that each terms are integrable, by (A-2)' and (A-3)'. Integrating by parts and using (12), we get (84). □

First we prove that the solution on (r_0, ∞) can be extended on $(0, \infty)$. We shall show that $u(r)$ must be singular under the assumption by the argument in [Pan1]. Let $\sum = \{\rho > 0 : u(r)$ is defined in $(\rho, +\infty)$, $u'(r) < 0$ and $u(r)$ satisfies (2)-(3)$\}$ and $\tau = \inf \sum$. Assume that $\tau > 0$. By (12), we have

$$
\begin{aligned}
r^{n-1}u'(r) &= R^{n-1}u'(R) + \int_r^R s^{n-1} f(s, u(s))ds \\
&> R^{n-1}u'(R) \text{ for any } r \in (\tau, R).
\end{aligned} \tag{85}
$$

Since $u'(R) < 0$,

$$
0 > u'(r) > \left(\frac{R}{r}\right)^{n-1} u'(R) > \left(\frac{R}{\tau}\right)^{n-1} u'(R) \equiv -M > -\infty. \tag{86}
$$

Using the basic formula on $[\tau + \epsilon, R]$ $(\epsilon > 0)$, we have

$$
u(R) - u(\tau + \epsilon) = \int_{\tau+\epsilon}^R u'(r)dr > -M\{R - (\tau + \epsilon)\}. \tag{87}
$$

Letting $\epsilon \to +0$, $u(\tau)$ remains bounded and

$$
u(\tau) = \lim_{\epsilon \to +0} u(\tau + \epsilon) < \infty. \tag{88}
$$

Then by (85), $u'(\tau)$ exists and

$$
\lim_{r \to \tau} u'(r) = \left(\frac{R}{\tau}\right)^{n-1} u'(R) + \left(\frac{1}{\tau}\right)^{n-1} \int_\tau^R s^{n-1} f(s, u(s))ds \tag{89}
$$

and we also get $u'(\tau) \le 0$. By the definition of τ, we must have $u'(\tau) = 0$. Hence by (84) and (A-2)',

$$
\begin{aligned}
&\int_\tau^\infty r^{n-1} \left\{ nF(r, u(r)) - \frac{n-2}{2} u(r) f(r, u(r)) + rF_r(r, u(r)) \right\} dr \\
&= -\tau^n F(\tau, u(\tau)) < 0.
\end{aligned} \tag{90}
$$

By (5), the left-hand side of (90) is non-negative and so this is a contradiction. Therefore $\tau = 0$. By Proposition 1.1, $u(r)$ can not be a 0-hit solution. If $u(r)$ is a regular solution, again by (84), we have

$$
\int_0^\infty r^{n-1} \left\{ nF(r, u(r)) - \frac{n-2}{2} u(r) f(r, u(r)) + rF_r(r, u(r)) \right\} dr \le 0. \tag{91}
$$

This is also a contradiction to our assumption (5). Hence we conclude that $u(r)$ must be a singular solution such that $\lim_{r \to +0} u(r) = +\infty$. $\square$

Example For $f(r,u) = u^p + K(r)u^q$, Pohozaev identity can be wrriten by

$$
(92) \qquad -\frac{n-2}{2}\tau^{n-1}u(\tau)u'(\tau) - \frac{1}{2}\tau^n u'(\tau)^2 - \tau^n\left(\frac{u(\tau)^{p+1}}{p+1} + \frac{K(\tau)u(\tau)^{q+1}}{q+1}\right)
$$
$$
= \int_\tau^\infty r^{n-1}\left[\left(\frac{n}{p+1} - \frac{n-2}{2}\right)u(r)^{p+1}\right.
$$
$$
\left. + \left\{\left(\frac{n}{q+1} - \frac{n-2}{2}\right)K(r) + \frac{rK'(r)}{q+1}\right\}u(r)^{q+1}\right]dr.
$$

Since $n/(p+1) - (n-2)/2 > 0$, we conclude that under the condition

$$
(93) \qquad K(r) \geq 0, \ \left(\frac{n}{q+1} - \frac{n-2}{2}\right)K(r) + \frac{rK'(r)}{q+1} \geq 0
$$

the slowly decaying soluiton $u(r)$ of (2)-(3) on (r_0, ∞) can be extended on $(0, \infty)$ as a singular solution. For example, If $q = (n+2)/(n-2)$ and $K'(r) \geq 0$, or $q > (n+2)/(n-2)$ and $K(r) = Ar^l$ with $c_2 \leq l < \sigma$, where $c_2 = \{(n-2)(q+1)/2\} - n, A \geq 0$, the condition (93) is satisfied. Note that $c_2 < \sigma$ always holds. On the other hand, by the result in [LN], for $f(r,u) = u^p + u^{2p-1}, n/(n-2) < p < (n+2)/(n-2)$, (1) has an exact positive radial solution

$$
\tilde{u}(r) = \frac{1}{\zeta}\left(\frac{\xi^2}{\xi^2 + r^2}\right)^{\frac{1}{p-1}},
$$

where

$$
\zeta = \left[\frac{n-2}{4p}\left(2p - 1 - \frac{n+2}{n-2}\right)\right]^{\frac{1}{p-1}}, \ \xi = \frac{n-2}{2(p-1)}\left(2p - 1 - \frac{n+2}{n-2}\right)\left(\frac{1}{p}\right)^{\frac{1}{2}}.
$$

Now, for any $q > p$, let

$$
K_1(r) = \left[\frac{1}{\zeta}\left(\frac{\xi^2}{\xi^2 + r^2}\right)^{\frac{1}{p-1}}\right]^{2p-1-q}.
$$

Then $\tilde{u}$ above is a regular solution to $-\Delta\tilde{u} = \tilde{u}^p + K_1(r)\tilde{u}^q$ in $\mathbb{R}^n$. So there is a family of regular slowly decaying solutions to (2)-(3).

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