Funkcialaj Ekvacioj
Volume 45, No. 3, pp 397--416

PERIODIC SOLUTIONS OF NONLINEAR EQUATIONS OF STRING WITH PERIODICALLY OSCILLATING BOUNDARIES

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Dedicated to Professor Kunihiko Kajitani on his sixties birthday

1. Introduction

In the study of the motions of nonlinear vibrating string with periodically oscillating ends, it seems to be interesting to investigate under which conditions periodic motions exist.

In this paper, we shall consider an oscillating string of finite length in the (x, u)-plane. Let the ends of the string move time-periodically on the (x, u)-plane and a nonlinear time-periodic vertical external force work on the string. We shall be concerned with *the existence of the time-periodic motions of the vibrating string under small vertical external forces.* This problem is mathematically formulated as the existence problem of periodic solutions of the Dirichlet boundary value problem for one-dimensional wave equation with a time-periodic nonlinear forcing term, where the boundaries oscillate periodically in t on the x-axis and the ends of the string are forced to move periodically in t in the vertical direction.

Let Ω be a time-periodic noncylindrical domain in (x, t)-plane defined by

$$a_1(t) < x < a_2(t), \quad t \in R^1.$$

Here $a_1(t)$ and $a_2(t)$ are periodic functions. The period is normalized to 1, for simplicity. Consider BVP (the boundary value problem) for a nonlinear one-dimensional wave equation :

$$\partial_t^2 u - \partial_x^2 u = \mu p(x,t) + f(x,t,u), \ \ (x,t) \in \Omega, \tag{1.1}$$

$$u(a_1(t),t) = \nu b_1(t), \quad u(a_2(t),t) = \nu b_2(t), \quad t \in R^1, \tag{1.2}$$

where $p(x, t)$ and $f(x, t, u)$, and $b_i(t)$, $i = 1, 2$, are periodic with period 1 in t, and $f(x, t, u)$ is of order more than or equal to 2 with respect to u. $p(x, t)$ and $f(x, t, u)$ satisfy some compatible boundary conditions (See (A4) later). As a typical example of f, if $b_i(t)$, $i = 1, 2$, identically vanish, then we give $f(x,t,u) = \pm u^m \, (m \geq 2)$. μ and ν are small parameters and are supposed to satisfy $\nu = \nu(\mu) = O(\mu)\,(\mu \to 0)$ continuous in μ. The above dependence of ν on μ is naturally imposed because we shall look for the small amplitude solutions and the external force working the whole string is of $O(\mu)\,(\mu \to 0)$. We assume that $a_1(t)$ and $a_2(t)$ satisfy $|\, a_i'(t) \,| < 1 \ (i = 1, 2)$. This condition is natural in the sense that the boundaries oscillate with slower speed than the eigenspeed 1 of waves by (1.1). Otherwise, the shock waves come out.

The aim of this paper is *to show the existence of time-periodic solutions with small amplitude of* BVP (1.1)-(1.2) *with the same period* 1 *as that of the given data.*

We define the following composed function A that is a fundamental tool in this research. Let A be a composed function defined by

$$A = A_1^{-1} \circ A_2, \quad A_i = (I + a_i) \circ (I - a_i)^{-1}, \quad i = 1, 2, \tag{1.3}$$

where I is an identity function, f^{-1} means the inverse function of f and $\circ$ means the composition operation of functions *i.e.* $f \circ g(x) = f(g(x))$. Geometrically A is a map naturally defined by the reflected characteristics in the (x,t)-plane. A is one dimensional periodic dynamical system. It is known in a series of works ([Ya1]-[Ya4], [Ya6]) that A and its rotation number $\rho(A)$ play an essential role in studying the qualitative behavior of solutions of IBVP and BVP in domain with periodically oscillating boundaries. For the definition of the rotation number, see *Notation and Definitions* in this section.

For the case where the ends of the string are fixed, BVP is of the form

$$\partial_t^2 u - \partial_x^2 u = F(x,t,u), \ \ (x,t) \in (0,a) \times R^1, \tag{1.4}$$

$$u(0,t) = u(a,t) = 0, \quad t \in R^1, \tag{1.5}$$

where a is a positive constant. In this case there are very many works on the existence of time-periodic solutions of BVP (1.4)-(1.5) (see [R1] [R2] [B-C-N] [W] etc. and see the references therein). It should be noted that the ratio of the period of the forcing term $F(x,t,u)$ to the length a of the interval $[\,0, a\,]$ plays an important role in the study of the behavior of the solution. That is, the behaviors depend on the rationality or irrationality of the ratio. As is shown in [Ya8], even in the linear case *i.e.*, $F(x,t,u) = F(x,t)$ in (1.4) it happens that there are no bounded solutions, as a matter of course, no periodic solutions of (1.1)-(1.2) if the Diophantine order of the irrational ratio is large and the differentiability of $F(x,t)$ is small. It is known that if the Diophantine order of a real number is large, the number is well-approximated by the rational numbers.

On the other hand, in our moving-boundary problem (1.1)-(1.2) the difficulty consists in the following. The length of the interval $[\,a_1(t), a_2(t)\,]$ varies continuously as time varies continuously. Hence the ratio takes both rational and irrational values as time proceeds. However, this difficulty is essentially overcome by introducing the rotation number of A. In a series of papers ([Ya4], [Ya6] and [Ya-Yo]) we clarified the interesting fact that the rotation number plays the same role as the length of the interval as the ends are fixed.

We shall show that under the Diophantine condition on the rotation number (See the assumption (A3) in this section) there exists a small 1-periodic solution of BVP (1.1)-(1.2) (Theorem 1.1). It is well-known in number theory ([Kh]) that all real numbers with periodic continued fraction expansions satisfy the above Diophantine condition. Especially the set of all algebraic numbers of degree 2 is equal to the above set.

Our steps to show the results on the existence of periodic solutions are as follows. First we shall reduce the function A to the affine function, using the Herman-Yoccoz reduction theorem ([H], [Yoc]) (see Proposition 2.1) :

$$H \circ A \circ H^{-1}(x) = x + \omega.$$

Here ω is the rotation number of A and H is a conjugate function that is one-dimensional periodic dynamical system of C^∞. Then, using the conjugate function H, we shall construct a domain transformation $\Phi : R^2 \to R^2$ in section 2 :

$$\begin{aligned} \xi &= \left(H \circ A_1^{-1}(x+t) - H(-x+t)\right)/2, \\ \tau &= \left(H \circ A_1^{-1}(x+t) + H(-x+t)\right)/2. \end{aligned}$$

Φ is the bijection of the noncylindrical domain Ω to a cylindrical domain $D = (0, \omega/2) \times R^1$, maps the boundaries of Ω, $x = a_1(t), x = a_2(t)$ onto the boundaries of D, $\xi = 0, \xi = \omega/2$ (resp.) and preserves the d'Alembertian form (Proposition 2.2). The last statement means that the transformed differential operator contains only d'Alembertian but has no lower order differential operators. Such transformations were developped in [Ya4], [Ya6] and [Ya-Yo]. It should be noted that the above d'Alembertian preserving property has good advantage to study the qualitative behavior of the solutions. Second, applying the domain transformation Φ to BVP (1.1)-(1.2), we shall obtain BVP in the cylindrical domain D :

$$\partial_\tau^2 v - \partial_\xi^2 v = \mu q(\xi,\tau) + g(\xi,\tau,v), \quad (\xi,\tau) \in D, \tag{1.6}$$

$$v(0,\tau) = \nu c_1(\tau), \quad v(\omega/2,\tau) = \nu c_2(\tau), \quad \tau \in R^1, \tag{1.7}$$

where $q(\xi,\tau)$ and $g(\xi,\tau,v)$, and $c_i(\tau)$, $i = 1,2$, are 1-periodic in τ, and $g(\xi,\tau,v)$ is of order more than or equal to 2 with respect to v. Then we shall show the existence of an 1-periodic solution of BVP (1.6)-(1.7) (Theorem 3.1). In case of $c_i(t) \equiv 0$, $i = 1,2$, the problem (1.6)-(1.7) was considered by [BN-Ma] and [Mc]. Under some monotonicity conditions and the Lipshitz condition on g and the Diophantine condition on the ratio of the length of the interval to the period of g, they showed the existence of periodic weak solution.

To show our results, first we shall decompose BVP (1.6)-(1.7) into two linear homogeneous BVPs

$$\partial_\tau^2 v_1 - \partial_\xi^2 v_1 = 0, \quad (\xi,\tau) \in D, \tag{1.8}$$

$$v_1(0,\tau) = c_1(\tau), \quad v_1(\omega/2,\tau) = 0, \quad \tau \in R^1, \tag{1.9}$$

$$\partial_\tau^2 v_2 - \partial_\xi^2 v_2 = 0, \quad (\xi,\tau) \in D, \tag{1.10}$$

$$v_2(0,\tau) = 0, \quad v_2(\omega/2,\tau) = c_2(\tau), \quad \tau \in R^1, \tag{1.11}$$

and nonlinear BVP

$$\partial_\tau^2 w - \partial_\xi^2 w = \mu q(\xi,\tau) + g(\xi,\tau,\nu(v_1+v_2)+w), \quad (\xi,\tau) \in D, \tag{1.12}$$

$$w(0,\tau) = 0, \quad w(\omega/2,\tau) = 0, \quad \tau \in R^1. \tag{1.13}$$

Then we shall show the existence of periodic solutions of BVP (1.8)-(1.9) and (1.10)-(1.11) (Proposition 3.1), using the method of [Ya3]. In order to show the existence of a periodic solution of BVP (1.12)-(1.13), we shall apply the standard contracting mapping principle in suitable function space to our BVP (1.6)-(1.7). This is similar to the existence theorem ([Ya5], pp.519 - 521) of periodic solutions of nonlinear evolution equations of second order. Then by the principle of superposition, $v = \nu(v_1 + v_2) + w$ is the 1-periodic C^2 solution of BVP (1.6)-(1.7). Finally, by operating the inverse Φ^{-1} of the domain transformation Φ to the above v, we shall obtain the desired 1-periodic solution of BVP (1.1)-(1.2).

Notation and Definitions.

Rotation Number. Let $F(x) = x+G(x)$ be one dimensional periodic dynamical system. This means that $F(x)$ is a continuous monotone increasing function and $G(x)$ is an 1-periodic function. We denote the set of such functions F by $D(T^1)$. $D^\infty(T^1)$ is the subgroup of $D(T^1)$ whose elements are of C^∞-class. According to H. Poincaré, the rotation number $\rho(F)$ of $F \in D(T^1)$ is defined by

$$\rho(F) = \lim_{n\to\pm\infty} \frac{F^n(x) - x}{n},$$

where $F^n(x)$ is the n-th iterate of $F(x)$. It is well-known ([H]) that $\rho(F)$ is independent of x and the convergence is uniform with respect to x. As we regard $\rho(F)$ as a functional of F, $\rho(F)$ is continuous with respect to $\sup_{0\le x\le 1} |F(x)|$. Note that *the rotation number has the conjugate-invariant property.* Namely, one has the following identity

$$\rho(F) = \rho(\phi \circ F \circ \phi^{-1})$$

for any $\phi \in D(T^1)$. Since clearly the rotation number of $R_\alpha(x) = x+\alpha$ (α : constant) is equal to α, it follows that $\rho(\phi \circ R_\alpha \circ \phi^{-1}) = \alpha$ for any $\phi \in D(T^1)$. For more details of the rotation numbers, see [H].

Some Function Spaces. Let s be a nonnegative integer. Let G be an open set in R^n. Let $L^2(G)$, $H^s(G)$ and $H^s_0(G)$ be the usual Lebesgue space and Sobolev spaces (resp.) with norms $|\cdot|_{L^2(G)}$ and $|\cdot|_{H^s(G)}$. $C^s(G)$ is defined as usual with norm $|\cdot|_{C^s(G)}$. We omit G in the norms if there is no confusion. We write $|\cdot|_{C^0}$ as $|\cdot|_C$.

Let $(0,\omega/2) \times R^1$ be denoted by D. Let $D^\infty_0(D)$ be a function space whose elements $f(x,t)$ are defined in D, of $C^\infty(D)$, 1-periodic in t and have the supports contained in D. We denote a set $(0,\omega/2) \times (0,1)$ by D_0. Let $K^s_0(D)$ be the completion of $D^\infty_0(D)$ with respect to norm $|\cdot|_{H^s(D_0)}$. We define function spaces $D^s_0(\Omega)$ and $K^s_0(\Omega)$ in the same way, where Ω is the noncylindrical domain defined by in section 1. In this paper, we write $H^s(D_0)$ and $L^2(D_0)$ as $H^s(D)$ and $L^2(D)$ (resp.). All the function spaces $K^s_0(D)$, $H^s(D)$, $K^s_0(\Omega)$ and $H^s(\Omega)$ are Hilbert spaces with the above norms.

Main Theorem

We formulate our main result. Assume the following conditions. Let s be an integer ≥ 4.

(A1) $a_i(t)$, $i = 1,2$, are of C^∞ and 1-periodic, and satisfy $a_1(t) < a_2(t)$ and $|a'_i(t)| < 1$ for $t \in R^1$.

(A2) $b_i(t)$, $i = 1, 2$, are of C^∞ and 1-periodic.

(A3) The rotation number ω of A satisfies the following Diophantine condition : There exists a positive constant C such that the Diophantine inequality

$$|n - m/\omega| \geq Cn^{-1}$$

holds for all n, $m \in N$.

(A4) $p(x, t)$ is of C^s-class with respect to $(x, t) \in \bar{\Omega}$ and 1-periodic in t. $f(x, t, u)$ is of C^{s+2}-class with respect to $(x, t, u) \in \bar{\Omega} \times R^1$ and 1-periodic in t and satisfies

$$f(x, t, 0) = \partial_u f(x, t, 0) = 0.$$

$p(x, t)$ and $f(x, t, u)$ satisfy compatible boundary conditions :

$$p(a_i(t), t) = 0, \quad i = 1, 2,$$

holds for all $t \in R^1$, and there exists a positive constant ν_0 such that for any ν with $|\nu| \leq \nu_0$,

$$f(a_i(t), t, \nu b_i(t)) = 0, \quad i = 1, 2,$$

holds for all $t \in R^1$.

Remark 1. It is well-known in number theory ([Kh]) that all numbers with periodic continued fraction expansion satisfy (A3). Note that the set of all algebraic numbers of degree 2 coinsides with the above set.

Remark 2. $f(x, t, u)$ satisfying (A4) is written of the form

$$f(x, t, u) = u^2 r(x, t, u),$$

where $r(x, t, u)$ is of C^s-class with respect to $(x, t, u) \in \bar{\Omega} \times R^1$. As an example of $f(x, t, u)$ that satisfies the compatible boundary condition in (A4), we can take $r(x, t, u)$ with $r(a_i(t), t, u) = 0$, $i = 1, 2$, for all $(t, u) \in R^1_t \times R^1_u$. f possibly depends on the parameter ν. As such an example we give $r(x, t, u) = R(x, t, (u - \nu b_1(t))(u - \nu b_2(t)))$, where $R(x, t, U)$ satisfies $R(a_i(t), t, 0) = 0$ for all $t \in R^1$.

Remark 3. If $a_1(t)$ and $a_2(t)$ are constants, *e.g.* $a_1(t) \equiv a$ and $a_2(t) \equiv b$, then we have $A_1(t) = t + 2a$ and $A_2(t) = t + 2b$, whence $A(t) = t + 2b - 2a$ and $\rho(A) = 2b - 2a$. This means that $\rho(A)/2$ is equal to the length of the interval.

The existence of the boundary functions that satisfy both of an analytical condition (A1) and a number-theoretic condition (A3) is assured by the following proposition.

Proposition 1.1. *Let* ω *be any real number. Then there exists* 1-*periodic* C^∞ *functions* $a_i(t)$, $i = 1, 2$, *such that* $\rho(A) = \omega$. *Here* A *is defined by* (1.3).

Proof. Note that the rotation number of $R_\alpha(x) = x + \alpha$ is equal to α and the rotation number is conjugate-invariant. For a given ω, we define A_ϕ by

$$A_\phi(x) = \phi \circ R_\omega \circ \phi^{-1}(x)$$

for $\phi \in D^\infty(T^1)$. Then clearly $\rho(A_\phi) = \omega$. For simplicity, we set $a_1(t) \equiv 0$. Then $A_1 = I$ and $A_\phi = A_2$. We set $a_2(t) = a(t)$, for simplicity. Then by (1.3) we have an equality

$$(I + a) \circ (I - a)^{-1}(t) = A_\phi(t) \equiv (I + B_\phi)(t),$$

where $B_\phi(t)$ is an 1-periodic C^∞ function and satisfy $|B'_\phi(t)| < 1$. Then setting $y = (I - a)^{-1}(t)$, we have

$$a(y) = (1/2)B_\phi \circ (I - a)(y).$$

We consider a function $G(y, a) = a - (1/2)B_\phi(y - a)$ of C^∞-class with respect to (y, a) and apply the implicit function theorem to a functional equation $G(y, a) = 0$. Since B_ϕ satisfies $|B'_\phi(t)| < 1$ for any $t \in R^1$ and hence we have

$$|G_a(y, a)| \geq 1 - (1/2)|B'_\phi(y - a)| > 1/2 > 0,$$

this functional equation has a C^∞ solution $a(y)$. Q.E.D.

Our main theorem is the following.

Theorem 1.1. *Assume* (A1), (A2), (A3) *and* (A4). *Then there exists a positive constant* μ_0 *such that for any* μ *satisfying* $|\mu| < \mu_0$ BVP (1.1)-(1.2) *has an* 1-*periodic solution of* C^2-*class with respect to* $(x, t) \in \bar{\Omega}$.

2. One Dimensional Periodic Dynamical Systems and Domain Transformation

In this section we shall construct a bijective transformation of $\bar{\Omega}$ to $\bar{D}$ that preserves the d'Alembertian. This is made by the conjugate function in the reduction theorem ([H],[Yoc]) of one dimensional periodic dynamical systems.

Consider one dimensional periodic dynamical system

$$F(x) = x + G(x), \quad x \in R^1. \tag{2.1}$$

Here $F(x)$ is a monotone increasing continuous function and $G(x)$ is a periodic function with period 1. We denote all such functions F by $D(T^1)$. It is clear that $D(T^1)$ ia a group with respect to composition of functions. Let $D^\infty(T^1)$ be a subgroup of $D(T^1)$ whose elements are of C^∞-class. Let $\rho(F)$ be the rotation number of F. Assume the following numerical condition.

(N) The rotation number $\rho(F)$ satisfies the following Diophantine condition : There exist positive constants C and $d > 1$ such that the Diophantine inequality

$$|n - m/\rho(F)| \geq Cn^{-d}$$

holds for all $n \in Z \backslash \{0\}$ and $m \in Z$.

Remark. Every number satisfying (N) is irrational. As is well-known, almost all real numbers satisfy the above condition (N). Here "almost all" is taken as the usual Lebesgue measure sense. According to the famous Roth Theorem of number theory, all algebraic numbers of degree ≥ 2 satisfy (N). The familiar transcendental numbers π and e also satisfy (N) with suitable d.

The following reduction theorem of one dimensional periodic dynamical systems is due to Herman and Yoccoz ([H], [Yoc]) and is important to construct the domain transformation.

Proposition 2.1 (Herman and Yoccoz). *Assume that F is an element of $D^\infty(T^1)$ and its rotation number satisfies* (N). *Then there exists a function ϕ in $D^\infty(T^1)$ such that F is conjugate to the rotation* :

$$\phi \circ F \circ \phi^{-1}(\xi) = \xi + \rho(F) \tag{2.2}$$

holds.

We apply this theorem to the composed function A. Clearly it follows from (A1) that A is an element of $D^\infty(T^1)$. By this and (A3) A satisfies the assumptions of Proposition 2.1. Hence there exists a function $H \in D^\infty(T^1)$ such that

$$H \circ A \circ H^{-1}(\xi) = \xi + \omega \tag{2.3}$$

holds, where ω is the rotation number of A. Using H, we define the domain transformation $\Phi : R^2 \to R^2$ by

$$\xi = \left(H \circ A_1^{-1}(x+t) - H(-x+t)\right)/2, \tag{2.4}$$

$$\tau = \left(H \circ A_1^{-1}(x+t) + H(-x+t)\right)/2. \tag{2.5}$$

for $(x,t) \in R^2$. The inverse transformation Φ^{-1} is written by

$$x = \left(A_1 \circ H^{-1}(\xi+\tau) - H^{-1}(-\xi+\tau)\right)/2, \tag{2.6}$$

$$t = \left(A_1 \circ H^{-1}(\xi+\tau) + H^{-1}(-\xi+\tau)\right)/2. \tag{2.7}$$

Then Φ has several natural properties to study our problem (*cf.* [Ya6], [Ya-Yo]).

Proposition 2.2. *Assume* (A1) *and* (A3). *Φ is the bijection of $\bar{\Omega}$ to $\bar{D}$, and maps the boundaries $x = a_1(t)$ and $x = a_2(t)$ onto the boundaries $\xi = 0$ and $\xi = \omega/2$* (resp.) *bijectively. Moreover the d'Alembert operator is preserved by Φ as follows. Let $u(x,t)$ be of C^2 in $(x,t) \in R^2$ and $v(\xi,\tau)$ be defined by $u(\Phi^{-1}(\xi,\tau))$. Then the following identity holds*

$$(\partial_t^2 - \partial_x^2)u(x,t) = K(\xi,\tau)(\partial_\tau^2 - \partial_\xi^2)v(\xi,\tau), \tag{2.8}$$

where $K(\xi,\tau)$ is defined by

$$4H' \circ H^{-1}(\xi+\tau)H' \circ H^{-1}(-\xi+\tau)(A_1^{-1})' \circ A_1 \circ H^{-1}(\xi+\tau).$$

$K(\xi,\tau)$ is 1-periodic in τ.

For the proof of this proposition, see [Ya6] (Proofs of Proposition 4.1 and 4.2).

3. BVP in Cylindrical Domain

In this section we show the existence of periodic solutions of BVP (1.6)-(1.7) in the cylindrical domain $D = (0,\omega/2) \times R^1$. Through this section, we assume that ω *is a positive number, but not necessarily the rotation number of A.* [Mc] and [BN-Ma] also treated the fixed ends case *i.e.*, $c_i(\tau) \equiv 0$, $i = 1,2$, in (1.6)-(1.7) and showed the existence of the weak periodic solutions of BVP, under the equivalent numerical condition as ours and some conditions on semilinear terms $q+g$. In our BVP, we shall deal with the case where the vertical external forces $c_i(\tau)$, $i = 1,2$, work on the string ends. By these external forces the resonance may happen. In order to make clear the representation of solutions of BVP (1.6)-(1.7), we shall independently treat this case. In this paper, we are concerned with classical solutions. In order to obtain classical solutions, we shall need several estimates of the semilinear term and its derivatives.

We consider BVP in the cylindrical domain $D = (0, \omega/2) \times R^1$:

$$\partial_\tau^2 v - \partial_\xi^2 v = \mu q(\xi,\tau) + g(\xi,\tau,v), \ \ (\xi,\tau) \in D, \tag{3.1}$$

$$v(0,\tau) = \nu c_1(\tau), \quad v(\omega/2,\tau) = \nu c_2(\tau), \quad \tau \in R^1. \tag{3.2}$$

We assume the following numerical condition on ω which is stated in (A3) in section 1.

(C0) ω satisfies the following condition : There exists a positive constant C such that a Diophantine inequality

$$|n - m/\omega| \geq C n^{-1} \tag{3.3}$$

holds for all $n \in N$ and all $m \in N$.

Remark. Note that (C0) implies (N).

We assume the following conditions. Let s be an integer ≥ 4.

(C1) $q(\xi,\tau)$ is of C^s-class in $\bar{D}$ and 1-periodic in τ. $g(\xi,\tau,v)$ is of C^{s+2}-class in $(\xi,\tau,v) \in \bar{D} \times R^1$, 1-periodic in τ and satisfies

$$g(\xi,\tau,0) = \partial_v g(\xi,\tau,0) = 0.$$

$q(\xi,\tau)$ and $g(\xi,\tau,v)$ satisfy boundary conditions :

$$q(0,\tau) = q(\omega/2,\tau) = 0$$

holds for all $\tau \in R^1$, and there exists a positive constant ν_0 such that for any ν with $|\nu| \leq \nu_0$

$$g(0,\tau,\nu c_1(\tau)) = g(\omega/2,\tau,\nu c_2(\tau)) = 0$$

holds for all $\tau \in R^1$.

(C2) $c_i(\tau)$, $i = 1, 2$, are 1-periodic and of C^∞-class.

Remark. As we stated in Remark 2 in section 1, $g(\xi,\tau,v)$ satisfying $g(\xi,\tau,0) = \partial_v g(\xi,\tau,0) = 0$ in (C1) is written in the form

$$g(\xi,\tau,v) = s(\xi,\tau,v) v^2, \tag{3.4}$$

where $s(\xi,\tau,v)$ is of C^s in $(\xi,\tau,v) \in \bar{D} \times R^1$. As an example of $g(\xi,\tau,v)$ that satisfies the compatible boundary condition in (C1), we give $s(\xi,\tau,v)$ with $s(0,\tau,v) = s(\omega/2,\tau,v) = 0$ for all $(\tau,v) \in R^2$. As will be seen in the proof of Theorem 3.1, g possibly depends on ν. As such an example we give $s(\xi,\tau,v) = J(\xi,\tau,(v - \nu c_1(\tau))(v - \nu c_2(\tau)))$, where $J(\xi,\tau,V)$ satisfies $J(0,\tau,0) = J(\omega/2,\tau,0) = 0$ for all $\tau \in R^1$.

Now we have the following theorem.

Theorem 3.1. *Assume* (C0), (C1) *and* (C2). *Then there exists a positive conatant* μ_1 *such that for any* μ *satisfying* $|\mu| < \mu_1$ BVP (3.1)-(3.2) *has an* 1*-periodic solution* v *locally unique in* $H^s(D) \cap K_0^1(D)$.

In order to prove this theorem, we decompose BVP (3.1)-(3.2) into the following three BVPs :

$$\partial_\tau^2 v_1 - \partial_\xi^2 v_1 = 0, \quad (\xi, \tau) \in D, \tag{3.5}$$

$$v_1(0,\tau) = c_1(\tau), \quad v_1(\omega/2, \tau) = 0, \quad \tau \in R^1, \tag{3.6}$$

$$\partial_\tau^2 v_2 - \partial_\xi^2 v_2 = 0, \quad (\xi, \tau) \in D, \tag{3.7}$$

$$v_2(0,\tau) = 0, \quad v_2(\omega/2, \tau) = c_2(\tau), \quad \tau \in R^1, \tag{3.8}$$

and

$$\partial_\tau^2 w - \partial_\xi^2 w = \mu q(\xi,\tau) + g(\xi, \tau, \nu(v_1 + v_2) + w), \quad (\xi, \tau) \in D, \tag{3.9}$$

$$w(0,\tau) = 0, \quad w(\omega/2, \tau) = 0, \quad \tau \in R^1. \tag{3.10}$$

If each BVP has an 1-periodic solution, then we have the periodic solution $u = \nu(v_1 + v_2) + w$ of BVP (3.1)-(3.2).

First we shall solve BVP (3.5)-(3.6) and (3.7)-(3.8). We have the following proposition.

Proposition 3.1. *Assume* (C0) *and* (C2). *Then each of* BVP (3.5)-(3.6) *and* BVP (3.7)-(3.8) *has a unique* 1*-periodic solution* v_1 *and* v_2 (resp.) *of* C^∞*-class in* $\bar{D}$ *with*

$$\left|\partial_\tau^j \partial_\xi^k v_i(\xi,\tau)\right| \leq Const. \sup_{0 \leq m \leq j+k+3,\, 0 \leq \tau \leq 1} |(d/d\tau)^m c_i(\tau)|$$

for any nonnegative integers j, k, *where* $Const.$ *depends on only* j, k.

Proof of Proposition 3.1. It is enough to deal with BVP (3.5)-(3.6). We rewrite c_1 and v_1 by c and v (resp.), and τ and ξ by t and x (resp.). By the d'Alembert formula we represent solutions of (3.5) as

$$v(x,t) = f(-x+t) + g(x+t). \tag{3.11}$$

From the boundary conditions (3.6) we have

$$f(t) + g(t) = c(t),$$

$$f(-\omega/2 + t) + g(\omega/2 + t) = 0.$$

By this we have

$$v(x,t) = f(-x+t) - f(x+t) + c(x+t) \tag{3.12}$$

and

$$f(-\omega/2+t)-f(\omega/2+t)+c(\omega/2+t)=0.$$

The latter equation becomes

$$f(t+\omega)-f(t)=c(t+\omega). \tag{3.13}$$

We look for a solution $f(t)$ of (3.13) of the form

$$f(t)=\alpha t+F(t),$$

where $F(t)$ is 1-periodic and satisfies $\int_0^1 F(t)dt=0$, and α is a constant. We expand $c(t+\omega)$ into the Fourier series

$$\sum_n c_n \exp(2\pi int). \tag{3.14}$$

We formally expand $F(t)$ into the Fourier series

$$F(t)=\sum_{n\neq 0} f_n \exp(2\pi int).$$

Substituting this and (3.14) into (3.13) and comparing the Fourier coefficients of both sides, we have

$$f_n=c_n/\left(\exp(2\pi in\omega)-1\right)\ \ (n\neq 0),\qquad \alpha=c_0/\omega. \tag{3.15}$$

Since $c(t+\omega)$ is of C^∞, the Fourier coefficients c_n decay to 0 with any polynomial order of n as n grows up to infinity. More precisely, by integrating the integral of c_n by parts, we have

$$|c_n|\leq C_d/(|n|+1)^d \tag{3.16}$$

for any $n\in Z$ and any fixed $d>0$. Here

$$C_d=Const.\sup_{0\leq j\leq d,\,0\leq t\leq 1}|c^{(j)}(t)|,$$

where $Const.$ depends only on d. By simple calculation we have

$$|\exp(2\pi in\omega)-1|\geq 4\,|n\omega-m|$$

for any $n\in Z\setminus\{0\}$ and some $m=m(n)\in Z$. By (C0) we have

$$|\exp(2\pi in\omega)-1|\geq 4C\,\omega\,|n|^{-1}. \tag{3.17}$$

It follows from (3.15)-(3.17) that

$$|f_n|\leq (1/4C\,\omega)\,C_d/(|n|+1)^{d-1}$$

for any $n\in Z$ and any $d>0$. This means that $f(t)$ is of C^∞. We have

$$\sum_n |n|^k|f_n|\leq (1/4C\,\omega)\,C_d\sum_n 1/(|n|+1)^{d-k-1}\leq Const.\,C_d$$

for $d > k+2$, where $Const.$ depends on only d. Hence it follows that

$$\left|F^{(k)}(t)\right| \leq Const. \sup_{0\leq j\leq d,\, 0\leq t\leq 1} \left|c^{(j)}(t)\right|$$

for $d > k+2$. Since v has the form

$$v(x,t) = -2(c_0/\omega)x + F(-x+t) - F(x+t) + c(x+t),$$

$v(x,t)$ is 1-periodic in t and has the estimate

$$\begin{aligned}\left|\partial_t^j\, \partial_x^k\, v(x,t)\right| &\leq Const.\, c_0 + 2 \sup_{0\leq t\leq 1} \left|F^{(j+k)}(t)\right| + \sup_{0\leq t\leq 1} \left|c^{(j+k)}(t)\right| \\ &\leq Const. \sup_{0\leq j\leq d,\, 0\leq t\leq 1} \left|c^{(j)}(t)\right|\end{aligned}$$

for nonnegative integers j, k satisfying $d > j+k+2$. This shows the estimates of v. It is clear from the construction that v is unique. Q.E.D.

In order to show the existence of periodic solutions of BVP (3.9)-(3.10), first let us deal with BVP for a linear nonhomogeous wave equation and obtain the necessary estimate of the solutions, by which the d'Alembertian has the bounded inverse in a suitable function space, for later use. Consider BVP

$$\partial_\tau^2\, \zeta - \partial_\xi^2\, \zeta = r(\xi,\tau), \quad (\xi,\tau) \in D, \tag{3.18}$$

$$\zeta(0,\tau) = \zeta(\omega/2,\tau) = 0, \quad \tau \in R^1. \tag{3.19}$$

We have the following proposition.

Proposition 3.2. *Assume* (C0). *Let $r(\xi,\tau)$ be of $H^s(D) \cap K_0^1(D)$ and* 1*-periodic in τ. Then* BVP (3.18)-(3.19) *has a unique* 1*-periodic solution ζ in $H^s(D) \cap K_0^1(D)$. The solution has the estimate :*

$$|\zeta|_{H^s} \leq C_0\, |r|_{H^s},$$

where C_0 is a constant dependent only on s.

To prove this proposition we prepare a lemma.

Lemma 3.1. *Let $\zeta(\xi,\tau)$ be an element of $H^1(D)$ satisfying the boundary condition $\zeta(0,\tau) = \zeta(\omega/2,\tau) = 0$ for almost all $\tau \in (0,1)$. Then ζ belongs to $K_0^1(D)$ and vice versa.*

Proof of Lemma 3.1. This is proved in the standard way. We set $a = \omega/2$, for simplicity. Let $\phi_n(\xi)$ be a function of $C_0^\infty(0,a)$ with

$0 \le \phi_n(\xi) \le 1$ and $|\phi_n'(\xi)| \le C\,n$

$$\begin{aligned}\phi_n(\xi) &= 0 \quad \left(0 \le \xi \le \frac{1}{n},\ a - \frac{1}{n} \le \xi \le a\right)\\ &= 1 \quad \left(\frac{2}{n} \le \xi \le a - \frac{2}{n}\right),\end{aligned}$$

where C is a constant. Then $\zeta_n(\xi,\tau) = \phi_n(\xi)\,\zeta(\xi,\tau)$ is an element of $K_0^1(D)$. We set $I_n = A_n^1 \cup A_n^2$, $A_n^1 = [0, 2/n]$ and $A_n^2 = [a - 2/n, a]$. We show

$$|\zeta_n - \zeta|_{H^1} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{3.20}$$

We first estimate

(3.21)

$$\begin{aligned}|\zeta_n - \zeta|_{L^2}^2 + |\partial_\tau(\zeta_n - \zeta)|_{L^2}^2 &= \int_{D_0} (1 - \phi_n(\xi))^2 \left(\zeta(\xi,\tau)^2 + \zeta_\tau(\xi,\tau)^2\right) d\xi d\tau\\ &\le \int_0^1 \int_{I_n} \left(\zeta(\xi,\tau)^2 + \zeta_\tau(\xi,\tau)^2\right) d\xi d\tau \to 0\end{aligned}$$

as $n \to \infty$. Second, we have

$$\begin{aligned}|\partial_\xi(\zeta_n - \zeta)|_{L^2} &= |-\phi_n'\zeta + (1 - \phi_n)\zeta_\xi|_{L^2}\\ &\le |\phi_n'\zeta|_{L^2} + |(1 - \phi_n)\zeta_\xi|_{L^2}\end{aligned} \tag{3.22}$$

The second term is estimated in the same way as (3.21) and tends to 0 as $n \to \infty$. For the first term, taking into (3.19) account we note that

$$\zeta(\xi,\tau) = \int_0^\xi \zeta_\xi(x,\tau)dx \quad \text{and} \quad \zeta(\xi,\tau) = \int_a^\xi \zeta_\xi(x,\tau)dx \tag{3.23}$$

for almost all $\tau \in (0,1)$. We calculate the following :

$$|\phi_n'\zeta|_{L^2}^2 = \int_0^1 \int_{I_n} \left(\phi_n'(\xi)\zeta(\xi,\tau)\right)^2 d\xi d\tau \equiv J_1 + J_2, \tag{3.24}$$

where

$$J_i = \int_0^1 \int_{A_n^i} \left(\phi_n'(\xi)\zeta(\xi,\tau)\right)^2 d\xi d\tau \quad (i = 1, 2).$$

We have, from (3.23) and by the Schwarz inequality,

$$\begin{aligned} J_1 &\le Cn^2 \int_0^1 \int_{A_n^1} \zeta(\xi,\tau)^2 d\xi d\tau \\ &\le Cn^2 \int_0^1 \int_{A_n^1} \left(\int_0^\xi \zeta_\xi(x,\tau) dx \right)^2 d\xi d\tau \\ &\le Cn^2 \int_0^1 \int_{A_n^1} \xi \left(\int_0^\xi \zeta_\xi(x,\tau)^2 dx \right) d\xi d\tau \\ &\le Cn^2 \int_0^1 \left(\int_0^{2/n} \zeta_\xi(x,\tau)^2 dx \right) \int_{A_n^1} \xi d\xi d\tau \\ &\le Const. \int_0^1 \int_0^{2/n} \zeta_\xi(x,\tau)^2 dx d\tau \to 0 \quad (n \to \infty). \end{aligned}$$

We can estimate J_2 in the same way. Hence from (3.21), (3.22) and (3.24), (3.20) is shown.

We show the converse. Let ζ be an element of $K_0^1(D)$. By the Fubini Theorem clearly $\zeta(\cdot,\tau)$ belongs to $H^1(0,a)$ for almost all τ, whence by the Sobolev embedding theorem $\zeta(\xi,\tau)$ is continuous in ξ for almost all $\tau \in (0,1)$. We take a sequence $\{f_n\} \subset D_0^\infty(D)$ so that $|f_n - \zeta|_{H^1(D)} \to 0$. It follows that there exists a measure-zero set N contained in $(0,1)$ and a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $|f_{n_j}(\cdot,\tau) - \zeta(\cdot,\tau)|_{H^1(0,a)} \to 0$ for any $\tau \in (0,1)\backslash N$. By the Sobolev embedding theorem we have

$$\max_{0\le\xi\le a} |f_{n_j}(\xi,\tau) - \zeta(\xi,\tau)| \le Const. \, |f_{n_j}(\cdot,\tau) - \zeta(\cdot,\tau)|_{H^1(0,a)}.$$

Since $f_{n_j}(0,\tau) = f_{n_j}(a,\tau) = 0$, we have $\zeta(0,\tau) = \zeta(\omega/2,\tau) = 0$ for $\tau \in (0,1)\backslash N$. Q.E.D.

Proof of Proposition 3.2. We denote $(2/\sqrt{\omega})\ \exp(2\pi i k\tau) \sin(2\pi/\omega) j\xi$ by e_{jk}. Since $\{e_{jk}\}$ is a complete orthonormal system in $L^2(D)$, we expand r into the Fourier series in $L^2(D)$:

$$r = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} r_{jk}\, e_{jk},$$

where $r_{jk} = (r, e_{jk})_{L^2(D)}$. We formally expand ζ into the Fourier series

$$\zeta = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \zeta_{jk}\, e_{jk}.$$

Substitute these into (3.18) formally and compare the coefficients in both sums. Then we have

$$\zeta_{jk} = (1/2\pi)^2 \frac{r_{jk}}{(j/\omega + k)(j/\omega - k)}.$$

Hence it follows from (C0) that

$$|\zeta_{jk}| \le Const.\, |r_{jk}|. \tag{3.25}$$

Hence it follows that

$$|\zeta|_{L^2(D)} \le Const.\, |r|_{L^2(D)}.$$

By diffrentiating (3.18) with respect to ξ, τ, we have

$$\partial_\tau^2 \, \zeta_{\xi^p\tau^q}(\xi,\tau) - \partial_\xi^2 \, \zeta_{\xi^p\tau^q}(\xi,\tau) = r_{\xi^p\tau^q}(\xi,\tau)$$

for $p+q \le s$. Since $r_{\xi^p\tau^q}$ belongs to $L^2(D)$, it follows that

$$|\zeta_{\xi^p\tau^q}|_{L^2(D)} \le Const.\, |r_{\xi^p\tau^q}|_{L^2(D)}.$$

This means $|\zeta|_{H^s} \le C_0 \, |r|_{H^s}$. It remains to show that ζ belongs to $K_0^1(D)$. It is shown that $\{e_{jk}\}$ is complete in $K_0^1(D)$. In fact, let f be an element of $K_0^1(D)$. Suppose that $(f, e_{jk})_{H^1(D)} = 0$ for all j, k. This means that

$$I_0 + I_1 + I_2 = (f, e_{jk})_{L^2(D)} + (\partial_\xi f, \partial_\xi e_{jk})_{L^2(D)} + (\partial_\tau f, \partial_\tau e_{jk})_{L^2(D)} = 0.$$

Integrating by parts and using Lemma 3.1, we have

$$I_1 = (2/\sqrt{\omega})(\partial_\xi f, (2\pi j/\omega) \exp 2\pi i k\tau \cos(2\pi/\omega) j\xi)_{L^2(D)} = (2\pi j/\omega)^2 I_0,$$
$$I_2 = (2/\sqrt{\omega})(\partial_\tau f, (2\pi i k) \exp 2\pi i k\tau \sin(2\pi/\omega) j\xi)_{L^2(D)} = (2\pi k)^2 I_0.$$

Hence we have

$$(f, e_{jk})_{H^1(D)} = a_{jk}\, (f, e_{jk})_{L^2(D)}, \tag{3.26}$$

where $a_{jk} = 1 + (2\pi j/\omega)^2 + (2\pi k)^2$. Thus we obtain $(f, e_{jk})_{L^2(D)} = 0$. By the completeness of $\{e_{jk}\}$ in $L^2(D)$ it follows that $f = 0$. Thus $\{\tilde{e}_{jk}\} = \{e_{jk}/a_{jk}^{1/2}\}$ is a complete orthonormal system in $K_0^1(D)$. Since r is an element of $K_0^1(D)$, it is expanded into the Fourier series in $K_0^1(D)$:

$$r = \sum_{k,j} \tilde{r}_{jk}\, \tilde{e}_{jk}.$$

Clearly it follows from (3.26) that $\tilde{r}_{jk} = a_{jk}\, r_{jk}$. Set $\tilde{\zeta}_{jk} = a_{jk}\, \zeta_{jk}$. Then we have, from (3.25)

$$|\tilde{\zeta}_{jk}| \le Const.\, |\tilde{r}_{jk}|.$$

Hence it follows that $\sum_{j,k} \tilde{\zeta}_{jk}\tilde{e}_{jk}$ converges to ζ in $K_0^1(D)$. Noting that ζ is continuous in D, by Lemma 3.1 (3.19) follows. Thus the proposition is proved. Q.E.D.

Theorem 3.1 is proved by applying the contraction mapping theorem. To this end, we first show the lemma on the nonlinear term.

Lemma 3.2. *Assume* (C1). *Then the followings hold.*
(1) *For* $v \in H^s(D)$ $g(\xi,\tau,v(\xi,\tau))$ *belongs to* $H^s(D)$. *If* $|v|_{H^s} \le L$ *and* $|v|_C \le M$ *hold, then*

$$|g(\xi,\tau,v(\xi,\tau))|_{H^s} \le c_1\, C_s(M,L)|v|_{H^s}^2. \tag{3.27}$$

(2) *For* $v_i \in H^s(D)$, $i = 1, 2$, *with* $|v_i|_{H^s} \le L$ *and* $|v_i|_C \le M$,

$$\begin{aligned}&|g(\xi,\tau,v_1) - g(\xi,\tau,v_2)|_{H^s}\\ &\le c_1\, C_s(M,L)\ \max\left(|v_1|_{H^s}, |v_2|_{H^s}\right)|v_1 - v_2|_{H^s}.\end{aligned} \tag{3.28}$$

holds. Here c_i *are positive constants dependent only on* s *and* $C_s(M,L)$ *is a positive constant dependent on* M, L *and* s.

Proof of Lemma 3.2. We note from Remark of (C1) that $g(\xi,\tau,v)$ is of the form $v^2 s(\xi,\tau,v)$, where $s(\xi,\tau,v)$ is of C^s-class.

We show the statement (1). Since for $s \ge 4$, $H^s(D)$ is a Banach algebra and $g(\xi,\tau,v(\xi,\tau))$, $v \in H^s$, belongs to $H^s(D)$ from Theorem, Appendix in [Ya7], it follows that

$$\begin{aligned}|g(\xi,\tau,v)|_{H^s} &= |v^2\, s(\xi,\tau,v)|_{H^s}\\ &\le |v|_{H^s}^2 |s(\xi,\tau,v)|_{H^s}.\end{aligned}$$

Using Theorem in [Ya7] again, we have, for v with $|v|_{H^s} \le L$ and $|v|_C \le M$,

$$\begin{aligned}|g(\xi,\tau,v)|_{H^s} &\le |v|_{H^s}^2 |s(\xi,\tau,v)|_{H^s}\\ &\le |v|_{H^s}^2\, c_s(M)(|v|_{H^s} + 1)\\ &\le C_s(M,L)|v|_{H^s}^2.\end{aligned}$$

This shows the statement of (1).

Next we show the statement (2). Let v_i, $i = 1, 2$, be elements of $H^s(D)$ with $|v_i|_{H^s} \le L$ and $|v_i|_C \le M$. Then using Theorem in [Ya7], we have

$$
\begin{aligned}
&|g(\xi,\tau,v_1)-g(\xi,\tau,v_2)|_{H^s}\\
&\qquad \le |v_1^2(s(\xi,\tau,v_1)-s(\xi,\tau,v_2))|_{H^s}+|(v_1^2-v_2^2)\,s(\xi,\tau,v_2)|_{H^s}\\
&\qquad \le |v_1|_{H^s}^2\,|s(\xi,\tau,v_1)-s(\xi,\tau,v_2)|_{H^s}+|v_1^2-v_2^2|_{H^s}\,|s(\xi,\tau,v_2)|_{H^s}\\
&\qquad \le |v_1|_{H^s}^2\,c_s'(M,L)\,|v_1-v_2|_{H^s}\\
&\qquad\qquad\quad +2\max(|v_1|_{H^s},|v_2|_{H^s})|v_1-v_2|_{H^s}(c_s'(M,L)L+1)\\
&\qquad \le Const.\max(|v_1|_{H^s},|v_2|_{H^s})\,|v_1-v_2|_{H^s},
\end{aligned}
$$

where $c_s'(M,L)$ are similar constants to $C_s(M,L)$. This shows the statement of (2). Q.E.D.

Now we shall prove Theorem 3.1.

Proof of Thorem 3.1. Let v_i, $i=1,2$, and ζ be the same functions in Proposition 3.1 and 3.2 (resp.). Here we take r in Proposition 3.2 as q in (C1). We take $w=\mu(\zeta+U)$. Then BVP (3.9)-(3.10) is transformed to the following BVP :

(3.29) $$\partial_\tau^2 U-\partial_\xi^2 U=\mu^{-1}g(\xi,\tau,\nu(v_1+v_2)+\mu(\zeta+U)),\quad (\xi,\tau)\in D,$$

(3.30) $$U(0,\tau)=U(\omega/2,\tau)=0,\quad \tau\in R^1.$$

Since $g(\xi,\tau,v)$ is equal to $s(\xi,\tau,v)v^2$ and $\nu=O(\mu)$ $(\mu\to 0)$, the right hand side is continuous and bounded in μ in some interval $(-\delta,\delta)$, $\delta>0$ and of $O(\mu)$ $(\mu\to 0)$. Set

(3.31) $$F(\xi,\tau,U,\mu)=\mu^{-2}g(\xi,\tau,\nu(v_1+v_2)+\mu(\zeta+U)).$$

Then $F(\xi,\tau,U,\mu)$ is 1-periodic in τ, of C^{s+2}-class with respect to (ξ,τ,U) and bounded and continuous in $\mu\in(-\delta,\delta)$ of $O(\mu)$ $(\mu\to 0)$. For simplicity, we abbreviate μ in F. By the boundary condition of (C1) F satisfies $F(0,\tau,0)=F(\omega/2,\tau,0)=0$. Let L be a fixed positive constant and define $B_L=\{U\in H^s(D)\cap K_0^1(D)\,|\,|U|_{H^s}\le L\}$. From Lemma 3.2, (1) we have, for $U\in B_L$

(3.32)
$$
\begin{aligned}
|F(\xi,\tau,U)|_{H^s}&\le|\mu|^{-2}c_1C_s(M)|\nu(v_1+v_2)+\mu(\zeta+U)|_{H^s}^2\\
&\le c_1C_s(M)C(\delta)(|v_1|_{H^s}+|v_2|_{H^s}+|\zeta|_{H^s}+|U|_{H^s})^2\\
&\le C(L,\delta)(1+|U|_{H^s}),
\end{aligned}
$$

where M is a constant = (Sobelev constant) × L. Similarly we have, for U, $V \in B_L$

$$
\begin{aligned}
(3.33) \qquad |F(\xi,\tau,U) - F(\xi,\tau,V)|_{H^s} &\le |\mu|^{-2} c_1 C_s(M) |\mu(U-V)|^2_{H^s} \\
&\le c_1 C_s(M)|U-V|_{H^s} \\
&\le C(L,\delta)|U-V|_{H^s}.
\end{aligned}
$$

(3.29) is rewritten :

$$
(3.29) \qquad \partial_\tau^2 U - \partial_\xi^2 U = \mu\, F(\xi,\tau,U,\mu), \quad (\xi,\tau) \in D.
$$

For simplicity, we denote the d'Alembertian $\partial_t^2 - \partial_x^2$ by A. Let R be the inverse operator of A. It follows from Proposition 3.2 that R is a linear bounded mapping of $H^s(D) \cap K_0^1(D)$ to $H^s(D) \cap K_0^1(D)$. We shall show that there exists a positive constant μ_1 such that for μ satisfying $|\mu| < \mu_1$, a nonlinear mapping $\mu\, R \circ F$ has a fixed point in B_L. In fact, we shall show that $\mu\, R \circ F$ is a contraction mapping of B_L to B_L. Set $Z = \mu\, R \circ F$. It follows from Lemma 3.1 and Lemma 3.2 that $F(\xi,\tau,U(\xi,\tau))$ belongs to $H^s(D) \cap K_0^1(D)$ for $U \in H^s(D) \cap K_0^1(D)$. Hence Z maps $H^s(D) \cap K_0^1(D)$ to $H^s(D) \cap K_0^1(D)$. It follows from Proposition 3.2 and (3.32) that for $U \in B_L$

$$
\begin{aligned}
|Z(U)|_{H^s} &\le |\mu|\, C_0\, |F(\xi,\tau,U)|_{H^s} \\
&\le |\mu|\, C_0\, C_2(L,\delta)(|U|_{H^s} + 1) \\
&\le |\mu|\, C_3(L,\delta),
\end{aligned}
$$

whence for any μ satisfying $|\mu| < \mu_1$ with $\mu_1 < L/C_3$, $|Z(U)|_{H^s} \le L$ holds. This shows that Z maps B_L to B_L. Next estimate $Z(U) - Z(V)$. From Proposition 3.2 and (3.33) we have

$$
\begin{aligned}
|Z(U) - Z(V)|_{H^s} &\le |\mu|\, C_0\, |F(\xi,\tau,U) - F(\xi,\tau,V)|_{H^s} \\
&\le |\mu|\, C_0\, C_2(L,\delta)\, |U-V|_{H^s} \\
&\le |\mu|\, C_3(L,\delta)|U-V|_{H^s}.
\end{aligned}
$$

Hence for μ_1 with $\mu_1 < 1/C_3$ the mapping Z is contracting. This shows the existence of the solution of BVP (3.9)-(3.10). Thus Theorem 3.1 is proved. Q.E.D.

Theorem 3.1 implies Theorem 1.1. In fact, it follows from Proposition 2.2 that BVP (1.1)-(1.2) is transformed to BVP (3.1)-(3.2) by the domain transformation Φ defined by (2.4)-(2.5). The transformed functions $q(\xi,\tau)$ and $g(\xi,\tau,v)$, and $c_i(\tau)$, $i = 1, 2$, satisfy (C1)and (C2) (resp.), and the rotation number ω of A satisfies (C0). We know that from Theorem 3.1 BVP (3.1)-(3.2) has an 1-periodic solution $v(\xi,\tau)$ of

C^2-class in D. Hence by the inverse transformation Φ^{-1} BVP (1.1)-(1.2) has an 1-periodic solution $u(x,t) = v \circ \Phi(x,t)$ of C^2-class in Ω.

Acknowledgement. The author would like to thank the referee for pointing out some mistakes on the compatible boundary condition on nonlinear term, and for appropriate and kind advices.

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