Funkcialaj Ekvacioj
Volume 45, No. 3, pp 387--395

Existence of global solutions to the Cauchy problem of Kirchhoff type quasilinear wave equation with weakly nonlinear dissipation

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January 30, 2003

1 Introduction

In this note we consider the existence of global solution to the initial value problem for the quasilinear wave equation:

$$u_{tt} - (1 + \|\nabla u\|^2)\Delta u + \rho(u_t) = 0 \quad \text{in} \quad R^N \times (0,\infty), \tag{1.1}$$

with

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x), \tag{1.2}$$

where $\|\cdot\|$ denotes L^2 norm in R^N and $\rho(v)$ is a differentiable function satisfying

$$0 < k_0 \le \rho'(v) \le k_1 < \infty \quad \text{and} \quad \rho(0) = 0. \tag{1.3}$$

For the Kirchhoff type quasilinear wave equation it is natural to seek for the solutions in the class $C([0,\infty); H_2) \cap C^1([0,\infty); H_1) \cap C^2([0,\infty); L^2)$ or a little weaker space $L^\infty([0,\infty); H_2) \cap W^{1,\infty}([0,\infty); H_1) \cap W^{2,\infty}([0,\infty); L^2)$ (cf. I. Lasiecka and J.Ong [2]), and we are interested in the global solution of the problem (1.1)-(1.2) in such a class, (we often call such a solution as H^2 solution). When $\rho(u_t) = u_t$, linear, we see $\rho(u_t)u = \frac{1}{2}\frac{d}{dt}|u|^2$ and by use of this fact we can easily derive the a priori estimates

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \le C(\|u_0\|_{H_1} + \|u_1\|_{L^2})(1+t)^{-1}$$

and

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|)(1+t)^{-2}$$

if $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ is small. These a priori estimates are sufficient for the desired global solution. Indeed, K. Mochizuki [5] has proved such result under a more general condition on the dissipation. However, the proof heavily depends on the linearity of the dissipation $\rho(u_t)$ and cannot be applied to the case of nonlinear dissipation. Y. Yamada [9] proved the existence of global solutions without direct use of the decay properties. But in [9] also, the linearity of the dissipation is essentially used. The object of this note is to prove the existence of global H^2-solution when $\rho(v)$ is weakly nonlinear as (1.3), though in our case, solution $u(t)$ itself belongs to $L^\infty_{loc}([0,\infty); L^2)$, not $L^\infty([0,\infty); L^2)$.

Our proof is based on the following observations.

First, we see for an assumed H^2-solution $u(t)$,

$$E(t) + \int_0^t \int_{R^N} \rho(u_t(s))u_t(s)dxds = E(0),$$

where

$$E(t) = \frac{1}{2}[\|u_t(t)\|^2 + (1 + \|\nabla u(t)\|^2)\|\nabla u(t)\|^2]$$

and hence

$$k_0 \int_0^\infty \|u_t(s)\|^2 ds \le E(0) < \infty. \tag{1.4}$$

Next, differentiating the equation we have

$$u_{ttt} - (1 + \|\nabla u\|^2)\Delta u_t - 2(\nabla u, \nabla u_t)\Delta u + \rho'(u_t)u_{tt} = 0,$$

which is rewritten as

$$U_{tt} - (1 + \|\nabla u\|^2)\Delta U + \rho'(u_t)U_t = 2(\nabla u, \nabla u_t)\Delta u$$

with $U = u_t$. Since $\rho'(u_t)U_t$ is linear in U_t, we can expect the decay estimate

$$\|U_t(t)\|^2 + \|\nabla U(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|)(1+t)^{-1} \tag{1.5}$$

if (u_0, u_1) is small. This estimate is weaker than the case of linear dissipation, but combining this with (1.4) we have a hope to get desired H_2-solutions.

We notice that if the problem is considered in a bounded domain Ω with the boundary condition $u|_{\partial\Omega} = 0$, it is easy to derive exponential decay

$$E(t) \le C(E(0))e^{-\lambda t}, \quad \lambda > 0,$$

and the global existence of H_2- solution is easily proved. In fact, more general problem have been treated by many authors (Lasiecka and Ong [2], T.

Mizumachi[4], Ono [7,8], Matsuyama and Ikehata [3], Nishihara and Yamada [6] etc.). But the problem in the whole space R^N is more delicate because of the lack of Poincaré inequality. When $\rho(u_t) \equiv 0$, there are proved deep results on global existence and scattering by Yamazaki [10], where Fourier integral method is employed.

2 Statement of result

We use only familiar function spaces and the definitions of them are omitted. We make the following hypothesis on $\rho(v)$ which is a little more general than (1.3) .

Hyp.A $\rho(v)$ is a differentiable function on R and satisfies

$$0 < k_0 \le \rho'(v) \le k_1(1+|v|^\alpha) \quad \text{and} \quad \rho(0) = 0, \quad \text{with some} \quad k_0, k_1 > 0, \tag{2.1}$$

where $0 \le \alpha \le 2/(N-2)^+ \quad (0 \le \alpha < \infty$ if $N = 1, 2)$.

We begin with a local existence theorem.

Theorem 1 *Let $(u_0, u_1) \in H_2 \times H_1$. Then there exists $T > 0$ such that the problem (1.1)-(1.2) admits a unique solution $u(t)$ in the class $X_2(T) \equiv W^{2,\infty}([0,T]; L^2) \cap W^{1,\infty}([0,T]; H_1) \cap L^\infty([0,T]; H^2)$.*

Note that $X_2(T) \subset C^1_w([0,T]; H_1) \cap C_w([0,T]; H_2)$ and hence returning to the equation, we see that the solution $u(t)$ belongs to $C^2_w([0,T]; L_2)$. Therefore $X_2(T) = C^2_w([0,T]; L_2) \cap C^1_w([0,T]; H_1) \cap C_w([0,T]; H_2)$. When $\rho(u_t)$ is linear, the local existence in $C^2([0,T]; L_2) \cap C^1([0,T]; H_1) \cap C^2([0,T); H_2)$ is proved by a contraction method (cf. Yamada [9], see also Mochizuki [5]). When $\rho(v)$ is nonlinear the argument is a little more delicate. However, by use of the space $X_2(K,T)$ defined by

$$X_2(K,T) = \{u \in X_2(T) | \sup_{0 \le t \le T} (\|u_{tt}(t)\| + \|\nabla u_t(t)\| + \|\Delta u(t)\|) \le K$$
$$\text{and} \quad u(0) = u_0, u_t(0) = u_1\}$$

and introducing the weaker metric in $X_2(K,T)$

$$d(u,v) = \sup_{0 \le t \le T} (\|u_t(t) - v_t(t)\| + \|\nabla(u(t) - v(t))\|)$$

we can prove Theorem 1. (Cf. Arosio and Garavaldi [1], Ono [6].)

To state our main result we introduce the space $X_{2,*}$ which the solutions belong to:

$$X_{2,*} = \{u \in L^\infty_{loc}([0,\infty); L^2) | D_t^i u(t), D_x^i u(t) \in L^\infty([0,\infty); L^2), i = 1,2\},$$

where D_t^i and D_x^i denote the differentiation(s) of order i with respect to t and $x_j, j = 1, \cdots, N$, respectively. In other words u belongs to $X_{2,*}$ if and only if all of the derivatives of u up to the second order except for u itself belongs to $L^\infty([0,\infty); L^2)$.

Our main result reads as follows.

Theorem 2 *Suppose* **Hyp.A**. *Then for each $K > 0$, there exists an open set $S_K \subset H_2 \times H_1$ including (0,0) such that if $(u_0, u_1) \in S_K$, the problem (1.1)-(1.2) admits a unique solution $u \in X_{2,*}$, satisfying*

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le K^2(1+t)^{-1}.$$

Remark The precise definition of S_K will be given in the course of proof. Roughly speaking, if $\|u_0\|_{H_2} + \|u_1\|_{H_1} < K$ and $\|u_0\|_{H_1} + \|u_1\|$ is sufficiently small, then $(u_0, u_1) \in S_K$.

3 Proof of Theorem 2

Relying on the local existence theorem it suffices for the global existence to derive the a priori estimates

$$\|u_{tt}(t)\| + \|\nabla u_t(t)\| + \|\Delta u(t)\| \le M < \infty, \ 0 \le t < T$$

and

$$\|u_t(t)\| + \|\nabla u(t)\| \le M < \infty, \ 0 \le t < T$$

for some $M > 0$ independent of T. (Note that

$$\|u(t)\| \le \|u_0\| + \int_0^T \|u_t(t)\| dt \le \|u_0\| + MT \quad \text{for any} \quad T > 0).$$

We set

$$E(t) = \frac{1}{2}[\|u_t(t)\|^2 + (1 + \|\nabla u(t)\|^2))\|\nabla u(t)\|^2]$$

and

$$E_1(t) = \frac{1}{2}[\|u_{tt}(t)\|^2 + (1 + \|\nabla u(t)\|^2))\|\nabla u_t(t)\|^2 + 2(\nabla u(t), \nabla u_t(t))^2].$$

The following is standard.

Proposition 1 *Let $u(t)$ be a local solution in $X_2(T)$. Then*

$$E(t) + k_0 \int_0^T \|u_t(s)\|^2 ds \le E(0). \tag{3.1}$$

Differentiating the equation (1.1) in t and setting $U = u_t$, we get

$$U_{tt} - (1 + \|\nabla u\|^2)\Delta U - 2(\nabla u, \nabla u_t)\Delta u + \rho'(u_t)U_t = 0. \tag{3.2}$$

Multiplying (3.2) by U_t and using **Hyp.A**, we obtain

$$\begin{aligned}\frac{d}{dt}E_1(t) + k_0\|U_t(t)\|^2 &\le 3|(\nabla u(t), \nabla u_t(t))|\|\nabla U(t)\|^2 \\ &\le 3\|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2. \end{aligned}\tag{3.3}$$

Next, multiplying (3.2) by U, we have

$$\frac{d}{dt}(U_t, U) - \|U_t\|^2 + (1+\|\nabla u\|^2)|\|\nabla U\|^2 + 2(\nabla u, \nabla u_t)^2 + \int_{R^N} \rho' U U_t dx = 0. \tag{3.4}$$

Here we note that

$$\int_{R^N} \rho' U U_t dx = \frac{d}{dt}\int_{R^N}\int_0^{U(x,t)} \rho'(\eta)\eta d\eta dx.$$

Then combining (3.3) and (3.4), we obtain the following.

Proposition 2 *Let $\lambda > \frac{1}{k_0}$. Then*

$$\begin{aligned}\frac{d}{dt}X(t) + (k_0\lambda - 1)\|U_t(t)\|^2 + (1 + \|\nabla u(t)\|^2)\|\nabla U(t)\|^2 + 2(\nabla u(t), \nabla u_t(t))^2 \\ \le 3\lambda\|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2, \end{aligned}\tag{3.5}$$

where

$$X(t) = \lambda E_1(t) + (U_t(t), U(t)) + \int_{R^N}\int_0^U \rho'(\eta)\eta d\eta dx. \tag{3.6}$$

Taking λ large we see that

$$\begin{aligned}\bar{k_0}(\|U_t(t)\|^2 + \|\nabla U(t)\|^2 + \|U(t)\|^2) \le X(t) \\ \le \bar{k_1}(1 + \|\nabla u(t)\|^2)(\|U_t(t)\|^2 + \|\nabla U(t)\|^2 + \|U(t)\|^2 + \|U(t)\|_{\alpha+2}^{\alpha+2}) \end{aligned}\tag{3.7}$$

with some $\bar{k_0}$, $\bar{k_1} > 0$.

To treat the right hand side of (3.5) we assume for a moment

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le K^2(1+t)^{-1}, \ 0 \le t < T. \tag{3.8}$$

Proposition 3 *Under the assumption (3.8), we have*

$$\int_0^T \|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2 dt \le Cq(K, E(0)), \tag{3.9}$$

$$\int_0^T \|U_t(t)\|^2 dt \le Cq(K, E(0)) + E_1(0) \tag{3.10}$$

and

$$\sup_{0<t<T} X(t) + C_1 \int_0^T E_1(t) dt \le Cq(K, E(0)) + X(0), \tag{3.11}$$

where we set

$$q(K, E(0)) \equiv K^2(K\sqrt{E(0)} + E(0) + K^{N\alpha}E(0)^{1-(N-2)\alpha/4}).$$

Proof We first note that

$$\|\Delta u(t)\| \le \|(1 + \|\nabla u(t)\|^2)\Delta u(t)\|$$

$$= \|u_{tt}(t) + \rho(u_t(t))\| \le \|u_{tt}(t)\| + k_1(\|u_t(t)\| + \|u_t(t)\|_{2(\alpha+1)}^{\alpha+1}).$$

Then

$$\int_0^T \|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2$$

$$\le \int_0^T \|u_{tt}(t)\|\|u_t(t)\|\|\nabla U(t)\|^2 + k_1 \int_0^T \|u_t(t)\|^2\|\nabla U(t)\|^2$$

$$+k_1 \int_0^T \|u_t(t)\|_{2(\alpha+1)}^{\alpha+1}\|u_t(t)\|\|\nabla U(t)\|^2 \equiv I_1 + I_2 + I_3. \tag{3.12}$$

Here, by use of the assumption (3.8),

$$I_1 \le K^3 \int_0^T (1+t)^{-\frac{3}{2}}\|u_t(t)\| dt$$

$$\le \frac{1}{2}K^3(\int_0^T \|u_t(t)\|^2 dt)^{\frac{1}{2}} \le CK^3\sqrt{E(0)}, \tag{3.13}$$

where we have used (3.1).

I_2 is estimated easily as follows;

$$I_2 \le CK^2 \int_0^T \|u_t(t)\|^2 dt \le CK^2 E(0). \tag{3.14}$$

To estimate I_3 we note that

$$\|u_t\|_{2(\alpha+1)} \le C\|u_t\|^{1-\theta}\|\nabla u_t\|^{\theta}$$

with $\theta = N\alpha/2(\alpha+1)$. Then,

$$I_3 \le CK^{N\alpha+2}\int_0^T (1+t)^{-1-N\alpha/4}\|u_t(t)\|^{2+(2-N)\alpha/2}dt \le CK^{N\alpha+2}E(0)^{1-(N-2)\alpha/4}. \tag{3.15}$$

The estimate (3.9) follow from (3.12) ,(3.13) and (3.15). Once (3.9) is proved, (3.10) and (3.11) follow immediately from (3.3) and (3.5), respectively.

On the basis of (3.10) and (3.11) we can derive the decay of $E_1(t)$, which is the final step of a priori estimates.

Proposition 4 *We have under (3.8)*

$$E_1(t) \le C(q(K,E(0)) + E_1(0))(1+t)^{-1}. \tag{3.16}$$

Proof By (3.3) we have

$$\frac{d}{dt}[(1+t)E_1(t)] = E_1(t) + (1+t)\frac{d}{dt}E_1(t)$$

$$\le E_1(t) + C(1+t)\|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2. \tag{3.17}$$

The integrability of $E_1(t)$ is already proved (see (3.11)). Further as in (3.11),

$$\int_0^T (1+t)\|\Delta u(t)\|\|u_t(t)\|\|\nabla U(t)\|^2 dt$$

$$\le \int_0^T (1+t)\|u_{tt}(t)\|\|u_t(t)\|\|\nabla U(t)\|^2 dt$$

$$+C\int_0^T (1+t)\|u_t(t)\|^2\|\nabla U(t)\|^2 dt + C\int_0^T (1+t)\|u_t\|_{2(\alpha+1)}^{\alpha+1}\|u_t(t)\|\|\nabla U(t)\|^2 dt$$

$$\equiv \bar{J}_1 + \bar{J}_2 + \bar{J}_3. \tag{3.18}$$

Here, by (3.10) and (3.1),

$$\bar{J}_1 \le K^2\int_0^T \|u_{tt}(t)\|\|u_t(t)\|dt$$

$$\le K^2\left(\int_0^T \|u_{tt}(t)\|^2 dt\right)^{\frac{1}{2}}\left(\int_0^T \|u_t(t)\|^2 dt\right)^{\frac{1}{2}} \le CK^2\sqrt{q(K,E(0)) + E_1(0)}\sqrt{E(0)}$$

$$\le CK^2(E(0) + q(K,E(0)) + E_1(0)). \tag{3.19}$$

and by (3.8) and (3.1)

$$\bar{J}_2 \le CK^2\int_0^T (1+t)(1+t)^{-1}\|u_t(t)\|^2 dt$$

$$\leq CK^2E(0) \tag{3.20}$$

and similarly, if $N \geq 3$, we have (see (3.15))

$$\bar{J}_3 \leq CK^{N\alpha+2}\int(1+t)^{-N\alpha/4}\|u_t\|^{2+(2-N)\alpha/2}dt$$

$$\leq CK^{N\alpha+2}(\int_0^T(1+t)^{-N/(N-2)}dt)^{(N-2)\alpha/4}(\int_0^T\|u_t\|^2dt)^{1-(N-2)\alpha/4}$$

$$\leq CK^{N\alpha+2}E(0)^{1+(2-N)\alpha/4}. \tag{3.21}$$

When $N = 1, 2$, (3.21) is proved more easily.

Thus, integrating (3.17) we obtain (3.16), where we note that $X(0) \geq CE_1(0)$.

Completion of the proof of Theorem 2

Set

$$Q(K, E(0), X(0)) = CK^2(E(0) + K^{N\alpha}E(0)^{1-(N-2)\alpha/4} + K\sqrt{E(0)}) + CX(0)$$

and

$$S_K \equiv \{(u_0, u_1) \in H_2 \cap H_1 \mid Q(K, E(0), X(0)) < K^2\}.$$

Since

$$E_1(0) = \frac{1}{2}\{\|u_{tt}(0)\|^2 + (1+\|\nabla u_0\|^2)\|\nabla u_1\|^2 + 2(\nabla u_0, \nabla u_1)^2\}$$

$$\leq (1+\|\nabla u_0\|^2)^2\|\Delta u_0\|^2 + k_1^2\|u_1\|^2 + \frac{1}{2}(1+\|\nabla u_0\|^2)\|\nabla u_1\|^2 + (\nabla u_0, \nabla u_1)^2$$

and

$$X(0) = \alpha E_1(0) + (u_{tt}(0), u_1) + \int_{R^N}\int_0^{u_1}\rho'(\eta)\eta d\eta dx \leq C(E_1(0) + \|u_1\|^2 + \|u_1\|^{\alpha+2})$$

we see that if $K > C(\|u_0\|_{H_2} + \|u_1\|_{H_1} + \|u_1\|_{H_1}^{\alpha+2})$ and $\|u_0\|_{H_1} + \|u_1\|$ is sufficiently small, then $(u_0, u_1) \in S_K$. Now we assume that $(u_0, u_1) \in S_K$. Then

$$E(t) \leq E(0) < \infty,$$

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \leq Q(1+t)^{-1} < K^2(1+t)^{-1}$$

and

$$\|\Delta u(t)\| \leq \|u_{tt}(t)\| + \|\rho(u_t(t))\| \leq \sqrt{Q} + k_1(\sqrt{E(0)} + \|u_1\|_{2(\alpha+1)}^{\alpha+1}) \quad \text{for} \quad 0 \leq t \leq T.$$

These estimates show that the assumption (3.8) is true as long as this solution exists and the local solution $u(t)$ can be continued, in fact, on the half interval $[0, \infty)$.

Acknowledgement

The authors would like to thank the referee for careful reading of the first version of the manuscript and giving useful comments.

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