The L^2-boundedness of Pseudodifferential Operators with Simple Symbols

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1. Introduction

In this paper we improve a sufficient condition for the L^2-boundedness of pseudodifferential operators with simple symbols $p(x,\xi)$ of $S^0_{1,0}$-type. We want to weaken the smoothness assumption with respect to ξ as much as possible under some continuity in x.

Concerning the known results and the references in this direction, we refer to the paper [5] of Yamazaki, who treated the boundedness of product-type pseudodifferential operators with simple or double symbols in the weighted $L^p(\mathbb{R}^n)$-space with a multiple modulus of growth and continuity. In this paper we discuss only the boundedness of the (non-product-type) pseudodifferential operators $p(X,D)$ with simple symbols $p(x,\xi)$ in the (unweighted) $L^2(\mathbb{R}^n)$ with a modulus of continuity. In this case the best result was first obtained by Muramatu and Nagase [4]. It was generalized by Yamazaki [5] to the case of product-type operators and so on. Roughly speaking, the sufficient condition obtained in [4] or [5] means that $p(x,\xi)$ has the continuity in x expressed by a modulus of continuity $\omega(t)$ and that $p(x,\xi)$ belongs to the $C^{n/2+\varepsilon}$-class in ξ for any $\varepsilon > 0$ and that the derivatives $\partial^\alpha_\xi p(x,\xi)$ satisfy some estimates.

The purpose of this paper is to show that the smoothness condition with respect to ξ can be relaxed. Our condition is expressed by a function $t^{n/2}\psi(t)$ defined on $(0,\infty)$, where $\psi(t)$ tends to 0 as $t \to 0$ more slowly than t^ε for any $\varepsilon > 0$. In our terminology the smoothness of the $C^{n/2+\varepsilon}$-class function corresponds to $t^{n/2+\varepsilon}$. Hence the smoothness corresponding to $t^{n/2}\psi(t)$ is weaker than that of the $C^{n/2+\varepsilon}$-class function.

The method to prove our result is similar to that Yamazaki [5] employed. First we decompose the symbol $p(x,\xi)$ into the sum of the functions $p_k(x,\xi)$ with compact support in ξ-space and further decompose $p_k(x,\xi)$ into the

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sum of a regularized part $p_{k,0}(x,\xi)$ and the remainder part $p_{k,1}(x,\xi)$ so that the Fourier transform of the function associated with $p_{k,0}(X,D)$ has compact support. Then we estimate the kernels of the pseudodifferential operators $p_{k,a}(X,D)$ $(a=0,1)$, which is related to the inverse Fourier transform of $p_{k,a}(x,\xi)$ with respect to ξ. Finally we apply the lemmas concerning the Littlewood-Paley decomposition.

There are two keys in the proof of our result. One key is the Sobolev space with function parameter due to Muarmatu [3], which enables us to express the smoothness condition on the symbol with respect to ξ by a function $t^{n/2}\psi(t)$. Another key is no use of the lemma concerning the strong maximal function which Yamazaki used in order to treat the L^p case. Instead of this lemma we use Fubini's theorem which is applicable only in the L^2 case.

The outline of this paper is as follows. In Sectin 2 we state the main theorem after introducing the weight function. In Section 3 we restate the smoothness condition in terms of difference operators. In Section 4 we decompose the symbol and estimate the integral kernel of the pseudodifferential operator with help of the Sobolev space with function parameter. Finally, in Section 5, we complete the proof of the main theorem.

2. Statement of the main theorem

Let $\mathbb{N}$ and $\mathbb{R}_+$ denote the set of non-negative integers and that of non-negative real numbers respectively. We often abbrebiate $L^2(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}$ to L^2 and $\int$ respectively. The Fourier transform is defined by $\mathcal{F}u(\xi) = \widehat{u}(\xi) = \int \exp(-ix\xi)u(x)\,dx$, and the inverse Fourier transform is defined by $\mathcal{F}^{-1}u(x) = c_n \int \exp(ix\xi)u(\xi)\,d\xi$, where $c_n = (2\pi)^{-n}$. We set $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

The pseudodifferential operator with simple symbol $p(x,\xi)$ is given by

$$p(X,D)u(x) = c_n \int e^{ix\xi} p(x,\xi)\widehat{u}(\xi)\,d\xi = c_n \iint e^{i(x-y)\xi} p(x,\xi)u(y)\,dy\,d\xi,$$

where the last integral is interpreted as an oscillatory integral. We define difference operators Δ_z and $\Delta_{2,\zeta}$ by

$$\Delta_z p(x,\xi) = p(x+z,\xi) - p(x,\xi),$$
$$\Delta_{2,\zeta} p(x,\xi) = p(x,\xi+\zeta) - p(x,\xi).$$

We regard Δ_z^0 and $\Delta_{2,\zeta}^0$ as the identity operators.

In our theorem the condition on the smoothness of $p(x,\xi)$ with respect to ξ is expressed by the weight function, which we introduce here briefly by following Muramatu [3]. We say that φ is a weight function on $(0,\infty)$ if φ is a positive-valued measurable function on $(0,\infty)$. We define the similarity ratio function $\widetilde{\varphi}$ of φ by

$$\widetilde{\varphi}(t) = \operatorname*{ess.\,sup}_{s>0} \frac{\varphi(ts)}{\varphi(s)}.$$

Throughout this paper we assume that $\widetilde{\varphi}(t)$ is finite for any $t>0$. Then the following two limits, called the left index and the right index respectively, exist and the inequalities

$$-\infty < \lim_{t\to 0} \frac{\log \widetilde{\varphi}(t)}{\log t} \le \lim_{t\to\infty} \frac{\log \widetilde{\varphi}(t)}{\log t} < \infty$$

hold. Furthermore, we assume that the left and the right indexes of φ coincide, and we denote them by $\operatorname{ind}\varphi$:

$$\operatorname{ind}\varphi = \lim_{t\to 0} \frac{\log \widetilde{\varphi}(t)}{\log t} = \lim_{t\to\infty} \frac{\log \widetilde{\varphi}(t)}{\log t}.$$

The following lemma will be often used.

Lemma 2.1. *Let φ be a weight function on $(0,\infty)$.*

(i) For any $s>0$ and any $t>0$ we have

$$\widetilde{\varphi}(t^{-1})^{-1} \le \frac{\varphi(ts)}{\varphi(s)} \le \widetilde{\varphi}(t).$$

(ii) Let $\alpha = ind\varphi$. For any $\varepsilon>0$ there exists $C>0$ such that

$$\begin{aligned} C^{-1}t^{\alpha+\varepsilon} &\le \varphi(t) \le Ct^{\alpha-\varepsilon} \quad \text{when } 0<t\le 1, \\ C^{-1}t^{\alpha-\varepsilon} &\le \varphi(t) \le Ct^{\alpha+\varepsilon} \quad \text{when } 1\le t. \end{aligned}$$

The above inequalities also hold if we replace φ with $\widetilde{\varphi}$.

Now we introduce the condition $(\mathrm{H}_{\omega,\psi})$ on the symbol $p(x,\xi)$, where $\omega(t)$ is an increasing and concave function defined on $\mathbb{R}_+$, and $\psi(t)$ is a weight function on $(0,\infty)$. $\omega(t)$ is called a modulus of continuity. Choose $m\in\mathbb{N}$ so that $m-1\le n/2<m$.

We say that $p(x,\xi)$ satisfies the condition $(\mathrm{H}_{\omega,\psi})$ if we have

$$|\Delta_{x'}^{a}\partial_\xi^\alpha p(x,\xi)| \le C\omega(|x'|)^a\langle\xi\rangle^{-|\alpha|} \tag{2.1}$$

for $a\in\{0,1\}$, $|\alpha|\le m-1$, and

$$|\Delta_{x'}^{a}\Delta_{2,\zeta}\partial_\xi^\alpha p(x,\xi)| \le C\omega(|x'|)^a|\zeta|^{n/2-m+1}\langle\xi\rangle^{-n/2}\psi(|\zeta|/\langle\xi\rangle) \tag{2.2}$$

for $a\in\{0,1\}$, $|\alpha|=m-1$, $|\zeta|\le\langle\xi\rangle/2$.

Main Theorem. *Assume that $p(x,\xi)$ satisfies the condition $(H_{\omega,\psi})$ with $\int_0^1 \omega(t)^2\, t^{-1}dt < \infty$, $0 \le ind\,\psi < m - n/2$ (≤ 1) and $\int_0^1 \psi(t)\, t^{-1}\, dt < \infty$. Then the pseudodifferential operator $p(X,D)$ is bounded on $L^2(\mathbb{R}^n)$.*

If we set $\psi(t) = t^\varepsilon$ for any $\varepsilon \in (0, m - n/2)$, then the assumption in the main theorem is fulfilled since $\widetilde{\psi}(t) = t^\varepsilon$ and $\operatorname{ind}\psi = \varepsilon$. This case corresponds to the known result obtained by Muramatu and Nagase [4].

The main theorem is essentially a new result when $\operatorname{ind}\psi = 0$ in view of Lemma 2.1. For example, let us set

$$\psi(t) = (1 + |\log t|)^{-1-\varepsilon}$$

for any $\varepsilon > 0$. In this case we have $\widetilde{\psi}(t) = (1 + |\log t|)^{1+\varepsilon}$, $\operatorname{ind}\psi = 0$ and $\int_0^1 \psi(t)\, t^{-1}dt = 1/\varepsilon$. The smoothness of the symbol $p(x,\xi)$ is weaker than that of the $C^{n/2+\varepsilon}$-class function for any $\varepsilon > 0$.

3. Smoothness of the symbol

For the proof of the main theorem it is convenient to express the smoothness of the symbol in terms of difference operators.

Lemma 3.1. *Under the assumption in the main theorem the condition $(H_{\omega,\psi})$ implies the estimates*

$$|\Delta^a_{x'} \Delta^j_{2,\zeta} p(x,\xi)| \le C\omega(|x'|)^a (|\zeta|/\langle\xi\rangle)^j \tag{3.1}$$

for $a \in \{0,1\}$, $0 \le j \le m-1$, $|\zeta| \le \langle\xi\rangle/2$, and

$$|\Delta^a_{x'} \Delta^m_{2,\zeta} p(x,\xi)| \le C\omega(|x'|)^a (|\zeta|/\langle\xi\rangle)^{n/2} \psi(|\zeta|/\langle\xi\rangle) \tag{3.2}$$

for $a \in \{0,1\}$, $|\zeta| \le \langle\xi\rangle/2$.

The proof of Lemma 3.1 is reduced to the following lemma with $\theta = n/2 - m + 1$.

Lemma 3.2. *Let f be a function defined on $\mathbb{R}^n$. Let $m \ge 1$ be an integer and $0 \le \theta < 1$ with $(m-1,\theta) \ne (0,0)$. Let ψ be a weight function on $(0,\infty)$ satisfying $0 \le ind\,\psi < 1 - \theta$ and $\int_0^1 \psi(t)\, t^{-1}\, dt < \infty$.*

Then the estimates

$$|\partial^\alpha f(x)| \le C\langle x\rangle^{-|\alpha|} \tag{3.3}$$

for $|\alpha| \leq m-1$, *and*

$$|\Delta_y \partial^\alpha f(x)| \leq C\langle x\rangle^{1-m}(|y|/\langle x\rangle)^\theta \psi(|y|/\langle x\rangle) \tag{3.4}$$

for $|\alpha| = m-1$, $|y| \leq \langle x\rangle/2$ *imply the estimates*

$$|\Delta_y^j f(x)| \leq C(|y|/\langle x\rangle)^j \tag{3.5}$$

for $0 \leq j \leq m-1$, $|y| \leq \langle x\rangle/2$, *and*

$$|\Delta_y^m f(x)| \leq C(|y|/\langle x\rangle)^{m+\theta-1}\psi(|y|/\langle x\rangle) \tag{3.6}$$

for $|y| \leq \langle x\rangle/2$.

In order to prove Lemma 3.2, we employ the integral representation by Muramatu. We introduce it briefly. The details are found in [2], [3].

For $j \in \mathbb{N}$ let $\mathcal{K}_j$ be the set of functions $K(x)$ which can be written

$$K(x) = \sum_{|\alpha|=j} \partial^\alpha(K_\alpha(x))$$

with some $K_\alpha \in C_0^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp} K_\alpha \subset \{x \in \mathbb{R}^n : |x| < 1\}$. For $t > 0$ and a function $K(x)$ on $\mathbb{R}^n$ we set $K_t(x) = t^{-n}K(x/t)$. We also use the convention $K_{\alpha,t}(x) = t^{-n}K_\alpha(x/t)$ for a function $K_\alpha(x)$ with subscript α.

Lemma 3.3. *Let $f(x)$ be a bounded continuous function on $\mathbb{R}^n$. For any $a > 0$ and any integer $m > 0$ there exist $K \in \mathcal{K}_0$ with $\int K(x)\,dx = 1$ and $M \in \mathcal{K}_m$ such that*

$$f(x) = K_a * f(x) + \int_0^a M_t * f(x)\,d_*t \tag{3.7}$$

for $a > 0$, *where* $d_*t = t^{-1}dt$.

The proof of Lemma 3.3 is given by choosing $k \in C_0^\infty(\mathbb{R}^n)$ with $\int k(x)\,dx = 1$, and setting

$$K(x) = \sum_{|\alpha|<m} \frac{1}{\alpha!}\partial^\alpha\{x^\alpha k(x)\}, \qquad M(x) = \sum_{|\alpha|=m} \frac{m}{\alpha!}\partial^\alpha\{x^\alpha k(x)\}.$$

By integration by parts it is seen that for $M \in \mathcal{K}_m$ and $0 \leq j \leq m$, there exist functions $M_\alpha \in \mathcal{K}_{m-j}$ such that

$$M_t * f(x) = t^j \sum_{|\alpha|=j} M_{\alpha,t} * \partial^\alpha f(x) \tag{3.8}$$

holds.

Lemma 3.4. *Let $j > 0$ be an integer and $\rho \in \mathcal{K}_j$. There exists $W \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ with*

$$\operatorname{supp} W \subset \{(x,y) : |x| < 1, |y| < 1\}$$

such that

$$\rho_t * f(x) = \iint t^{-2n} W\left(\frac{x-y}{t}, \frac{x-y-z}{t}\right) \Delta_z^j f(y)\, dy\, dz.$$

We are now ready to prove Lemma 3.2. It is sufficient to show (3.5) and (3.6) when $|y| \le \langle x\rangle/4m$, since (3.5) and (3.6) for $\langle x\rangle/4m \le |y| \le \langle x\rangle/2$ can be derived from (3.3) with $|\alpha| = 0$.

First, we will show (3.6). We begin with the integral representation (3.7). Using (3.8) and Lemma 3.4, we have

$$\begin{aligned} M_t * f(x) &= t^{m-1} \sum_{|\alpha|=m-1} M_{\alpha,t} * \partial^\alpha f(x) \\ &= t^{m-1} \sum_{|\alpha|=m-1} \iint t^{-2n} W_\alpha\Big(\frac{x-y}{t}, \frac{x-y-z}{t}\Big) \Delta_z \partial^\alpha f(y)\, dy\, dz \end{aligned} \tag{3.9}$$

with some $M_\alpha \in \mathcal{K}_1$ and some $W_\alpha \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ whose supports are contained in $\{(x,y); |x| < 1, |y| < 1\}$. Let $\eta(t) = \min\{t, 1\}$. It is easily seen that

$$\left| \Delta_w^j W_\alpha \left(\frac{x-y}{t}, \frac{x-y-z}{t}\right) \right| \le C\eta\left(\frac{|w|}{t}\right)^j \tag{3.10}$$

for $0 \le j \le m$, where Δ_w^j is regarded as an operator to the function of x. The support of the function of the left side in (3.10) as a function of (y,z) is contained in

$$\Omega(j,w,t) = \bigcup_{i=0}^{j} \{(y,z);\, |x+iw-y| \le t,\, |x+iw-y-z| \le t\}.$$

In the following we assume that $a = \langle x\rangle/4$, $0 < t \le a$ and $|w| \le \langle x\rangle/4m$. Then if $(y,z) \in \Omega(j,w,t)$, we have

$$\frac{1}{2}\langle x\rangle \le \langle y\rangle \le \frac{3}{2}\langle x\rangle.$$

Using (3.4), (3.9) and (3.10), and noting

$$\psi\Big(\frac{|z|}{\langle y\rangle}\Big) \le \psi\Big(\frac{|z|}{\langle x\rangle}\Big)\widetilde{\psi}\Big(\frac{\langle x\rangle}{\langle y\rangle}\Big) \le \sup_{2/3 \le t \le 2} \widetilde{\psi}(t) \cdot \psi\Big(\frac{|z|}{\langle x\rangle}\Big),$$

we have

$$
\begin{aligned}
&|\Delta_w^j M_t * f(x)| \\
&\le t^{m-1} \iint_{\Omega(j,w,t)} t^{-2n} \eta\Big(\frac{|w|}{t}\Big)^j |z|^\theta \langle y\rangle^{-m-\theta+1} \psi\Big(\frac{|z|}{\langle y\rangle}\Big)\, dy\, dz \\
&\le C t^{m-2n-1} \langle x\rangle^{-m-\theta+1} \eta\left(\frac{|w|}{t}\right)^j \int_{|y|\le t} dy \int_{|y+z|\le t} |z|^\theta \psi\left(\frac{|z|}{\langle x\rangle}\right) dz \\
&\le C\eta\left(\frac{|w|}{t}\right)^j \left(\frac{t}{\langle x\rangle}\right)^{m+\theta-1} \psi\left(\frac{t}{\langle x\rangle}\right) \int_{|y|\le 1} dy \int_{|y+z|\le 1} |z|^\theta \widetilde{\psi}(|z|)\, dz \\
&\le C\eta\left(\frac{|w|}{t}\right)^j \left(\frac{t}{\langle x\rangle}\right)^{m+\theta-1} \psi\left(\frac{t}{\langle x\rangle}\right)
\end{aligned}
\tag{3.11}
$$

for $0 \le j \le m$, which gives

$$
\begin{aligned}
&\left|\Delta_w^m \int_0^a M_t * f(x)\, d_*t\right| \\
&\qquad \le C \int_0^a \eta\Big(\frac{|w|}{t}\Big)^m \Big(\frac{t}{\langle x\rangle}\Big)^{m+\theta-1} \psi\left(\frac{t}{\langle x\rangle}\right) d_*t \\
&\qquad \le C \int_0^{a/|w|} \eta\Big(\frac{1}{t}\Big)^m \Big(\frac{t|w|}{\langle x\rangle}\Big)^{m+\theta-1} \psi\Big(\frac{t|w|}{\langle x\rangle}\Big)\, d_*t \\
&\qquad \le C\Big(\frac{|w|}{\langle x\rangle}\Big)^{m+\theta-1} \psi\Big(\frac{|w|}{\langle x\rangle}\Big) \int_0^{a/|w|} \eta\Big(\frac{1}{t}\Big)^m t^{m+\theta-1} \widetilde{\psi}(t)\, d_*t \\
&\qquad \le C\Big(\frac{|w|}{\langle x\rangle}\Big)^{m+\theta-1} \psi\Big(\frac{|w|}{\langle x\rangle}\Big),
\end{aligned}
\tag{3.12}
$$

where the last integral is finite due to Lemma 2.1.

We also have, for $0 \le j \le m$,

$$
\begin{aligned}
\left|\Delta_w^j K_a * f(x)\right| &\le C\eta\left(\frac{|w|}{a}\right)^j \sum_{i=0}^j \int_{|x+iw-y|\le a} a^{-n} |f(y)|\, dy \\
&\le C\eta\left(\frac{|w|}{a}\right)^j \le C\left(\frac{|w|}{\langle x\rangle}\right)^j.
\end{aligned}
\tag{3.13}
$$

Combining (3.7), (3.12) and (3.13) with $j = m$, we get (3.6) for $|y| \le \langle x\rangle/4m$.

Next, we will show (3.5) for $0 \le j \le m-1$. From (3.11) we have

$$\begin{aligned}\left|\Delta_w^j \int_0^a M_t * f(x)\, d_*t\right| &\le C \int_0^a \eta\Big(\frac{|w|}{t}\Big)^j \Big(\frac{t}{\langle x\rangle}\Big)^{m+\theta-1} \psi\Big(\frac{t}{\langle x\rangle}\Big)\, d_*t \\ &\le C \int_0^{1/4} \eta\Big(\frac{|w|}{t\langle x\rangle}\Big)^j t^{m+\theta-1}\psi(t)\, d_*t \\ &\le C\Big(\frac{|w|}{\langle x\rangle}\Big)^j \int_0^{1/4} t^{m-j+\theta-1}\psi(t)\, d_*t \\ &\le C\Big(\frac{|w|}{\langle x\rangle}\Big)^j \int_0^{1/4} \psi(t)\, d_*t \le C\Big(\frac{|w|}{\langle x\rangle}\Big)^j.\end{aligned}$$

This combined with (3.7) and (3.13) gives (3.5) for $|y| \le \langle x\rangle/4m$. We complete the proof of Lemma 3.2.

4. Estimates of integral kernels

In this section we decompose $p(x,\xi)$ and estimate the integral kernel of the associated pseudodifferential operator. To do so we prepare two lemmas.

Lemma 4.1. *Let $\omega(t)$ be as in the main theorem. Then*

$$\omega(\lambda x) \le \omega(\lambda)(1+x)$$

holds for $x > 0$ and $\lambda > 0$.

Proof. Since ω is increasing, we have $\omega(x) \le \omega(\lambda)$ when $0 \le x \le \lambda$. On the other hand, the concavity and $\omega(0) = 0$ give $\omega(x) \le \omega(\lambda)x/\lambda$ when $x \ge \lambda$. Combining these inequalities, we get $\omega(x) \le \omega(\lambda)(1 + x/\lambda)$ for $x > 0$, from which the lemma follows. □

We introduce the Sobolev space with function parameter (see [3]), and characterize its Fourier transform. Let φ be a weight function on $(0,\infty)$ satisfying $0 < \operatorname{ind}\varphi < m$. We define $H^\varphi(\mathbb{R}^n)$ as the set of functions $f \in L^2(\mathbb{R}^n)$ satisfying

$$|f|_{H^\varphi}^2 = \int \left(\frac{\|\Delta_y^m f\|_{L^2}}{\varphi(|y|)}\right)^2 \frac{dy}{|y|^n} < \infty,$$

and set

$$\|f\|_{H^\varphi}^2 = \|f\|_{L^2}^2 + |f|_{H^\varphi}^2.$$

Lemma 4.2. *For $f \in L^2(\mathbb{R}^n)$ we have*

$$f \in H^\varphi(\mathbb{R}^n) \Longleftrightarrow \varphi(|\xi|^{-1})^{-1}\mathcal{F}f(\xi) \in L^2(\mathbb{R}^n),$$
$$C^{-1}\|f\|_{H^\varphi} \le \|\{1+\varphi(|\xi|^{-1})^{-1}\}\mathcal{F}f(\xi)\|_{L^2} \le C\|f\|_{H^\varphi}.$$

Proof. See [1, Theorem 3.1]. □

We fix an infinitely differentiable function $\chi(t)$ defined on $\mathbb{R}_+$ which satisfies $0 \le \chi(t) \le 1$ $(t \in \mathbb{R}_+)$, $\chi(t) = 1$ $(0 \le t \le 1)$ and $\chi(t) = 0$ $(t \ge 4/3)$. We set $\Phi_k(\xi) = \chi(2^{-k}|\xi|) - \chi(2^{1-k}|\xi|)$ for $k \ge 1$, $\Phi_0(\xi) = \chi(|\xi|)$ and $\Phi_k(\xi) = 0$ for $k < 0$. Then we note that $\{\Phi_k(\xi)\}_{k\in\mathbb{N}}$ is a partition of unity, that is, $\sum_{k=0}^\infty \Phi_k(\xi) = 1$. For $a \in \{0, 1\}$ and $k \in \mathbb{N}$ we set

$$\varphi_{k,0}(x) = \mathcal{F}^{-1}[\chi(2^{-k+2}|\xi|)](x), \qquad \varphi_{k,1}(x) = \delta(x) - \mathcal{F}^{-1}[\chi(2^{-k+2}|\xi|)](x)$$

and

$$p_k(x, \xi) = p(x, \xi)\Phi_k(\xi), \qquad p_{k,a}(x, \xi) = \int \varphi_{k,a}(x')p_k(x - x', \xi)\, dx'.$$

Clearly, we have $p_{k,0}(x, \xi) + p_{k,1}(x, \xi) = p_k(x, \xi)$.

Lemma 4.3. *Suppose that $p(x, \xi)$ satisfies the condition in the main theorem. Then there exist a sequence of fucntions $\{G_k(z)\}_{k\in\mathbb{N}}$ defined on $\mathbb{R}^n$ and that of functions $\{H_k(x, z)\}_{k\in\mathbb{N}}$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ which satisfy* (i) $\sup\{\|G_k\|_{L^2(\mathbb{R}^n)};\, k \in \mathbb{N}\} < \infty$, (ii) $\sup\{\|H_k(x, \cdot)\|_{L^2(\mathbb{R}^n)};\, k \in \mathbb{N}, x \in \mathbb{R}^n\} < \infty$ *and* (iii) *the inequality*

$$|\mathcal{F}^{-1}[p_{k,a}(x, \cdot)](z)| \le C\omega_k^a G_k(z)H_k(x, z),$$

where $\omega_k = \omega(2^{-k})$ and C is independent of k, x and z. Moreover, $\{G_k\}_{k\in\mathbb{N}}$ depends only on $\psi(t)$.

Proof. Since $\int \varphi_{k,0}(x)\, dx = \chi(2^{-k+2}|0|) = 1$, we have

$$\begin{aligned} p_{k,1}(x, \xi) &= p_k(x, \xi) - \int \varphi_{k,0}(x')p_k(x - x', \xi)\, dx' \\ &= \int \varphi_{k,0}(x')\Delta_{x'}p_k(x - x', \xi)\, dx', \end{aligned}$$

and therefore

$$p_{k,a}(x, \xi) = \int \varphi_{k,0}(x')\Delta_{x'}^a p_k(x - x', \xi)\, dx' \tag{4.1}$$

for $a = 0$, 1. Since $\operatorname{supp}\Phi_k \subset \{\xi \in \mathbb{R}^n;\, 2^{k-1} \le |\xi| \le 2^{k+1}\}$ for $k \ge 1$ and $\operatorname{supp}\Phi_0 \subset \{\xi \in \mathbb{R}^n;\, |\xi| \le 2\}$, it is easily seen that $|\partial_\xi^\alpha \Phi_k(\xi)| \le C\langle\xi\rangle^{-|\alpha|}$ for $k \in \mathbb{N}$, and therefore that $|\Delta_{2,\zeta}^j \Phi_k(\xi)| \le C(|\zeta|/\langle\xi\rangle)^j$ when $|\zeta| \le \langle\xi\rangle/2m$. Use of the condition $(\mathrm{H}_{\omega,\varphi})$, Lemma 3.1 and the identity

$$\Delta_{2,\zeta}^m(p_k(x, \xi)) = \sum_{j=0}^m \binom{m}{j} \Delta_{2,\zeta}^j p(x, \xi) \cdot \tau_\zeta^j \Delta_{2,\zeta}^{m-j}\Phi_k(\xi),$$

where τ_ζ is defined by $\tau_\zeta \Phi(\xi) = \Phi(\xi+\zeta)$, gives

$$\begin{aligned}
&|\Delta^a_{1,x'}\Delta^m_{2,\zeta} p_k(x,\xi)| \\
&\quad \le C\omega(|x'|)^a \left\{ (|\zeta|/\langle\xi\rangle)^{n/2}\psi(|\zeta|/\langle\xi\rangle) + (|\zeta|/\langle\xi\rangle)^m \right\} \chi_k(\xi) \\
&\quad \le C\omega(|x'|)^a \psi_0(|\zeta|/\langle\xi\rangle)\chi_k(\xi)
\end{aligned} \tag{4.2}$$

when $|\zeta| \le \langle\xi\rangle/2m$, where $\psi_0(t) = t^{n/2}\psi(t)$, and $\chi_k(\xi)$ is the characteristic function of the set $\{\xi \in \mathbb{R}^n;\ 2^{k-3} \le |\xi| \le 2^{k+2}\}$ for $k \ge 1$ and $\{\xi \in \mathbb{R}^n;\ |\xi| \le 4\}$ for $k=0$. Noting $\varphi_{k,0}(x) = 2^{kn}\varphi_{0,0}(2^k x)$ for $k \in \mathbb{N}$ and applying Lemma 4.1, we have

$$\begin{aligned}
\int \varphi_{k,0}(x')\omega(|x'|)^a\,dx' &= \int \varphi_{0,0}(x')\omega(2^{-k}|x'|)^a\,dx' \\
&\le \int \varphi_{0,0}(x')\{\omega(2^{-k})(1+|x'|)\}^a\,dx' \le C\omega_k^a
\end{aligned} \tag{4.3}$$

for $k \in \mathbb{N}$.

Combining (4.1) - (4.3), we have

$$\begin{aligned}
\left|\Delta^m_{2,\zeta} p_{k,a}(x,\xi)\right| &\le C\int |\varphi_{k,0}(x')|\,\omega(|x'|)^a \psi_0(|\zeta|/\langle\xi\rangle)\chi_k(\xi)\,dx' \\
&\le C\omega_k^a\,\psi_0(|\zeta|/\langle\xi\rangle)\chi_k(\xi)
\end{aligned}$$

when $|\zeta| \le \langle\xi\rangle/2m$. Noting that $\widetilde{\psi}_0(s) = s^{n/2}\widetilde{\psi}(s)$ and that

$$\psi_0(|\zeta|/\langle\xi\rangle)/\psi_0(|\zeta|/2^k) \le \widetilde{\psi}_0(2^k/\langle\xi\rangle) \le \sup_{1/5\le s\le 3} \widetilde{\psi}_0(s)$$

for $\xi \in \operatorname{supp}\chi_k$, we get

$$\left\|\Delta^m_{2,\zeta} p_{k,a}(x,\,\cdot\,)\right\|_{L^2} \le C\omega_k^a 2^{kn/2}\psi_0(2^{-k}|\zeta|) \tag{4.4}$$

when $|\zeta| \le 2^{k-4}/m$. Similary we get

$$\|p_{k,a}(x,\,\cdot\,)\|_{L^2} \le C\omega_k^a\,2^{kn/2} \tag{4.5}$$

and

$$\left\|\Delta^m_{2,\zeta} p_{k,a}(x,\,\cdot\,)\right\|_{L^2} \le C\omega_k^a\,2^{kn/2} \qquad \text{for } \zeta \in \mathbb{R}^n. \tag{4.6}$$

Let us set $\psi_1(t) = t^{n/2}\psi(t)^{1/2}$ and apply Lemma 4.2 with $\varphi(t) = \psi_1(2^{-k}t)$.

In view of (4.4) - (4.6) we have

$$
\begin{aligned}
&\big\|\{1+\psi_1(2^{-k}|z|^{-1})^{-1}\}\mathcal{F}^{-1}[p_{k,a}(x,\,\cdot\,)](z)\big\|_{L^2}^2 \\
&\quad \le C\|p_{k,a}(x,\,\cdot\,)\|_{L^2}^2 + C\int \frac{\|\Delta_{2,\zeta}^m p_{k,a}(x,\,\cdot\,)\|_{L^2}^2}{\psi_1(2^{-k}|\zeta|)^2}\frac{d\zeta}{|\zeta|^n} \\
&\quad \le C\omega_k^{2a}2^{kn} + C\int_{|\zeta|\le 2^{k-4}/m} \omega_k^{2a}2^{kn}\frac{\psi_0(2^{-k}|\zeta|)^2}{\psi_1(2^{-k}|\zeta|)^2}\frac{d\zeta}{|\zeta|^n} \\
&\qquad\qquad + C\int_{|\zeta|\ge 2^{k-4}/m} \omega_k^{2a}2^{kn}\frac{1}{\psi_1(2^{-k}|\zeta|)^2}\frac{d\zeta}{|\zeta|^n} \\
&\quad \le C\omega_k^{2a}2^{kn}\left\{1+\int_0^{1/16m}\psi(t)\,\frac{dt}{t}+\int_{1/16m}^\infty \frac{1}{t^{n+1}\psi(t)}\,dt\right\} \\
&\quad \le C\omega_k^{2a}2^{kn},
\end{aligned}
$$

where the last integral is finite due to Lemma 2.1. Set

$$
G_k(z) = 2^{kn/2}\{1+\psi_1(2^{-k}|z|^{-1})^{-1}\}^{-1},
$$

then we have

$$
\begin{aligned}
\|G_k\|_{L^2}^2 &= \int 2^{kn}\{1+\psi_1(2^{-k}|z|^{-1})^{-1}\}^{-2}\,dz \\
&= C\int_0^\infty \{1+\psi_1(t^{-1})^{-1}\}^{-2}t^{n-1}\,dt \\
&= C\int_0^\infty (1+t^{-n/2}\psi(t)^{-1/2})^{-2}t^{-n-1}\,dt \\
&\le C\int_0^1 \frac{\psi(t)}{t}\,dt + C\int_1^\infty \frac{dt}{t^{n+1}} < \infty.
\end{aligned}
$$

Finally setting

$$
H_k(x,z) = \max_{a=0,1}\big|\mathcal{F}^{-1}[p_{k,a}(x,\,\cdot\,)](z)\big|/\omega_k^a G_k(z),
$$

we get the lemma. □

5. Proof of the main theorem

We cite two lemmas concerning the Littlewood-Paley decomposition from [5] in simpler forms.

Lemma 5.1. *There exists a contant $C>0$ such that, for $u\in L^2(\mathbb{R}^n)$,*

$$
\sum_{k\in\mathbb{N}} \|\mathcal{F}^{-1}[\Phi_k\widehat{u}]\|_{L^2}^2 \le C\|u\|_{L^2}^2.
$$

Lemma 5.2. *Fix $A > 2$ and put $I_k = \{\xi \in \mathbb{R}^n;\ A^{-1}2^k \le |\xi| \le A \cdot 2^k\}$ for $k \ge 1$ and $I_0 = \{\xi \in \mathbb{R}^n;\ |\xi| \le A\}$. Suppose that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of functions in $L^2(\mathbb{R}^n)$ satisfying* $\operatorname{supp} \widehat{u}_k \subset I_k$ *and $\sum_{k\in\mathbb{N}} \|u_k\|_{L^2}^2 < \infty$. Then the series $u = \sum_{k\in\mathbb{N}} u_k$ converges strongly in $L^2(\mathbb{R}^n)$ and there exists a constant $C > 0$ independent of $\{u_k\}_{k\in\mathbb{N}}$ such that $\|u\|_{L^2}^2 \le C \sum_{k\in\mathbb{N}} \|u_k\|_{L^2}^2$ holds.*

We are now ready to prove the main theorem. We set

$$u_k(x) = \sum_{|k'|\le 1} \mathcal{F}^{-1}[\Phi_{k+k'}\widehat{u}](x).$$

Since $\Phi_k(\xi) = \Phi_k(\xi) \sum_{|k'|\le 1} \Phi_{k+k'}(\xi)$, we can formally write

$$p(X,D)u(x) = \sum_{a=0}^{1} \sum_{k\in\mathbb{N}} p_{k,a}(X,D)u(x) = \sum_{a=0}^{1} \sum_{k\in\mathbb{N}} p_{k,a}(X,D)u_k(x).$$

Hence it is sufficient to prove

$$\| \sum_{k\in\mathbb{N}} p_{k,a}(X,D)u_k \|_{L^2} \le C\|u\|_{L^2}.$$

Applying Lemma 4.3 and the Schwarz inequality, we have

$$\begin{aligned}
|p_{k,a}(X,D)u_k(x)| &= \Big| \int \mathcal{F}^{-1}[p_{k,a}(x,\,\cdot\,)](y) u_k(x-y)\, dy \Big| \\
&\le C\omega_k^a \int H_k(x,y) G_k(y) |u_k(x-y)|\, dy \\
&\le C\omega_k^a \Big(\int H_k(x,y)^2 dy \Big)^{1/2} \Big(\int G_k(y)^2 |u_k(x-y)|^2 dy \Big)^{1/2} \\
&\le C\omega_k^a \Big(\int G_k(y)^2 |u_k(x-y)|^2 dy \Big)^{1/2}.
\end{aligned}$$

This combined with Fubini's theorem gives

$$\|p_{k,a}(X,D)u_k\|_{L^2} \le C\omega_k^a \|G_k\|_{L^2} \|u_k\|_{L^2} \le C\omega_k^a \|u_k\|_{L^2}. \tag{5.1}$$

First, we consider the case $a = 1$. We note that

$$\sum_{k=1}^{\infty} \omega_k^2 < \infty,$$

since

$$\int_0^1 \omega(t)^2 t^{-1}\, dt = \sum_{k=1}^{\infty} \int_{2^{-k}}^{2^{-k+1}} \omega(t)^2 t^{-1}\, dt \ge \sum_{k=1}^{\infty} \omega_k^2 \log 2.$$

From (5.1), the Schwarz inequality and Lemma 5.1 we have

$$\|\sum_{k\in\mathbb{N}} p_{k,1}(X,D)u_k\|_{L^2} \le \sum_{k\in\mathbb{N}} \|p_{k,1}(X,D)u_k\|_{L^2} \le C\sum_{k\in\mathbb{N}} \omega_k \|u_k\|_{L^2}$$
$$\le C\Big(\sum_{k\in\mathbb{N}} \omega_k^2\Big)^{1/2}\Big(\sum_{k\in\mathbb{N}} \|u_k\|_{L^2}^2\Big)^{1/2} \le C\|u\|_{L^2}.$$

Next, we consider the case $a = 0$. We will see that $\mathcal{F}[p_{k,0}(X,D)u_k](\zeta)$ has compact support which satisfies the condition in Lemma 5.2. Without loss of generality we may assume that $p_{k,0}(x,\xi)$ has compact support as a function of x by replacing $p_{k,0}(x,\xi)$ with $p_{k,0,j}(x,\xi) = \chi(2^{-j}|x|)p_{k,0}(x,\xi)$ if necessary, because $\mathcal{F}[p_{k,0,j}(\,\cdot\,,\xi)](\zeta)$ converges to $\mathcal{F}[p_{k,0}(\,\cdot\,,\xi)](\zeta)$ in $\mathcal{S}'(\mathbb{R}^n_\zeta)$ uniformly in ξ. Set $\check{p}(\zeta,\xi) = \mathcal{F}^{-1}[p(\,\cdot\,,\xi)](\zeta)$. Then we have

$$\begin{aligned}
&\mathcal{F}[p_{k,0}(X,D)u_k](\zeta)\\
&\quad = c_n \iint \exp(ix(-\zeta+\xi))\varphi_k(x')p(x-x',\xi)\Phi_k(\xi)\widehat{u}_k(\xi)\,d\xi\,dx'\,dx\\
&\quad = \int \chi(2^{-k+2}|\xi-\zeta|)\check{p}(\xi-\zeta,\xi)\Phi_k(\xi)\widehat{u}_k(\xi)\,d\xi,
\end{aligned}$$

which yields

$$\begin{aligned}
&\operatorname{supp}\mathcal{F}[p_{k,0}(X,D)u_k](\zeta)\\
&\quad\subset \{\zeta\in\mathbb{R}^n;\ |\xi-\zeta|\le 3^{-1}2^k \text{ and } 2^{k-1}\le|\xi|\le 2^{k+1}\}\\
&\quad\subset \{\zeta\in\mathbb{R}^n;\ 6^{-1}2^k\le|\zeta|\le 6\cdot 2^k\}
\end{aligned}$$

for $k \ge 1$. Similarly we have

$$\operatorname{supp}\mathcal{F}[p_{0,0}(X,D)u_0](\zeta) \subset \{\zeta\in\mathbb{R}^n;\ |\zeta|\le 6\}.$$

Applying Lemma 5.2 with $A = 6$, we get from (5.1),

$$\|\sum_{k\in\mathbb{N}} p_{k,0}(X,D)u_k\|_{L^2}^2 \le C\sum_{k\in\mathbb{N}} \|p_{k,0}(X,D)u_k\|_{L^2}^2$$
$$\le C\sum_{k\in\mathbb{N}} \|u_k\|_{L^2}^2 \le C\|u\|_{L^2}^2.$$

This completes the proof of the main theorem.

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