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Decomposition of Variation of Constants Formula for Abstract Functional Differential Equations

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1. Introduction

In this paper we are concerned with the linear functional differential equation

$$\dot{u}(t) = Au(t) + F(t)u_t + f(t) \tag{1}$$

on a phase space $\mathcal{B} = \mathcal{B}((-\infty, 0]; \mathbb{X})$ satisfying some fundamental axioms listed in Section 2.1, where A is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on a Banach space $\mathbb{X}$, u_t is an element of $\mathcal{B}$ defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in (-\infty, 0]$, $F(t)$ is a bounded linear operator mapping $\mathcal{B}$ into $\mathbb{X}$ which depends strongly continuously and periodically on t, and f is an $\mathbb{X}$-valued bounded and continuous function.

In a recent paper [24], Murakami, Naito and Nguyen have established a variation of constants formula (VCF) in the phase space for Eq. (1). The formula has been then applied to extend a classical theorem of Massera [22] on the existence of periodic solutions of linear ordinary differential equations to almost periodic solutions for Eq. (1).

A key point in [24] is to analyze difference equations associated with Eq. (1), which are derived naturally from the formula.

In this paper we will continue to study the subject, and establish several sharper results on the existence of almost periodic and quasiperiodic solutions for Eq. (1). Our approach employed in this paper is different from the one in [24]. Indeed, we will decompose the variation of constants formula into two parts referred to as the *stable part* of VCF and the *unstable part* of VCF (Theorem 3.1), and study each part of VCF to ensure the existence of almost periodic solutions and quasiperiodic solutions for Eq. (1). There are several advantages in our approach. Among them, we point out the following two facts: Roughly speaking, some spectral properties of the function f is inherited to the stable part of VCF (Theorem 3.3). Meanwhile, the unstable part of VCF is reduced to an ordinary

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differential equations (Theorem 3.5), and via this fact some spectral properties of f is inherited also to the unstable part of VCF (Theorem 3.6).

As an appendix, we refer to the Riesz representation for each element belonging to the dual space of $\mathcal{B}((-\infty]; \mathbb{X})$ and Favard's type argument to ensure the existence of almost periodic solutions of ordinary differential equations with a discontinuous forcing term. They seem to be unknown in the general situation and will be indispensable in our approach.

2. Abstract Functional Differential Equations

Throughout this paper, we will use the following notation. $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, integers, real numbers and complex numbers, respectively. Also, $C(J, \mathbb{X})$ denotes the space of all $\mathbb{X}$-valued continuous functions on J, and $BC(J, \mathbb{X})$ denotes the subspace of $C(J, \mathbb{X})$ consisting of all bounded and continuous functions on J.

We now consider the abstract functional differential equation

$$\frac{du(t)}{dt} = Au(t) + F(t)u_t + f(t), \tag{1}$$

where A is the generator of a semigroup of linear operators on a Banach space $\mathbb{X}$, $F(t)$ is a bounded linear operator from $\mathcal{B}$ into $\mathbb{X}$ which is periodic in t with period 1, where $\mathcal{B}$ is a fading memory phase space of Eq. (1) with infinite delay satisfying the axioms listed below and $f \in BC(\mathbb{R}, \mathbb{X})$. We emphasize that the assumption that the period of F is 1 does not constitute any restrictions on the obtained results.

2.1. Fading Memory Phase Spaces.

We will give a precise definition of the notion of fading memory space for Eq. (1) in this subsection. Let us denote the norm of $\mathbb{X}$ by $\|\cdot\|_{\mathbb{X}}$. For any function $x : (-\infty, a) \mapsto \mathbb{X}$ and $t < a$, we define a function $x_t : \mathbb{R}^- := (-\infty, 0] \mapsto \mathbb{X}$ by $x_t(s) = x(t+s)$ for $s \in \mathbb{R}^-$. A Banach space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ which consists of functions $\psi : (-\infty, 0] \mapsto \mathbb{X}$ is called a *fading memory space* if it satisfies the following axioms:

(A1) There exist a positive constant N and locally bounded functions $K(\cdot)$ and $M(\cdot)$ on $\mathbb{R}^+$ with the property that if $x : (-\infty, a) \mapsto \mathbb{X}$ is continuous on $[\sigma, a)$ with $x_\sigma \in \mathcal{B}$ for some $\sigma < a$, then for all $t \in [\sigma, a)$,

- (i) $x_t \in \mathcal{B}$,
- (ii) x_t is continuous in t (w.r.t. $\|\cdot\|_{\mathcal{B}}$),
- (iii) $N\|x(t)\|_{\mathbb{X}} \le \|x_t\|_{\mathcal{B}} \le K(t-\sigma)\sup_{\sigma \le s \le t}\|x(s)\|_{\mathbb{X}} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$,

(A2) If $\{\phi^k\}$, $\phi^k \in \mathcal{B}$, converges to ϕ uniformly on any compact set in $\mathbb{R}^-$ and if $\{\phi^k\}$ is a Cauchy sequence in $\mathcal{B}$, then $\phi \in \mathcal{B}$ and $\phi^k \to \phi$ in $\mathcal{B}$.

A fading memory space $\mathcal{B}$ is called a *uniform fading memory space*, if it satisfies (A1) and (A2) with $K(\cdot) \equiv K$ (a constant) and $M(\beta) \to 0$ as $\beta \to \infty$ in (A1). A typical example of uniform fading memory spaces is the following one:

$$C_\gamma := C_\gamma(\mathbb{X}) = \{\phi \in C(\mathbb{R}^-;\mathbb{X}) : \lim_{\theta\to-\infty} \frac{\|\phi(\theta)\|_{\mathbb{X}}}{e^{\gamma\theta}} = 0\}$$

which is equipped with norm $\|\phi\|_{C_\gamma} = \sup_{\theta\le 0} \|\phi(\theta)\|_{\mathbb{X}}/e^{\gamma\theta}$, where γ is a negative constant.

It is known [9, Lemma 3.2] that if $\mathcal{B}$ is a uniform fading memory space, then BC $:= BC(\mathbb{R}^-;\mathbb{X}) \subset \mathcal{B}$ and the inclusion map from BC into $\mathcal{B}$ is continuous. For other properties of fading memory spaces and uniform fading memory spaces, we refer the reader to the book [14].

2.2. **A Variation of Constants Formula for FDE.** We consider now the abstract functional differential equation (1) with the fading memory phase space $\mathcal{B}$. Throughout the paper we shall assume that $F(t)\phi$ is continuous in $(t,\phi) \in \mathbb{R}\times\mathcal{B}$ and that the operator $F(t)$ is of the form:

$$F(t)(\phi) = \int_{-\infty}^{0} [d_\theta \eta(t,\theta)]\phi(\theta), \qquad \phi \in C_{00}, \tag{2}$$

where $\eta(t,\theta)$ is an $L(\mathbb{X})$-valued function of locally bounded variation in θ on $\mathbb{R}^-$; here C_{00} denotes the subspace of $C(\mathbb{R}^-,\mathbb{X})$ consisting of functions with compact support, and $L(\mathbb{X})$ is the space of all bounded linear operators on $\mathbb{X}$. Without loss of generality, one may assume by Lemma 5.2 in Appendix that the function η in (2) is normalized; that is, $\eta(t,\theta)$ is left continuous in $\theta < 0$ with $\eta(t,0) = 0$.

Furthermore, we always assume that for any $R > 0$ the total variation $Var(\eta(t,\cdot) : [-R,0])$ of $\eta(t,\cdot)$ on $[-R,0]$ as a function of t is dominated by a locally integrable function m_R:

$$Var(\eta(t,\cdot) : [-R,0]) \le m_R(t), \qquad t \in \mathbb{R}.$$

For any $(\sigma,\phi) \in \mathbb{R}\times\mathcal{B}$, there exists a (unique) function $u : \mathbb{R} \mapsto \mathbb{X}$ such that $u_\sigma = \phi$, u is continuous on $[\sigma,\infty)$ and the following relation holds:

$$u(t) = T(t-\sigma)\phi(0) + \int_\sigma^t T(t-s)\{F(s)u_s + f(s)\}ds, \qquad t \ge \sigma.$$

The function u is called a *(mild) solution* of (1) through (σ,ϕ) on $[\sigma,\infty)$, and denoted by $u(\cdot,\sigma,\phi;f)$. Also, a function $v \in C(\mathbb{R},\mathbb{X})$ is called a *(mild) solution of Eq. (1) on* $\mathbb{R}$, if $v_t \in \mathcal{B}$ for all $t \in \mathbb{R}$ and it satisfies $u(t,\sigma,v_\sigma;f) = v(t)$ for all t and σ with $t \ge \sigma$. For any $t \ge s$, we define an operator $V(t,s)$ on $\mathcal{B}$ by

$$V(t,s)\phi = u_t(s,\phi;0), \qquad \phi \in \mathcal{B}.$$

We can easily see that under the assumption on the strong continuity and periodicity of $F(t)$, the two-parameter family $(V(t,s))_{t\ge s}$ is a strongly continuous evolutionary process on $\mathcal{B}$, which is called the *solution process* of (1). By a strongly continuous evolutionary process in a Banach space $\mathbb{Y}$ we mean a two-parameter family of bounded linear operators

$(\mathbb{V}(t,s))_{t\geq s}, (-\infty < s \leq t < \infty)$ from $\mathbb{Y}$ to $\mathbb{Y}$ such that the following conditions are satisfied:

i) $\mathbb{V}(t,t) = I, \quad \forall t \in \mathbb{R}$,
ii) $\mathbb{V}(t,s)\mathbb{V}(s,r) = \mathbb{V}(t,r), \quad \forall t \geq s \geq r$,
iii) For every fixed $y \in \mathbb{Y}$ the following map is continuous:

$$\{(\eta,\xi) \in \mathbb{R}^2 : \eta \geq \xi\} \ni (t,s) \to \mathbb{V}(t,s)y,$$

iv) There exist positive constants N, ω such that

$$\|\mathbb{V}(t,s)\| \leq Ne^{\omega(t-s)}, \quad \forall t \geq s, t, s \in \mathbb{R}.$$

By the principle of superposition, we get the relation

$$\begin{aligned} u_t(\sigma,\phi;f) &= u_t(\sigma,\phi;0) + u_t(\sigma,0;f) \\ &= V(t,\sigma)\phi + u_t(\sigma,0;f). \end{aligned} \tag{3}$$

In what follows, we shall give a representation of $u_t(\sigma,0;f)$ in terms of f and the solution process $(V(t,s))_{t\geq s}$. To this end, we introduce a function Γ^n defined by

$$\Gamma^n(\theta) = \begin{cases} (n\theta+1)I, & -1/n \leq \theta \leq 0 \\ \\ 0, & \theta < -1/n, \end{cases}$$

where n is any positive integer and I is the identity operator on $\mathbb{X}$. It follows from (A1) that if $x \in \mathbb{X}$, then $\Gamma^n x \in \mathcal{B}$ with $\|\Gamma^n x\|_{\mathcal{B}} \leq K(1)\|x\|_{\mathbb{X}}$. Moreover, since the process $(V(t,s))_{t\geq s}$ is strongly continuous, the $\mathcal{B}$-valued function $V(t,s)\Gamma^n f(s)$ is continuous in $s \in (-\infty, t]$ whenever $f \in \mathrm{BC}(\mathbb{R},\mathbb{X})$.

The following theorem yields a representation formula for solutions of (1) in the phase space:

Theorem 2.1. *The segment $u_t(\sigma,\phi;f)$ of solution $u(\cdot,\sigma,\phi,f)$ of (1) satisfies the following relation in $\mathcal{B}$:*

$$u_t(\sigma,\phi;f) = V(t,\sigma)\phi + \lim_{n\to\infty}\int_\sigma^t V(t,s)\Gamma^n f(s)ds, \qquad t \geq \sigma. \tag{4}$$

Moreover, the above limit exists uniformly for bounded $|t-\sigma|$.

In what follows, we give an outline of the proof of Theorem 2.1, because several facts established in the proof will be essentially used in subsequent sections. For a complete proof of the theorem, we refer the reader to [24].

Outline of the proof of Theorem 2.1. For any $(\alpha, x) \in \mathbb{R} \times \mathbb{X}$ we consider the integral equation

$$y(t) = P(t,\alpha;x) + \int_\alpha^t T(t-s)F(s)y_s ds, \quad t \geq \alpha, \tag{5}$$

with the condition $y_\alpha = 0$, where

$$P(t,\alpha;x) = T(t-\alpha)x - x - \int_\alpha^t T(t-s)\eta(s,\alpha-s)x ds.$$

By Picard's iteration method, we see that the above equation for y possesses a unique solution which we denote by $y(t,\alpha,x)$. In fact, there exists a unique $L(\mathbb{X})$-valued function $Y(t,\alpha)$ such that:

i) $Y(t,\alpha)$ is strongly continuous in t with $t \geq \alpha$ and is strongly left continuous in α with $\alpha \leq t$,
ii) $Y(t,\alpha)x =: y(t)$ satisfies (5) with the condition $y_\alpha = 0$,
iii) on each separable subspace of $\mathbb{X}$, $Y(t,\alpha)$ is a strong limit of some sequence of operators in $L(\mathbb{X})$ which is strongly continuous in (t,α) with $t \geq \alpha$ and uniformly bounded for bounded (t,α) with $t \geq \alpha$.

Now we consider a function $v(t,\alpha;x)$ defined by

$$v(t,\alpha;x) = x + Y(t,\alpha)x \qquad (x \in \mathbb{X})$$

for $t \geq \alpha$. The function $v(t,\alpha;x)$ is continuous in t with $t \geq \alpha$ and is left continuous in α with $\alpha \leq t$, and the following relation holds:

$$(6) \qquad v(t,\alpha;x) = T(t-\alpha)x + \int_\alpha^t T(t-s)\Big(\int_{\alpha-s}^0 [d_\theta \eta(s,\theta)]v(s+\theta,\alpha;x)\Big)ds, \quad t \geq \alpha.$$

We extend $v(t,\alpha;x)$ by putting $v(t,\alpha;x) = 0$ for $t < \alpha$. The function $v(t,\alpha;x)$ intimately concerns the solution of the equation (1) as follows:

i) $\displaystyle\int_\sigma^t v(t,s;h(s))ds \equiv u(t,\sigma,0;h) \qquad (\forall h \in BC(\mathbb{R},\mathbb{X}))$,
ii) $\lim_{n\to\infty} u(t,\alpha,\Gamma^n x;0) = v(t,\alpha;x)$ uniformly for each bounded $(t,x) \in [\alpha,\infty) \times \mathbb{X}$.

In order to prove the theorem, it suffices to establish the following relation:

$$(7) \qquad \lim_{n\to\infty}\Big[\sup_{\sigma \leq t \leq \sigma+L} \|\int_\sigma^t V(t,s)\Gamma^n f(s)ds - u_t(\sigma,0;f)\|_{\mathcal{B}}\Big] = 0 \quad (\forall L > 0).$$

The integration $\displaystyle\int_\sigma^t V(t,s)\Gamma^n f(s)ds$ is written as the limit of a Riemann sum of the form $\phi^\Delta := \sum_k V(t,s_k)\Gamma^n f(s_k)\Delta s_k$ in $\mathcal{B}$. Observe that $\phi^\Delta(\theta) = \sum_k u(t+\theta,s_k,\Gamma^n f(s_k);0)\Delta s_k$ is a Riemann sum of the integration

$$\int_\sigma^t u(t+\theta,s,\Gamma^n f(s);0)ds =: \xi^n(\theta)$$

and it converges to the above integral uniformly on any compact set in $\mathbb{R}^-$ because of the uniform continuity of $u(t+\theta,s,\Gamma^n f(s);0)$ as a function of (θ,s) on $\mathbb{R}^- \times [\sigma,t]$. Since

$\xi^n(\theta)$ is continuous in $\theta \le 0$ with $\xi^n(\theta) = 0$ for $\theta \le \sigma - t - 1/n$, it follows from (A1)-(i) that $\xi^n \in \mathcal{B}$. Moreover, we get

$$\|\xi^n - \phi^\Delta\|_{\mathcal{B}} \le K_1 \cdot \sup_{\sigma - t - 1/n \le \theta \le 0} \|\xi^n(\theta) - \phi^\Delta(\theta)\|_{\mathbb{X}}$$

by (A1)-(iii), where $K_1 = K(t - \sigma + 1)$. Thus ϕ^Δ converges to ξ^n in $\mathcal{B}$, and hence

$$\| \int_\sigma^t V(t,s)\Gamma^n f(s)ds - \xi^n\|_{\mathcal{B}} = 0.$$

Using (A1)-(iii) again, we get

$$(8) \qquad \begin{aligned} &\|u_t(\sigma, 0; f) - \int_\sigma^t V(t,s)\Gamma^n f(s)ds\|_{\mathcal{B}} = \|u_t(\sigma, 0; f) - \xi^n\|_{\mathcal{B}} \\ \le\ & K_1 \cdot \sup_{\sigma - t - 1/n \le \theta \le 0} \|u(t+\theta, \sigma, 0; f) - \xi^n(\theta)\|_{\mathbb{X}}. \end{aligned}$$

Observe that

$$\begin{aligned} & \sup_{\sigma - t - 1/n \le \theta \le \sigma - t} \|u(t+\theta, \sigma, 0; f) - \xi^n(\theta)\|_{\mathbb{X}} \\ = & \sup_{\sigma - t - 1/n \le \theta \le \sigma - t} \| \int_\sigma^t u(t+\theta, s, \Gamma^n f(s); 0)ds\|_{\mathbb{X}} \\ = & \sup_{\sigma - t - 1/n \le \theta \le \sigma - t} \| \int_\sigma^t \Gamma^n(t+\theta - s) f(s)ds\|_{\mathbb{X}} \\ \le & (1/2n) \sup_{\sigma \le s \le t} \|f(s)\|_{\mathbb{X}} \to 0 \end{aligned}$$

as $n \to \infty$. Also, we get

$$\begin{aligned} & \sup_{\sigma - t \le \theta \le 0} \|u(t+\theta, \sigma, 0; f) - \xi^n(\theta)\|_{\mathbb{X}} \\ = & \sup_{\sigma \le \tau \le t} \|u(\tau, \sigma, 0; f) - \int_\sigma^\tau u(\tau, s, \Gamma^n f(s); 0)ds - \int_\tau^t u(\tau, s, \Gamma^n f(s); 0)ds\|_{\mathbb{X}} \\ = & \sup_{\sigma \le \tau \le t} \|u(\tau, \sigma, 0; f) - \int_\sigma^\tau u(\tau, s, \Gamma^n f(s); 0)ds - \int_\tau^t \Gamma^n(\tau - s) f(s)ds\|_{\mathbb{X}} \\ \le & \sup_{\sigma \le \tau \le t} \int_\sigma^\tau \|v(\tau, s; f(s)) - u(\tau, s, \Gamma^n f(s); 0)\|_{\mathbb{X}} ds + (1/2n) \sup_{\sigma \le s \le t} \|f(s)\|_{\mathbb{X}}. \end{aligned}$$

We can easily show that

$$\begin{aligned} \lim_{n \to \infty} \Big[\sup\{ \int_\sigma^\tau \|v(\tau, s; f(s)) - u(\tau, s, \Gamma^n f(s); 0)\|_{\mathbb{X}} ds : \\ -L \le \sigma \le \tau \le L,\ \sup_{t \in \mathbb{R}} \|f(t)\|_{\mathbb{X}} \le L\} \Big] \\ = 0. \end{aligned}$$

It follows from this observation and Relation (8) that

$$\lim_{n\to\infty}\Big[\sup\{\|\int_\sigma^t V(t,s)\Gamma^n f(s)ds - u_t(\sigma,0;f)\|_{\mathcal{B}} : \\ -L\le \sigma\le t\le L,\ \sup_{\tau\in\mathbb{R}}\|f(\tau)\|_{\mathbb{X}}\le L\}\Big] = 0$$

for each $L>0$. Let p be the integer such that $p\le\sigma<p+1$. Since $F(t)$ is 1-periodic, one can easily see that $u_t(\sigma,0;f)=u_{t-p}(\sigma-p,0;f^p)$, where $f^p(\cdot)=f(p+\cdot)$. Then we get Relation (7), because of the relations $\sup_{p\in\mathbb{Z},\ t\in\mathbb{R}}\|f^p(t)\|_{\mathbb{X}}=\sup_{t\in\mathbb{R}}\|f(t)\|_{\mathbb{X}}<\infty$ and

$$\int_\sigma^t V(t,s)\Gamma^n f(s)ds - u_t(\sigma,0;f) \\ = \int_{\sigma-p}^{t-p} V(t-p,s)\Gamma^n f^p(s)ds - u_{t-p}(\sigma-p,0;f^p).$$

This completes the proof of the theorem.

3. Decomposition of Variation of constants formula

Throughout the rest of this paper we will make as a *standing assumption* that $\mathcal{B}$ is a uniform fading memory space, A is the generator of a compact semigroup $(T(t))_{t\ge0}$ and F is a 1-periodic function satisfying the assumptions stated in Section 2.

3.1. **Stable Part and Unstable Part of VCF.** In this subsection, correponding to the spectrum of the solution process of (1), we shall decompose the phase space $\mathcal{B}$ as a direct sum of two closed subspaces, and moreover decompose the formula obtained in Theorem 2.1 into two formulas.

First, by virtue of [36, Theorem 4.8], we see that the essential spectrum radius $r_e(V(1,0))$ of the operator $V(1,0)$ is less than one. Choose a negative number γ so that $\gamma>\log(r_e(V(1,0)))$, and set $\mathbb{C}_\gamma=\{\mu\in\mathbb{C}:|\mu|\ge e^\gamma\}$. ¿From the well known results on the periodic evolutionary process (see e.g. [7, 12, 14]) it follows that the set $\Lambda:=\sigma(V(s+1,s))\cap\mathbb{C}_\gamma$ is independent of $s\in\mathbb{R}$ and it consists of points of normal eigenvalues of $V(s+1,s)$, where $\sigma(V(s+1,s))$ denotes the spectrum of the bounded linear operator $V(s+1,s)$. Moreover, for any $s\in\mathbb{R}$ the space $\mathcal{B}$ is decomposed as a direct sum of closed subspaces $S(s)$ and $U(s)$

$$\mathcal{B}=S(s)\oplus U(s)\qquad (s\in\mathbb{R})$$

with the following properties:

i) $\dim U(s)\ (=:d)<\infty$ is independent of $s\in\mathbb{R}$;
ii) $S(t+1)\equiv S(t),\qquad U(t+1)\equiv U(t)$;
iii) $V(t,s)S(s)\subset S(t),\qquad V(t,s)U(s)\subset U(t)\qquad (\forall t\ge s)$;
iv) $\sigma(V(s+1,s)|_{U(s)})=\Lambda\qquad \sigma(V(s+1,s)|_{S(s)})=\sigma(V(s+1,s))\setminus\Lambda$.

Henceforth, we use the notation

$$V(t,s)|_{S(s)} = V^S(t,s), \qquad V(t,s)|_{U(s)} = V^U(t,s).$$

Also, we denote by $\Pi^S(s)$ (or $\Pi^U(s)$) the projection from $\mathcal{B}$ onto $S(s)$ (or $U(s)$ respectively) along to the above decomposition. In fact, the projection $\Pi^U(s)$ is represented as

$$\Pi^U(s) = \frac{1}{2\pi i}\int_{\mathcal{C}} (\lambda I - V(s+1,s))^{-1} d\lambda,$$

where $\mathcal{C}$ is a closed rectifiable Jordan curve which is disjointed with $\sigma(V(s+1,s))$ and contains Λ in its interior but no points of $\sigma(V(s+1,s))\backslash\Lambda$ ([12]). From this it follows that the operator valued function $\Pi^U(s)$ is 1-periodic and strongly continuous in s. In particular, we see that the norm of $\Pi^U(s)$, together with the norm of $\Pi^S(s)$, is uniformly bounded on $\mathbb{R}$:

$$\sup_{s\in\mathbb{R}} \|\Pi^S(s)\| + \sup_{s\in\mathbb{R}} \|\Pi^U(s)\| =: C_0 < \infty.$$

Corresponding to the above decomposition of the space $\mathcal{B}$, we consider the following two equations:

$$\xi(t) = V^S(t,\sigma)\xi(\sigma) + \lim_{n\to\infty}\int_\sigma^t V^S(t,s)\Pi^S(s)\Gamma^n f(s)ds, \qquad t \geq \sigma. \tag{9}$$

$$\eta(t) = V^U(t,\sigma)\eta(\sigma) + \lim_{n\to\infty}\int_\sigma^t V^U(t,s)\Pi^U(s)\Gamma^n f(s)ds, \qquad t \geq \sigma. \tag{10}$$

Henceforth, we refer to Eq. (9) and Eq. (10) as *the stable part of VCF* and *the unstable part of VCF*, respectively. Note that they correspond to the S-component and the U-component of the variation-of-constants formula (VCF) in Theorem 2.1, respectively.

Since the projections $\Pi^S(s)$ and $\Pi^U(s)$ are continuous on $\mathcal{B}$, the former part of the following theorem follows from Theorem 2.1. Also, the latter part can be proved by the same reasoning as in the proof of Theorem 4.2.9 of [14, p.121].

Theorem 3.1. *Assume that $\mathcal{B}$ is decomposed as cited above. Then, for the solution $u(\cdot,\sigma,\phi,f)$ of (1) the S-component $\Pi^S(t)u_t(\sigma,\phi;f)$ and the U-component $\Pi^U(t)u_t(\sigma,\phi;f)$ of the segment $u_t(\sigma,\phi;f)$ satisfy the stable part of VCF and the unstable part of VCF, respectively.*

Conversely, if the functions ξ and η on $\mathbb{R}$ with $\xi(t) \in S(t)$ and $\eta(t) \in U(t)$ satisfy the stable part of VCF and the unstable part of VCF , respectively, for all (t,σ) with $t \geq \sigma > -\infty$, then the function $u(t)$ defined by $u(t) = [\xi(t) + \eta(t)](0)$ for $t \in \mathbb{R}$ is a solution of Eq. (1) on $\mathbb{R}$, and satisfies $u_t = \xi(t) + \eta(t)$ in $\mathcal{B}$.

3.2. **Solutions for Stable Part of VCF.** In this subsection we shall investigate solutions of the stable part of VCF. Let us recall that the spectrum of $V^S(t,s)$ which is the restriction of $V(t,s)$ to $S(s)$ is evaluated as

$$\sigma(V^S(s+1,s)|_{S(s)}) = \sigma(V(s+1,s))\backslash\Lambda.$$

Therefore the spectrum radius of $V^S(s+1,s)$ is less than one, and the norm of the operator $V^S(t,s)$ decays exponentially as $t \to \infty$;

$$\exists C > 0, \alpha > 0 : ||V^S(t,s)|| \le Ce^{-\alpha(t-s)} \qquad (\forall t \ge s). \tag{11}$$

Let $f \in BC(\mathbb{R}, \mathbb{X})$ be given. In virtue of the estimate (11), it is straightforward to check that the integral $\int_{-\infty}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau$ converges in $\mathcal{B}$. Moreover, by the estimate (11) and Theorem 2.1 one can see that $\{\int_{-\infty}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\}_{n\ge1}$ is a Cauchy sequence in $\mathcal{B}$, and hence the limit $\lim_{n\to\infty}\int_{-\infty}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau$ exists in $\mathcal{B}$ by virtue of the completeness of $\mathcal{B}$. We set

$$\begin{aligned}\mathcal{Y}(t) &= \lim_{n\to\infty}\int_{-\infty}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau \\ &= \lim_{n\to\infty}\int_0^\infty V^S(t,t-s)\Pi^S(t-s)(\Gamma^n f(t-s))ds.\end{aligned}$$

Theorem 3.2. *Let $f \in BC(\mathbb{R}, \mathbb{X})$. Then the function $\mathcal{Y}(t)$ is $\mathcal{B}$-bounded (that is, $\sup_{t\in\mathbb{R}} \|\mathcal{Y}(t)\|_{\mathcal{B}} < \infty$) and continuous on $\mathbb{R}$ and satisfies the stable part of VCF for all (t,σ) with $t \ge \sigma > -\infty$. Moreover, if $\bar{\mathcal{Y}}$ is $\mathcal{B}$-bounded and continuous on $\mathbb{R}$ and satisfies the stable part of VCF for all (t,σ) with $t \ge \sigma > -\infty$, then $\mathcal{Y}(t) \equiv \bar{\mathcal{Y}}(t)$ in $\mathcal{B}$ for all $t \in \mathbb{R}$.*

Proof. We first assert that for any $\epsilon > 0$ there exist $L_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in\mathbb{R}} \|\int_{-\infty}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^m f(\tau))d\tau - \int_{t-L}^t V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\|_{\mathcal{B}} < \epsilon$$

for $\forall m, n \geq n_0$ and $\forall L \geq L_0$. Indeed

$$\begin{aligned}
&\|\int_{-\infty}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^m f(\tau))d\tau - \int_{t-L}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\|_{\mathcal{B}} \\
\leq\ & \|\int_{-\infty}^{t-L_0} V^S(t,\tau)\Pi^S(\tau)(\Gamma^m f(\tau))d\tau\|_{\mathcal{B}} \\
&\quad +\|\int_{t-L}^{t-L_0} V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\|_{\mathcal{B}} \\
&+\|\int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^m f(\tau))d\tau - \int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\|_{\mathcal{B}} \\
\leq\ & (2CC_0K(1)/\alpha)||f||e^{-\alpha L_0} + C_0 \cdot \|\int_{t-L_0}^{t} V(t,\tau)\Gamma^m f(\tau)d\tau \\
&\quad - \int_{t-L_0}^{t} V(t,\tau)\Gamma^n f(\tau)d\tau\|_{\mathcal{B}}
\end{aligned}$$

by (A1)-(iii) and the estimate (11), where $\|f\| = \sup_{t\in\mathbb{R}} \|f(t)\|_{\mathbb{X}}$. Choose $L_0 > 0$ so that

$$(2CC_0K(1)/\alpha)||f||e^{-\alpha L_0} < \epsilon/2.$$

Also, by Theorem 2.1 one can choose n_0 so that

$$C_0 \cdot \|\int_{t-L_0}^{t} V(t,\tau)\Gamma^n f(\tau)d\tau - u_t(t-L_0, 0; f)\|_{\mathcal{B}} < \epsilon/4 \quad (\forall n \geq n_0).$$

Then the assertion follows from the above inequality.

Now, letting $m \to \infty$ in the inequality of the assertion, we get

$$\sup_{t\in\mathbb{R}} \|\mathcal{Y}(t) - \int_{t-L}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau\|_{\mathcal{B}} < \epsilon \quad (\forall n \geq n_0,\ \forall L \geq L_0). \tag{12}$$

Since $t \mapsto \int_{t-L}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))d\tau$ is continuous, we see that $\mathcal{Y}(t)$ also is continuous. Moreover, $\mathcal{Y}$ is $\mathcal{B}$-bounded on $\mathbb{R}$ because of

$$\begin{aligned}
\|\mathcal{Y}(t)\|_{\mathcal{B}} &\leq \sup_n \int_{-\infty}^{t} \|V^S(t,\tau)\Pi^S(\tau)(\Gamma^n f(\tau))\|_{\mathcal{B}} d\tau \\
&\leq \int_{-\infty}^{t} Ce^{-\alpha(t-\tau)}d\tau \cdot C_0K(1)\|f\| \\
&= (CC_0/\alpha)K(1)\|f\|.
\end{aligned}$$

If $t \geq \sigma > -\infty$, then

$$\begin{aligned}
& V^S(t,\sigma)\mathcal{Y}(\sigma) + \lim_{n\to\infty}\int_\sigma^t V^S(t,s)\Pi^S(s)(\Gamma^n f(s))ds \\
= & \lim_{n\to\infty} V^S(t,\sigma)\int_{-\infty}^\sigma V^S(\sigma,s)\Pi^S(s)(\Gamma^n f(s))ds \\
& \quad + \lim_{n\to\infty}\int_\sigma^t V^S(t,s)\Pi^S(s)(\Gamma^n f(s))ds \\
= & \lim_{n\to\infty}\{\int_{-\infty}^\sigma V^S(t,s)\Pi^S(s)(\Gamma^n f(s))ds + \int_\sigma^t V^S(t,s)\Pi^S(s)(\Gamma^n f(s))ds\} \\
= & \lim_{n\to\infty}\int_{-\infty}^t V^S(t,s)\Pi^S(s)(\Gamma^n f(s))ds \\
= & \mathcal{Y}(t).
\end{aligned}$$

Thus $\mathcal{Y}(t)$ satisfies the stable part of VCF for all (t,σ) with $t \geq \sigma > -\infty$.

If $\bar{\mathcal{Y}}$ is another $\mathcal{B}$-bounded and continuous function satisfying the stable part of VCF, then $\mathcal{Y}(t) - \bar{\mathcal{Y}}(t) = V^S(t,\sigma)(\mathcal{Y}(\sigma) - \bar{\mathcal{Y}}(\sigma))$ for all $t \geq \sigma > -\infty$. Hence

$$\|\mathcal{Y}(t) - \bar{\mathcal{Y}}(t)\|_{\mathcal{B}} \leq Ce^{-\alpha(t-\sigma)}\{\sup_{\tau\in\mathbb{R}}\|\mathcal{Y}(\tau)\|_{\mathcal{B}} + \sup_{\tau\in\mathbb{R}}\|\bar{\mathcal{Y}}(\tau)\|_{\mathcal{B}}\} \to 0$$

as $\sigma \to -\infty$, and consequently $\mathcal{Y}(t) \equiv \bar{\mathcal{Y}}(t)$ in $\mathcal{B}$. This completes the proof of the theorem. □

We recall that $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be *almost periodic* (in the sense of Bohr), if for any positive ϵ the set $T(f,\epsilon)$ defined by

$$T(f,\epsilon) = \{\tau \in \mathbb{R} : \sup_{t\in\mathbb{R}}\|f(t+\tau) - f(t)\|_{\mathbb{X}} < \epsilon\}$$

is relatively dense in $\mathbb{R}$ ([19]). If f is an almost periodic function, the limit

$$a(\lambda; f) = \lim_{l\to\infty}\frac{1}{2l}\int_{-l}^l f(t)e^{-i\lambda t}dt, \quad \forall\lambda \in \mathbb{R} \tag{13}$$

exists. $a(\lambda; f)$ is called the Bohr transformation of f. We denote by $\sigma_b(f)$ the set which consists of real numbers λ satisfying $a(\lambda; f) \neq 0$, and call $\sigma_b(f)$ the Bohr spectrum of f. It is known that the set $\sigma_b(f)$ is at most countable. We denote by $m_b(f)$ the module generated by $\sigma_b(f)$;

$$m_b(f) = \{\sum_{\text{finite}} l_i\lambda_i : l_i \in \mathbb{Z},\ \lambda_i \in \sigma_b(f)\}.$$

An almost periodic function f is called a quasiperiodic function if $m_b(f)$ has a finite integer basis (cf. [19, p.26 and p.48]). For properties of almost periodic functions and quasiperiodic functions, see [19].

The following result will be needed later to study of existence of almost periodic solutions and quasiperiodic solutions for Eq. (1).

Theorem 3.3. *Let f be an almost periodic function. Then the function $\mathcal{Y}(t)$ is an almost periodic function, and the following relation holds:*

$$\sigma_b(\mathcal{Y}) \subset \sigma_b(f) + 2\pi\mathbb{Z}.$$

Proof. Since the set $\mathbb{Z} \cap T(f, \epsilon)$ is relatively dense in $\mathbb{R}$ (cf. pp. 163–164 in [5]) and $V^S(t, t-s)$ is 1-periodic in t, we can easily see by (11) that $\mathcal{Y}(t)$ is almost periodic. We will prove the inclusion relation on $\sigma_b(\mathcal{Y})$. To do this, it suffices to establish that

$$a(\lambda_0; \mathcal{Y}) := \lim_{l\to\infty} \frac{1}{2l} \int_{-l}^{l} e^{-i\lambda_0 t}\mathcal{Y}(t)dt = 0$$

in $\mathcal{B}$ under the assumption that $\lambda_0 \notin \sigma_b(f) + 2\pi\mathbb{Z}$. Let ϵ be any positive number. By the approxmation theorem for almost periodic functions ([19], pp.22–24), there exists a function $P(t)$ such that

$$\sup_{t\in\mathbb{R}} \|P(t) - f(t)\|_{\mathbb{X}} < \epsilon,$$

where $P(t)$ is a function of the form

$$P(t) = \sum_{j=1}^{m} x_j e^{i\mu_j t} \quad (x_j \in \mathbb{X},\ \mu_j \in \sigma_b(f)). \tag{14}$$

Then

$$\begin{aligned} & \| \int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0} f(\tau))d\tau - \int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0} P(\tau))d\tau \|_{\mathcal{B}} \\ \le \;& K(1)CC_0\epsilon \int_{t-L_0}^{t} e^{-\alpha(t-\tau)}d\tau \\ = \;& C_1\epsilon, \end{aligned}$$

where $C_1 = K(1)CC_0/\alpha$, and n_0 and L_0 is the number ensured in the proof of Theorem 3.2. Hence we get

$$\sup_{t\in\mathbb{R}} \|\mathcal{Y}(t) - \int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0} P(\tau))d\tau\|_{\mathcal{B}} < (1 + C_1)\epsilon =: C_2\epsilon$$

by (12), and hence

$$\begin{aligned} & \|\frac{1}{2l}\int_{-l}^{l} \mathcal{Y}(t)e^{-i\lambda_0 t}dt \\ & -\frac{1}{2l}\int_{-l}^{l} \Big(\int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0} P(\tau))d\tau \Big) e^{-i\lambda_0 t}dt\|_{\mathcal{B}} < C_2\epsilon. \end{aligned} \tag{15}$$

Notice that

$$\begin{aligned}
&\frac{1}{2l}\int_{-l}^{l}\left(\int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0}P(\tau))d\tau\right)e^{-i\lambda_0 t}dt \\
=\;& \frac{1}{2l}\int_{-l}^{l}\left(\int_{0}^{L_0} V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}P(t-s))ds\right)e^{-i\lambda_0 t}dt \\
=\;& \sum_{j=1}^{m}\int_{0}^{L_0}\left(\frac{1}{2l}\int_{-l}^{l} V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}x_j)e^{-i(\lambda_0-\mu_j)t}dt\right)e^{-i\mu_j s}ds,
\end{aligned}$$

and that the function

$$\mathcal{H}: t \mapsto V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}x_j)$$

is continuous and 1-periodic. Since $\lambda_0 - \mu_j \notin 2\pi\mathbb{Z}$, we get $a(\lambda_0-\mu_j;\mathcal{H}) = 0$ or

$$\lim_{l\to\infty}\|\frac{1}{2l}\int_{-l}^{l} V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}x_j)e^{-i(\lambda_0-\mu_j)t}dt\|_{\mathcal{B}} = 0$$

for each $j = 1,\cdots m$ and $s \in [0, L_0]$. By the estimate (11) it follows that

$$\|\frac{1}{2l}\int_{-l}^{l} V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}x_j)e^{-i(\lambda_0-\mu_j)t}dt\|_{\mathcal{B}} \le \frac{1}{2l}\int_{-l}^{l} e^{\alpha s}C_3 dt = C_3 e^{\alpha s},$$

where C_3 is a constant independent of l and s. Then Lebesgue's convergence theorem implies that

$$\lim_{l\to\infty}\|\int_{0}^{L_0}\left(\frac{1}{2l}\int_{-l}^{l} V^S(t,t-s)\Pi^S(t-s)(\Gamma^{n_0}x_j)e^{-i(\lambda_0-\mu_j)t}dt\right)e^{-i\mu_j s}ds\|_{\mathcal{B}} = 0$$

for each $j = 1,\cdots m$, and hence

$$\lim_{l\to\infty}\|\frac{1}{2l}\int_{-l}^{l}\left(\int_{t-L_0}^{t} V^S(t,\tau)\Pi^S(\tau)(\Gamma^{n_0}P(\tau))d\tau\right)e^{-i\lambda_0 t}dt\|_{\mathcal{B}} = 0.$$

Therefore, letting $l \to \infty$ in (15), we get $\|a(\lambda_0,\mathcal{Y})\|_{\mathcal{B}} \le C_2\epsilon$. Since ϵ is an arbitrary positive number, it follows $\|a(\lambda_0;\mathcal{Y})\|_{\mathcal{B}} = 0$, as required.

□

3.3. **Solutions for Unstable Part of VCF.** In what follows we shall analyze the unstable part of VCF, and deduce that the unstable part of VCF is reduced to an ordinary differential equation.

If $\{\phi_1(s),\cdots,\phi_d(s)\}$ is a basis of $U(s)$, then $\Phi(s) = (\phi_1(s),\cdots,\phi_d(s))$ is called a basis vector of $U(s)$. Let $\Phi = (\phi_1,\cdots,\phi_d)$ be a basis vector of $U(0)$. Since $V(1,0)$ is a linear transform on $U(0)$, there is a $d\times d$ matrix M such that $V(1,0)\Phi = \Phi M$. Notice that $\sigma(M) = \sigma(V(1,0)|_{U(0)}) = \Lambda$ does not contain zero. Therefore there exists a $d\times d$ matrix G such that $M = e^G$. We set

$$\Phi(t) = V(t,0)\Phi e^{-Gt}$$

for $t \geq 0$. One can easily see that $\Phi(t+1) \equiv \Phi(t)$ for all $t \geq 0$, and that $\Phi(t)$ is a basis vector of $U(t)$. We extend $\Phi(t)$ for $t < 0$ periodically (which we denote by $\Phi(t)$ again). Then

$$V(t,s)\Phi(s) = \Phi(t)e^{G(t-s)} \qquad (\forall t \geq s).$$

Now, for the above basis vector $\Phi(t) = (\phi_1(t), \cdots, \phi_d(t))$ of $U(t)$, there exist uniquely d-elements $\psi_1(t), \cdots, \psi_d(t)$ in $\mathcal{B}^*$ (the dual space of $\mathcal{B}$) such that $\langle \psi_i(t), \phi_j(t) \rangle = 1$ if $i = j$, 0 if $i \neq j$, and that $\psi_i(t) = 0$ on $S(t)$. Here and hereafter, $\langle \ , \ \rangle$ denotes the canonical pairing between the dual space and the original space. It is clear that $\{\psi_1(t), \cdots, \psi_d(t)\}$ is linearly independent. We denote by $\Psi(t)$ the transpose of $(\psi_1(t), \cdots, \psi_d(t))$ to use the expression $\langle \Psi(t), \Phi(t) \rangle = I_d$ (here I_d is the $d \times d$ unit matrix), and call $\Psi(t)$ the dual vector associated with $\Phi(t)$. It follows from the uniqueness that $\Psi(t)$ is 1-periodic; that is, $\Psi(t+1) \equiv \Psi(t)$. The projection operator $\Pi^U(t)$ is represented by means of $\Phi(t)$ and $\Psi(t)$ as follows:

$$\Pi^U(t)\phi = \Phi(t)\langle \Psi(t), \phi \rangle, \qquad \phi \in \mathcal{B}.$$

By the same reasoning as in Theorem 6.1.2 and Proposition 6.2.2 in [14] one can see that

$$V^*(t,s)\Psi(t) = e^{G(t-s)}\Psi(s) \qquad (\forall t \geq s), \tag{16}$$

and that $\Psi(t)$ is weak-star continuous in t; that is, $\langle \Psi(t), \phi \rangle$ is continuous in t for each $\phi \in \mathcal{B}$. Consequently

$$\sup_{t \in \mathbb{R}} \|\Psi(t)\| =: C_4 < \infty. \tag{17}$$

It follows from Proposition 5.1 in Appendix that for any $\psi \in \mathcal{B}^*$ there is a unique $\tilde{\psi} \in \mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; \mathbb{X}^*)$ such that

$$\langle \psi, \phi \rangle = \int_{-\infty}^{0} [d\tilde{\psi}(\theta)]\phi(\theta) \qquad (\forall \phi \in C_{00}),$$

where $\mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; \mathbb{X}^*)$ denotes the space of all $\mathbb{X}^*$-valued functions $\chi(\theta)$ of locally bounded variation on $\mathbb{R}^-$ which is left continuous at each $\theta < 0$ and $\chi(0) = 0$. For the above dual vector $\Psi(t)$ we denote by $\tilde{\Psi}(t)$ the transpose of $(\tilde{\psi}_1(t), \cdots, \tilde{\psi}_d(t))$. It follows from the proof of Proposition 5.1 in Appendix that

$$Var(\tilde{\Psi}(t)(\cdot) : [-r, 0]) \leq K(r+1)\|\Psi(t)\| \leq C_4 K(r+1)$$

by (17). Also, from (29) in the proof of Lemma 5.2 in Appendix we know that the left limit

$$\tilde{\Psi}(t)(0^-) = \lim_{\theta < 0, \ \theta \to 0} \tilde{\Psi}(t)(\theta)$$

converges in the norm topology of X^*.

Lemma 3.4. *For any $f \in BC(\mathbb{R}, \mathbb{X})$ the function $\langle \tilde{\Psi}(t)(0^-), f(t) \rangle$ is locally integrable in $t \in \mathbb{R}$. Also, $\tilde{\Psi}(t)(0^-)$ is weak-star left continuous in $t \in \mathbb{R}$.*

Proof. For any $n \in \mathbb{N}$ and $t \in \mathbb{R}$ we consider a linear functional $x \in \mathbb{X} \mapsto \langle \Psi(t), \Gamma^n x \rangle$. Then the functional is bounded on $\mathbb{X}$ because of the inequality

$$\begin{aligned} \|\langle \Psi(t), \Gamma^n x \rangle\| &\leq \|\Psi(t)\| \|\Gamma^n x\|_{\mathcal{B}} \\ &\leq K(1) \|\Psi(t)\| \|x\|_{\mathbb{X}}. \end{aligned}$$

We denote this functional by $x_n^*(t)$:

$$\langle x_n^*(t), x \rangle = \langle \Psi(t), \Gamma^n x \rangle.$$

Thus $x_n^*(t)$ belongs to the space of d-copies of X^* and

$$\|x_n^*(t)\| \leq K(1) \|\Psi(t)\| \leq C_4 K(1) < \infty.$$

It follows that

$$\begin{aligned} \langle x_n^*(t), x \rangle &= \langle \Psi(t), \Gamma^n x \rangle \\ &= \int_{-1/n}^{0} [d_\theta \tilde{\Psi}(t)(\theta)](n\theta + 1)x \\ &= \Big[\tilde{\Psi}(t)(\theta)(n\theta + 1)x \Big]_{-1/n}^{0} - n \int_{-1/n}^{0} \tilde{\Psi}(t)(\theta) d\theta \cdot x \\ &= -n \int_{-1/n}^{0} \tilde{\Psi}(t)(\theta) d\theta \cdot x. \end{aligned}$$

Observe that

$$\|\tilde{\Psi}(t)(0^-) - n \int_{-1/n}^{0} \tilde{\Psi}(t)(\theta) d\theta\| \leq n \int_{-1/n}^{0} \|\tilde{\Psi}(t)(0^-) - \tilde{\Psi}(t)(\theta)\| d\theta \to 0$$

as $n \to \infty$, by Lebesgue's convergence theorem. Therefore, combining these facts we see that

$$\lim_{n \to \infty} \|x_n^*(t) + \tilde{\Psi}(t)(0^-)\| = 0. \tag{18}$$

Now, let $f \in BC(\mathbb{R}, \mathbb{X})$ be given. It follows from (18) that

$$-\langle \tilde{\Psi}(t)(0^-), f(t) \rangle = \lim_{n \to \infty} \langle x_n^*(t), f(t) \rangle = \lim_{n \to \infty} \langle \Psi(t), \Gamma^n f(t) \rangle$$

for all $t \in \mathbb{R}$. Observe that for each $n \in \mathbb{N}$ the function $\langle \Psi(t), \Gamma^n f(t) \rangle$ is continuous because of the weak-star continuity of $\Psi(t)$ and the continuity of $f(t)$. Then the function $\langle \tilde{\Psi}(t)(0^-), f(t) \rangle$ is locally integrable as a pointwise limit of a family of (uniformly bounded) continuous functions.

Next we shall establish the weak-star left continuity of $\tilde{\Psi}(t)(0^-)$. To do this, it suffices to establish the weak-star continuity at negative points because of the periodicity of

$\tilde{\Psi}(t)(0^-)$. Let $t<0$. Then it follows from (16) and (18) that for any $x \in \mathbb{X}$,

$$\begin{aligned}
-\langle \tilde{\Psi}(t)(0^-), x\rangle &= \lim_{n\to\infty} \langle x_n^*(t), x\rangle = \lim_{n\to\infty} \langle \Psi(t), \Gamma^n x\rangle \\
&= \lim_{n\to\infty} \langle e^{Gt} V^*(0,t)\Psi(0), \Gamma^n x\rangle \\
&= e^{Gt} \lim_{n\to\infty} \langle \Psi(0), V(0,t)\Gamma^n x\rangle \\
&= e^{Gt} \lim_{n\to\infty} \int_{-\infty}^0 [d_\theta \tilde{\Psi}(0)(\theta)] u(\theta, t, \Gamma^n x; 0) \\
&= e^{Gt} \lim_{n\to\infty} \{ \int_{t-1/n}^t [d_\theta \tilde{\Psi}(0)(\theta)](n\theta - nt + 1)x \\
&\qquad + \int_t^0 [d_\theta \tilde{\Psi}(0)(\theta)] u(\theta, t, \Gamma^n x; 0)\} \\
&= e^{Gt} \lim_{n\to\infty} \{[\tilde{\Psi}(0)(t) - n\int_{t-1/n}^t \tilde{\Psi}(0)(\theta) d\theta] x \\
&\qquad + \int_t^0 [d_\theta \tilde{\Psi}(0)(\theta)] u(\theta, t, \Gamma^n x; 0)\} \\
&= e^{Gt} \{[\tilde{\Psi}(0)(t) - \tilde{\Psi}(0)(t^-)]x + \int_t^0 [d_\theta \tilde{\Psi}(0)(\theta)] v(\theta, t; x)\} \\
&= e^{Gt} \int_t^0 [d_\theta \tilde{\Psi}(0)(\theta)] v(\theta, t; x),
\end{aligned}$$

where we used the left continuity of $\tilde{\Psi}(0)(t)$ at $t<0$ and the fact that $\lim_{n\to\infty} u(\theta, t, \Gamma^n x; 0) = v(\theta, t; x)$ for $\theta \geq t$ which has already been explained in the outline of the proof of Theorem 2.1. Notice that $v(\theta, t; x)$ is left continuous in $t \leq \theta$. Then the weak-star left continuity of $\tilde{\Psi}(t)(0^-)$ at $t<0$ follows from the left continuity of $v(\theta, t; x)$ in t and the left continuity of $\tilde{\Psi}(0)(\theta)$ in $\theta < 0$. This completes the proof of the lemma. □

Now we are in a position to state the following theorem which shows that the unstable part of VCF of (1) is reduced to an ordinary differential equation.

Theorem 3.5. *Let G, $\Phi(t)$, $\Psi(t)$, $\tilde{\Psi}(t)$ be the ones cited above. Then a $\mathcal{B}$-valued function $\eta(t)$ with $\eta(t) \in U(t)$ satisfies the unstable part of VCF of (1) if and only if the function $z(t)$ determined by $\Phi(t)z(t) = \eta(t)$ is locally absolutely continuous in $t \geq \sigma$ and satisfies the following ordinary differential equation*

$$\dot{z}(t) = Gz(t) - \langle \tilde{\Psi}(t)(0^-), f(t)\rangle \quad \text{a.e. in } t \geq \sigma. \tag{19}$$

Proof. Let $\eta(t)$ be a $\mathcal{B}$-valued function with $\eta(t) \in U(t)$ satisfying the unstable part of VCF of (1). Since $z(t) = \langle \Psi(t), \eta(t) \rangle$, we get

$$\begin{aligned} z(t) &= \langle \Psi(t), V(t,\sigma)\eta(\sigma) + \lim_{n\to\infty} \int_\sigma^t V(t,s)\Pi^U(s)(\Gamma^n f(s))ds \rangle \\ &= \langle V^*(t,\sigma)\Psi(t), \eta(\sigma)\rangle + \lim_{n\to\infty} \int_\sigma^t \langle V^*(t,s)\Psi(t), \Pi^U(s)(\Gamma^n f(s))\rangle ds \\ &= \langle e^{G(t-\sigma)}\Psi(\sigma), \eta(\sigma)\rangle + \lim_{n\to\infty} \int_\sigma^t \langle e^{G(t-s)}\Psi(s), \Gamma^n f(s)\rangle ds \\ &= e^{G(t-\sigma)} z(\sigma) + \lim_{n\to\infty} \int_\sigma^t e^{G(t-s)} \langle x_n^*(s), f(s)\rangle ds \\ &= e^{G(t-\sigma)} z(\sigma) - \int_\sigma^t e^{G(t-s)} \langle \tilde{\Psi}(s)(0^-), f(s)\rangle ds \end{aligned}$$

by (16) and (18), where we used Lebesgue's convergence theorem in the last equality. By virtue of Lemma 3.4, the function $\langle \tilde{\Psi}(s)(0^-), f(s)\rangle ds$ is locally integrable, and hence it follows from the above relation that $z(t)$ is locally absolutely continuous and satisfies Eq. (19).

Conversely, let $z(t)$ be a locally absolutely continuous function which satisfies Eq. (19). Then

$$z(t) = e^{G(t-\sigma)} z(\sigma) - \int_\sigma^t e^{G(t-s)} \langle \tilde{\Psi}(s)(0^-), f(s)\rangle ds \qquad (\forall t \geq \sigma).$$

Then $\eta(t) := \Phi(t)z(t)$ satisfies the unstable part of VCF, because

$$\begin{aligned} \eta(t) &= \Phi(t)\Big(e^{G(t-\sigma)} z(\sigma) - \int_\sigma^t e^{G(t-s)} \langle \tilde{\Psi}(s)(0^-), f(s)\rangle ds\Big) \\ &= \Phi(t)\Big(e^{G(t-\sigma)} z(\sigma) + \lim_{n\to\infty} \int_\sigma^t e^{G(t-s)} \langle \Psi(s), \Gamma^n f(s)\rangle ds\Big) \\ &= \Phi(t)e^{G(t-\sigma)} z(\sigma) + \lim_{n\to\infty} \int_\sigma^t \Phi(t)e^{G(t-s)} \langle \Psi(s), \Gamma^n f(s)\rangle ds \\ &= V^U(t,\sigma)\Phi(\sigma)z(\sigma) + \lim_{n\to\infty} \int_\sigma^t V^U(t,s)\Phi(s)\langle \Psi(s), \Gamma^n f(s)\rangle ds \\ &= V^U(t,\sigma)\eta(\sigma) + \lim_{n\to\infty} \int_\sigma^t V^U(t,s)\Pi^U(s)(\Gamma^n f(s))ds. \end{aligned}$$

□

As the following theorem shows, it is advantageous to treat Eq. (19) instead of the unstable part of VCF.

Theorem 3.6. *Let f be an almost periodic function, and assume that Eq. (19) has a bounded solution on $\mathbb{R}^+ := [0, \infty)$. Then there exists an almost periodic solution z of Eq.*

(19) such that

$$\sigma_b(z) \subset \sigma_b(f) + 2\pi\mathbb{Z}. \tag{20}$$

Proof. We first note that in order to ensre the existence of almost periodic solutions of Eq. (19), Favard's theorem (e.g., [5]) in the theory of ordinary differential equations is not applicable, because the term $\langle \tilde{\Psi}(t)(0^-), f(t)\rangle$ in Eq. (19) is not continuous in t. But, we can establish the existence of an almost periodic solution of Eq. (19) by modifying the argument due to Favard. In Proposition 5.4 of Appendix, we will give a complete proof for the existence of an almost periodic solution of Eq. (19).

Now let $p(t)$ be an almost periodic solution of Eq. (19). We first assert that

$$(G - i\lambda)a(\lambda; p) = 0 \qquad (\forall \lambda \notin \sigma_b(f) + 2\pi\mathbb{Z}). \tag{21}$$

Indeed, we get

$$\begin{aligned} (G - i\lambda)a(\lambda; p) &= \lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} (G - i\lambda)p(t)e^{-i\lambda t}dt \\ &= \lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} \{\dot{p}(t) + \langle \tilde{\Psi}(t)(0^-), f(t)\rangle - i\lambda p(t)\}e^{-i\lambda t}dt \\ &= \lim_{l\to\infty} \frac{1}{2l}\left(\Big[p(t)e^{-i\lambda t}\Big]_{-l}^{l} + \int_{-l}^{l} \langle \tilde{\Psi}(t)(0^-), f(t)\rangle e^{-i\lambda t}dt\right) \\ &= \lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} \langle \tilde{\Psi}(t)(0^-), f(t)\rangle e^{-i\lambda t}dt. \end{aligned}$$

We claim that

$$\lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} \langle \tilde{\Psi}(t)(0^-), f(t)\rangle e^{-i\lambda t}dt = 0 \qquad (\forall \lambda \notin \sigma_b(f) + 2\pi\mathbb{Z}). \tag{22}$$

Then the assertion (21) is a direct consequence of (22). We now prove (22). Let ϵ be any positive number. By the approximation theorem for almost periodic functions, there exists a function $P(t)$ such that

$$\sup_{t\in\mathbb{R}} \|f(t) - P(t)\|_{\mathbb{X}} < \epsilon,$$

where $P(t)$ is a function of the form (14). Then

$$\begin{aligned} &\left|\frac{1}{2l}\int_{-l}^{l} e^{-i\lambda t}\langle \tilde{\Psi}(t)(0^-), f(t)\rangle dt - \sum_{j=1}^{m}\left(\frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}\langle \tilde{\Psi}(t)(0^-), x_j\rangle dt\right)\right| \\ &\le \sup_{t\in\mathbb{R}} \|\tilde{\Psi}(t)(0^-)\|\epsilon \le C_5\epsilon, \end{aligned} \tag{23}$$

where C_5 is a constant independent of l. For any $j = 1, \cdots, m$ we set

$$\varphi(t) = \langle \tilde{\Psi}(t)(0^-), x_j\rangle.$$

Then $\varphi(t)$ is a d-column vector valued function on $\mathbb{R}$ which is left continuous bounded and 1-periodic. Let us expand $\varphi(t)$ into the Fourier series as

$$\varphi(t) \sim \sum_{n\in\mathbb{Z}} c_n e^{2n\pi it},$$

and consider the N-partial sum $S_N(t) = \sum_{|n|\le N} c_n e^{2n\pi it}$. Then it follows from Parseval's equality that $\int_0^1 |\varphi(t) - S_N(t)|^2 dt = \sum_{|n|>N} |c_n|^2$, and consequently

$$\lim_{N\to\infty} \int_0^1 |\varphi(t) - S_N(t)|^2 dt = 0. \tag{24}$$

Therefore, if $l \ge 1$, then

$$\begin{aligned} |\frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}(\varphi(t) - S_n(t))dt|^2 &\le \frac{1}{2l}\int_{-l}^{l} |\varphi(t) - S_n(t)|^2 dt \\ &\le \frac{[l]+1}{l}\int_0^1 |\varphi(t) - S_n(t)|^2 dt \\ &\le 2\int_0^1 |\varphi(t) - S_n(t)|^2 dt, \end{aligned}$$

where $[l]$ denotes the largest integer which does not exceed l. By virtue of this inequality and (24), one can choose a positive integer N_0 so that

$$\sup_{l\ge 1} |\frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}(\varphi(t) - S_{N_0}(t))dt| < \epsilon. \tag{25}$$

Notice that $\lambda - \mu_j - 2n\pi \ne 0$ for all $n \in \mathbb{Z}$ because of $\lambda \notin \sigma_b(f) + 2\pi\mathbb{Z}$. Then

$$\begin{aligned} |\frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t} S_{N_0}(t)dt| &= |\sum_{|n|\le N_0} c_n \left(\frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j-2n\pi)t}\right) dt| \\ &\le \sum_{|n|\le N_0} \frac{|c_n|}{l(2n\pi + \mu_j - \lambda)} \to 0 \end{aligned}$$

as $l \to \infty$. This observation and (25) lead to $\lim_{l\to\infty} |(1/2l)\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}\varphi(t)dt| \le \epsilon$. Since ϵ is an arbitrary positive number, we get

$$\lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}\varphi(t)dt = 0,$$

or

$$\lim_{l\to\infty} \frac{1}{2l}\int_{-l}^{l} e^{-i(\lambda-\mu_j)t}\langle\tilde{\Psi}(t)(0^-), x_j\rangle dt = 0$$

for $j = 1, \cdots, m$. Then, letting $l \to \infty$ in (23) we get

$$|\frac{1}{2l}\int_{-l}^{l} e^{-i\lambda t}\langle\tilde{\Psi}(t)(0^-), f(t)\rangle dt| \le C_5\epsilon,$$

or the desired relation (22) because ϵ is arbitrary.

We now set $\sigma_i(G) = \{\xi \in \mathbb{R} : i\xi \in \sigma(G)\}$. By virtue of the relation (21) one can easily see that $a(\lambda; p) = 0$ for all $\lambda \notin \sigma_i(G) \cup (\sigma_b(f) + 2\pi\mathbb{Z})$. Consider a function $z(t)$ defined by

$$z(t) = p(t) - q(t),$$

where $q(t) = \sum_{\lambda \in \Delta} a(\lambda; p)e^{i\lambda t}$ and $\Delta = \sigma_i(G) \backslash (\sigma_b(f) + 2\pi\mathbb{Z})$. It is obvious that $\sigma_b(z) \subset \sigma_b(f) + 2\pi\mathbb{Z}$. Also, we get

$$\dot{q}(t) = \sum_{\lambda \in \Delta} i\lambda a(\lambda; p)e^{i\lambda t} = \sum_{\lambda \in \Delta} Ga(\lambda; p)e^{i\lambda t} = Gq(t)$$

by (21), and consequently $z(t)$ is a solution of Eq. (19). Thus $z(t)$ is a required one. This completes the proof of the theorem.

□

4. Almost Periodic Solutions and Quasi Periodic Solutions

In this section we shall study the existence of almost periodic solutions and quasiperiodic solutions of Eq. (1), and give some results which extends the classical theorem due to Massera [22] on the existence of periodic solutions for linear periodic ordinary differential equations.

Theorem 4.1. *Let the standing assumption stated in Section 3 be met, and assume that f be almost periodic. If Eq. (1) has a bounded solution on $\mathbb{R}^+$, then it has an almost periodic solution u on $\mathbb{R}$ such that*

$$\sigma_b(u) \subset \sigma_b(f) + 2\pi\mathbb{Z}. \quad (26)$$

Proof. Let v be a bounded solution of Eq. (1) on $\mathbb{R}^+$. Since $\mathcal{B}$ is a uniform fading memory space, the segment v_t is $\mathcal{B}$ bounded on $\mathbb{R}^+$. By virtue of Theorems 3.1, 3.5 and 3.6, there exists an almost periodic solution z of Eq. (19) such that $\sigma_b(z) \subset \sigma_b(f) + 2\pi\mathbb{Z}$. Set $\mathcal{Z}(t) = \Phi(t)z(t)$. By virtue of Theorem 3.5, $\mathcal{Z}(t)$ satisfies the unstable part of VCF of (1). Since $\Phi(t)$ is continuous and 1-periodic, $\mathcal{Z}(t)$ is a $\mathcal{B}$-valued almost periodic function, and moreover by almost the same argument as in the proof of Theorem 3.3 we see that $\sigma_b(\mathcal{Z}) \subset \sigma_b(z) + 2\pi\mathbb{Z} \subset \sigma_b(f) + 2\pi\mathbb{Z}$. Let us recall that $\mathcal{Y}(t)$ is the one introduced in the subsection 3.2. Then $u(t) := [\mathcal{Y}(t) + \mathcal{Z}(t)](0)$ satisfies the required properties by Theorems 3.1 and 3.3. □

The relation (26) implies the following relation on the modules of u and f:

$$m_b(u) \subset m_b(f) + 2\pi\mathbb{Z}.$$

Therefore, if the module $m_b(f)$ of f has an integer and finite basis, then $m_b(u)$ has an integer and finite basis, too. This observation leads to the following result on the existence of quasiperiodic solutions which extends Theorem 4.12 of [24] for autonomous equations to periodic equations.

Corollary 4.2. *Let the standing assumption be met, and assume that f be quasiperiodic. If Eq. (1) has a bounded solution on $\mathbb{R}^+$, then it has a quasiperiodic solution u on $\mathbb{R}$ such that $\sigma_b(u) \subset \sigma_b(f) + 2\pi\mathbb{Z}$.*

Under the existence of a bounded solution *on the whole line* $\mathbb{R}$, we can derive the following result finer than the above results.

Theorem 4.3. *Let the standing assumption be met, and assume that $v(t)$ is a bounded solution of Eq. (1) on $\mathbb{R}$. Then $v(t)$ itself is almost periodic (or quasiperiodic) if f is almost periodic (or quasiperiodic, respectively).*

Proof. By virtue of Theorems 3.1 and 3.2, we get $\Pi^S v_t \equiv \mathcal{Y}(t)$ on $\mathbb{R}$. On the one hand, the function w defined by $\Pi^U(t)v_t = \Phi(t)w(t)$ is a bounded solution of Eq. (19) on $\mathbb{R}$, and hence $w(t) - z(t)$ is a solution of the homogeneous linear ordinary equation $\dot{y}(t) = Gy(t)$ on $\mathbb{R}$, where $z(t)$ is the one ensured in the proof of Theorem 4.1. Since $w(t) - z(t)$ is bounded on $\mathbb{R}$, we see that the function $w(t) - z(t)$ is quasiperiodic. Then the conlusion of the theorem follows immdiately from Theorem 3.3 and the above observation, because of $v(t) \equiv [\mathcal{Y}(t) + \Phi(t)w(t)](0)$ on $\mathbb{R}$. □

5. Appendix

In this appendix we will state several results on the Riesz representation of bounded linear functionals on the fading memory space $\mathcal{B} := \mathcal{B}(\mathbb{R}^-; \mathbb{X})$ explained in Section 2 and the existence of almost periodic solutions of Eq. (19) by using Favard's type method.

5.1. **Representation of Bounded Linear Functionals.** In what follows we will represent any bounded linear functional on $\mathcal{B}$ by means of the Stieltjes integration. Let us recall that $\mathcal{B}^*$ denotes the dual space of $\mathcal{B}$, and that $\mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; \mathbb{X}^*)$ denotes the space of all $\mathbb{X}^*$-valued functions $\zeta(\theta)$ of locally bounded variation on $\mathbb{R}^-$ which is left continuous at each $\theta < 0$ and $\zeta(0) = 0$. Also, we often denote by $\langle \phi^*, \phi \rangle$ or $\phi^*\phi$ the canonical pairing $\phi^*(\phi)$ between an element ϕ^* of the dual space and an element ϕ of the original space.

Proposition 5.1. *Let $\phi^* \in \mathcal{B}^*$ be given. Then there exists a unique $\zeta \in \mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; \mathbb{X}^*)$ such that*

$$\langle \phi^*, \phi \rangle = \int_{-\infty}^{0} [d\zeta(\theta)]\phi(\theta) \qquad (\forall \phi \in C_{00}).$$

This result may be derived by combining several results in Dinculeanu's book ([4]). However the above result is not referred to in the book, explicitly. In what follows we will give a rather elementary proof of the result for completeness, without relying on the results in Dinculeanu's book. To do this, we need some lemmas. Throughout the rest of the paper, we use the notation $L(\mathbb{X}, \mathbb{Y})$ and $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^-; L(\mathbb{X}, \mathbb{Y}))$ to denote the space of all bounded linear operators mapping $\mathbb{X}$ into $\mathbb{Y}$ and the space of all $L(\mathbb{X}, \mathbb{Y})$-valued functions of locally bounded variation on $\mathbb{R}^-$, respectively, where $\mathbb{X}$ and $\mathbb{Y}$ are any Banach spaces.

Lemma 5.2. *Let $F \in L(\mathcal{B}, \mathbb{Y})$ be the one satisfying*

$$F(\phi) = \int_{-\infty}^{0} [d\eta(\theta)]\phi(\theta) \qquad (\forall \phi \in C_{00}), \tag{27}$$

where $\eta \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^-; L(\mathbb{X}, \mathbb{Y}))$ and $\eta(0) = 0$. Then there exists a unique $\zeta \in \mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; L(\mathbb{X}, \mathbb{Y}))$ such that

$$F(\phi) = \int_{-\infty}^{0} [d\zeta(\theta)]\phi(\theta) \qquad (\forall \phi \in C_{00}). \tag{28}$$

Proof. We first assert that the $L(\mathbb{X}, \mathbb{Y})$-valued function η is continuous except at most at countable points. To establish this assertion, we set $\rho(\theta) = Var(\eta(\cdot) : [\theta, 0])$ if $\theta < 0$, and $\rho(0) = 0$. Then the function $\rho : \mathbb{R}^- \mapsto \mathbb{R}^+$ is nonincreasing, and hence ρ is continuous except at most at countable points. Then the assertion follows from this observation and the inequality $\|\eta(\theta) - \eta(\theta_0)\| \le |\rho(\theta) - \rho(\theta_0)|$ for $\theta, \theta_0 \le 0$.

We next assert that for any $\theta_0 \le 0$ the left limit

$$\lim_{\theta < \theta_0,\ \theta \to \theta_0} \eta(\theta) =: \eta(\theta_0^-) \tag{29}$$

exists in $L(\mathbb{X}, \mathbb{Y})$. Suppose this assertion is false. Then there exist $\epsilon_0 > 0$ and increasing sequences $\{\theta_1^{(n)}\}$ and $\{\theta_2^{(n)}\}$ such that $\theta_0 - 1 < \theta_1^{(1)} < \theta_2^{(1)} < \theta_1^{(2)} < \theta_2^{(2)} < \cdots < \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_0$ and that

$$\inf_{n \in \mathbb{N}} \|\eta(\theta_1^{(n)}) - \eta(\theta_2^{(n)})\| \ge \epsilon_0.$$

Then $Var(\eta(\cdot) : [\theta_0 - 1, \theta_0]) \ge \sum_{n=1}^{N} \|\eta(\theta_1^{(n)}) - \eta(\theta_2^{(n)})\| \ge N\epsilon_0 \to \infty$ as $N \to \infty$. This is a contradiction, because η is an $L(\mathbb{X}, \mathbb{Y})$-valued function of locally bounded variation.

Now we put $\zeta(\theta) = \eta(\theta^-)$ if $\theta < 0$, and $\zeta(0) = 0$. It is straightforward to see that $\zeta \in \mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; L(\mathbb{X}, \mathbb{Y}))$. We shall establish the equality (28). Let $\phi \in C_{00}$ be given. Then $L(\phi) = \displaystyle\int_{-\infty}^{0} [d\eta(\theta)]\phi(\theta)$ and the integration is represented as a limit of Riemann sum $\displaystyle\sum_j [\eta(\theta_j) - \eta(\theta_{j-1})]\phi(\theta_j)$, where $\Delta : -a = \theta_0 < \theta_1 < \cdots\cdots < \theta_N = 0$ is a partition of the intetval $[-a, 0]$ which contains the support of ϕ, and all $\theta_j,\ j = 0, 1, \cdots, N-1$ are points at which η is continuous. Observe that $\zeta(\theta_j) = \eta(\theta_j)$ for $j = 0, 1, \cdots, N$. This observation leads to (28), as required.

Finally we shall show the uniqueness of ζ. To do this, it is sufficient to certify that $\chi(s) \equiv 0$ $(\forall s < 0)$ under the assumption that $\chi \in \mathrm{NBV}_{\mathrm{loc}}(\mathbb{R}^-; L(\mathbb{X}, \mathbb{Y}))$ and $\displaystyle\int_{-\infty}^{0} [d\chi(\theta)]\phi(\theta) = 0$ $(\forall \phi \in C_{00})$. Let $s < 0$ be given, and consider a function $f^{n,s}$ defined by

$$f^{n,s}(\theta) = \begin{cases} 0, & \theta < s - 1/n \\ (n\theta - ns + 1)I, & s - 1/n \le \theta \le s \\ I, & s < \theta \le 0. \end{cases}$$

It follows that $f^{n,s}x \in C_{00}$ for any $x \in \mathbb{X}$, and hence

$$\begin{aligned} 0 &= \int_{-\infty}^{0} [d\chi(\theta)] f^{n,s}(\theta)x \\ &= \int_{s-1/n}^{s} [d\chi(\theta)]\{n(\theta - s) + 1\}x + \int_{s}^{0} [d\chi(\theta)]x \\ &= -n \int_{s-1/n}^{s} \chi(\theta)d\theta \cdot x, \end{aligned}$$

or

$$n \int_{s-1/n}^{s} \chi(\theta)d\theta \cdot x = 0.$$

Letting $n \to \infty$ in the above, we get $0 = \chi(s^-)x = \chi(s)x$ for any $x \in \mathbb{X}$, and consequently $\chi(s) = 0$, as required. This completes the proof. $\square$

Let $r > 0$. We denote by $\mathrm{BV}([-r,0];\mathbb{X}^*)$ the space of all $\mathbb{X}^*$-valued functions η of bounded variation on $[-r,0]$, and moreover by $\mathrm{NBV}([-r,0];\mathbb{X}^*)$ the space of all $\zeta \in \mathrm{BV}([-r,0];\mathbb{X}^*)$ which are left continuous at each $\theta \in (-r,0)$ and $\zeta(0) = 0$.

Lemma 5.3. *Let $r > 0$, and let $f \in C([-r,0],\mathbb{X})^*$. Then there exists a unique $\zeta \in \mathrm{NBV}([-r,0],\mathbb{X}^*)$ such that*

$$f(\phi) = \int_{-r}^{0} [d\zeta(\theta)]\phi(\theta) \qquad (\forall \phi \in C([-r,0],\mathbb{X}). \tag{30}$$

Proof. Let $M([-r,0],\mathbb{X})$ be the Banach space of all $\mathbb{X}$-valued bounded functions on $[-r,0]$ with the supremum norm. Since $C([-r,0],\mathbb{X})$ is a closed subspace of $M([-r,0],\mathbb{X})$, the Hahn-Banach theorem implies that there is an $F \in M([-r,0],\mathbb{X})^*$ which is an extension of f with $\|F\| = \|f\|$.

Now, for any $\theta \in [-r,0]$ we consider a linear functional $\eta(\theta)$ on $\mathbb{X}$ defined by

$$\eta(\theta)x = \begin{cases} -F(\chi_{[\theta,0]}x), & \theta < 0 \\ 0, & \theta = 0 \end{cases}$$

for $x \in \mathbb{X}$, where χ_J is the characteristic function of the interval J. Then $\eta(\theta) \in \mathbb{X}^*$, because of the inequality

$$|\eta(\theta)x| = |F(\chi_{[\theta,0]}x)| \le \|F\|\|\chi_{[\theta,0]}x\| \le \|F\|\|x\|.$$

We claim that $\eta \in \mathrm{BV}([-r,0],\mathbb{X}^*)$. For any partition $\Delta : -r = \theta_0 < \theta_1 < \cdots\cdots < \theta_N = 0$ we set

$$V(\eta;\Delta) = \sum_{j=1}^{N} \|\eta(\theta_j) - \eta(\theta_{j-1})\|.$$

In what follows we shall evaluate $V(\eta;\Delta)$. Let any $\epsilon > 0$ be given. For each $j = 1,\cdots,N$, there exists an $x_j \in \mathbb{X}$ with $\|x_j\|_{\mathbb{X}} \le 1$ such that

$$\|\eta(\theta_j) - \eta(\theta_{j-1})\| < \epsilon/N + [\eta(\theta_j) - \eta(\theta_{j-1})]x_j.$$

Notice that

$$\begin{aligned} F(\chi_{[\theta_{j-1},\theta_j)}x_j) &= F(\chi_{[\theta_{j-1},0]}x_j) - F(\chi_{[\theta_j,0]}x_j) \\ &= [\eta(\theta_j) - \eta(\theta_{j-1})]x_j, \forall\ j = 1, \cdots, N-1, \end{aligned}$$

and

$$\begin{aligned} F(\chi_{[\theta_{N-1},\theta_N]}x_N) &= -\eta(\theta_{N-1})x_N \\ &= [\eta(\theta_N) - \eta(\theta_{N-1})]x_N. \end{aligned}$$

Put

$$\psi = \sum_{j=1}^{N-1} \chi_{[\theta_{j-1},\theta_j)}x_j + \chi_{[\theta_{N-1},\theta_N]}x_N.$$

Then $\psi \in M([-r,0],\mathbb{X})$ and $\|\psi\| \le 1$, and hence

$$\begin{aligned} V(\eta;\Delta) &< \epsilon + \sum_{j=1}^{N} [\eta(\theta_j) - \eta(\theta_{j-1})]x_j \\ &= \epsilon + F(\psi) \\ &\le \epsilon + \|F\| = \epsilon + \|f\|. \end{aligned}$$

Therefore

$$Var(\eta;[-r,0]) = \sup_{\Delta} V(\eta;\Delta) \le \epsilon + \|f\|,$$

which shows that $\eta \in \mathrm{BV}([-r,0],\mathbb{X}^*)$ with $Var(\eta;[-r,0]) \le \|f\|$, because ϵ is arbitrary.

Next, we shall establish the equality

$$f(\phi) = \int_{-r}^{0} [d\eta(\theta)]\phi(\theta) \qquad (\forall \phi \in C([-r,0],\mathbb{X}).$$

Let $\phi \in C([-r,0],\mathbb{X})$ be given. The integral $\int_{-r}^{0} [d\eta(\theta)]\phi(\theta)$ is represented as a limit of Riemann sum $\sum_{j=1}^{N} [\eta(\theta_j) - \eta(\theta_{j-1})]\phi(\theta_j)$, where $\Delta : -r = \theta_0 < \theta_1 < \cdots\cdots < \theta_N = 0$. Let us consider a function $\phi^\Delta \in M([-r,0],\mathbb{X})$ defined by

$$\phi^\Delta = \sum_{j=1}^{N-1} \phi(\theta_j)\chi_{[\theta_{j-1},\theta_j)} + \phi(\theta_N)\chi_{[\theta_{N-1},\theta_N]}.$$

It follows from the uniform continuity of ϕ that $\|\phi^\Delta - \phi\| \to 0$ as $|\Delta| \to 0$. Since $F(\phi^\Delta) = \sum_{j=1}^{N} [\eta(\theta_j) - \eta(\theta_{j-1})]\phi(\theta_j)$, we get

$$\int_{-r}^{0} [d\eta(\theta)]\phi(s) = \lim_{\Delta \to 0} F(\phi^\Delta) = F(\phi) = f(\phi),$$

as required.

Finally, repeating almost the same argument as in the proof of Lemma 5.2, one can normalize η to get $\zeta \in \mathrm{NBV}([-r,0],\mathbb{X}^*)$ which satisfies Relation (30). We shall show the uniqueness of ζ. To do this, it is sufficient to certify that $\chi(s) \equiv 0$ $(\forall s \in [-r,0])$ under the assumption that $\chi \in \mathrm{NBV}([-r,0];\mathbb{X}^*)$ and

$$\int_{-r}^{0} [d\chi(\theta)]\phi(\theta) = 0 \qquad (\forall \phi \in C([-r,0];\mathbb{X})). \tag{31}$$

By the same reasoning as in the proof of Lemma 5.2, we see that $\chi(s) = 0$ for all $s \in (-r,0]$. Then $\int_{-r}^{0} [d\chi(\theta)]\phi(\theta) = -\chi(-r)\phi(-r)$, and hence $\chi(-r) = 0$ by (31). This completes the proof. □

We are now in a position to give the proof of Proposition 5.1.

Proof. For any $r > 0$ we consider a functional $f^r \in C([-r,0],\mathbb{X})^*$ defined by

$$f^r(\psi) = \langle \phi^*, \hat{\psi}^r \rangle \qquad (\forall \psi \in C([-r,0],\mathbb{X}),$$

where

$$\hat{\psi}^r(s) = \begin{cases} \psi(s), & -r \le s \le 0 \\ \psi(-r)(s+r+1), & -r-1 \le s \le -r \\ 0, & s < -r-1. \end{cases}$$

Indeed, f^r is well-defined because $\hat{\psi}^r \in C_{00}$ with $\|\hat{\psi}^r\|_{\mathcal{B}} \le K(r+1)\|\psi\|$, and it satisfies $\|f^r\| \le K(r+1)\|\phi^*\|$. By virtue of Lemma 5.3, there exists a $\zeta^r \in \mathrm{NBV}([-r,0],\mathbb{X}^*)$ such that

$$\langle \phi^*, \hat{\psi}^r \rangle = f^r(\psi) = \int_{-r}^{0} [d\zeta^r(\theta)]\psi(\theta) \qquad (\forall \psi \in C([-r,0],\mathbb{X}).$$

Let $l > r$. Then there exists $\zeta^l \in \mathrm{NBV}([-l,0],\mathbb{X}^*)$ such that

$$\langle \phi^*, \hat{\psi}^l \rangle = \int_{-l}^{0} [d\zeta^l(\theta)]\psi(\theta) \qquad (\forall \psi \in C([-l,0],\mathbb{X})).$$

In particular, for any $\psi \in C([-l,0],\mathbb{X})$ whose support is contained in $(-r,0]$ we get $\hat{\psi}^l \equiv \widehat{\psi|_{[-r,0]}}^{\,r}$, and hence

$$\int_{-l}^{0} [d\zeta^l(\theta)]\psi(\theta) = \int_{-r}^{0} [d\zeta^r(\theta)]\psi(\theta)$$

or

$$\int_{-r}^{0} [d\chi(\theta)]\psi(\theta) = 0,$$

where $\chi(\theta) := \zeta^l(\theta) - \zeta^r(\theta)$ for $\theta \in [-r,0]$. Therefore, by the same reasoning as in the proof of Lemma 5.3 one can see that $\chi(\theta) = 0$ or $\zeta^l(\theta) = \zeta^r(\theta)$ for all $\theta \in (-r,0]$. Define ζ by $\zeta(\theta) = \zeta^r(\theta)$ if $\theta \in (-r,0]$. Then ζ is well defined, and it satisfies all requirements in the proposition. □

5.2. **Existence of Almost Periodic Solutions.** In this subsection we shall give a proof for the existence of almost periodic solutions of Eq. (19),

$$\dot{z}(t) = Gz(t) - \langle \tilde{\Psi}(t)(0^-), f(t) \rangle \qquad \text{a.e.},$$

under the assumption that Eq. (19) has a bounded (locally absolutely continuous) solution on $\mathbb{R}^+$ to completeness the proof of Theorem 3.6. Let us recall that G is a $d \times d$ matrix, $f(t)$ is an $\mathbb{X}$-valued almost periodic function and $\tilde{\Psi}(t)(0^-)$ is a d-column vector whose components are $\mathbb{X}^*$-valued functions. Moreover $\tilde{\Psi}(t)(0^-)$ is 1-periodic and weak-star left continuous in $t \in \mathbb{R}$ with $\sup_{t\in\mathbb{R}} |\tilde{\Psi}(t)(0^-)| \le K(1) \sup_{t\in\mathbb{R}} \|\Psi(t)\| =: C_6 < \infty$ by (17).

Proposition 5.4. *If Eq. (19) has a bounded solution on $\mathbb{R}^+$, then it has an almost periodic solution on $\mathbb{R}$.*

Before proving the proposition, we prepare a lemma.

Lemma 5.5. *If Eq. (19) has a bounded solution on $\mathbb{R}^+$, then it has a bounded solution on $\mathbb{R}$.*

Proof. Let $z(t)$ be a bounded solution of Eq. (19) on $\mathbb{R}^+$. From Lemma 3.4 we know that $\langle \tilde{\Psi}(t)(0^-), f(t) \rangle$ is locally integrable in $t \in \mathbb{R}$, and hence $z(t)$ satisfies the integral relation

$$z(t) = z(s) + \int_s^t \{Gz(\xi) - \langle \tilde{\Psi}(\xi)(0^-), f(\xi) \rangle\} d\xi \qquad (\forall t, s \in \mathbb{R}^+). \tag{32}$$

Therefore it follows that $z(t)$ is uniformly Lipschitz continuous on $\mathbb{R}^+$.

Now let $\{t_n\}$ be any sequence such that $\lim_{n\to\infty} t_n = \infty$. Since f is almost periodic, taking a subsequence if necessary, one can assume that $\lim_{n\to\infty} f(t+t_n) = g(t)$ uniformly on $\mathbb{R}$, for some almost periodic function g. Let $t_n = [t_n] + \tau_n$, $0 \le \tau_n < 1$, where $[t_n]$ denotes the largest integer which does not exceed t_n. Considering a subsequence, one can assume that $\lim_{n\to\infty} \tau_n = \tau$ for some $\tau \in [0, 1]$. Set

$$\hat{\tau}_n = \min\{\tau, \tau_n\} - 1/n, \qquad \check{\tau}_n = \max\{\tau, \tau_n\} + 1/n.$$

Then $\hat{\tau}_n < \tau < \check{\tau}_n$ and $\lim_{n\to\infty} \hat{\tau}_n = \lim_{n\to\infty} \check{\tau}_n = \tau$. Moreover we set

$$[\hat{t}]_n = [t_n] + \hat{\tau}_n, \qquad [\check{t}]_n = [t_n] + \check{\tau}_n.$$

Let us consider a family of functions $\{z^n(t)\}$ which is defined by

$$z^n(t) = z(t + [\hat{t}]_n) \qquad (\forall t \ge -[\hat{t}]_n).$$

The family $\{z^n(t)\}$ is uniformly bounded and equicontinuous on any bounded interval in $\mathbb{R}$. Therefore, considering a subsequence if necessary, one can assume that $\{z^n(t)\}$ converges to some $y \in BC(\mathbb{R}, \mathbb{C}^d)$ uniformly on any bounded interval in $\mathbb{R}$. Notice that $\lim_{n\to\infty} f(t+[\hat{t}]_n) = \lim_{n\to\infty} f(t_n+t+\hat{\tau}_n-\tau_n) = g(t)$ uniformly on $\mathbb{R}$. Since $\lim_{n\to\infty} \tilde{\Psi}(t+$

$\hat{\tau}_n)(0^-) = \tilde{\Psi}(t+\tau)(0^-)$ in the weak-star topology of X^* by the weak-star left continuity of $\tilde{\Psi}(t)(0^-)$, we get

$$\begin{aligned}
& |\langle \tilde{\Psi}(t+\hat{[t]}_n)(0^-), f(t+\hat{[t]}_n)\rangle - \langle \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle| \\
= \ & |\langle \tilde{\Psi}(t+\hat{\tau}_n)(0^-), f(t+\hat{[t]}_n)\rangle - \langle \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle| \\
\leq \ & |\langle \tilde{\Psi}(t+\hat{\tau}_n)(0^-), f(t+\hat{[t]}_n) - g(t)\rangle| \\
& + |\langle \tilde{\Psi}(t+\hat{\tau}_n)(0^-) - \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle| \\
\leq \ & C_6 \|f(t+\hat{[t]}_n) - g(t)\|_{\mathbb{X}} + |\langle \tilde{\Psi}(t+\hat{\tau}_n)(0^-) - \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle| \to 0
\end{aligned}$$

as $n \to \infty$, and consequently $\lim_{n\to\infty} \langle \tilde{\Psi}(t+\hat{[t]}_n)(0^-), f(t+\hat{[t]}_n)\rangle = \langle \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle$. ¿From (32) it follows that

$$z^n(t) = z^n(s) + \int_s^t \{Gz^n(\xi) - \langle \tilde{\Psi}(\xi+\hat{[t]}_n)(0^-), f(\xi+\hat{[t]}_n)\rangle\} d\xi \qquad (\forall t, s \geq -\hat{[t]}_n).$$

Letting $n \to \infty$ in the above, one gets by Lebesgue's convergence theorem that

$$y(t) = y(s) + \int_s^t \{Gy(\xi) - \langle \tilde{\Psi}(\xi+\tau)(0^-), g(\xi)\rangle\} d\xi \qquad (\forall t, s \in \mathbb{R}),$$

which shows that y is absolutely continuous and satisfies the equation

$$\dot{y}(t) = Gy(t) - \langle \tilde{\Psi}(t+\tau)(0^-), g(t)\rangle \qquad \text{a.e.}$$

in $t \in \mathbb{R}$.

Next we set $y^n(t) = y(t - \check{[t]}_n)$ for any $t \in \mathbb{R}$. Since $\{y^n(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$, one can assume that $\{y^n(t)\}$ converges to some $p \in BC(\mathbb{R}, \mathbb{C}^d)$ uniformly on any bounded interval in $\mathbb{R}$. Observe that $\lim_{n\to\infty} g(t - \check{[t]}_n) = f(t)$ uniformly on $\mathbb{R}$ and that $\lim_{n\to\infty} \tilde{\Psi}(t+\tau-\check{[t]}_n)(0^-) = \tilde{\Psi}(t)(0^-)$ in the weak-star topology of X^*. Then, by the same reasoning as in the preceding paragraph, we see that $\dot{p}(t) = Gp(t) - \langle \tilde{\Psi}(t)(0^-), f(t)\rangle$ a.e. in $t \in \mathbb{R}$. Thus p is a bounded solution of Eq. (19) on $\mathbb{R}$. □

Now we shall prove Proposition 5.4.

Proof. For any bounded solution z of Eq. (19) on $\mathbb{R}$, we set $\|z\| = \sup_{t\in\mathbb{R}} |z(t)|$, where $|\cdot|$ is the Euclidean norm of $\mathbb{C}^d$. Moreover we set

$$\lambda = \inf\{\|z\| : z \in \mathcal{K}\},$$

where $\mathcal{K}$ is the set of all bounded solutions of Eq. (19) on $\mathbb{R}$. By Lemma 5.5 we get $0 \leq \lambda < \infty$. We shall show that $\lambda = \|p\|$ for some $p \in \mathcal{K}$ (such a p is called *a minimal solution* of Eq. (19)). Indeed, one can choose a sequence $\{z^n\}$ in $\mathcal{K}$ so that $\lambda \leq \|z^n\| \leq \lambda + 1/n$ for all $n \in \mathbb{N}$. Since $\{z^n(t)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$, one can assume that $\{z^n(t)\}$ converges to some $p \in BC(\mathbb{R}, \mathbb{C}^d)$ uniformly on any bounded interval in $\mathbb{R}$. By the same reasoning as in the proof of Lemma 5.5, we see that $p \in \mathcal{K}$. Letting $n \to \infty$ in the relation $|z^n(t)| \leq \|z^n\| \leq \lambda + 1/n$, we get $|p(t)| \leq \lambda$ $(\forall t \in \mathbb{R})$ or $\|p\| = \lambda$.

Now we shall prove that a minimal solution p of Eq. (19) is almost periodic. To do this, it is sufficient to establish that if $\{t_n\}$ is any sequence in $\mathbb{R}$, then $\{p(t+t_n)\}$ contains a subsequence which converges uniformly on $\mathbb{R}$ (e.g., see [5], Theorem 1.14). In what follows, we will prove this by a contradiction argument. Let us assume that $\{p(t+t_n)\}$ does not contain any subsequence which converges uniformly on $\mathbb{R}$. Then there exist a constant $\delta > 0$ and sequences $\{t_n^{(1)}\}$, $\{t_n^{(2)}\} \subset \{t_n\}$ and $\{s_n\} \subset \mathbb{R}$ such that

$$\inf_{n\in\mathbb{N}} |p(t_n^{(1)}+s_n) - p(t_n^{(2)}+s_n)| \geq \delta.$$

By taking a subsequence if necessary, one can assume that $\lim_{n\to\infty} f(t+t_n) = g(t)$ and $\lim_{n\to\infty} g(t+s_n) = h(t)$ uniformly on $\mathbb{R}$ for some almost periodic functions g and h. Then $\lim_{n\to\infty} f(t+t_n^{(j)}+s_n) = h(t)$ uniformly on $\mathbb{R}$. In the following we use the notation $[t_n]$, $[\hat{t}]_n$ and so on which were introduced in the proof of Lemma 5.5. Let $t_n = [t_n]+\tau_n$ and $s_n = [s_n]+\mu_n$ with $\tau_n, \mu_n \in [0,1)$. One can assume that $\lim_{n\to\infty}\tau_n = \tau$ and $\lim_{n\to\infty}\mu_n = \mu$ for some $\tau, \mu \in [0,1]$. Since $\lim_{n\to\infty}(t_n^{(j)} + s_n - [\hat{t}]_n^{(j)} - [\hat{s}]_n) = 0$ for $j = 1,2$, from the uniform continuity of p it follows that

$$\inf_{n\geq n_0} |p([\hat{t}]_n^{(1)} + [\hat{s}]_n) - p([\hat{t}]_n^{(2)} + [\hat{s}]_n)| \geq \delta/2 \tag{33}$$

for some $n_0 \in \mathbb{N}$. For $j = 1,2$ the family of functions $\{p(t+[\hat{t}]_n^{(j)}+[\hat{s}]_n)\}$ is uniformly bounded and equicontinuous on $\mathbb{R}$, and hence one can assume that $\lim_{n\to\infty} p(t+[\hat{t}]_n^{(j)} + [\hat{s}]_n) = q^{(j)}(t)$ uniformly on any bounded interval in $\mathbb{R}$, for some $q^{(j)} \in BC(\mathbb{R}, \mathbb{C}^d)$. Especially, we get

$$|q^{(1)}(0) - q^{(2)}(0)| > 0 \tag{34}$$

by (33). Observe that $\lim_{n\to\infty} f(t+[\hat{t}]_n^{(j)}+[\hat{s}]_n) = h(t)$ uniformly on $\mathbb{R}$ and that $\lim_{n\to\infty}\tilde{\Psi}(t+[\hat{t}]_n^{(j)} + [\hat{s}]_n)(0^-) = \tilde{\Psi}(t+\tau+\mu)(0^-)$ in the weak-star topology of X^*. Then, by the same reasoning as in the preceding paragraph, we see that the functions $q^{(1)}$ and $q^{(2)}$ satisfy the equation

$$\dot{z}(t) = Gz(t) - \langle\tilde{\Psi}(t+\tau+\mu)(0^-), h(t)\rangle \qquad \text{a.e.}, \tag{35}$$

in $t \in \mathbb{R}$ with $\|q^{(1)}\| = \|q^{(2)}\| = \lambda$. Put

$$q(t) = \frac{q^{(1)}(t)+q^{(2)}(t)}{2}, \qquad w(t) = \frac{q^{(1)}(t)-q^{(2)}(t)}{2}.$$

Then q satisfies Eq. (35) in a.e. $t\in\mathbb{R}$. On the one hand, w is a bounded solution of the equation $\dot{w} = Gw$ on $\mathbb{R}$, and hence it must be almost periodic. Therefore it follows from Theorem 5.7 in [5] that $\inf_{t\in\mathbb{R}} |w(t)| > 0$ because w is nontrivial by (34). Thus we get

$$|q(t)|^2 + |w(t)|^2 = \frac{1}{2}(|q^{(1)}(t)|^2 + |q^{(2)}(t)|^2) \leq \lambda^2$$

or

$$\|q\| \leq \left(\lambda^2 - \inf_{t\in\mathbb{R}} |w(t)|^2\right)^{1/2} < \lambda.$$

By the same reasoning as in the last paragraph of the proof of Lemma 5.5, one can assume that $\lim_{n\to\infty} q(t - [\check{t}]_n^{(1)} - [\check{s}]_n) = r(t)$ for some $r \in \mathcal{K}$. Notice that $\|r\| = \|q\| < \lambda = \|p\|$. This is a contradiction to the fact that p is a minimal solution of Eq. (19). This completes the proof of the proposition. □

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