On White Hole Solutions of a Class of Nonlinear Ordinary Differential Equations of the Second Order

By
Jaroslav JAROŠ and KUSANO Takaŝi
(Comenius University, Slovakia and Fukuoka University, Japan)

1. Introduction

Consider the nonlinear ordinary differential equation

$$(|y'|^{\alpha})' + q(t)|y|^{\beta} = 0 \tag{A}$$

where $\alpha, \beta \in R, \alpha \neq 0$, are constants and $q : [a, \infty) \to (0, \infty), a \geq 0$, is a continuous function.

By a solution of (A) on an interval $J = [t_0, T), a \leq t_0 < T \leq \infty$, we mean a function $y \in C^1(J)$ which has the property $|y'|^{\alpha} \in C^1(J)$ and satisfies (A) at each point of J. If we denote by T_y the maximal existence time of y, then we say that $y(t)$ is *proper* if $T_y = \infty$ and

$$\sup\{|y(t)| : t \in [\tau, \infty)\} > 0 \quad \text{for} \quad \text{any} \quad \tau \geq t_0.$$

A solution $y(t)$ is called *singular* if either $T_y < \infty$ or $T_y = \infty$ and there exists $T \in [t_0, \infty)$ such that

$$\max\{y(s) : t \leq s \leq T\} > 0 \quad \text{for} \quad t \in (t_0, T)$$

and $y(t) = 0$ for $t \geq T$. In the later case, the interval $[t_0, T)$ is called the *support* of the solution $y(t)$.

Our main objective here is to investigate the structure of the solution set of (A) in the case $\alpha > 0$ and to show that nonlinear equations of the form (A) may have singular solutions of a new type satisfying

$$\lim_{t \to T_y - 0} y(t) = \text{const} \neq 0 \quad \text{and} \quad \lim_{t \to T_y - 0} y'(t) = 0 \tag{1}$$

at the (finite) right end-point of the maximal interval of existence. By analogy with the concept of "black hole" solutions, that is, singular solutions defined on $[t_0, T_y)$ and satisfying

$$\lim_{t\to T_y-0} y(t) = \text{const} \neq 0 \quad \text{and} \quad \lim_{t\to T_y-0} |y'(t)| = \infty, \tag{2}$$

introduced by the present authors in [3] (see also [4], [7] and [8]), positive solutions of (A) satisfying (1) as t approaches the maximal existence time $T_y < \infty$ are called *white hole* (singular) *solutions.*

An example of a nonlinear equation of the form (A) (with $\beta = 0$) which possesses singular solutions of this new type is

$$(|y'|^\alpha)' + \left(\frac{\alpha+1}{\alpha}\right)^\alpha = 0, \tag{3}$$

where $\alpha > 0$. Indeed, for any given $T > a$ and $c > 0$, the function $y(t) = c + (T-t)^{\frac{\alpha+1}{\alpha}}$ defined and positive on $[a, T)$ is a decreasing solution of (3) with a singularity of white hole type at T.

Similarly, for any $c > 0$, the function $y(t) = c - (T-t)^{\frac{\alpha+1}{\alpha}}$ provides an example of a 'local' increasing white hole solution of (3) which is defined and positive in some sufficiently small left neighborhood of the maximal existence time T.

Another simple example of an equation of the form (A) having white hole singular solutions is the following "almost linear" equation

$$(|y'|)' + |y| = 0. \tag{4}$$

As easily seen, for any real T and any $c > 0$, the function $y(t) = c\cos(T-t)$ defined and positive on $[t_0, T), t_0 \geq T - \frac{\pi}{2}$, is an increasing singular solution of (4) which is of the white hole type.

While the existence and asymptotic theory for quasilinear second-order differential equations of the form

$$(p(t)|y'|^\alpha \text{sgn} y')' + q(t)|y|^\beta \text{sgn} y = 0, \quad t \geq a, \tag{B}$$

and for singular equations

$$(C) \qquad (p(t)|y'|^{\alpha}\text{sgn}y')' + q(t)y^{-\beta} = 0, \quad t \geq a,$$

where $\alpha > 0$ and $\beta > 0$ are constants and $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, is well developed (see, for example, the papers [1], [5-6] and [9-11]), according to our knowledge there are no papers concerning the existence of singular and proper solutions for the nonlinear equation (A) (with $\alpha > 0$) in our setting. This observation was one of the motivations for the present paper.

The plan of this paper is as follows. In Section 2 we first show that, regardless of positivity or negativity of β, Eq. (A) always has singular solutions of white hole type iff $\alpha > 0$. Our procedure for establishing the existence of such solutions for (A) is based on the solution of appropriate nonlinear integral equation via the Schauder fixed point theorem. Secondly, we investigate the existence of another type of singular solutions of (A) named “extinct solutions”. More specifically, it is shown that if either $0 < \alpha \leq 1$ and $|\beta| < \alpha$ or $\alpha > 1$ and $-1 < \beta < \alpha$, then Eq. (A) possesses a singular solution which extincts (together with its first derivative) at an arbitrarily prescribed extinction point $T > a$.

In Sections 3-5 our consideration is focused on the set of proper solutions of (A), i.e., solutions which exist on some interval $[t_0, \infty) \subset [a, \infty)$ and are not identically zero in any neighborhood of infinity. Although the equation (A) has a relativelly simple form, the totality of proper solutions of (A) has surprisingly rich structure. This is demonstrated in Section 3 where the set of all possible proper solutions is classified into eight different types according to their asymptotic behavior as $t \to \infty$.

In Section 4 we establish conditions guaranteeing the existence of increasing proper solutions of each of the types (IV)-(VI) appearing in the general classification scheme given in Section 3. We prove in particular that if $0 < \beta < \alpha$, then for any given $y_0 > 0$, Eq. (A) has a ‘global’ solution y (i.e., a proper solution existing on the whole interval $[a, \infty)$) satisfying $y(a) = y_0$ and growing to infinity as fast as a constant multiple of t as $t \to \infty$ if and only if the function $t^{\beta}q(t)$ is integrable on $[a, \infty)$. The next theorem in Section 4 presents sufficient conditions under which Eq. (A) possesses an increasing proper solution which grows at infinity like a positive constant multiple of $t^{(\alpha+\sigma+1)/(\alpha-\beta)}$ for some $\sigma \in R$ with $0 < (\alpha+\sigma+1)/(\alpha-\beta) < 1$.

As regards increasing proper solutions of (A) which remain bounded as $t \to \infty$, the ('local') existence of such solutions satisfying $\lim_{t\to\infty} y(t) = \omega_0$ for arbitrarily prescribed terminal value $\omega_0 > 0$ is characterized by the integral condition (38) below.

Finally, in Section 5, the existence of decreasing proper solutions of (A) is discussed. First, it is shown that the necessary and sufficient condition for the existence of a decreasing proper solution of (A) which remains positive as long as it exists and tends to a given positive constant ω_0 as $t \to \infty$ is the same as the condition characterizing the existence of bounded increasing proper solutions for (A). Next, we establish conditions guaranteeing the existence of positive proper solutions which decay to zero at infinity like a positive constant multiple of the function $t^{(\alpha+\sigma+1)/(\alpha-\beta)}$ for some $\sigma < -1$ where $(\alpha+\sigma+1)/(\alpha-\beta) < 0$. We end Section 5 with the existence result characterizing the situation in which Eq. (A) (with $0 < \beta < \alpha$) possesses an eventually negative decreasing proper solution emanating from a given point $(a, y_0), y_0 > 0$, with specific asymptotic behavior as $t \to \infty$.

2. Existence of "white hole" solutions

From (A) with $\alpha > 0$ we easily see that if $\beta < 0$ and $y(t)$ is a singular solution of (A), then $T_y < \infty$ and either $y'(t) > 0$ and $y'(t)$ is decreasing on $J = [t_0, T_y)$ or $y'(t) < 0$ and $y'(t)$ is increasing on J. Thus, the following three cases are possible: either

$$\text{(I)} \qquad \lim_{t\to T_y-0} y(t) = 0 \quad \text{and} \quad \lim_{t\to T_y-0} y'(t) = 0,$$

or

$$\text{(II)} \qquad \lim_{t\to T_y-0} y(t) = \text{const} \neq 0 \quad \text{and} \quad \lim_{t\to T_y-0} y'(t) = 0,$$

or

$$\text{(III)} \qquad \lim_{t\to T_y-0} y(t) = 0 \quad \text{and} \quad \lim_{t\to T_y-0} y'(t) = \text{const} \neq 0.$$

A solution of the type (I) (resp. (III)) is usually referred to as an *extinct* solution of the *first kind* (resp. an *extinct* solution of the *third kind*), while a solution of the type (II) is a singular solution of a new type that we suggest to call a *white hole* singular solution.

We remark that by *extinct* solutions of *the second kind* we understand singular solutions satisfying

$$\lim_{t \to T_y - 0} y(t) = 0 \quad \text{and} \quad \lim_{t \to T_y - 0} |y'(t)| = \infty,$$

where $T_y < \infty$ is the right end-point of the maximal interval of existence of y which may exist for equations of the form (A) with $\alpha < 0$. Clearly, Eq. (A) with $\alpha > 0$ and positive q cannot have extinct solutions of this kind.

If $\beta \geq 0$, then along with singular solutions of the new type (II), the equation (A) may have also "usual" singular solutions which are defined in some neighborhood of infinity (i.e., $T_y = \infty$) and are identically zero for all large t. If $[t_0, T)$ is the support of such solution, then obviously (I) holds with T_y replaced by T and we may again call such a singular solution an *extinct* solution of the *first kind* (with the extinction point T).

Our first result in this section concerns the existence of increasing white hole solutions of (A) which are eventually positive, that is, singular solutions satisfying

$$\lim_{t \to T_y - 0} y(t) = c > 0 \quad \text{and} \quad \lim_{t \to T_y - 0} y'(t) = 0 \tag{5}$$

at the right end-point of the maximal interval of existence. One can characterize the existence of such singular solutions, as the following theorem shows.

Theorem 1. *A necessary and sufficient condition for Eq.* (A) *to have, for any given $T > a$ and $c > 0$, an increasing white hole solution defined in some left neighborhood of T and satisfying* (5) *is that $\alpha > 0$.*

Proof. (The "only if" part.) Let $y(t)$ be a positive increasing solution of (A) with dom(y) $= [t_0, T_y), a \leq t_0 < T_y < \infty$, satisfying (5). Assume to the contrary that $\alpha < 0$.

From the integrated form of Eq. (A)

$$|y'(t)|^\alpha = |y'(t_0)|^\alpha - \int_{t_0}^{t} q(s)|y(s)|^\beta ds, \quad t \in [t_0, T_y), \tag{6}$$

we see, in passing to the limit as $t \to T_y - 0$, that the left hand side tends to ∞, whereas the right hand side has a finite limit. This contradiction shows that α must be positive.

(The "if" part.) Suppose that $\alpha > 0$. Let $T > a$ and $c > 0$ be given arbitrarily. Put $q^*(t,T) = \max_{s\in[t,T]} q(s), a \le t \le T$, and choose $t_0 \in [a,T)$ so large that

$$\frac{\alpha}{\alpha+1}[q^*(t_0,T)]^{1/\alpha}(T-t_0)^{\frac{\alpha+1}{\alpha}} \le j^{-\frac{\beta}{\alpha}} \left(\frac{c}{2}\right)^{1-\frac{\beta}{\alpha}} \tag{7}$$

where $j = 1$ if $\beta < 0$ and $j = 2$ if $\beta \ge 0$.

Consider the set $Y \subset C[t_0,T]$ and the mapping $F : Y \to C[t_0,T]$ defined by

$$Y = \{y \in C[t_0,T] : \frac{c}{2} \le y(t) \le c, \quad t \in [t_0,T]\}$$

and

$$(Fy)(t) = c - \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr\right]^{1/\alpha} ds, \quad t \in [t_0,T].$$

It is easily verified that F is continuous and maps Y into a compact subset of Y, and so F has a fixed element $y \in Y$ by the Schauder fixed point theorem. Differentiating the integral equation $y(t) = (Fy)(t)$, we see that $y(t)$, when restricted to $[t_0,T)$, is the solution of (A) which satisfies (5). This completes the proof of Theorem 1.

An inspection of the sufficiency part of the proof of Theorem 1 shows that if $t_0 = a$ and $T > a$ are fixed and $c > 0$ is taken so that (7) is satisfied (with $j = 1$ if $\beta < 0$ and $j = 2$ if $\beta \ge 0$ and $\alpha \ne \beta$), then the desired increasing white hole solution is guaranteed to exist on the entire given interval $[a,T)$ and Theorem 1 can be re-formulated as follows.

Theorem 1'. ('Global' existence) *Let $\alpha \neq \beta$. For any $T > a$, Eq.* (A) *has an increasing white hole solution defined on* $[a, T)$ *and satisfying* (5) *for some $c > 0$ if and only if $\alpha > 0$.*

Similarly, it can be shown that if $\alpha > 0$, $T > a$ and $c > 0$ are given arbitrarily and $t_0 \in [a, T)$ is chosen such that

$$\frac{\alpha}{\alpha+1}(q^*(t_0,T))^{1/\alpha}(T-t_0)^{\frac{\alpha+1}{\alpha}} \leq j^{-\frac{\beta}{\alpha}} c^{1-\frac{\beta}{\alpha}}$$

where $j = 1$ if $\beta < 0$ and $j = 2$ if $\beta \geq 0$, then the mapping F defined by

$$(Fy)(t) = c + \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr\right]^{1/\alpha} ds, \quad t \in [t_0, T],$$

has a fixed element in the set Y of continuous functions defined by

$$Y = \{y \in C[t_0, T] : c \leq y(t) \leq 2c, \quad t \in [t_0, T]\}.$$

That this fixed point $y = y(t)$, when restricted to $[t_0, T)$, is a decreasing singular solution of (A) with desired properties follows from differentiation of the integral equation $y(t) = (Fy)(t)$.

Since the necessity of the condition $\alpha > 0$ for the existence of decreasing singular solutions of white hole type can be proved in a similar manner as in the proof of Theorem 1, we have the following

Theorem 2. *For any $T > a$ and any $c > 0$, Eq.* (A) *possesses a decreasing white hole solution defined on some interval $[t_0, T), a \leq t_0 < T$, and satisfying* (5) *at T if and only if $\alpha > 0$.*

If we are interested in 'global' existence of decreasing solutions of white hole type, then the above result can be re-formulated as follows.

Theorem 2'. *Let $\alpha \neq \beta$. For any $T > a$, Eq.* (A) *has a 'global' decreasing white hole solution defined on* $[a, T)$ *and satisfying* (5) *for some $c > 0$ if and only if $\alpha > 0$.*

Our next result in this part is the following theorem which shows that there is a class of equations of the form (A) for which one can construct extinct solutions of the first kind with arbitrarily prescribed extinction points.

Theorem 3. *Let $0 \le \beta < \alpha$. Then, for any $T > a$, Eq.* (A) *has an extinct solution of the first kind defined on $[a, \infty)$ with the support $[a, T)$.*

Proof. Let $T > a$ be given arbitrarily. Denote

$$q_* = \min_{t \in [a,T]} q(t), \quad q^* = \max_{t \in [a,T]} q(t),$$

and put

$$K = \left(\frac{\alpha - \beta}{\alpha + 1}\right) \left[\frac{\alpha - \beta}{\alpha(\beta + 1)}\right]^{1/\alpha}. \tag{8}$$

Define

$$c_1 = (q_* K^\alpha)^{\frac{1}{\alpha-\beta}} \quad \text{and} \quad c_2 = (q^* K^\alpha)^{\frac{1}{\alpha-\beta}}. \tag{9}$$

Since $0 \le \beta < \alpha$ implies $c_1 \le c_2$, we can consider the subset Y of the set of continuous functions $C[a, T]$ and the integral operator $F : Y \to C[a, T]$ defined by

$$Y = \{y \in C[a, T] : c_1(T - t)^{\frac{\alpha+1}{\alpha-\beta}} \le y(t) \le c_2(T - t)^{\frac{\alpha+1}{\alpha-\beta}}, t \in [a, T]\}$$

and

$$(Fy)(t) = \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr\right]^{1/\alpha} ds, \quad t \in [a, T].$$

It is easily checked that (i) $F(Y) \subset Y$, (ii) F is continuous, and (iii) $F(y)$ is compact in $C[a, T]$. Therefore, there exists a fixed element $y \in Y$ of F by the Schauder fixed point theorem. This fixed element $y = y(t)$ satisfies the integral equation

$$y(t) = \int_t^T \left[\int_s^T q(r)(y(r))^\beta dr\right]^{1/\alpha} ds, \quad t \in [a, T],$$

which implies that $y(t)$ is a solution of (A) on $[a, T)$ and satisfies

$$\lim_{t \to T-0} y(t) = 0 \quad \text{and} \quad \lim_{t \to T-0} y'(t) = 0.$$

Since this $y(t)$ may be continued to the right of T by putting $y(t) = 0$ on $[T, \infty)$, this establishes the existence of the desired extinct solution of the first kind for (A) defined on $[a, \infty)$ with the support $[a, T)$.

Theorem 3'. *Let either $0 < \alpha \le 1$ and $-\alpha < \beta < 0$ or $\alpha > 1$ and $-1 < \beta < 0$. Then, for any $T > a$, Eq.* (A) *has an extinct solution of the first kind with $[a, T)$ as its maximal interval of existence.*

Proof. Let $T > a$ be given arbitrarily. Define

$$c_1 = (K_1^{\beta} K_2^{\alpha})^{\frac{\alpha}{\alpha^2-\beta^2}} \quad \text{and} \quad c_2 = (K_2^{\beta} K_1^{\alpha})^{\frac{\alpha}{\alpha^2-\beta^2}} \tag{10}$$

where $K_1 = (q^*)^{1/\alpha} K, \quad K_2 = (q_*)^{1/\alpha} K$ and q_*, q^* and K are as above. If either $0 < \alpha \le 1$ and $-\alpha < \beta < 0$ or $\alpha > 1$ and $-1 < \beta < 0$, then $c_1 \le c_2$ and we can consider the set $Y \subset C[a, T]$ and the mapping $F : Y \to C[a, T]$ given as in the proof of Theorem 3. Since the rest of the proof is similar to that of the case $0 \le \beta < \alpha$, we can again conclude by the Schauder fixed point theorem that the operator F has a fixed element $y \in Y$ which (when reduced to $[a, T)$) gives the desired extinct solution of the first kind for (A) with the maximal interval of existence $[a, T)$.

Example 1. Consider the equation

$$(|y'|^{\alpha})' + \gamma |y|^{\beta} = 0 \tag{11}$$

where $\alpha > 0, \beta \in R$ and $\gamma > 0$ are constants. According to Theorem 3, if $0 \le \beta < \alpha$, then, for any $T > a$, Eq. (11) has an extinct solution of the first kind defined on $[a, \infty)$ with the support $[a, T)$. Indeed, for any $T > a$, the function y^* defined by

$$y^*(t) = c(T-t)^{\frac{\alpha+1}{\alpha-\beta}}, \text{ if } t \in [a, T), \text{ and } y^*(t) = 0, \text{ if } t \in [T, \infty), \tag{12}$$

where

$$c = \left[\frac{\gamma(\alpha-\beta)}{\alpha(\beta+1)} \left| \frac{\alpha-\beta}{\alpha+1} \right|^{\alpha} \right]^{\frac{1}{\alpha-\beta}}, \tag{13}$$

is the extinct solution of the first kind of the equation (11) if

$$\frac{\alpha+1}{\alpha-\beta} > 1 \tag{14}$$

which is clearly satisfied if $0 \le \beta < \alpha$ holds.

Similarly, if either $0 < \alpha \le 1$ and $-\alpha < \beta < 0$, or $\alpha > 1$ and $-1 < \beta < 0$, then, for any $T > a$, the function

$$y(t) = c(T-t)^{\frac{\alpha+1}{\alpha-\beta}},$$

where c is given by (13), is the extinct solution of the first kind of (11) with the maximal interval of existence $[a, T)$.

The following conjecture has been motivated by the gap between (14) and the conditions of Theorems 3 and 3'.

Conjecture 1. For any $T > a$, Eq. (A) has an extinct solution of the first kind (defined on $[a, \infty)$ or $[a, T)$ according to whether $\beta \ge 0$ or $\beta < 0$) with the extinction point at T if and only if (14) holds.

As we remarked in the introductory part of this section, if $\beta < 0$, then Eq. (A) may have also extinct solutions of the third kind, that is, singular solutions satisfying (III) at the (finite) right end-point of the maximal interval of existence. The following theorem establishes the existence of such singular solutions.

Theorem 4. *Suppose that $-1 < \beta < 0$. Then, for any $T > a$, Eq.* (A) *has an extinct solution of the third kind with $[a, T)$ as its maximal interval of existence.*

Proof. Let $T > a$ be given arbitrarily. Put $q^* = \max_{t \in [a,T]} q(t)$ and select $c > 0$ so that

$$\frac{1}{\beta+1} c^{\beta-\alpha} q^* (T-a)^{\beta+1} \le 2^\alpha - 1. \tag{15}$$

Consider the set $Y \subset C[a,T]$ and the mapping $F : Y \to C[a,T]$ defined by

$$Y = \{y \in C[a,T] : c(T-t) \le y(t) \le 2c(T-t), \quad t \in [a,T]\}$$

and

$$(16) \qquad (Fy)(t) = \int_t^T \left[c^\alpha + \int_s^T q(r)(y(r))^\beta dr \right]^{\frac{1}{\alpha}} ds, \quad t \in [a,T].$$

That F maps Y into itself is an immediate consequence of (15). One easily verifies that F sends Y continuously into a compact subset of Y. The Schauder fixed point theorem then guarantees the existence of a fixed element $y \in Y$ of F, which in view of (16) satisfies

$$(17) \qquad y(t) = \int_t^T \left[c^\alpha + \int_s^T q(r)(y(r))^\beta dr \right]^{\frac{1}{\alpha}} ds, \quad t \in [a,T].$$

From (17) it follows that this function $y(t)$, when restricted to $[a,T)$, gives a positive extinct solution of the third kind of (A) with extinction point at $t = T$. This completes the proof of Theorem 4.

It may happen that the equation (A) has no other singular solutions except for white hole ones, as is demonstrated by the following theorem.

Theorem 5. *Suppose that $\alpha > 0$ and $q \in C^1[a,\infty)$. If $\beta < -1$, then the only decreasing singular solutions of* (A) *are white hole solutions.*

Proof. Let $y(t)$ be any decreasing singular solution of (A) on $J = [t_0, T_y)$. Assume that it is not a white hole solution. Then $\lim_{t \to T_y - 0} y(t) = 0$ and since $y(t)$ is decreasing, we may assume that $y(t) > 0$ on J. Define the function $W[y](t)$ by

$$(18) \qquad W[y](t) = -\frac{\alpha}{\alpha+1}(-y'(t))^{\alpha+1} + \frac{q(t)}{\beta+1}(y(t))^{\beta+1}, \quad t \in J.$$

We easily have

$$\frac{d}{dt} W[y](t) = \frac{q'(t)}{\beta+1}(y(t))^{\beta+1}, \quad t \in J,$$

from which, noting that $W[y](t) < 0, t \in J$, and

$$-\frac{q'(t)}{\beta+1}(y(t))^{\beta+1} \le -\frac{q'_+(t)}{\beta+1}(y(t))^{\beta+1} \le \frac{q'_+(t)}{q(t)}|W[y](t)|, \quad t \in J,$$

we find that

$$\frac{d}{dt}|W[y](t)| \le \frac{q'_+(t)}{q(t)}|W[y](t)|, \quad t \in J,$$

where $q'_+(t) = \max\{q'(t), 0\}$, which implies that

$$|W[y](t)| \le |W[y](t_0)| \exp\left(\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J.$$

Thus,

$$-\frac{q(t)}{\beta+1}(y(t))^{\beta+1} \le |W[y](t_0)| \exp\left(\int_{t_0}^{t} \frac{q'_+(s)}{q(s)} ds\right), \quad t \in J,$$

which is a contradiction with the assumption $\lim_{t\to T_y-0} y(t) = 0$. Thus, $y(t)$ must be a white hole solution.

3. Classification of proper solutions of (A)

Our purpose in this section is to classify the set of all possible proper solutions of (A) with $\alpha > 0$ and $q(t)$ positive on $[a, \infty)$ according to their asymptotic behavior as $t \to \infty$ and to derive nonlinear integral equations for proper solutions of each of the classified types. These integral equations will play a crucial role in establishing the existence of proper solutions of various kinds in the next section.

Let $y(t)$ be a proper solution of (A) on $[t_0, \infty), t_0 \ge a$, and suppose first that this solution is positive and increasing for $t \ge t_0$. Then, by (A), $y'(t)$ is decreasing on $[t_0, \infty)$ and the limit $\lim_{t\to\infty} y'(t) \ge 0$ exists and is finite. It follows that there are three possibilities for the asymptotic behavior of $y(t)$ as $t \to \infty$: either

(IV) $$\lim_{t\to\infty} y(t) = \infty \quad \text{and} \quad 0 < \lim_{t\to\infty} y'(t) < \infty$$

or

(V) $$\lim_{t\to\infty} y(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0$$

or

(VI) $$0 < \lim_{t\to\infty} y(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0.$$

We call a proper solution satisfying (IV), (V) or (VI) a *dominant,* an *intermediate* or a *subdominant* solution, respectively.

From two integrations of (A) it follows that if $y(t)$ is an increasing proper solution with $y(t_0) = y_0 > 0, y(\infty) = \lim_{t\to\infty} y(t) = \infty$ and $y'(\infty) = \lim_{t\to\infty} y'(t) = \omega_1 \geq 0$ (i.e., a proper solution of dominant or intermediate type), then it solves the nonlinear integral equation

(19) $$y(t) = y_0 + \int_{t_0}^{t} \left[\omega_1^\alpha + \int_s^\infty q(r)(y(r))^\beta dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0.$$

From (19) it follows in particular that the function $q(t)(y(t))^\beta$ is integrable on $[t_0, \infty)$ for both types (IV) and (V), and that $[\int_t^\infty q(s)(y(s))^\beta ds]^{1/\alpha}$ is non-integrable if $y(t)$ is a proper solution of type (V).

Similarly, if $y(t)$ is a positive increasing proper solution defined on $[t_0, \infty)$ with "terminal values" $y(\infty) = \omega_0 > 0$ and $y'(\infty) = 0$ (i.e., a positive proper solution of subdominant type), then

(20) $$y(t) = \omega_0 - \int_t^\infty \left[\int_s^\infty q(r)(y(r))^\beta dr\right]^{\frac{1}{\alpha}} ds,$$

for $t \geq t_0$. An immediate consequence of (20) is that both $q(t)(y(t))^\beta$ and $\left[\int_t^\infty q(s)(y(s))^\beta ds\right]^{1/\alpha}$ are integrable on $[t_0, \infty)$ for a proper solution $y(t)$ of this type.

Now let $y(t)$ be a proper solution of (A) on $[t_0, \infty)$ with $y(t_0)0$ and $y'(t_0) < 0$. Then, by (A), such a solution is decreasing on the whole $[t_0, \infty)$

and so either $y(t)$ remains positive on $[t_0, \infty)$ or there exists $t_1 \geq t_0$ such that $y(t) < 0$ for $t \geq t_1$. (Clearly, the latter case is possible only when $\beta \geq 0$.)

If $y(t)$ is a positive decreasing solution of (A) on $[t_0, \infty)$, then either

$$\text{(VII)} \qquad 0 < \lim_{t\to\infty} y(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0$$

or

$$\text{(VIII)} \qquad \lim_{t\to\infty} y(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0.$$

We call positive solution $y(t)$ a *weakly decreasing* proper solution if it satisfies (VII) and a *strongly decreasing* (or *decaying*) proper solution if (VIII) holds.

Integrating (A) twice from t to ∞, we obtain that if a positive decreasing function $y(t)$ defined on $[t_0, \infty)$ is a decreasing proper solution of (A) with terminal values $y(\infty) = \omega_0 \geq 0$ and $y'(\infty) = 0$ (i.e., weakly or strongly decreasing proper solution according to whether $\omega_0 > 0$ or $\omega_0 = 0$), then it satisfies the integral equation

$$\text{(21)} \qquad y(t) = \omega_0 + \int_t^\infty \left[\int_s^\infty q(r)(y(r))^\beta dr \right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0.$$

From (21) it is clear that the functions $q(t)(y(t))^\beta$ and $[\int_t^\infty q(s)(y(s))^\beta ds]^{1/\alpha}$ are integrable on $[t_0, \infty)$ for proper solutions of types (VII) and (VIII).

Classification of the set of eventually negative decreasing proper solutions emanating from a point (t_0, y_0) with $y_0 > 0$ can be done in a similar manner as in the case of positive increasing solutions, and so we have that either

$$\text{(IX)} \qquad \lim_{t\to\infty} y(t) = -\infty \quad \text{and} \quad -\infty < \lim_{t\to\infty} y'(t) < 0$$

or

$$\text{(X)} \qquad \lim_{t\to\infty} y(t) = -\infty \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0$$

or

$$\text{(XI)} \qquad -\infty < \lim_{t\to\infty} y(t) < 0 \quad \text{and} \quad \lim_{t\to\infty} y'(t) = 0.$$

From repeated integration of (A) it follows that if $y(t)$ is a decreasing eventually negative proper solution of (A) emanating from the point (t_0, y_0) with $y_0 > 0$ and satisfying $\lim_{t\to\infty} y'(t) = -\omega_1 \le 0$, then

$$y(t) = y_0 - \int_{t_0}^{t} \left[\omega_1^\alpha + \int_s^\infty q(r)|y(r)|^\beta dr\right]^{\frac{1}{\alpha}} ds, \quad t \ge t_0. \tag{22}$$

From (22) we see that the function $q(t)|y(t)|^\beta$ is integrable on $[t_0, \infty)$ and that $[\int_t^\infty q(s)|y(s)|^\beta ds]^{1/\alpha}$ is non-integrable if $y(\infty) = -\infty$ and $\omega_1 = 0$, that is, if $y(t)$ is an eventually negative proper solution of type (X).

Finally, if $y(t)$ is an evenually negative decreasing proper solution of (A) defined on $[t_0, \infty)$ with terminal values $y(\infty) = -\omega_0 < 0$ and $y'(\infty) = 0$ (i.e., a proper solution of type (XI)), then it satisfies

$$y(t) = -\omega_0 + \int_t^\infty \left[\int_s^\infty q(r)|y(r)|^\beta dr\right]^{\frac{1}{\alpha}} ds, t \ge t_0. \tag{23}$$

Clearly, $q(t)|y(t)|^\beta$ and $[\int_t^\infty q(s)|y(s)|^\beta ds]^{1/\alpha}$ are integrable on $[t_0, \infty)$ in this case.

4. Existence of increasing proper solutions

We start with the existence of increasing proper solutions of dominant type.

Theorem 6. *Let $0 < \beta < \alpha$. For any given $y_0 > 0$ there exists a positive increasing proper solution satisfying $y(a) = y_0$ and the asymptotic relations* (IV) *if and only if*

$$\int_a^\infty t^\beta q(t) dt < \infty. \tag{24}$$

Proof. (The "only if" part.) Assume that (A) has a positive increasing solution $y(t)$ of dominant type defined on $[a,\infty)$. Integrating (A) over $[t,\infty), t \geq a$, we obtain

$$(y'(\infty))^\alpha - (y'(t))^\alpha + \int_t^\infty q(s)(y(s))^\beta ds = 0,$$

which implies in particular that

$$\int_t^\infty q(s)(y(s))^\beta ds < \infty, \quad t \geq a. \tag{25}$$

Since $\lim_{t\to\infty} y'(t) = \lim_{t\to\infty}[y(t)/t] = \text{const} > 0$ by (IV), there exist positive constants c_1, c_2 and $t_1 > a$ such that $c_1 t \leq y(t) \leq c_2 t$ for $t \geq t_1$. Combining this observation with (25) yields

$$c_1^\beta \int_{t_1}^\infty s^\beta q(s) ds < \infty,$$

which verifies the truth of (24).

(The "if" part.) Assume that $0 < \beta < \alpha$ and (24) is satisfied. Let $y_0 > 0$ be given arbitrarily and choose ω_1 so that

$$\int_a^\infty q(t)[y_0 + j\omega_1(t-a)]^\beta dt \leq (2^\alpha - 1)\omega_1^\alpha. \tag{26}$$

Consider the set $Y \subset C[a,\infty)$ and the integral operator $F : Y \to C[a,\infty)$ defined by

$$Y = \{y \in C[a,\infty) : y_0 + \omega_1(t-a) \leq y(t) \leq y_0 + 2\omega_1(t-a), t \geq a\} \tag{27}$$

and

$$(Fy)(t) = y_0 + \int_a^t \left[\omega_1^\alpha + \int_s^\infty q(r)(y(r))^\beta dr\right]^{\frac{1}{\alpha}} ds, t \geq a. \tag{28}$$

The requirement (26) ensures that F maps Y into Y. It can be shown routinely that F is a continuous mapping and that $F(Y)$ is a relatively compact subset of $C[a,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $y \in Y$ such that $y = Fy$, that is,

$$y(t) = y_0 + \int_a^t \left[\omega_1^\alpha + \int_s^\infty q(r)(y(r))^\beta \right]^{\frac{1}{\alpha}} ds, t \geq a. \tag{29}$$

From (29) we easily see that $y(t)$ is an increasing proper solution of (A) which is positive on $[a,\infty)$ and satisfies $y(a) = y_0$ and $\lim_{t\to\infty} y'(t) = \omega_1 > 0$. This completes the proof.

The next theorem concerns positive increasing solutions of intermediate type, that is, the proper solutions $y(t)$ satisfying $\lim_{t\to\infty} y(t) = \infty$ and $\lim_{t\to\infty} y'(t) = 0$.

Theorem 7. *Suppose that there exist constants $q_1, q_2 > 0$ and $\sigma \in R$ such that*

$$q_1 t^\sigma \leq q(t) \leq q_2 t^\sigma \tag{30}$$

for all large t. Moreover, let either

(i) $\sigma \geq -1$ *and* $-\alpha < \beta < -\sigma - 1$;

or

(ii) $\sigma < -1$ *and one of the following two cases hold:*

a) $0 \leq \beta < -\sigma - 1 < \alpha$,

b) $0 < -\beta < \alpha, \ -\sigma - 1 < \alpha$.

Then Eq.(A) *has an intermediate increasing proper solution.*

Proof. Let $t_1 \geq a$ be so large that (30) is satisfied for $t \geq t_1$. Put

$$L = \frac{\alpha - \beta}{\alpha + \sigma + 1} \left[-\frac{\alpha - \beta}{\alpha(\beta + \sigma + 1)} \right]^{\frac{1}{\alpha}}$$

and define

$$c_1 = (q_1 L^\alpha)^{\frac{1}{\alpha-\beta}}, \quad c_2 = (q_2 L^\alpha)^{\frac{1}{\alpha-\beta}}, \tag{31}$$

if $\beta \geq 0$, that is, the case (ii)(a) holds, and

$$c_1 = (L_1^\alpha L_2^\beta)^{\frac{\alpha}{\alpha^2-\beta^2}}, \quad c_2 = (L_2^\alpha L_1^\beta)^{\frac{\alpha}{\alpha^2-\beta^2}}, \tag{32}$$

where $L_1 = (q_1)^{1/\alpha} L$ and $L_2 = (q_2)^{1/\alpha} L$, if $\beta < 0$ (i.e., (i) or (ii)(b) is satisfied). Then $c_1 \leq c_2$ and we can define the set Y by

$$Y = \{y \in C[t_1, \infty) : c_1 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}} \leq y(t) \leq c_2 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}}, \quad t \geq t_1\}, \tag{33}$$

which is clearly a closed convex subset of the locally convex space $C[t_1, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$. Define the integral operator $F : Y \to C[t_1, \infty)$ by

$$(Fy)(t) = c_1 t_1^{\frac{\alpha+\sigma+1}{\alpha-\beta}} + \int_{t_1}^{t} \left[\int_s^\infty q(r)(y(r))^\beta dr \right]^{\frac{1}{\alpha}} ds, \quad t \geq t_1. \tag{34}$$

From (30) and (34) we see that, for any $y \in Y$ and $t \geq t_1$,

$$\begin{aligned}
(Fy)(t) &\leq c_1 t_1^{\frac{\alpha+\sigma+1}{\alpha-\beta}} + (c_i^\beta q_2)^{1/\alpha} \int_{t_1}^{t} \left[\int_s^\infty r^\sigma r^{\beta \frac{\alpha+\sigma+1}{\alpha-\beta}} dr \right]^{\frac{1}{\alpha}} ds \\
&= [c_1 - (c_i^\beta q_2)^{1/\alpha} L] t_1^{\frac{\alpha+\sigma+1}{\alpha-\beta}} + (c_i^\beta q_2)^{1/\alpha} L t^{\frac{\alpha+\sigma+1}{\alpha-\beta}} \\
&\leq (c_i^\beta q_2)^{1/\alpha} L t^{\frac{\alpha+\sigma+1}{\alpha-\beta}} = c_2 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}},
\end{aligned}$$

where $i = 1$ if $\beta < 0$ and $i = 2$ if $\beta \geq 0$, and

$$\begin{aligned}
(Fy)(t) &\geq c_1 t_1^{\frac{\alpha+\sigma+1}{\alpha-\beta}} + (c_i^\beta q_1)^{1/\alpha} \int_{t_1}^{t} \left[\int_s^\infty r^\sigma r^{\beta \frac{\alpha+\sigma+1}{\alpha-\beta}} dr \right]^{\frac{1}{\alpha}} ds \\
&= (c_i^\beta q_1)^{1/\alpha} L t^{\frac{\alpha+\sigma+1}{\alpha-\beta}} = c_1 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}},
\end{aligned}$$

where $i = 2$ if $\beta < 0$ and $i = 1$ if $\beta \geq 0$, which implies that $Fy \in Y$, and hence F maps Y into itself.

We can prove routinely that F is continuous and $F(Y)$ is a relatively compact subset of $C[t_1, \infty)$. Therefore, the Schauder-Tychonoff fixed point theorem guarantees that F has a fixed point $y \in Y$, giving rise to the desired positive increasing intermediate proper solution $y(t)$ of (A).

Example 2. Applying Theorem 7 to the equation

$$(|y'|^{\alpha})' + \gamma t^{\sigma} |y|^{\beta} = 0, \quad t \ge a > 0, \tag{35}$$

where $\alpha > 0, \beta, \sigma \in R$ and $\gamma > 0$ are constants, we obtain that (35) has an intermediate proper solution if either $\sigma \ge -1$ and $-\alpha < \beta < -\sigma - 1$, or $\sigma < -1$ and one of the conditions (a) and (b) of Theorem 7 is satisfied. Clearly, the function $y(t)$ given by

$$y(t) = ct^{(\alpha+\sigma+1)/(\alpha-\beta)}, \quad t \ge a > 0, \tag{36}$$

where

$$c = \left[\frac{\gamma}{\alpha} \left(\frac{-\alpha+\beta}{\beta+\sigma+1} \right) \left| \frac{\alpha-\beta}{\alpha+\sigma+1} \right|^{\alpha} \right]^{1/(\alpha-\beta)},$$

is an increasing proper solution of type (V) for the equation (35) if

$$0 < \frac{\alpha+\sigma+1}{\alpha-\beta} < 1. \tag{37}$$

As easily seen, any of the conditions (i), (ii)(a) or (ii)(b) implies (37).

The above example motivated us to believe that the following conjecture is true.

Conjecture 2. Suppose that there exist constants $q_1, q_2 > 0$ and $\sigma \in R$ such that (30) holds for all large t. Then Eq. (A) has an intermediate proper solution if and only if (37) is satisfied.

In our next result we present a necessary and sufficient condition for the existence of increasing proper solutions of (A) of type (VI) with arbitrarily prescibed positive terminal value ω_0. Proper solutions of this type can be regarded also as increasing solutions with finite nonzero "white holes" at infinity.

Theorem 8. *For any given $\omega_0 > 0$, Eq.* (A) *has an increasing proper solution defined on some interval* $[t_0, \infty) \subset [a, \infty)$ *and satisfying* (VI) *with* $\lim_{t\to\infty} y(t) = \omega_0$ *if and only if*

$$\int_a^\infty \left[\int_t^\infty q(s)ds\right]^{1/\alpha} dt < \infty. \tag{38}$$

Proof. (The "only if" part.) Let $y(t)$ be a positive proper solution of (A) defined on $[t_0, \infty), t_0 \geq a$, which is of type (VI). Then, as pointed out in the preceding section, both $q(t)(y(t))^\beta$ and $[\int_t^\infty q(s)(y(s))^\beta ds]^{1/\alpha}$ are integrable on $[t_0, \infty)$. This fact combined with the inequality $y(t) \geq y(t_0)0, t \geq t_0$, if $\beta \geq 0$ (resp. the inequality $y(t) \leq y(\infty), t \geq t_0$, if $\beta < 0$) implies (38).

(The "if" part.) Assume that (38) holds. Let $\omega_0 > 0$ be fixed arbitrarily and choose $t_0 \geq a$ so large that

$$\int_{t_0}^\infty \left[\int_t^\infty q(s)ds\right]^{\frac{1}{\alpha}} dt \leq j^{-\frac{\beta}{\alpha}} \left(\frac{\omega_0}{2}\right)^{1-\frac{\beta}{\alpha}}, \tag{39}$$

where $j = 1$ if $\beta < 0$ and $j = 2$ if $\beta \geq 0$.

Consider the set $Y \subset C[t_0, \infty)$ and the mapping $F : Y \to C[t_0, \infty)$ defined by

$$Y = \{y \in C[t_0, \infty) : \frac{1}{2}\omega_0 \leq y(t) \leq \omega_0, t \in [t_0, \infty)\}$$

and

$$(Fy)(t) = \omega_0 - \int_t^\infty \left[\int_s^\infty q(r)(y(r))^\beta dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_0.$$

It is a matter of simple computation to show that F is a continuous operator mapping Y into a relatively compact subset of Y. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $y \in Y$ such that $y = Fy$, i.e., $y(t)$ satisfies the integral equation (20) for $t \geq t_0$. From (20) it is clear that $y(t)$ is a positive increasing solution of (A) defined on $[t_0, \infty)$ and satisfying $\lim_{t\to\infty} y(t) = \omega_0$, which completes the proof.

Theorem 8 is “local near ∞” in the sense that it guarantees the existence of a desired subdominant proper solution $y(t)$ for arbitrarily given terminal value $\omega_0 > 0$ only in a sufficiently “small” neighborhood of infinity. A close look at the proof of the above theorem shows that under the additional assumption $\alpha \neq \beta$, the initial time $t_0 = a$ may be fixed and $\omega_0 > 0$ can then be selected so that the condition (39) is fulfilled. Since the rest of the proof is the same as before, we have the following global existence version of Theorem 8.

Theorem 8'. *Let $\alpha \neq \beta$. The equation* (A) *has an increasing proper solution defined on the whole given interval $[a, \infty)$ and satisfying* (VI) *with $y(\infty) = \omega_0$ for some $\omega_0 > 0$ if and only if* (38) *is satisfied.*

5. Existence of decreasing proper solutions

Now we turn our attention to the problem of existence of decreasing proper solutions of (A). As we already know from Section 3, if $y(t)$ is one such solution, then it either remains positive as long as it exists and belongs to one of the types (VII) and (VIII), or it is eventually negative and satisfies one of the relations (IX), (X) or (XI).

Our consideration will first be focused on the set of weakly decreasing proper solutions of (A). More specifically, we intend to show that the ‘local’ existence of such solutions with arbitrarily prescribed terminal value $\omega_0 > 0$ can be characterized by the same integral condition (38) as the existence of increasing proper solutions of subdominat type established in the preceding section.

Theorem 9. *For any given $\omega_0 > 0$, Eq.* (A) *has a decreasing proper solution which exists on some interval $[t_0, \infty) \subset [a, \infty)$ and satisfies* (VII) *with $\lim_{t\to\infty} y(t) = \omega_0$ if and only if* (38) *is satisfied.*

Proof. (The “only if” part.) If $y(t)$ is a weakly decreasing solution of (A) on $[t_0, \infty), t_0 \geq a$, satisfying $y(t_0) > 0$ and (VII), then from the integrability of $q(t)(y(t))^{\beta}$ and $[\int_t^{\infty} q(s)(y(s))^{\beta} ds]^{1/\alpha}$ on $[t_0, \infty)$ combined with the inequalities $y(t_0) \geq y(t) \geq y(\infty), t \geq t_0$, we easily obtain (38).

(The “if” part.) Suppose that (38) is satisfied. If $\omega_0 > 0$ is given arbitrarily and $t_0 \geq a$ is chosen so large that

$$(40) \qquad \int_{t_0}^{\infty} \left[\int_t^{\infty} q(s)ds \right]^{\frac{1}{\alpha}} dt \le j^{-\frac{\beta}{\alpha}} (\omega_0)^{1-\frac{\beta}{\alpha}},$$

where $j = 1$ if $\beta < 0$ and $j = 2$ if $\beta \ge 0$, then it can be shown that the mapping F defined by

$$(Fy)(t) = \omega_0 + \int_t^{\infty} \left[\int_s^{\infty} q(r)(y(r))^{\beta} dr \right]^{\frac{1}{\alpha}} ds, \quad t \in [t_0, \infty).$$

has a fixed element in the set Y of continuous functions defined by

$$Y = \{y \in C[t_0, \infty) : \omega_0 \le y(t) \le 2\omega_0, t \in [t_0, \infty)\}$$

which gives rise to a desired weakly decreasing proper solution $y(t)$ with the property $\lim_{t\to\infty} y(t) = \omega_0$. This sketches the proof. The detailed verification is left to the reader.

As in the case of increasing proper solutions of subdominant type, if we assume that $\alpha \ne \beta$, then we can modify the sufficiency part of the proof slightly so that we fix $t_0 = a$ and take $\omega_0 > 0$ so large that the condition (40) is satisfied. This modification of the proof enables us to re-formulate Theorem 9 as the following 'global' existence result.

Theorem 9'. *Let $\alpha \ne \beta$. Then Eq.* (A) *possesses a decreasing proper solution of type* (VII) *defined on* $[a, \infty)$ *and satisfying* $\lim_{t\to\infty} y(t) = \omega_0$ *for some* $\omega_0 > 0$ *if and only if* (38) *holds.*

Our next task is to construct proper solutions of type (VIII), that is, strongly decreasing positive solutions of (A). The existence of such proper solutions is guaranteed by the following theorem.

Theorem 10. *Suppose that there exist constants $q_1, q_2 > 0$ and $\sigma < -1$ such that* (30) *is satisfied for all large t. Moreover, let either*

(i) $0 \le \beta < \alpha < -\sigma - 1$,

or

(ii) $\sigma + 1 < -\alpha < \beta < 0$.

Then Eq. (A) *has a strongly decreasing proper solution.*

Proof. Assume that $t_1 \geq a$ is such that (30) holds for $t \geq t_1$. Put

$$L = -\frac{\alpha-\beta}{\alpha+\sigma+1}\left[-\frac{\alpha-\beta}{\alpha(\beta+\sigma+1)}\right]^{\frac{1}{\alpha}}$$

and with this L define the constants c_1 and c_2 by (31) or (32) according as $\beta \geq 0$ or $\beta < 0$. With such a pair of constants c_1 and c_2, which clearly satisfy $c_1 \leq c_2$, define the set Y by

$$Y = \{y \in C[t_1,\infty) : c_1 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}} \leq y(t) \leq c_2 t^{\frac{\alpha+\sigma+1}{\alpha-\beta}}, t \geq t_1\}$$

and the integral operator $F : Y \to C[t_1,\infty)$ by

$$(Fy)(t) = \int_t^\infty \left[\int_s^\infty q(r)(y(r))^\beta dr\right]^{\frac{1}{\alpha}} ds, \quad t \geq t_1.$$

It can be verified in a routine manner that $F(Y) \subset Y$, F is continuous and $F(Y)$ is a relatively compact subset of $C[t_1,\infty)$. Therefore there exists $y \in Y$ such that $y = Fy$ by the Schauder-Tychonoff fixed point theorem, that is, $y(t)$ satisfies the integral equation

$$y(t) = \int_t^\infty \left[\int_s^\infty q(r)(y(r))^\beta dr\right]^{1/\alpha} ds$$

for $t \geq t_1$. Since obviously $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0$, it follows that $y(t)$ is a desired strongly decreasing solution of Eq. (A). This completes the proof.

Example 3. Consider again the equation (35) with a positive constant coefficient γ. Theorem 10 ensures the existence of strongly decreasing proper solutions for (35) if $\sigma < -1$ and $\sigma + 1 < -\alpha < \beta < \alpha < -\sigma - 1$.

As is easily verified, if

$$\frac{\alpha+\sigma+1}{\alpha-\beta} < 0 \tag{41}$$

which is obviously satisfied if the conditions of Theorem 10 hold, then the function $y(t)$ given by (36) is a strongly decreasing proper solution of (35).

The following conjecture has been motivated by the gap between (41) and the conditions of Theorem 10.

Conjecture 3. Let there exist constants $q_1, q_2 > 0$ and $\sigma \in R$ such that (30) holds for all large t. Then Eq. (A) has a strongly decreasing proper solution if and only if (41) is satisfied.

Our final theorem establishes the (global) existence of eventually negative decreasing proper solutions of (A) satisfying (IX) and $y(a) = y_0$ with arbitrarily prescribed initial value $y_0 > 0$.

Theorem 11. *Let* $0 < \beta < \alpha$. *For any given* $y_0 > 0$, *Eq.* (A) *has an eventually negative decreasing proper solution satisfying* $y(a) = y_0$ *and* (IX) *if and only if*

$$\int_a^\infty t^\beta q(t)dt < \infty. \tag{42}$$

.

Proof. (The "only if" part.) Suppose that (A) has a decreasing proper solution $y(t)$ on $[a, \infty)$ with $y(a) > 0$ which is eventually negative and satisfies (IX). From (IX) it follows that there exist constants $k_1, k_2 > 0$ and $t_1 > a$ such that $-k_1 t \le y(t) \le -k_2 t$ for $t \ge t_1$. Combining these inequalities with a known fact that the function $q(t)|y(t)|^\beta$ is integrable on $[t_1, \infty)$ for decreasing proper solutions of type (IX), we easily obtain (42).

(The "if" part.) Suppose that $0 < \beta < \alpha$ and (42) holds. Let $\omega_1 > 0$ be a constant such that

$$\int_a^\infty (y_0 + 2\omega_1(t-a))^\beta q(t)dt \le (2^\alpha - 1)\omega_1^\alpha \tag{43}$$

and consider the set $Y \subset C[a, \infty)$ and the mapping $F : Y \to C[a, \infty)$ defined by

$$Y = \{y \in C[a, \infty) : y_0 - 2\omega_1(t-a) \le y(t) \le y_0 - \omega_1(t-a), t \ge a\}$$

and

$$(Fy)(t) = y_0 - \int_a^t \left[\omega_1^\alpha + \int_s^\infty q(r)|y(r)|^\beta dr\right]^{1/\alpha} ds, \quad t \geq a.$$

Then, as is easily verified, the Schauder-Tychonoff fixed point theorem applies, and there exists an element $y \in Y$ such that $y = Fy$, that is, (22) holds for $t \geq a$. It follows that $y(t)$ is a decreasing proper solution of (A) on $[a, \infty)$ which is eventually negative and satisfies $y(a) = y_0$ and $\lim_{t\to\infty} y'(t) = -\omega_1 < 0$. This completes the proof.

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T. Kusano, Department of Applied Mathematics, Faculty of Science, Fukuoka University, Nanakuma, Jonan-ku, Fukuoka 814-80, Japan

J. Jaroš, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Comenius University, 842 48 Bratislava, Slovakia