

Notes on Space-Time Decay Properties of Nonstationary Incompressible Navier-Stokes Flows in \mathbb{R}^n

By

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Dedicated to Professor Shinnosuke Oharu on his 60th birthday

1. Introduction and results

Consider the nonstationary Navier-Stokes system in \mathbb{R}^n , $n \geq 2$, in the form of the integral equation :

$$(IE) \quad u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A}P\nabla \cdot (u \otimes u)(s)ds.$$

Here $\{e^{-tA}\}_{t \geq 0}$ is the heat semigroup, P is the bounded projection from the space \mathbf{L}^q , $1 < q < \infty$, of vector fields to the subspace of \mathbf{L}^q consisting of all divergence-free vector fields. As is shown in [4], the operator $e^{-tA}P\nabla \cdot$ has the kernel function $F = (F_{\ell,jk})_{j,k,\ell=1}^n$ with

$$F_{\ell,jk}(x, t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j \partial_k \partial_\ell E_{s+t}(x)ds, \quad \partial_j = \partial/\partial x_j,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $t > 0$, and $E_t(x) = (4\pi t)^{-\frac{n}{2}}e^{-|x|^2/4t}$.

Inspired by Takahashi [9], we proved in [4] the following

Theorem 1. *Fix $1 \leq \gamma \leq n + 1$ and let a be divergence-free, satisfying*

$$(1.1) \quad |(e^{-tA}a)(x)| \leq c(1 + |x|)^{-\gamma}, \quad |(e^{-tA}a)(x)| \leq c(1 + t)^{-\frac{\gamma}{2}}.$$

If $c > 0$ is small, then (IE) possesses a solution u such that, with another $c > 0$,

$$|u(x, t)| \leq c(1 + |x|)^{-\gamma}, \quad |u(x, t)| \leq c(1 + t)^{-\frac{\gamma}{2}}.$$

However, in proving Theorem 1, the Hardy space theory was used to find relevant decay estimates in t . In this paper we show that Theorem 1 can be deduced by a more elementary method, i.e., without using the Hardy space theory.

In [4] we also gave a class of initial data a for which $e^{-tA}a$ satisfies (1.1). But, the assumption on a imposed in [4] was too complicated and restrictive. So, we here give a

simpler version of the assumption on a that ensures the validity of (1.1). More specifically, we shall show the following

Theorem 2. *Let a be divergence-free and satisfy*

$$c_0 = \sup_y (1 + |y|)^\gamma |a(y)| < \infty \quad \text{for some } 0 < \gamma \leq n + 1.$$

Suppose further that

$$c_1 = \int |a(y)| dy < \infty \quad \text{when } \gamma = n, \quad c_1 = \int |y| |a(y)| dy < \infty \quad \text{when } \gamma = n + 1.$$

Then

$$(1.2) \quad |(e^{-tA}a)(x)| \leq cd(1 + |x|)^{-\gamma}, \quad |(e^{-tA}a)(x)| \leq cd(1 + t)^{-\frac{\gamma}{2}}.$$

Here, $d = c_0 + c_1$ when $\gamma = n$ or $\gamma = n + 1$; and $d = c_0$ otherwise.

Starting from Theorems 1 and 2, we deduced in [3] a space-time asymptotic expansion of solutions u in the case $\gamma = n + 1$, which was then applied in [8] to find a lower bound estimate of rates of energy decay for weak solutions of (IE).

Finally, we supplement the recent result of Brandolese [2] on the existence of solutions which decay more rapidly than those treated in Theorem 1. Consider the divergence-free vector fields a such that

- (a) a_j is odd in x_j and is even in each of the other variables.
- (b) a is cyclically symmetric in the sense that

$$a_1(x_1, \dots, x_n) = a_2(x_n, x_1, \dots, x_{n-1}) = \dots = a_n(x_2, \dots, x_n, x_1).$$

Brandolese [2] shows the existence of a solution u satisfying (a) and (b) above, with the estimates

$$(1.3) \quad |u(x, t)| \leq c(1 + |x|)^{-n-2}, \quad |u(x, t)| \leq c(1 + t)^{-\frac{n+2}{2}}.$$

Observe that the solutions above decay more rapidly than those treated in Theorem 1. But, estimate (1.3) seems not optimal. Indeed, we shall prove

Theorem 3. *Suppose a satisfies (a) and (b), and*

$$(1.4) \quad c_0 = \sup (1 + |y|)^{n+3} |a(y)| < \infty, \quad c_1 = \int |y|^3 |a(y)| dy < \infty.$$

- (i) *The function $x \mapsto e^{-tA}a(x)$ satisfies (a) and (b), and there hold the estimates*

$$|(e^{-tA}a)(x)| \leq c(c_0 + c_1)(1 + |x|)^{-n-3}, \quad |(e^{-tA}a)(x)| \leq c(c_0 + c_1)(1 + t)^{-\frac{n+3}{2}}.$$

(ii) If $c_0 + c_1$ is small, then (IE) has a strong solution u satisfying (a), (b), and

$$(1.5) \quad |u(x, t)| \leq c(1 + |x|)^{-n-3}, \quad |u(x, t)| \leq c(1 + t)^{-\frac{n+3}{2}}.$$

Brandolese [2] proves the existence of solutions which satisfy (1.3), assuming

$$(1.6) \quad c'_0 = \sup(1 + |y|)^{n+2}|a(y)| < \infty, \quad c'_1 = \int |y|^2|a(y)|dy < \infty.$$

This is the reason why (1.3) is deduced in [2] instead of (1.5). However, the result of [3] implies that the corresponding solution u has to satisfy

$$\|u(t)\|_\infty = o(t^{-\frac{n+2}{2}}) \quad \text{as } t \rightarrow \infty$$

whenever a satisfies (a), (b) and (1.6). Thus, (1.3) is not optimal even under the assumption (1.6). On the other hand, in view of the result given in [7], estimate (1.5) seems to be optimal for general solutions satisfying (a) and (b). Namely, one can reasonably expect the existence of a solution u satisfying (a) and (b) such that

$$\|u(t)\|_\infty \geq ct^{-\frac{n+3}{2}} \quad \text{for large } t > 0,$$

even when a is in \mathcal{S} and satisfies (a), (b) and $\int y^\alpha a(y)dy = 0$ for every multi-index α .

2. Proof of Theorem 2

Observe first that

$$|(e^{-tA}a)(x)| = |(E_t * a)(x)| \leq \|a\|_\infty \|E_t\|_1 = \|a\|_\infty \leq c_0,$$

and so $|(e^{-tA}a)(x)|$ is bounded in x and $t > 0$. So we assume from now on $|x| > 1$ and $t > 1$. Recall the estimates

$$t^{-\frac{n+m}{2}} e^{-c|x|^2/t} \leq \begin{cases} c_m |x|^{-n-m}, \\ t^{-\frac{n+m}{2}}, \end{cases} \quad m = 0, 1, 2, \dots,$$

which will be systematically applied below.

(i) Suppose $\gamma = n + 1$. Since $a \in \mathbf{L}^1$, condition $\nabla \cdot a = 0$ implies (see [5])

$$(2.1) \quad \int a(y)dy = 0.$$

We write

$$(e^{-tA}a)(x) = (E_t * a)(x) = \left(\int_{|y| < |x|/2} + \int_{|y| > |x|/2} \right) E_t(x - y)a(y)dy \equiv I_1 + I_2.$$

Then (2.1) implies

$$\begin{aligned}
I_1 &= \int_{|y| < |x|/2} [E_t(x-y) - E_t(x)]a(y)dy + E_t(x) \int_{|y| < |x|/2} a(y)dy \\
&= - \int_0^1 \int_{|y| < |x|/2} (y \cdot \nabla E_t)(x-y\theta)a(y)d\theta - E_t(x) \int_{|y| > |x|/2} a(y)dy \\
&\equiv I_{11} + I_{12}.
\end{aligned}$$

Since $|x-y\theta| \geq |x| - \theta|y| \geq |x| - |y| > |x|/2$ whenever $|y| < |x|/2$, it follows that

$$\begin{aligned}
|I_{11}| &\leq ct^{-\frac{n+1}{2}} \int_0^1 \int_{|y| < |x|/2} e^{-c|x-y\theta|^2/t} |y| |a(y)| dy d\theta \\
&\leq ct^{-\frac{n+1}{2}} e^{-c|x|^2/t} \int |y| |a(y)| dy \leq cc_1 |x|^{-n-1}.
\end{aligned}$$

Moreover,

$$|I_{12}| \leq cc_0 t^{-\frac{n}{2}} e^{-c|x|^2/t} \int_{|y| > |x|/2} (1+|y|)^{-n-1} dy \leq cc_0 |x|^{-n} \times |x|^{-1} = cc_0 |x|^{-n-1}.$$

Hence

$$(2.2) \quad |I_1| \leq c(c_0 + c_1)(1+|x|)^{-n-1}.$$

On the other hand,

$$|I_2| \leq cc_0 t^{-\frac{n}{2}} \int_{|y| > |x|/2} e^{-|x-y|^2/4t} (1+|y|)^{-n-1} dy \leq cc_0 (1+|x|)^{-n-1}.$$

This, together with (2.2), implies

$$|(e^{-tA}a)(x)| \leq c(c_0 + c_1)(1+|x|)^{-n-1}.$$

Now, (2.1) yields

$$(e^{-tA}a)(x) = \int [E_t(x-y) - E_t(x)]a(y)dy = - \int_0^1 \int (y \cdot \nabla E_t)(x-y\theta)a(y)d\theta$$

and so

$$|(e^{-tA}a)(x)| \leq ct^{-\frac{n+1}{2}} \int_0^1 \int e^{-c|x-y\theta|^2/t} |y| |a(y)| dy d\theta \leq cc_1 t^{-\frac{n+1}{2}}.$$

This proves (1.2) in case $\gamma = n+1$.

(ii) Let $n \leq \gamma < n+1$. Since $a \in \mathbf{L}^1$ and $\nabla \cdot a = 0$, (2.1) holds for a . So we can decompose

$$e^{-tA}a = I_1 + I_2 = I_{11} + I_{12} + I_2$$

in the same way as in (i). Since $|x-y\theta| > |x|/2$ whenever $|y| < |x|/2$, we obtain

$$\begin{aligned}
|I_{11}| &\leq cc_0 t^{-\frac{n+1}{2}} \int_0^1 \int_{|y| < |x|/2} e^{-c|x-y\theta|^2/t} (1+|y|)^{1-\gamma} dy d\theta \\
&\leq cc_0 t^{-\frac{n+1}{2}} e^{-c|x|^2/t} \times |x|^{n+1-\gamma} \leq cc_0 |x|^{-\gamma}.
\end{aligned}$$

When $n < \gamma < n + 1$, we have

$$\begin{aligned} |I_{12}| &\leq cc_0 t^{-\frac{n}{2}} e^{-c|x|^2/t} \int_{|y|>|x|/2} (1+|y|)^{-\gamma} dy \leq cc_0 |x|^{-n} \times |x|^{n-\gamma} = cc_0 |x|^{-\gamma}, \\ |I_2| &\leq cc_0 t^{-\frac{n}{2}} \int_{|y|>|x|/2} e^{-c|x-y|^2/t} (1+|y|)^{-\gamma} dy \leq cc_0 (1+|x|)^{-\gamma}. \end{aligned}$$

When $\gamma = n$, we see that

$$\begin{aligned} |I_{12}| &\leq ct^{-\frac{n}{2}} e^{-c|x|^2/t} \int |a(y)| dy \leq cc_1 |x|^{-n}, \\ |I_2| &\leq cc_0 t^{-\frac{n}{2}} \int_{|y|>|x|/2} e^{-c|x-y|^2/t} (1+|y|)^{-n} dy \leq cc_0 (1+|x|)^{-n}. \end{aligned}$$

Hence

$$|(e^{-tA}a)(x)| \leq cd(1+|x|)^{-\gamma}.$$

On the other hand, since $\int a(y) dy = 0$,

$$\begin{aligned} (e^{-tA}a)(x) &= \left(\int_{|y|<\sqrt{t}} + \int_{|y|>\sqrt{t}} \right) E_t(x-y)a(y) dy \\ &= - \int_0^1 \int_{|y|<\sqrt{t}} (y \cdot \nabla E_t)(x-y\theta)a(y) dy d\theta \\ &\quad - E_t(x) \int_{|y|>\sqrt{t}} a(y) dy + \int_{|y|>\sqrt{t}} E_t(x-y)a(y) dy \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

Obviously,

$$|J_2| \leq \begin{cases} cc_0 t^{-\frac{n}{2}} \int_{|y|>\sqrt{t}} |y|^{-\gamma} dy = cc_0 t^{-\frac{n}{2}} \times t^{\frac{n-\gamma}{2}} = cc_0 t^{-\frac{\gamma}{2}} & (n < \gamma < n + 1), \\ ct^{-\frac{n}{2}} \int_{|y|>\sqrt{t}} |a(y)| dy \leq cc_1 t^{-\frac{n}{2}} & (\gamma = n), \end{cases}$$

and, in exactly the same way,

$$|J_3| \leq cc_1 t^{-\frac{n}{2}} \quad (\gamma = n), \quad |J_3| \leq cc_0 t^{-\frac{\gamma}{2}} \quad (n < \gamma < n + 1).$$

Furthermore, since $-n < 1 - \gamma \leq 1 - n$, we have

$$|J_1| \leq cc_0 t^{-\frac{n+1}{2}} \int_{|y|<\sqrt{t}} |y|^{1-\gamma} dy \leq cc_0 t^{-\frac{n+1}{2}} \times t^{\frac{n-\gamma+1}{2}} = cc_0 t^{-\frac{\gamma}{2}}.$$

We thus obtain

$$|(e^{-tA}a)(x)| \leq cd(1+t)^{-\frac{\gamma}{2}}.$$

This proves (1.2) in case $n \leq \gamma < n + 1$.

(iii) Suppose finally $0 < \gamma < n$. From

$$|(e^{-tA}a)(x)| \leq cc_0 t^{-\frac{n}{2}} \int e^{-c|x-y|^2/t} |y|^{-\gamma} dy = \int_{|y| < |x|/2} + \int_{|y| > |x|/2} \equiv I_1 + I_2,$$

we get

$$\begin{aligned} |I_1| &\leq cc_0 |x|^{-n} \int_{|y| < |x|/2} |y|^{-\gamma} dy = cc_0 |x|^{-n} \times |x|^{n-\gamma} = cc_0 |x|^{-\gamma}, \\ |I_2| &\leq cc_0 |x|^{-\gamma} t^{-\frac{n}{2}} \int e^{-c|x-y|^2/t} dy = cc_0 |x|^{-\gamma}. \end{aligned}$$

Therefore,

$$|(e^{-tA}a)(x)| \leq cc_0 (1 + |x|)^{-\gamma}.$$

Furthermore, the boundedness of the Fourier transforms of L^1 -functions gives

$$|(e^{-tA}a)(x)| \leq cc_0 \int E_t(x-y) |y|^{-\gamma} dy \leq cc_0 \int e^{-t|\xi|^2} |\xi|^{\gamma-n} d\xi = cc_0 t^{-\frac{\gamma}{2}}$$

and so

$$|(e^{-tA}a)(x)| \leq cc_0 (1+t)^{-\frac{\gamma}{2}}.$$

The proof of Theorem 2 is complete.

3. Proof of Theorem 1

Let $1 \leq \gamma \leq n+1$ and

$$(3.1) \quad \mathcal{S}[u, v](t) = - \int_0^t e^{-(t-s)A} P \nabla \cdot (u \otimes v)(s) ds.$$

As shown in [4], the kernel function $F_{\ell, jk}$ of $e^{-tA} P \nabla \cdot$ is written as

$$F_{\ell, jk}(x, t) = \partial_\ell E_t(x) \delta_{jk} + \int_0^\infty \partial_j \partial_k \partial_\ell E_{t+s}(x) ds,$$

so that

$$\mathcal{S}[u, v](t) = - \int_0^t \mathbf{F}_{k\ell}(t-s) * (u_k v_\ell)(s) ds, \quad \mathbf{F}_{k\ell}(x, t) = (F_{\ell, jk}(x, t))_{j=1}^n,$$

where $*$ means convolution of functions defined on \mathbb{R}^n . We first prepare

Lemma 3.1. *We have*

$$\begin{aligned} |\nabla_x^m F_{\ell, jk}(x, t)| &\leq c_m |x|^{\alpha-n-1-m} t^{-\frac{\alpha}{2}}, \quad 0 \leq \alpha \leq n+1+m, \quad m = 0, 1, 2, \dots, \\ \|\nabla_x^m F(\cdot, t)\|_q &= c_{m,q} t^{-\frac{n}{2}(1+\frac{1+m}{n}-\frac{1}{q})}, \quad 1 \leq q \leq \infty, \quad m = 0, 1, 2, \dots \end{aligned}$$

Proof. Obviously,

$$|\nabla_x^{m+1} E_t(x)| \leq c_m t^{-\frac{n+1+m}{2}} e^{-c_m |x|^2/t} \leq \begin{cases} c_m |x|^{-n-1-m}, \\ c_m t^{-\frac{n+1+m}{2}}. \end{cases}$$

On the other hand, the m -th order derivatives of the integral in the definition of $F_{\ell,jk}$ is estimated in absolute value as follows: Firstly,

$$\leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} e^{-c_m|x|^2/s} ds \leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} ds = c_m t^{-\frac{n+1+m}{2}}.$$

Secondly, via the change of the variable $\tau = |x|^2/s$,

$$\leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} e^{-c_m|x|^2/s} ds = c_m |x|^{-n-1-m} \int_0^{|x|^2/t} e^{-c_m \tau} \tau^{\frac{n+m-1}{2}} d\tau \leq c_m |x|^{-n-1-m}.$$

Moreover, direct calculation gives

$$\nabla_x^m F(x, t) = t^{-\frac{n+1+m}{2}} K_m(xt^{-\frac{1}{2}}), \quad K_m(x) = \nabla_x^m \partial_\ell E_1(x) + \int_1^\infty \nabla_x^m \partial_j \partial_k \partial_\ell E_s(x) ds,$$

and

$$\|K_m\|_q \leq c_{m,q} + c'_{m,q} \int_1^\infty s^{-\frac{n+3+m}{2}} \|e^{-c|x|^2/s}\|_q ds = c_{m,q} + c'_{m,q} \int_1^\infty s^{-\frac{n}{2}(1+\frac{3+m}{n}-\frac{1}{q})} ds = c''_{m,q}.$$

Hence

$$\|\nabla_x^m F\|_q = t^{-\frac{n+1+m}{2}} \times t^{\frac{n}{2q}} \|K_m\|_q = c_{m,q} t^{-\frac{n}{2}(1+\frac{1+m}{n}-\frac{1}{q})}.$$

This proves Lemma 3.1.

Lemma 3.2. *Let $1 \leq \gamma \leq n+1$ and*

$$X_\gamma = \{u(y, s) : \nabla \cdot u = 0, \quad \|u\|_\gamma \equiv \sup(1 + |y|)^{\gamma-\alpha} (1+s)^{\frac{\alpha}{2}} |u(y, s)| < \infty\}$$

where the supremum is taken over $y \in \mathbb{R}^n$, $s > 0$ and $0 \leq \alpha \leq \gamma$. For $u \in X_\gamma$ and $v \in X_\gamma$, the function $\mathcal{S}[u, v]$ defined by (3.1) belongs to X_γ and satisfies

$$(3.2) \quad \|\mathcal{S}[u, v]\|_\gamma \leq c \|u\|_\gamma \cdot \|v\|_\gamma.$$

Proof. Since $\mathcal{S}[\cdot, \cdot]$ is bilinear on $X_\gamma \times X_\gamma$, we may assume $\|u\|_\gamma = \|v\|_\gamma = 1$. From

$$|(u \otimes v)(y, s)| \leq (1 + |y|)^{1-2\gamma} (1+s)^{-\frac{1}{2}} \leq (1+s)^{-\frac{1}{2}}$$

and Lemma 3.1, we get

$$|\mathcal{S}[u, v]| \leq \int_0^t \|F(t-s)\|_1 \|(u \otimes v)(s)\|_\infty ds \leq c \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = c;$$

and so $|\mathcal{S}[u, v]|$ is bounded in x and $t > 0$. Thus, we assume below $|x| > 1$ and $t > 1$.

(i) Suppose first $\gamma = n+1$. We write

$$\begin{aligned} |\mathcal{S}[u, v]| &\leq \int_0^t \left(\int_{|x-y| < |x|/2} + \int_{|x-y| > |x|/2} \right) |F(x-y, t-s)| |(u \otimes v)(y, s)| dy ds \\ &\equiv \mathcal{S}_1 + \mathcal{S}_2. \end{aligned}$$

From $|x - y| < |x|/2$ we get $|y| \geq |x| - |x - y| > |x|/2$; so

$$\begin{aligned}\mathcal{S}_1 &\leq c \int_0^t \int_{|x-y| < |x|/2} (t-s)^{-\frac{3}{4}} |x-y|^{\frac{1}{2}-n} (1+|y|)^{-\frac{3}{2}-2n} (1+s)^{-\frac{1}{4}} dy ds \\ &\leq c \int_{|x-y| < |x|/2} |x-y|^{\frac{1}{2}-n} (1+|y|)^{-\frac{3}{2}-2n} dy \\ &\leq c |x|^{\frac{1}{2}} (1+|x|)^{-\frac{3}{2}-2n} \leq c (1+|x|)^{-1-2n} \leq c (1+|x|)^{-n-1}.\end{aligned}$$

Furthermore,

$$\mathcal{S}_2 \leq c \int_0^t \int_{|x-y| > |x|/2} |x-y|^{-n-1} (1+|y|)^{1-2n} (1+s)^{-\frac{3}{2}} dy ds \leq c |x|^{-n-1}.$$

Hence

$$|\mathcal{S}[u, v]| \leq c (1+|x|)^{-n-1}.$$

We next write

$$|\mathcal{S}[u, v]| \leq \left(\int_0^{t/2} + \int_{t/2}^t \right) |F(t-s)| * |(u \otimes v)(s)| ds \equiv \mathcal{S}_3 + \mathcal{S}_4.$$

Then

$$\mathcal{S}_3 \leq \int_0^{t/2} \|F(t-s)\|_\infty \|(u \otimes v)(s)\|_1 ds \leq ct^{-\frac{n+1}{2}} \int_0^{t/2} \|(u \otimes v)(s)\|_1 ds.$$

Since $|(u \otimes v)(y, s)| \leq (1+|y|)^{-n-1} (1+s)^{-\frac{n+1}{2}}$, it follows that

$$\mathcal{S}_3 \leq ct^{-\frac{n+1}{2}} \int_0^\infty (1+s)^{-\frac{n+1}{2}} ds = ct^{-\frac{n+1}{2}}.$$

Moreover,

$$\mathcal{S}_4 \leq \int_{t/2}^t \|F(t-s)\|_1 \|(u \otimes v)(s)\|_\infty ds = c \int_{t/2}^t (t-s)^{-\frac{1}{2}} \|(u \otimes v)(s)\|_\infty ds.$$

From $|(u \otimes v)(y, s)| \leq (1+s)^{-n-1}$, we get

$$\mathcal{S}_4 \leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} s^{-n-1} ds \leq ct^{-n-\frac{1}{2}} \leq ct^{-\frac{n+1}{2}}.$$

Therefore,

$$|\mathcal{S}[u, v]| \leq c (1+t)^{-\frac{n+1}{2}}.$$

This proves the result.

(ii) Suppose next $n < \gamma < n+1$. In this case,

$$\begin{aligned}\mathcal{S}_1 &\leq c \int_0^t \int_{|x-y| < |x|/2} (t-s)^{-\frac{2+n-\gamma}{2}} |x-y|^{1-\gamma} (1+|y|)^{-n-\gamma} (1+s)^{-\frac{\gamma-n}{2}} dy ds \\ &\leq c |x|^{n+1-\gamma} (1+|x|)^{-n-\gamma} \leq c (1+|x|)^{1-2\gamma} \leq c (1+|x|)^{-\gamma},\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2 &\leq c \int_0^t \int_{|x-y|>|x|/2} (t-s)^{-\frac{n+1-\gamma}{2}} |x-y|^{-\gamma} (1+|y|)^{1-n-\gamma} (1+s)^{-\frac{1+\gamma-n}{2}} dy ds \\ &\leq c \int_{|x-y|>|x|/2} |x-y|^{-\gamma} (1+|y|)^{1-n-\gamma} dy \leq c|x|^{-\gamma}.\end{aligned}$$

On the other hand,

$$\mathcal{S}_3 \leq \int_0^{t/2} \|F(t-s)\|_\infty \|(u \otimes v)(s)\|_1 ds \leq ct^{-\frac{n+1}{2}} \int_0^{t/2} \|(u \otimes v)(s)\|_1 ds.$$

Since $|(u \otimes v)(y, s)| \leq (1+|y|)^{-\gamma} (1+s)^{-\frac{\gamma}{2}}$ and $\gamma/2 > n/2 \geq 1$, we see that

$$\mathcal{S}_3 \leq ct^{-\frac{n+1}{2}} \int_0^\infty (1+s)^{-\frac{\gamma}{2}} ds = ct^{-\frac{n+1}{2}} \leq ct^{-\frac{\gamma}{2}}.$$

Furthermore, $|(u \otimes v)(y, s)| \leq (1+s)^{-\gamma}$ implies

$$\begin{aligned}\mathcal{S}_4 &\leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} \|(u \otimes v)(s)\|_\infty ds \leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} (1+s)^{-\gamma} ds \\ &\leq c(1+t)^{\frac{1}{2}-\gamma} \leq c(1+t)^{-\frac{\gamma}{2}}.\end{aligned}$$

Hence

$$|\mathcal{S}[u, v]| \leq c(1+t)^{-\frac{\gamma}{2}}.$$

This shows the desired result.

(iii) Suppose now $\gamma = n$. In this case, we have

$$|(u \otimes v)(y, s)| \leq (1+|y|)^{2\alpha-2n} (1+s)^{-\alpha} \quad 0 \leq \alpha \leq n.$$

So we get

$$\begin{aligned}\mathcal{S}_1 &\leq c \int_0^t \int_{|x-y|<|x|/2} (t-s)^{-\frac{3}{4}} |x-y|^{\frac{1}{2}-n} (1+|y|)^{\frac{1}{2}-2n} s^{-\frac{1}{4}} dy ds \\ &\leq c(1+|x|)^{\frac{1}{2}-2n} |x|^{\frac{1}{2}} \leq c|x|^{1-2n} \leq c|x|^{-n}\end{aligned}$$

and

$$\mathcal{S}_2 \leq c \int_0^t \int_{|x-y|>|x|/2} (t-s)^{-\frac{1}{2}} |x-y|^{-n} (1+|y|)^{1-2n} s^{-\frac{1}{2}} dy ds \leq c|x|^{-n}.$$

Moreover,

$$\mathcal{S}_3 \leq \int_0^{t/2} \|F(t-s)\|_\infty \|(u \otimes v)(s)\|_1 ds \leq ct^{-\frac{n+1}{2}} \int_0^{t/2} (1+s)^{-\frac{1}{2}} ds \leq ct^{-\frac{n}{2}},$$

and

$$\begin{aligned}\mathcal{S}_4 &\leq \int_{t/2}^t \|F(t-s)\|_1 \|(u \otimes v)(s)\|_\infty ds \leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} (1+s)^{-n} ds \\ &\leq c(1+t)^{\frac{1}{2}-n} \leq c(1+t)^{-\frac{n}{2}}.\end{aligned}$$

This proves the result.

(iv) Suppose finally $1 \leq \gamma < n$. We take $0 < \varepsilon < 1$ so that $\gamma + \varepsilon < n$ to get

$$\begin{aligned} |\mathcal{S}[u, v]| &\leq c \int_0^t \int (t-s)^{-\frac{1+\varepsilon}{2}} |x-y|^{\varepsilon-n} (1+|y|)^{-(\gamma+\varepsilon)} (1+s)^{-\frac{\gamma-\varepsilon}{2}} dy ds \\ &\leq c \int |x-y|^{\varepsilon-n} (1+|y|)^{-(\gamma+\varepsilon)} dy \leq c|x|^{-\gamma}. \end{aligned}$$

Next, from $|(u \otimes v)(y, s)| \leq (1+|y|)^{-\gamma} (1+s)^{-\frac{\gamma}{2}}$ we get $\|(u \otimes v)(s)\|_{n/\gamma, w} \leq c(1+s)^{-\frac{\gamma}{2}}$; so

$$(3.3) \quad \|\mathcal{S}[u, v]\|_{n/\gamma, w} \leq c \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{\gamma}{2}} ds \leq c$$

since $\gamma \geq 1$. Hereafter $\|\cdot\|_{n/\gamma, w}$ denotes the $L^{n/\gamma, w}$ -norm (see [1,3]). Next we write

$$\mathcal{S}[u, v](t) = \mathcal{S}_3[u, v](t) + \mathcal{S}_4[u, v](t)$$

where

$$\begin{aligned} \mathcal{S}_3[u, v](t) &= - \int_0^{t/2} e^{-(t-s)A} P \nabla \cdot (u \otimes v)(s) ds = e^{-tA/2} \mathcal{S}[u, v](t/2), \\ \mathcal{S}_4[u, v](t) &= - \int_{t/2}^t e^{-(t-s)A} P \nabla \cdot (u \otimes v)(s) ds. \end{aligned}$$

Applying (3.3) and the estimate

$$(3.4) \quad \|e^{-tA} a\|_\infty \leq ct^{-\frac{\gamma}{2}} \|a\|_{n/\gamma, w},$$

which will be proved below, we get

$$\|\mathcal{S}_3[u, v](t)\|_\infty \leq ct^{-\frac{\gamma}{2}} \|\mathcal{S}[u, v](t/2)\|_{n/\gamma, w} \leq ct^{-\frac{\gamma}{2}}.$$

Since $\gamma \geq 1$, from $|(u \otimes v)(y, s)| \leq (1+s)^{-\gamma}$ we obtain

$$\|\mathcal{S}_4[u, v](t)\|_\infty \leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} (1+s)^{-\gamma} ds \leq c(1+t)^{\frac{1}{2}-\gamma} \leq c(1+t)^{-\frac{\gamma}{2}}.$$

The proof of Lemma 3.2 is complete.

We can now prove Theorem 1. Let

$$\Phi(u) = e^{-tA} a + \mathcal{S}[u, u] \equiv u_0 + \mathcal{S}[u, u].$$

Lemma 3.1 and Lemma 3.2 together imply

$$(3.5) \quad \begin{aligned} \|\Phi(u)\|_\gamma &\leq c\|u_0\|_\gamma + c'\|u\|_\gamma^2, \\ \|\Phi(u) - \Phi(v)\|_\gamma &\leq c'(\|u\|_\gamma + \|v\|_\gamma)\|u - v\|_\gamma. \end{aligned}$$

Assuming

$$\|u_0\|_\gamma < \frac{1}{4cc'},$$

we define

$$(3.6) \quad M = \frac{1 - \sqrt{1 - 4cc' \|u_0\|_\gamma}}{2c'}.$$

By (3.5), $\|u\|_\gamma \leq M$ implies

$$\|\Phi(u)\|_\gamma \leq c\|u_0\| + c'M^2 = M$$

and

$$\|\Phi(u) - \Phi(v)\|_\gamma \leq 2c'M\|u - v\|_\gamma \quad \text{whenever } \|u\|_\gamma \leq M \text{ and } \|v\|_\gamma \leq M.$$

Since $2c'M < 1$ by (3.6), we can apply the contraction mapping principle to find a unique $u \in X_\gamma$ such that $\|u\|_\gamma \leq M$ and $u = \Phi(u)$. This completes the proof of Theorem 1.

Proof of (3.4). The $L^{n/\gamma, w}$ -norm of a measurable function f is given by (see [1,6])

$$\|f\|_{n/\gamma, w} = \sup_E |E|^{\gamma/n-1} \int_E |f| dx,$$

where $|E|$ denotes Lebesgue measure of the measurable sets $E \subset \mathbb{R}^n$. Using this, we can easily deduce (3.4). The details are given in [6], but we here give a proof for the reader's convenience. The definition of Lebesgue integral gives

$$|(e^{-tA}a)(x)| \leq ct^{-\frac{n}{2}} \int e^{-c|x-y|^2/t} |a(y)| dy = ct^{-\frac{n}{2}} \int_0^1 \mu[\{y : e^{-c|x-y|^2/t} > s\}] ds,$$

where $\mu = |a(y)| dy$. But, for $0 < s < 1$, $\{y : e^{-c|x-y|^2/t} > s\} = B(x, ct^{\frac{1}{2}}(\log(1/s))^{\frac{1}{2}})$, where $B(x, r)$ is the open ball with radius r centered at x . So we have

$$\mu[B(x, t^{\frac{1}{2}}(\log(1/s))^{\frac{1}{2}})] \leq c\|a\|_{n/\gamma, w} t^{\frac{n}{2}(1-\frac{\gamma}{n})} (\log(1/s))^{\frac{n}{2}(1-\frac{\gamma}{n})}.$$

Thus,

$$|(e^{-tA}a)(x)| \leq c\|a\|_{n/\gamma, w} t^{-\frac{\gamma}{2}} \int_0^1 [\log(1/s)]^{\frac{n}{2}(1-\frac{\gamma}{n})} ds = ct^{-\frac{\gamma}{2}} \|a\|_{n/\gamma, w}$$

with $c > 0$ independent of x ; and (3.4) is proved.

4. Proof of Theorem 3

The following results are due to [2], but we give a proof for the reader's convenience.

Lemma 4.1. (i) *If a satisfies (a) and (b), so does the function $e^{-tA}a$.*

(ii) *If $u(x, t)$ and $v(x, t)$ satisfy (a) and (b) for all $t \geq 0$, so does the function*

$$w(t) = \mathcal{S}[u, v](t) = - \int_0^t e^{-(t-s)A} P \nabla \cdot (u \otimes v)(s) ds.$$

(iii) *Let a be divergence-free and satisfy*

$$\int (1 + |y|) |a(y)| dy < \infty.$$

Then the matrix $(\int y_j a_k(y) dy)$ is skew-symmetric.

Proof. (i) Applying the Fourier transform to $u_0(t) = e^{-tA}a$, we get

$$\widehat{u}_0(\xi, t) = e^{-t|\xi|^2} \widehat{a}(\xi).$$

The result is then obvious, since the parity of a in x_j is the same as that of \widehat{a} in ξ_j .

(ii) We also apply the Fourier transform to get

$$\begin{aligned} \widehat{w}_j(\xi, t) &= -i \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (\widehat{u_k v_\ell})(\xi, s) ds \\ &= -i \sum_{\ell=1}^n \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} (\widehat{u_j v_\ell})(\xi, s) ds \\ &\quad + i \sum_{k \neq \ell} \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} \frac{\xi_j \xi_k}{|\xi|^2} (\widehat{u_k v_\ell})(\xi, s) ds \\ &\quad + i \sum_{\ell=1}^n \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} \frac{\xi_j \xi_\ell}{|\xi|^2} (\widehat{u_\ell v_\ell})(\xi, s) ds \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

When $\ell \neq j$, $(\widehat{u_j v_\ell})$ is odd in ξ_j , odd in ξ_ℓ , and even in each of the other components of ξ . So, $\xi_\ell (\widehat{u_j v_\ell})$ is odd in ξ_j and even in each of the other components of ξ . On the other hand, $\xi_j (\widehat{u_j v_j})$ is odd in ξ_j and even in each of the other components of ξ . Hence J_1 satisfies (a). As for J_2 , we see that $\xi_\ell \xi_k (\widehat{u_k v_\ell})$ is even in each component of ξ . Hence J_2 satisfies (a). Since $\xi_\ell^2 (\widehat{u_\ell v_\ell})$ is obviously even in each component of ξ , J_3 satisfies (a).

Next, let $\eta = (\eta_1, \dots, \eta_n) = (\xi_n, \xi_1, \dots, \xi_{n-1})$. One can easily verify that

$$(\widehat{u_{k+1} v_{\ell+1}})(\eta, s) = (\widehat{u_k v_\ell})(\xi, s),$$

with the understanding that $n+1$ equals 1. Thus,

$$\begin{aligned} \widehat{w}_{j+1}(\eta, t) &= -i \int_0^t \eta_\ell e^{-(t-s)|\eta|^2} \left(\delta_{j+1, k} - \frac{\eta_{j+1} \eta_k}{|\eta|^2} \right) (\widehat{u_k v_\ell})(\eta, s) ds \\ &= -i \int_0^t \xi_{\ell-1} e^{-(t-s)|\xi|^2} \left(\delta_{j+1, k} - \frac{\xi_j \xi_{k-1}}{|\xi|^2} \right) (\widehat{u_{k-1} v_{\ell-1}})(\xi, s) ds \\ &= -i \int_0^t \xi_{\ell-1} e^{-(t-s)|\xi|^2} \delta_{j+1, k} (\widehat{u_{k-1} v_{\ell-1}})(\xi, s) ds \\ &\quad + i \int_0^t \frac{\xi_j \xi_{k-1}}{|\xi|^2} (\widehat{u_{k-1} v_{\ell-1}})(\xi, s) ds \\ &= -i \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} (\widehat{u_j v_\ell})(\xi, s) ds + i \int_0^t \frac{\xi_j \xi_k}{|\xi|^2} (\widehat{u_k v_\ell})(\xi, s) ds \end{aligned}$$

$$\begin{aligned}
&= -i \int_0^t \xi_\ell e^{-(t-s)|\xi|^2} \left(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (\widehat{u_k v_\ell})(\xi, s) ds \\
&= \widehat{w}_j(\xi, t).
\end{aligned}$$

This shows that w satisfies (b).

(iii) Note that the assumption implies $\widehat{a} \in C^1$. Since $\nabla \cdot a = 0$, we have $\xi \cdot \widehat{a}(\xi) = 0$. Differentiating this with respect to ξ_j gives

$$(4.1) \quad \widehat{a}_j(\xi) + \xi \cdot (\partial_j \widehat{a})(\xi) = 0, \quad j = 1, \dots, n.$$

On the other hand, condition $\xi \cdot \widehat{a}(\xi) = 0$ implies $\widehat{a}(0) = 0$; so

$$(4.2) \quad \widehat{a}_j(\xi) = \xi \cdot \int_0^1 (\nabla \widehat{a}_j)(t\xi) dt, \quad j = 1, \dots, n.$$

From (4.1) and (4.2) we get, for any fixed j and k ,

$$\begin{aligned}
&\xi_k (\partial_j \widehat{a}_k)(0, \dots, 0, \xi_k, 0, \dots, 0) + \widehat{a}_j(0, \dots, 0, \xi_k, 0, \dots, 0) = 0, \\
&\widehat{a}_j(0, \dots, 0, \xi_k, 0, \dots, 0) = \xi_k \int_0^1 (\partial_k \widehat{a}_j)(0, \dots, 0, t\xi_k, 0, \dots, 0) dt,
\end{aligned}$$

and therefore

$$\int_0^1 (\partial_k \widehat{a}_j)(0, \dots, 0, t\xi_k, 0, \dots, 0) dt + (\partial_j \widehat{a}_k)(0, \dots, 0, \xi_k, 0, \dots, 0) = 0.$$

Letting $\xi_k \rightarrow 0$ gives $(\partial_k \widehat{a}_j)(0) + (\partial_j \widehat{a}_k)(0) = 0$ and this proves the desired result. The proof of Lemma 4.1 is complete.

Proof of Theorem 3. First we show that

$$(4.3) \quad |(e^{-tA}a)(x)| \leq c(c_0 + c_1)(1 + |x|)^{-n-3}, \quad |(e^{-tA}a)(x)| \leq c(c_0 + c_1)(1 + t)^{-\frac{n+3}{2}}.$$

Since a is divergence-free and satisfies (a), (b) and (1.4), it follows that

$$(4.4) \quad \widehat{a}_j(0) = \int a_j(y) dy = 0, \quad \int y_k y_\ell a_j(y) dy = 0, \quad j, k, \ell = 1, \dots, n.$$

Moreover, by Lemma 4.1 (iii), the matrix $(\int y_k a_j(y) dy)$ is skew-symmetric. This, together with (a), implies that

$$(4.5) \quad \int y_k a_j(y) dy = 0, \quad j, k = 1, \dots, n.$$

Now let

$$(e^{-tA}a)(x) = \left(\int_{|y| < |x|/2} + \int_{|y| > |x|/2} \right) E_t(x - y) a(y) dy \equiv K_1 + K_2.$$

We easily see that

$$|K_2| \leq \sup_{|y| > |x|/2} |a(y)| \leq cc_0(1 + |x|)^{-n-3}.$$

Next, Taylor's formula, (4.4) and (4.5) together yield

$$\begin{aligned}
K_1 &= \int_{|y| < |x|/2} [E_t(x-y) - \sum_{|\gamma| \leq 2} \frac{(-y)^\gamma}{\gamma!} (\partial_x^\gamma E_t)(x)] a(y) dy \\
&\quad + \sum_{|\gamma| \leq 2} \frac{1}{\gamma!} (\partial_x^\gamma E_t)(x) \int_{|y| < |x|/2} (-y)^\gamma a(y) dy \\
&= 3 \sum_{|\gamma|=3} \int_0^1 \int_{|y| < |x|/2} (1-\theta)^2 \frac{(-y)^\gamma}{\gamma!} (\partial_x^\gamma E_t)(x-y\theta) a(y) dy \\
&\quad - \sum_{|\gamma| \leq 2} \frac{1}{\gamma!} (\partial_x^\gamma E_t)(x) \int_{|y| > |x|/2} (-y)^\gamma a(y) dy \\
&\equiv K_{11} + K_{12}.
\end{aligned}$$

Since $|x-y\theta| > |x|/2$ whenever $|y| < |x|/2$, it follows that

$$|K_{11}| \leq ct^{-\frac{n+3}{2}} e^{-c|x|^2/t} \int_{|y| < |x|/2} |y|^3 |a(y)| dy \leq c|x|^{-n-3} \int |y|^3 |a(y)| dy = cc_1|x|^{-n-3}.$$

Moreover, direct calculation gives

$$|K_{12}| \leq cc_0 \sum_{|\gamma| \leq 2} |x|^{-n-|\gamma|} \int_{|y| > |x|/2} (1+|y|)^{-n-3+|\gamma|} dy \leq cc_0|x|^{-n-3}.$$

We thus obtain the first estimate of (4.3). On the other hand, Taylor's formula, (4.4) and (4.5) together imply

$$(e^{-tA}a)(x) = 3 \sum_{|\gamma|=3} \int_0^1 (1-\theta)^2 \int \frac{(-y)^\gamma}{\gamma!} (\partial_x^\gamma E_t)(x-y\theta) a(y) dy d\theta$$

so that

$$|(e^{-tA}a)(x)| \leq ct^{-\frac{n+3}{2}} \int e^{-c|x-y\theta|^2/t} |y|^3 |a(y)| dy \leq cc_1 t^{-\frac{n+3}{2}}.$$

This, together with $|e^{-tA}a)(x)| \leq \|E_t\|_1 \|a\|_\infty \leq c_0$, gives the second estimate of (4.3).

Consider next the function $w = (w_1, \dots, w_n)$ with

$$w_j(t) = - \int_0^t F_{\ell,jk}(t-s) * (u_k v_\ell)(s) ds.$$

Suppose that u and v satisfy (a), (b), and

$$\begin{aligned}
(4.6) \quad |u(y, s)| &\leq c(1+|y|)^{\alpha-n-3}(1+s)^{-\frac{\alpha}{2}}, \\
|v(y, s)| &\leq c(1+|y|)^{\alpha-n-3}(1+s)^{-\frac{\alpha}{2}},
\end{aligned} \quad \text{for all } 0 \leq \alpha \leq n+3.$$

We shall show that

$$(4.7) \quad |w(x, t)| \leq c(1+|x|)^{-n-3}, \quad |w(x, t)| \leq c(1+t)^{-\frac{n+3}{2}}.$$

Observe that (a) and (b) together imply

$$(4.8) \quad \int (u_k v_\ell)(y, s) dy = \lambda(s) \delta_{k\ell}, \quad \int y_j (u_k v_\ell)(y, s) dy = 0.$$

We write

$$w_j = - \left(\int_0^{t/2} + \int_{t/2}^t \right) F_{\ell, jk}(t-s) * (u_k v_\ell)(s) ds \equiv I_1 + I_2.$$

By (4.8), Taylor's formula and the fact that $F_{\ell, j\ell} \equiv 0$, we see that

$$I_1 = -2 \int_0^1 (1-\theta) \int_0^{t/2} \int \sum_{|\gamma|=2} \frac{1}{\gamma!} (\partial_x^\gamma F_{\ell, jk})(x-y\theta, t-s) y^\gamma (u_k v_\ell)(y, s) dy ds d\theta.$$

Direct calculation using (4.6) gives

$$|I_2| \leq c \int_{t/2}^t (t-s)^{-\frac{1}{2}} (1+s)^{-n-3} ds \leq c(1+t)^{-n-5/2} \leq c(1+t)^{-\frac{n+3}{2}}$$

and

$$|I_1| \leq ct^{-\frac{n+3}{2}} \int_0^{t/2} \int |y|^2 |u(y, s)| |v(y, s)| dy ds.$$

Since $|y|^2 |u(y, s)| |v(y, s)| \leq c(1+|y|)^{-1-2n} (1+s)^{-\frac{3}{2}}$, it follows that

$$|I_1| \leq ct^{-\frac{n+3}{2}} \int_0^\infty \int |y|^2 |u(y, s)| |v(y, s)| dy ds = ct^{-\frac{n+3}{2}}.$$

This proves the second estimate of (4.7). To deduce the first estimate of (4.7), we write

$$w_j = I'_1 + I'_2,$$

where

$$I'_1 = - \int_0^t \int_{|y| < |x|/2} F_{\ell, jk}(x-y, t-s) (u_k v_\ell)(y, s) dy ds,$$

$$I'_2 = - \int_0^t \int_{|y| > |x|/2} F_{\ell, jk}(x-y, t-s) (u_k v_\ell)(y, s) dy ds.$$

Direct calculation using (4.6) gives

$$|I'_2| \leq c \int_0^t \int_{|y| > |x|/2} |F(x-y, t-s)| (1+|y|)^{-2n-5} (1+s)^{-\frac{1}{2}} dy ds$$

$$\leq c(1+|x|)^{-2n-5} \int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}} ds \leq c(1+|x|)^{-n-3}.$$

On the other hand, by Taylor's formula we get

$$I'_1 = -2 \int_0^1 \int_0^t \int_{|y| < |x|/2} (1-\theta) \sum_{|\gamma|=2} \frac{1}{\gamma!} (\partial_x^\gamma F_{\ell, jk})(x-y\theta, t-s) y^\gamma (u_k v_\ell)(y, s) dy ds d\theta$$

$$- \int_0^t \int_{|y| < |x|/2} [F_{\ell, jk}(x, t-s) - (\partial_m F_{\ell, jk})(x, t-s) y_m] (u_k v_\ell)(y, s) dy ds.$$

Since (4.8) implies

$$\int F_{\ell,jk}(x, t-s)(u_k v_\ell)(y, s) dy = \lambda(s) F_{\ell,j\ell}(x, t-s) = 0,$$

it follows that

$$\int_{|y| < |x|/2} F_{\ell,jk}(x, t-s)(u_k v_\ell)(y, s) dy = - \int_{|y| > |x|/2} F_{\ell,jk}(x, t-s)(u_k v_\ell)(y, s) dy.$$

Inserting this in the above yields

$$\begin{aligned} I'_1 &= -2 \int_0^1 \int_0^t \int_{|y| < |x|/2} (1-\theta) \sum_{|\gamma|=2} \frac{1}{\gamma!} (\partial_x^\gamma F_{\ell,jk})(x-y\theta, t-s) y^\gamma (u_k v_\ell)(y, s) dy ds d\theta \\ &\quad + \int_0^t \int_{|y| > |x|/2} F_{\ell,jk}(x, t-s)(u_k v_\ell)(y, s) dy ds \\ &\quad - \int_0^t \int_{|y| > |x|/2} (\partial_m F_{\ell,jk})(x, t-s) y_m (u_k v_\ell)(y, s) dy ds \\ &\equiv I'_{11} + I'_{12} + I'_{13}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} |I'_{11}| &\leq c|x|^{-n-3} \int_0^\infty \int |y|^2 |u(y, s)| |v(y, s)| dy ds = c|x|^{-n-3}, \\ |I'_{12}| &\leq c|x|^{-n-1} \int_0^t \int_{|y| > |x|/2} (1+|y|)^{-2n-3} (1+s)^{-\frac{3}{2}} dy ds \leq c|x|^{-n-3}, \\ |I'_{13}| &\leq c|x|^{-n-2} \int_0^t \int_{|y| > |x|/2} (1+|y|)^{-2n-2} (1+s)^{-\frac{3}{2}} dy ds \leq c|x|^{-n-3}. \end{aligned}$$

This proves the first estimate of (4.7).

Due to Lemma 4.1 and the above calculation, we can now complete the proof of Theorem 3. Namely, we consider the operator

$$\Phi(u) = e^{-tA} a + \mathcal{S}[u, u]$$

on the space of divergence-free vector fields $u(y, s)$ satisfying (a), (b) and

$$\|u\|_{n+3} \equiv \sup(1+|y|)^{n+3-\alpha} (1+s)^{\frac{\alpha}{2}} |u(y, s)| < \infty,$$

where the supremum is taken over $y \in \mathbb{R}^n$, $s \geq 0$ and $0 \leq \alpha \leq n+3$. The foregoing calculation then shows that

$$\|\mathcal{S}[u, v]\|_{n+3} \leq c \|u\|_{n+3} \cdot \|v\|_{n+3}.$$

So we can apply the contraction mapping principle, assuming that c_0 and c_1 in (1.4) be sufficiently small, to get the desired solution of the equation $u = \Phi(u)$. The details are completely parallel to the final step of the proof of Theorem 1 and so omitted here.

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