Notes on Space-Time Decay Properties of Nonstationary Incompressible Navier-Stokes Flows in $\mathbb{R}^n$

By

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Dedicated to Professor Shinnosuke Oharu on his 60th birthday

1. Introduction and results

Consider the nonstationary Navier-Stokes system in $\mathbb{R}^n$, $n \geq 2$, in the form of the integral equation:

$$\text{(IE)} \qquad u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes u)(s)ds.$$

Here $\{e^{-tA}\}_{t\geq 0}$ is the heat semigroup, P is the bounded projetion from the space $\boldsymbol{L}^q$, $1 < q < \infty$, of vector fields to the subspace of $\boldsymbol{L}^q$ consisting of all divergence-free vector fields. As is shown in [4], the operator $e^{-tA}P\nabla\cdot$ has the kernel function $F = (F_{\ell,jk})_{j,k,\ell=1}^n$ with

$$F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j\partial_k\partial_\ell E_{s+t}(x)ds, \qquad \partial_j = \partial/\partial x_j,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $t > 0$, and $E_t(x) = (4\pi t)^{-\frac{n}{2}} e^{-|x|^2/4t}$.

Inspired by Takahashi [9], we proved in [4] the following

Theorem 1. *Fix $1 \leq \gamma \leq n+1$ and let a be divergence-free, satisfying*

$$(1.1) \qquad |(e^{-tA}a)(x)| \leq c(1+|x|)^{-\gamma}, \qquad |(e^{-tA}a)(x)| \leq c(1+t)^{-\frac{\gamma}{2}}.$$

If $c > 0$ is small, then (IE) *possesses a solution u such that, with another $c > 0$,*

$$|u(x,t)| \leq c(1+|x|)^{-\gamma}, \qquad |u(x,t)| \leq c(1+t)^{-\frac{\gamma}{2}}.$$

However, in proving Theorem 1, the Hardy space theory was used to find relevant decay estimates in t. In this paper we show that Theorem 1 can be deduced by a more elementary method, i.e., without using the Hardy space theory.

In [4] we also gave a class of initial data a for which $e^{-tA}a$ satisfies (1.1). But, the assumption on a imposed in [4] was too complicated and restrictive. So, we here give a simpler version of the assumption on a that ensures the validity of (1.1). More specifically, we shall show the following

Theorem 2. *Let a be divergence-free and satisfy*

$$c_0 = \sup_y (1+|y|)^{\gamma}|a(y)| < \infty \qquad \text{for some } 0 < \gamma \le n+1.$$

Suppose further that

$$c_1 = \int |a(y)|dy < \infty \quad \text{when } \gamma = n, \qquad c_1 = \int |y||a(y)|dy < \infty \quad \text{when } \gamma = n+1.$$

Then

$$|(e^{-tA}a)(x)| \le cd(1+|x|)^{-\gamma}, \qquad |(e^{-tA}a)(x)| \le cd(1+t)^{-\frac{\gamma}{2}}. \tag{1.2}$$

Here, $d = c_0 + c_1$ when $\gamma = n$ or $\gamma = n+1$; and $d = c_0$ otherwise.

Starting from Theorems 1 and 2, we deduced in [3] a space-time asymptotic expansion of solutions u in the case $\gamma = n+1$, which was then applied in [8] to find a lower bound estimate of rates of energy decay for weak solutions of (IE).

Finally, we supplement the recent result of Brandolese [2] on the existence of solutions which decay more rapidly than those treated in Theorem 1. Consider the divergence-free vector fields a such that

(a) a_j is odd in x_j and is even in each of the other variables.

(b) a is cyclically symmetric in the sense that

$$a_1(x_1, \dots, x_n) = a_2(x_n, x_1, \dots, x_{n-1}) = \cdots = a_n(x_2, \dots, x_n, x_1).$$

Brandolese [2] shows the existence of a solution u satisfying (a) and (b) above, with the estimates

$$|u(x,t)| \le c(1+|x|)^{-n-2}, \qquad |u(x,t)| \le c(1+t)^{-\frac{n+2}{2}}. \tag{1.3}$$

Observe that the solutions above decay more rapidly than those treated in Theorem 1. But, estimate (1.3) seems not optimal. Indeed, we shall prove

Theorem 3. *Suppose a satisfies (a) and (b), and*

$$c_0 = \sup (1+|y|)^{n+3}|a(y)| < \infty, \qquad c_1 = \int |y|^3|a(y)|dy < \infty. \tag{1.4}$$

(i) *The function $x \mapsto e^{-tA}a(x)$ satisfies (a) and (b), and there hold the estimates*

$$|(e^{-tA}a)(x)| \le c(c_0+c_1)(1+|x|)^{-n-3}, \qquad |(e^{-tA}a)(x)| \le c(c_0+c_1)(1+t)^{-\frac{n+3}{2}}.$$

(ii) *If* $c_0 + c_1$ *is small, then* (IE) *has a strong solution* u *satisfying* (a), (b), *and*

$$|u(x,t)| \le c(1+|x|)^{-n-3}, \qquad |u(x,t)| \le c(1+t)^{-\frac{n+3}{2}}. \tag{1.5}$$

Brandolese [2] proves the existence of solutions which satisfy (1.3), assuming

$$c_0' = \sup(1+|y|)^{n+2}|a(y)| < \infty, \qquad c_1' = \int |y|^2|a(y)|dy < \infty. \tag{1.6}$$

This is the reason why (1.3) is deduced in [2] instead of (1.5). However, the result of [3] implies that the corresponding solution u has to satisfy

$$\|u(t)\|_\infty = o(t^{-\frac{n+2}{2}}) \qquad \text{as } t \to \infty$$

whenever a satisfies (a), (b) and (1.6). Thus, (1.3) is not optimal even under the assumption (1.6). On the other hand, in view of the result given in [7], estimate (1.5) seems to be optimal for general solutions satisfying (a) and (b). Namely, one can reasonably expect the existence of a solution u satisfying (a) and (b) such that

$$\|u(t)\|_\infty \ge ct^{-\frac{n+3}{2}} \qquad \text{for large } \ t > 0,$$

even when a is in $\mathcal{S}$ and satisfies (a), (b) and $\int y^\alpha a(y)dy = 0$ for every multi-index α.

2. Proof of Theorem 2

Observe first that

$$|(e^{-tA}a)(x)| = |(E_t * a)(x)| \le \|a\|_\infty \|E_t\|_1 = \|a\|_\infty \le c_0,$$

and so $|(e^{-tA}a)(x)|$ is bounded in x and $t > 0$. So we assume from now on $|x| > 1$ and $t > 1$. Recall the estimates

$$t^{-\frac{n+m}{2}} e^{-c|x|^2/t} \le \begin{cases} c_m|x|^{-n-m}, \\ t^{-\frac{n+m}{2}}, \end{cases} \qquad m = 0, 1, 2, \ldots,$$

which will be systematically applied below.

(i) Suppose $\gamma = n + 1$. Since $a \in \boldsymbol{L}^1$, condition $\nabla \cdot a = 0$ implies (see [5])

$$\int a(y)dy = 0. \tag{2.1}$$

We write

$$(e^{-tA}a)(x) = (E_t * a)(x) = \left(\int_{|y|<|x|/2} + \int_{|y|>|x|/2}\right) E_t(x-y)a(y)dy \equiv I_1 + I_2.$$

Then (2.1) implies

$$\begin{aligned} I_1 &= \int_{|y|<|x|/2} [E_t(x-y) - E_t(x)]a(y)dy + E_t(x)\int_{|y|<|x|/2} a(y)dy \\ &= -\int_0^1\int_{|y|<|x|/2} (y\cdot\nabla E_t)(x-y\theta)a(y)dyd\theta - E_t(x)\int_{|y|>|x|/2} a(y)dy \\ &\equiv I_{11} + I_{12}. \end{aligned}$$

Since $|x-y\theta| \geq |x| - \theta|y| \geq |x| - |y| > |x|/2$ whenever $|y| < |x|/2$, it follows that

$$\begin{aligned} |I_{11}| &\leq ct^{-\frac{n+1}{2}}\int_0^1\int_{|y|<|x|/2} e^{-c|x-y\theta|^2/t}|y||a(y)|dyd\theta \\ &\leq ct^{-\frac{n+1}{2}}e^{-c|x|^2/t}\int |y||a(y)|dy \leq cc_1|x|^{-n-1}. \end{aligned}$$

Moreover,

$$|I_{12}| \leq cc_0t^{-\frac{n}{2}}e^{-c|x|^2/t}\int_{|y|>|x|/2}(1+|y|)^{-n-1}dy \leq cc_0|x|^{-n}\times|x|^{-1} = cc_0|x|^{-n-1}.$$

Hence

$$|I_1| \leq c(c_0+c_1)(1+|x|)^{-n-1}. \tag{2.2}$$

On the other hand,

$$|I_2| \leq cc_0t^{-\frac{n}{2}}\int_{|y|>|x|/2} e^{-|x-y|^2/4t}(1+|y|)^{-n-1}dy \leq cc_0(1+|x|)^{-n-1}.$$

This, together with (2.2), implies

$$|(e^{-tA}a)(x)| \leq c(c_0+c_1)(1+|x|)^{-n-1}.$$

Now, (2.1) yields

$$(e^{-tA}a)(x) = \int[E_t(x-y) - E_t(x)]a(y)dy = -\int_0^1\int(y\cdot\nabla E_t)(x-y\theta)a(y)dyd\theta$$

and so

$$|(e^{-tA}a)(x)| \leq ct^{-\frac{n+1}{2}}\int_0^1\int e^{-c|x-y\theta|^2/t}|y||a(y)|dyd\theta \leq cc_1t^{-\frac{n+1}{2}}.$$

This proves (1.2) in case $\gamma = n+1$.

(ii) Let $n \leq \gamma < n+1$. Since $a \in \boldsymbol{L}^1$ and $\nabla\cdot a = 0$, (2.1) holds for a. So we can decompose

$$e^{-tA}a = I_1 + I_2 = I_{11} + I_{12} + I_2$$

in the same way as in (i). Since $|x-y\theta| > |x|/2$ whenever $|y| < |x|/2$, we obtain

$$\begin{aligned} |I_{11}| &\leq cc_0t^{-\frac{n+1}{2}}\int_0^1\int_{|y|<|x|/2} e^{-c|x-y\theta|^2/t}(1+|y|)^{1-\gamma}dyd\theta \\ &\leq cc_0t^{-\frac{n+1}{2}}e^{-c|x|^2/t}\times|x|^{n+1-\gamma} \leq cc_0|x|^{-\gamma}. \end{aligned}$$

When $n < \gamma < n+1$, we have

$$|I_{12}| \le cc_0 t^{-\frac{n}{2}} e^{-c|x|^2/t} \int_{|y|>|x|/2} (1+|y|)^{-\gamma} dy \le cc_0 |x|^{-n} \times |x|^{n-\gamma} = cc_0 |x|^{-\gamma},$$

$$|I_2| \le cc_0 t^{-\frac{n}{2}} \int_{|y|>|x|/2} e^{-c|x-y|^2/t} (1+|y|)^{-\gamma} dy \le cc_0 (1+|x|)^{-\gamma}.$$

When $\gamma = n$, we see that

$$|I_{12}| \le ct^{-\frac{n}{2}} e^{-c|x|^2/t} \int |a(y)| dy \le cc_1 |x|^{-n},$$

$$|I_2| \le cc_0 t^{-\frac{n}{2}} \int_{|y|>|x|/2} e^{-c|x-y|^2/t} (1+|y|)^{-n} dy \le cc_0 (1+|x|)^{-n}.$$

Hence

$$|(e^{-tA}a)(x)| \le cd(1+|x|)^{-\gamma}.$$

On the other hand, since $\int a(y)dy = 0$,

$$\begin{aligned}
(e^{-tA}a)(x) &= \left(\int_{|y|<\sqrt{t}} + \int_{|y|>\sqrt{t}} \right) E_t(x-y) a(y) dy \\
&= -\int_0^1 \int_{|y|<\sqrt{t}} (y \cdot \nabla E_t)(x - y\theta) a(y) dy d\theta \\
&\quad - E_t(x) \int_{|y|>\sqrt{t}} a(y) dy + \int_{|y|>\sqrt{t}} E_t(x-y) a(y) dy \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}$$

Obviously,

$$|J_2| \le \begin{cases} cc_0 t^{-\frac{n}{2}} \int_{|y|>\sqrt{t}} |y|^{-\gamma} dy = cc_0 t^{-\frac{n}{2}} \times t^{\frac{n-\gamma}{2}} = cc_0 t^{-\frac{\gamma}{2}} & (n < \gamma < n+1), \\ ct^{-\frac{n}{2}} \int_{|y|>\sqrt{t}} |a(y)| dy \le cc_1 t^{-\frac{n}{2}} & (\gamma = n), \end{cases}$$

and, in exactly the same way,

$$|J_3| \le cc_1 t^{-\frac{n}{2}} \quad (\gamma = n), \qquad |J_3| \le cc_0 t^{-\frac{\gamma}{2}} \quad (n < \gamma < n+1).$$

Furthermore, since $-n < 1 - \gamma \le 1 - n$, we have

$$|J_1| \le cc_0 t^{-\frac{n+1}{2}} \int_{|y|<\sqrt{t}} |y|^{1-\gamma} dy \le cc_0 t^{-\frac{n+1}{2}} \times t^{\frac{n-\gamma+1}{2}} = cc_0 t^{-\frac{\gamma}{2}}.$$

We thus obtain

$$|(e^{-tA}a)(x)| \le cd(1+t)^{-\frac{\gamma}{2}}.$$

This proves (1.2) in case $n \le \gamma < n+1$.

(iii) Suppose finally $0 < \gamma < n$. From

$$|(e^{-tA}a)(x)| \le cc_0 t^{-\frac{n}{2}} \int e^{-c|x-y|^2/t}|y|^{-\gamma}dy = \int_{|y|<|x|/2} + \int_{|y|>|x|/2} \equiv I_1 + I_2,$$

we get

$$|I_1| \le cc_0|x|^{-n}\int_{|y|<|x|/2}|y|^{-\gamma}dy = cc_0|x|^{-n} \times |x|^{n-\gamma} = cc_0|x|^{-\gamma},$$
$$|I_2| \le cc_0|x|^{-\gamma}t^{-\frac{n}{2}}\int e^{-c|x-y|^2/t}dy = cc_0|x|^{-\gamma}.$$

Therefore,

$$|(e^{-tA}a)(x)| \le cc_0(1+|x|)^{-\gamma}.$$

Furthermore, the boundedness of the Fourier transforms of L^1-functions gives

$$|(e^{-tA}a)(x)| \le cc_0\int E_t(x-y)|y|^{-\gamma}dy \le cc_0\int e^{-t|\xi|^2}|\xi|^{\gamma-n}d\xi = cc_0 t^{-\frac{\gamma}{2}}$$

and so

$$|(e^{-tA}a)(x)| \le cc_0(1+t)^{-\frac{\gamma}{2}}.$$

The proof of Theorem 2 is complete.

3. Proof of Theorem 1

Let $1 \le \gamma \le n+1$ and

$$\mathcal{S}[u,v](t) = -\int_0^t e^{-(t-s)A}P\nabla\cdot(u\otimes v)(s)ds. \tag{3.1}$$

As shown in [4], the kernel function $F_{\ell,jk}$ of $e^{-tA}P\nabla\cdot$ is written as

$$F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_0^\infty \partial_j\partial_k\partial_\ell E_{t+s}(x)ds,$$

so that

$$\mathcal{S}[u,v](t) = -\int_0^t \boldsymbol{F}_{k\ell}(t-s) * (u_k v_\ell)(s)ds, \qquad \boldsymbol{F}_{k\ell}(x,t) = (F_{\ell,jk}(x,t))_{j=1}^n,$$

where $*$ means convolution of functions defined on $\mathbb{R}^n$. We first prepare

Lemma 3.1. *We have*

$$|\nabla_x^m F_{\ell,jk}(x,t)| \le c_m|x|^{\alpha-n-1-m}t^{-\frac{\alpha}{2}}, \qquad 0 \le \alpha \le n+1+m, \quad m = 0,1,2,\dots,$$
$$\|\nabla_x^m F(\cdot,t)\|_q = c_{m,q}t^{-\frac{n}{2}(1+\frac{1+m}{n}-\frac{1}{q})}, \qquad 1 \le q \le \infty, \quad m = 0,1,2,\dots$$

Proof. Obviously,

$$|\nabla_x^{m+1}E_t(x)| \le c_m t^{-\frac{n+1+m}{2}}e^{-c_m|x|^2/t} \le \begin{cases} c_m|x|^{-n-1-m}, \\ c_m t^{-\frac{n+1+m}{2}}. \end{cases}$$

On the other hand, the m-th order derivatives of the integral in the definition of $F_{\ell,jk}$ is estimated in absolute value as follows: Firstly,

$$\leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} e^{-c_m|x|^2/s} ds \leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} ds = c_m t^{-\frac{n+1+m}{2}}.$$

Secondly, via the change of the variable $\tau = |x|^2/s$,

$$\leq c_m \int_t^\infty s^{-\frac{n+3+m}{2}} e^{-c_m|x|^2/s} ds = c_m |x|^{-n-1-m} \int_0^{|x|^2/t} e^{-c_m\tau} \tau^{\frac{n+m-1}{2}} d\tau \leq c_m |x|^{-n-1-m}.$$

Moreover, direct calculation gives

$$\nabla_x^m F(x,t) = t^{-\frac{n+1+m}{2}} K_m(xt^{-\frac{1}{2}}), \qquad K_m(x) = \nabla_x^m \partial_\ell E_1(x) + \int_1^\infty \nabla_x^m \partial_j \partial_k \partial_\ell E_s(x) ds,$$

and

$$\|K_m\|_q \leq c_{m,q} + c'_{m,q} \int_1^\infty s^{-\frac{n+3+m}{2}} \|e^{-c|x|^2/s}\|_q ds = c_{m,q} + c'_{m,q} \int_1^\infty s^{-\frac{n}{2}(1+\frac{3+m}{n}-\frac{1}{q})} ds = c''_{m,q}.$$

Hence

$$\|\nabla_x^m F\|_q = t^{-\frac{n+1+m}{2}} \times t^{\frac{n}{2q}} \|K_m\|_q = c_{m,q} t^{-\frac{n}{2}(1+\frac{1+m}{n}-\frac{1}{q})}.$$

This proves Lemma 3.1.

Lemma 3.2. *Let* $1 \leq \gamma \leq n+1$ *and*

$$X_\gamma = \{u(y,s) \,:\, \nabla \cdot u = 0, \quad \|u\|_\gamma \equiv \sup (1+|y|)^{\gamma-\alpha}(1+s)^{\frac{\alpha}{2}} |u(y,s)| < \infty\}$$

where the supremum is taken over $y \in \mathbb{R}^n$, $s > 0$ *and* $0 \leq \alpha \leq \gamma$. *For* $u \in X_\gamma$ *and* $v \in X_\gamma$, *the function* $\mathcal{S}[u,v]$ *defined by* (3.1) *belongs to* X_γ *and satisfies*

$$\|\mathcal{S}[u,v]\|_\gamma \leq c\|u\|_\gamma \cdot \|v\|_\gamma. \tag{3.2}$$

Proof. Since $\mathcal{S}[\cdot,\cdot]$ is bilinear on $X_\gamma \times X_\gamma$, we may assume $\|u\|_\gamma = \|v\|_\gamma = 1$. From

$$|(u \otimes v)(y,s)| \leq (1+|y|)^{1-2\gamma}(1+s)^{-\frac{1}{2}} \leq (1+s)^{-\frac{1}{2}}$$

and Lemma 3.1, we get

$$|\mathcal{S}[u,v]| \leq \int_0^t \|F(t-s)\|_1 \|(u\otimes v)(s)\|_\infty ds \leq c \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds = c\,;$$

and so $|\mathcal{S}[u,v]|$ is bounded in x and $t > 0$. Thus, we assume below $|x| > 1$ and $t > 1$.

(i) Suppose first $\gamma = n+1$. We write

$$\begin{aligned} |\mathcal{S}[u,v]| &\leq \int_0^t \left(\int_{|x-y|<|x|/2} + \int_{|x-y|>|x|/2} \right) |F(x-y,t-s)||(u\otimes v)(y,s)| dyds \\ &\equiv \mathcal{S}_1 + \mathcal{S}_2. \end{aligned}$$

From $|x-y| < |x|/2$ we get $|y| \geq |x| - |x-y| > |x|/2$; so

$$\begin{aligned}\mathcal{S}_1 &\leq c\int_0^t\int_{|x-y|<|x|/2}(t-s)^{-\frac{3}{4}}|x-y|^{\frac{1}{2}-n}(1+|y|)^{-\frac{3}{2}-2n}(1+s)^{-\frac{1}{4}}dyds \\ &\leq c\int_{|x-y|<|x|/2}|x-y|^{\frac{1}{2}-n}(1+|y|)^{-\frac{3}{2}-2n}dy \\ &\leq c|x|^{\frac{1}{2}}(1+|x|)^{-\frac{3}{2}-2n} \leq c(1+|x|)^{-1-2n} \leq c(1+|x|)^{-n-1}.\end{aligned}$$

Furthermore,

$$\mathcal{S}_2 \leq c\int_0^t\int_{|x-y|>|x|/2}|x-y|^{-n-1}(1+|y|)^{1-2n}(1+s)^{-\frac{3}{2}}dyds \leq c|x|^{-n-1}.$$

Hence

$$|\mathcal{S}[u,v]| \leq c(1+|x|)^{-n-1}.$$

We next write

$$|\mathcal{S}[u,v]| \leq \left(\int_0^{t/2} + \int_{t/2}^t\right)|F(t-s)| * |(u\otimes v)(s)|ds \equiv \mathcal{S}_3 + \mathcal{S}_4.$$

Then

$$\mathcal{S}_3 \leq \int_0^{t/2}\|F(t-s)\|_\infty\|(u\otimes v)(s)\|_1 ds \leq ct^{-\frac{n+1}{2}}\int_0^{t/2}\|(u\otimes v)(s)\|_1 ds.$$

Since $|(u\otimes v)(y,s)| \leq (1+|y|)^{-n-1}(1+s)^{-\frac{n+1}{2}}$, it follows that

$$\mathcal{S}_3 \leq ct^{-\frac{n+1}{2}}\int_0^\infty(1+s)^{-\frac{n+1}{2}}ds = ct^{-\frac{n+1}{2}}.$$

Moreover,

$$\mathcal{S}_4 \leq \int_{t/2}^t\|F(t-s)\|_1\|(u\otimes v)(s)\|_\infty ds = c\int_{t/2}^t(t-s)^{-\frac{1}{2}}\|(u\otimes v)(s)\|_\infty ds.$$

From $|(u\otimes v)(y,s)| \leq (1+s)^{-n-1}$, we get

$$\mathcal{S}_4 \leq c\int_{t/2}^t(t-s)^{-\frac{1}{2}}s^{-n-1}ds \leq ct^{-n-\frac{1}{2}} \leq ct^{-\frac{n+1}{2}}.$$

Therefore,

$$|\mathcal{S}[u,v]| \leq c(1+t)^{-\frac{n+1}{2}}.$$

This proves the result.

(ii) Suppose next $n < \gamma < n+1$. In this case,

$$\begin{aligned}\mathcal{S}_1 &\leq c\int_0^t\int_{|x-y|<|x|/2}(t-s)^{-\frac{2+n-\gamma}{2}}|x-y|^{1-\gamma}(1+|y|)^{-n-\gamma}(1+s)^{-\frac{\gamma-n}{2}}dyds \\ &\leq c|x|^{n+1-\gamma}(1+|x|)^{-n-\gamma} \leq c(1+|x|)^{1-2\gamma} \leq c(1+|x|)^{-\gamma},\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_2 &\le c\int_0^t\int_{|x-y|>|x|/2}(t-s)^{-\frac{n+1-\gamma}{2}}|x-y|^{-\gamma}(1+|y|)^{1-n-\gamma}(1+s)^{-\frac{1+\gamma-n}{2}}dyds\\ &\le c\int_{|x-y|>|x|/2}|x-y|^{-\gamma}(1+|y|)^{1-n-\gamma}dy\le c|x|^{-\gamma}.\end{aligned}$$

On the other hand,

$$\mathcal{S}_3\le\int_0^{t/2}\|F(t-s)\|_\infty\|(u\otimes v)(s)\|_1ds\le ct^{-\frac{n+1}{2}}\int_0^{t/2}\|(u\otimes v)(s)\|_1ds.$$

Since $|(u\otimes v)(y,s)|\le(1+|y|)^{-\gamma}(1+s)^{-\frac{\gamma}{2}}$ and $\gamma/2>n/2\ge 1$, we see that

$$\mathcal{S}_3\le ct^{-\frac{n+1}{2}}\int_0^\infty(1+s)^{-\frac{\gamma}{2}}ds=ct^{-\frac{n+1}{2}}\le ct^{-\frac{\gamma}{2}}.$$

Furthermore, $|(u\otimes v)(y,s)|\le(1+s)^{-\gamma}$ implies

$$\begin{aligned}\mathcal{S}_4 &\le c\int_{t/2}^t(t-s)^{-\frac{1}{2}}\|(u\otimes v)(s)\|_\infty ds\le c\int_{t/2}^t(t-s)^{-\frac{1}{2}}(1+s)^{-\gamma}ds\\ &\le c(1+t)^{\frac{1}{2}-\gamma}\le c(1+t)^{-\frac{\gamma}{2}}.\end{aligned}$$

Hence

$$|\mathcal{S}[u,v]|\le c(1+t)^{-\frac{\gamma}{2}}.$$

This shows the desired result.

(iii) Suppose now $\gamma=n$. In this case, we have

$$|(u\otimes v)(y,s)|\le(1+|y|)^{2\alpha-2n}(1+s)^{-\alpha}\qquad 0\le\alpha\le n.$$

So we get

$$\begin{aligned}\mathcal{S}_1 &\le c\int_0^t\int_{|x-y|<|x|/2}(t-s)^{-\frac{3}{4}}|x-y|^{\frac{1}{2}-n}(1+|y|)^{\frac{1}{2}-2n}s^{-\frac{1}{4}}dyds\\ &\le c(1+|x|)^{\frac{1}{2}-2n}|x|^{\frac{1}{2}}\le c|x|^{1-2n}\le c|x|^{-n}\end{aligned}$$

and

$$\mathcal{S}_2\le c\int_0^t\int_{|x-y|>|x|/2}(t-s)^{-\frac{1}{2}}|x-y|^{-n}(1+|y|)^{1-2n}s^{-\frac{1}{2}}dyds\le c|x|^{-n}.$$

Moreover,

$$\mathcal{S}_3\le\int_0^{t/2}\|F(t-s)\|_\infty\|(u\otimes v)(s)\|_1ds\le ct^{-\frac{n+1}{2}}\int_0^{t/2}(1+s)^{-\frac{1}{2}}ds\le ct^{-\frac{n}{2}},$$

and

$$\begin{aligned}\mathcal{S}_4 &\le \int_{t/2}^t\|F(t-s)\|_1\|(u\otimes v)(s)\|_\infty ds\le c\int_{t/2}^t(t-s)^{-\frac{1}{2}}(1+s)^{-n}ds\\ &\le c(1+t)^{\frac{1}{2}-n}\le c(1+t)^{-\frac{n}{2}}.\end{aligned}$$

This proves the result.

(iv) Suppose finally $1 \le \gamma < n$. We take $0 < \varepsilon < 1$ so that $\gamma + \varepsilon < n$ to get

$$\begin{aligned} |\mathcal{S}[u,v]| &\le c\int_0^t\!\!\int (t-s)^{-\frac{1+\varepsilon}{2}}|x-y|^{\varepsilon-n}(1+|y|)^{-(\gamma+\varepsilon)}(1+s)^{-\frac{\gamma-\varepsilon}{2}}dyds \\ &\le c\int |x-y|^{\varepsilon-n}(1+|y|)^{-(\gamma+\varepsilon)}dy \le c|x|^{-\gamma}. \end{aligned}$$

Next, from $|(u\otimes v)(y,s)| \le (1+|y|)^{-\gamma}(1+s)^{-\frac{\gamma}{2}}$ we get $\|(u\otimes v)(s)\|_{n/\gamma,w} \le c(1+s)^{-\frac{\gamma}{2}}$; so

$$\|\mathcal{S}[u,v]\|_{n/\gamma,w} \le c\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{\gamma}{2}}ds \le c \tag{3.3}$$

since $\gamma \ge 1$. Hereafter $\|\cdot\|_{n/\gamma,w}$ denotes the $L^{n/\gamma,w}$-norm (see [1,3]). Next we write

$$\mathcal{S}[u,v](t) = \mathcal{S}_3[u,v](t) + \mathcal{S}_4[u,v](t)$$

where

$$\begin{aligned} \mathcal{S}_3[u,v](t) &= -\int_0^{t/2} e^{-(t-s)A}P\nabla\cdot(u\otimes v)(s)ds = e^{-tA/2}\mathcal{S}[u,v](t/2), \\ \mathcal{S}_4[u,v](t) &= -\int_{t/2}^t e^{-(t-s)A}P\nabla\cdot(u\otimes v)(s)ds. \end{aligned}$$

Applying (3.3) and the estimate

$$\|e^{-tA}a\|_\infty \le ct^{-\frac{\gamma}{2}}\|a\|_{n/\gamma,w}, \tag{3.4}$$

which will be proved below, we get

$$\|\mathcal{S}_3[u,v](t)\|_\infty \le ct^{-\frac{\gamma}{2}}\|\mathcal{S}[u,v](t/2)\|_{n/\gamma,w} \le ct^{-\frac{\gamma}{2}}.$$

Since $\gamma \ge 1$, from $|(u\otimes v)(y,s)| \le (1+s)^{-\gamma}$ we obtain

$$\|\mathcal{S}_4[u,v](t)\|_\infty \le c\int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-\gamma}ds \le c(1+t)^{\frac{1}{2}-\gamma} \le c(1+t)^{-\frac{\gamma}{2}}.$$

The proof of Lemma 3.2 is complete.

We can now prove Theorem 1. Let

$$\Phi(u) = e^{-tA}a + \mathcal{S}[u,u] \equiv u_0 + \mathcal{S}[u,u].$$

Lemma 3.1 and Lemma 3.2 together imply

$$\begin{aligned} &\|\Phi(u)\|_\gamma \le c\|u_0\|_\gamma + c'\|u\|_\gamma^2, \\ &\|\Phi(u)-\Phi(v)\|_\gamma \le c'(\|u\|_\gamma + \|v\|_\gamma)\|u-v\|_\gamma. \end{aligned} \tag{3.5}$$

Assuming

$$\|u_0\|_\gamma < \frac{1}{4cc'},$$

we define

$$M = \frac{1 - \sqrt{1 - 4cc'\|u_0\|_\gamma}}{2c'}. \tag{3.6}$$

By (3.5), $\|u\|_\gamma \le M$ implies

$$\|\Phi(u)\|_\gamma \le c\|u_0\| + c'M^2 = M$$

and

$$\|\Phi(u) - \Phi(v)\|_\gamma \le 2c'M\|u - v\|_\gamma \qquad \text{whenever } \|u\|_\gamma \le M \text{ and } \|v\|_\gamma \le M.$$

Since $2c'M < 1$ by (3.6), we can apply the contraction mapping principle to find a unique $u \in X_\gamma$ such that $\|u\|_\gamma \le M$ and $u = \Phi(u)$. This completes the proof of Theorem 1.

Proof of (3.4). The $L^{n/\gamma,w}$-norm of a measurable function f is given by (see [1,6])

$$\|f\|_{n/\gamma,w} = \sup_E |E|^{\gamma/n-1} \int_E |f|dx,$$

where $|E|$ denotes Lebesgue measure of the measurable sets $E \subset \mathbb{R}^n$. Using this, we can easily deduce (3.4). The details are given in [6], but we here give a proof for the reader's convenience. The definition of Lebesgue integral gives

$$|(e^{-tA}a)(x)| \le ct^{-\frac{n}{2}} \int e^{-c|x-y|^2/t}|a(y)|dy = ct^{-\frac{n}{2}} \int_0^1 \mu[\{y : e^{-c|x-y|^2/t} > s\}]ds,$$

where $\mu = |a(y)|dy$. But, for $0 < s < 1$, $\{y : e^{-c|x-y|^2/t} > s\} = B(x, ct^{\frac{1}{2}}(\log(1/s))^{\frac{1}{2}})$, where $B(x,r)$ is the open ball with radius r centered at x. So we have

$$\mu[B(x, t^{\frac{1}{2}}(\log(1/s))^{\frac{1}{2}})] \le c\|a\|_{n/\gamma,w} t^{\frac{n}{2}(1-\frac{\gamma}{n})}(\log(1/s))^{\frac{n}{2}(1-\frac{\gamma}{n})}.$$

Thus,

$$|(e^{-tA}a)(x)| \le c\|a\|_{n/\gamma,w} t^{-\frac{\gamma}{2}} \int_0^1 [\log(1/s)]^{\frac{n}{2}(1-\frac{\gamma}{n})} ds = ct^{-\frac{\gamma}{2}}\|a\|_{n/\gamma,w}$$

with $c > 0$ independent of x; and (3.4) is proved.

4. Proof of Theorem 3

The following results are due to [2], but we give a proof for the reader's convenience.

Lemma 4.1. (i) *If a satisfies* (a) *and* (b), *so does the function $e^{-tA}a$.*

(ii) *If $u(x,t)$ and $v(x,t)$ satisfy* (a) *and* (b) *for all $t \ge 0$, so does the function*

$$w(t) = \mathcal{S}[u,v](t) = -\int_0^t e^{-(t-s)A} P\nabla \cdot (u \otimes v)(s)ds.$$

(iii) *Let a be divergence-free and satisfy*

$$\int (1 + |y|)|a(y)|dy < \infty.$$

Then the matrix $(\int y_j a_k(y)dy)$ is skew-symmetric.

Proof. (i) Applying the Fourier transform to $u_0(t) = e^{-tA}a$, we get

$$\widehat{u}_0(\xi, t) = e^{-t|\xi|^2}\widehat{a}(\xi).$$

The result is then obvious, since the parity of a in x_j is the same as that of $\widehat{a}$ in ξ_j.

(ii) We also apply the Fourier transform to get

$$\begin{aligned}
\widehat{w}_j(\xi, t) &= -i\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\left(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\right)(\widehat{u_k v_\ell})(\xi, s)ds \\
&= -i\sum_{\ell=1}^n \int_0^t \xi_\ell e^{-(t-s)|\xi|^2}(\widehat{u_j v_\ell})(\xi, s)ds \\
&\quad + i\sum_{k\neq\ell}\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\frac{\xi_j\xi_k}{|\xi|^2}(\widehat{u_k v_\ell})(\xi, s)ds \\
&\quad + i\sum_{\ell=1}^n\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\frac{\xi_j\xi_\ell}{|\xi|^2}(\widehat{u_\ell v_\ell})(\xi, s)ds \\
&\equiv J_1 + J_2 + J_3.
\end{aligned}$$

When $\ell \neq j$, $(\widehat{u_j v_\ell})$ is odd in ξ_j, odd in ξ_ℓ, and even in each of the other components of ξ. So, $\xi_\ell(\widehat{u_j v_\ell})$ is odd in ξ_j and even in each of the other components of ξ. On the other hand, $\xi_j(\widehat{u_j v_j})$ is odd in ξ_j and even in each of the other components of ξ. Hence J_1 satisfies (a). As for J_2, we see that $\xi_\ell\xi_k(\widehat{u_k v_\ell})$ is even in each component of ξ. Hence J_2 satisfies (a). Since $\xi_\ell^2(\widehat{u_\ell v_\ell})$ is obviously even in each component of ξ, J_3 satisfies (a).

Next, let $\eta = (\eta_1, \ldots, \eta_n) = (\xi_n, \xi_1, \ldots, \xi_{n-1})$. One can easily verify that

$$(\widehat{u_{k+1}v_{\ell+1}})(\eta, s) = (\widehat{u_k v_\ell})(\xi, s),$$

with the understanding that $n+1$ equals 1. Thus,

$$\begin{aligned}
\widehat{w}_{j+1}(\eta, t) &= -i\int_0^t \eta_\ell e^{-(t-s)|\eta|^2}\left(\delta_{j+1,k} - \frac{\eta_{j+1}\eta_k}{|\eta|^2}\right)(\widehat{u_k v_\ell})(\eta, s)ds \\
&= -i\int_0^t \xi_{\ell-1} e^{-(t-s)|\xi|^2}\left(\delta_{j+1,k} - \frac{\xi_j\xi_{k-1}}{|\xi|^2}\right)(\widehat{u_{k-1}v_{\ell-1}})(\xi, s)ds \\
&= -i\int_0^t \xi_{\ell-1} e^{-(t-s)|\xi|^2}\delta_{j+1,k}(\widehat{u_{k-1}v_{\ell-1}})(\xi, s)ds \\
&\quad + i\int_0^t \frac{\xi_j\xi_{k-1}}{|\xi|^2}(\widehat{u_{k-1}v_{\ell-1}})(\xi, s)ds \\
&= -i\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}(\widehat{u_j v_\ell})(\xi, s)ds + i\int_0^t \frac{\xi_j\xi_k}{|\xi|^2}(\widehat{u_k v_\ell})(\xi, s)ds
\end{aligned}$$

$$= \quad -i\int_0^t \xi_\ell e^{-(t-s)|\xi|^2}\left(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\right)(\widehat{u_k v_\ell})(\xi,s)ds$$

$$= \quad \widehat{w}_j(\xi,t).$$

This shows that w satisfies (b).

(iii) Note that the assumption implies $\widehat{a}\in C^1$. Since $\nabla\cdot a=0$, we have $\xi\cdot\widehat{a}(\xi)=0$. Differentiating this with respect to ξ_j gives

$$\widehat{a}_j(\xi)+\xi\cdot(\partial_j\widehat{a})(\xi)=0, \qquad j=1,\dots,n. \tag{4.1}$$

On the other hand, condition $\xi\cdot\widehat{a}(\xi)=0$ implies $\widehat{a}(0)=0$; so

$$\widehat{a}_j(\xi)=\xi\cdot\int_0^1(\nabla\widehat{a}_j)(t\xi)dt, \qquad j=1,\dots,n. \tag{4.2}$$

From (4.1) and (4.2) we get, for any fixed j and k,

$$\xi_k(\partial_j\widehat{a}_k)(0,\dots,0,\xi_k,0,\dots,0)+\widehat{a}_j(0,\dots,0,\xi_k,0,\dots,0)=0,$$

$$\widehat{a}_j(0,\dots,0,\xi_k,0,\dots,0)=\xi_k\int_0^1(\partial_k\widehat{a}_j)(0,\dots,0,t\xi_k,0,\dots,0)dt,$$

and therefore

$$\int_0^1(\partial_k\widehat{a}_j)(0,\dots,0,t\xi_k,0,\dots,0)dt+(\partial_j\widehat{a}_k)(0,\dots,0,\xi_k,0,\dots,0)=0.$$

Letting $\xi_k\to 0$ gives $(\partial_k\widehat{a}_j)(0)+(\partial_j\widehat{a}_k)(0)=0$ and this proves the desired result. The proof of Lemma 4.1 is complete.

Proof of Theorem 3. First we show that

$$|(e^{-tA}a)(x)|\le c(c_0+c_1)(1+|x|)^{-n-3}, \qquad |(e^{-tA}a)(x)|\le c(c_0+c_1)(1+t)^{-\frac{n+3}{2}}. \tag{4.3}$$

Since a is divergence-free and satisfies (a), (b) and (1.4), it follows that

$$\widehat{a}_j(0)=\int a_j(y)dy=0, \qquad \int y_k y_\ell a_j(y)dy=0, \qquad j,k,\ell=1,\dots,n. \tag{4.4}$$

Moreover, by Lemma 4.1 (iii), the matrix $(\int y_k a_j(y)dy)$ is skew-symmetric. This, together with (a), implies that

$$\int y_k a_j(y)dy=0, \qquad j,k=1,\dots,n. \tag{4.5}$$

Now let

$$(e^{-tA}a)(x)=\left(\int_{|y|<|x|/2}+\int_{|y|>|x|/2}\right)E_t(x-y)a(y)dy\equiv K_1+K_2.$$

We easily see that

$$|K_2|\le\sup_{|y|>|x|/2}|a(y)|\le cc_0(1+|x|)^{-n-3}.$$

Next, Taylor's formula, (4.4) and (4.5) together yield

$$
\begin{aligned}
K_1 &= \int_{|y|<|x|/2} [E_t(x-y) - \sum_{|\gamma|\le 2} \frac{(-y)^\gamma}{\gamma!}(\partial_x^\gamma E_t)(x)]a(y)dy \\
&\quad + \sum_{|\gamma|\le 2} \frac{1}{\gamma!}(\partial_x^\gamma E_t)(x) \int_{|y|<|x|/2} (-y)^\gamma a(y)dy \\
&= 3\sum_{|\gamma|=3} \int_0^1 \int_{|y|<|x|/2} (1-\theta)^2 \frac{(-y)^\gamma}{\gamma!}(\partial_x^\gamma E_t)(x-y\theta)a(y)dy \\
&\quad - \sum_{|\gamma|\le 2} \frac{1}{\gamma!}(\partial_x^\gamma E_t)(x) \int_{|y|>|x|/2} (-y)^\gamma a(y)dy \\
&\equiv K_{11} + K_{12}.
\end{aligned}
$$

Since $|x-y\theta| > |x|/2$ whenever $|y| < |x|/2$, it follows that

$$
|K_{11}| \le ct^{-\frac{n+3}{2}} e^{-c|x|^2/t} \int_{|y|<|x|/2} |y|^3|a(y)|dy \le c|x|^{-n-3} \int |y|^3|a(y)|dy = cc_1|x|^{-n-3}.
$$

Moreover, direct calculation gives

$$
|K_{12}| \le cc_0 \sum_{|\gamma|\le 2} |x|^{-n-|\gamma|} \int_{|y|>|x|/2} (1+|y|)^{-n-3+|\gamma|}dy \le cc_0|x|^{-n-3}.
$$

We thus obtain the first estimate of (4.3). On the other hand, Taylor's formula, (4.4) and (4.5) together imply

$$
(e^{-tA}a)(x) = 3\sum_{|\gamma|=3} \int_0^1 (1-\theta)^2 \int \frac{(-y)^\gamma}{\gamma!}(\partial_x E_t)(x-y\theta)a(y)dyd\theta
$$

so that

$$
|(e^{-tA}a)(x)| \le ct^{-\frac{n+3}{2}} \int e^{-c|x-y\theta|^2/t}|y|^3|a(y)|dy \le cc_1 t^{-\frac{n+3}{2}}.
$$

This, together with $|e^{-tA}a)(x)| \le \|E_t\|_1\|a\|_\infty \le c_0$, gives the second estimate of (4.3).

Consider next the function $w = (w_1, \dots, w_n)$ with

$$
w_j(t) = -\int_0^t F_{\ell,jk}(t-s) * (u_k v_\ell)(s)ds.
$$

Suppose that u and v satisfy (a), (b), and

$$
\begin{aligned}
&|u(y,s)| \le c(1+|y|)^{\alpha-n-3}(1+s)^{-\frac{\alpha}{2}}, \\
&|v(y,s)| \le c(1+|y|)^{\alpha-n-3}(1+s)^{-\frac{\alpha}{2}},
\end{aligned}
\qquad \text{for all } 0 \le \alpha \le n+3. \tag{4.6}
$$

We shall show that

$$
|w(x,t)| \le c(1+|x|)^{-n-3}, \qquad |w(x,t)| \le c(1+t)^{-\frac{n+3}{2}}. \tag{4.7}
$$

Observe that (a) and (b) together imply

$$\int (u_k v_\ell)(y,s)dy = \lambda(s)\delta_{k\ell}, \qquad \int y_j (u_k v_\ell)(y,s)dy = 0. \tag{4.8}$$

We write

$$w_j = -\left(\int_0^{t/2} + \int_{t/2}^t\right) F_{\ell,jk}(t-s) * (u_k v_\ell)(s)ds \equiv I_1 + I_2.$$

By (4.8), Taylor's formula and the fact that $F_{\ell,j\ell} \equiv 0$, we see that

$$I_1 = -2\int_0^1 (1-\theta)\int_0^{t/2}\int \sum_{|\gamma|=2} \frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x-y\theta, t-s)y^\gamma (u_k v_\ell)(y,s)dydsd\theta.$$

Direct calculation using (4.6) gives

$$|I_2| \le c\int_{t/2}^t (t-s)^{-\frac{1}{2}}(1+s)^{-n-3}ds \le c(1+t)^{-n-5/2} \le c(1+t)^{-\frac{n+3}{2}}$$

and

$$|I_1| \le ct^{-\frac{n+3}{2}}\int_0^{t/2}\int |y|^2|u(y,s)||v(y,s)|dyds.$$

Since $|y|^2|u(y,s)||v(y,s)| \le c(1+|y|)^{-1-2n}(1+s)^{-\frac{3}{2}}$, it follows that

$$|I_1| \le ct^{-\frac{n+3}{2}}\int_0^{\infty}\int |y|^2|u(y,s)||v(y,s)|dyds = ct^{-\frac{n+3}{2}}.$$

This proves the second estimate of (4.7). To deduce the first estimate of (4.7), we write

$$w_j = I_1' + I_2',$$

where

$$I_1' = -\int_0^t\int_{|y|<|x|/2} F_{\ell,jk}(x-y,t-s)(u_k v_\ell)(y,s)dyds,$$
$$I_2' = -\int_0^t\int_{|y|>|x|/2} F_{\ell,jk}(x-y,t-s)(u_k v_\ell)(y,s)dyds.$$

Direct calculation using (4.6) gives

$$\begin{aligned} |I_2'| &\le c\int_0^t\int_{|y|>|x|/2} |F(x-y,t-s)|(1+|y|)^{-2n-5}(1+s)^{-\frac{1}{2}}dyds \\ &\le c(1+|x|)^{-2n-5}\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds \le c(1+|x|)^{-n-3}. \end{aligned}$$

On the other hand, by Taylor's formula we get

$$\begin{aligned} I_1' &= -2\int_0^1\int_0^t\int_{|y|<|x|/2}(1-\theta)\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)y^\gamma(u_k v_\ell)(y,s)dydsd\theta \\ &\quad -\int_0^t\int_{|y|<|x|/2}[F_{\ell,jk}(x,t-s) - (\partial_m F_{\ell,jk})(x,t-s)y_m](u_k v_\ell)(y,s)dyds. \end{aligned}$$

Since (4.8) implies

$$\int F_{\ell,jk}(x,t-s)(u_k v_\ell)(y,s)dy = \lambda(s)F_{\ell,j\ell}(x,t-s) = 0,$$

it follows that

$$\int_{|y|<|x|/2} F_{\ell,jk}(x,t-s)(u_k v_\ell)(y,s)dy = -\int_{|y|>|x|/2} F_{\ell,jk}(x,t-s)(u_k v_\ell)(y,s)dy.$$

Inserting this in the above yields

$$\begin{aligned} I_1' &= -2\int_0^1\int_0^t\int_{|y|<|x|/2}(1-\theta)\sum_{|\gamma|=2}\frac{1}{\gamma!}(\partial_x^\gamma F_{\ell,jk})(x-y\theta,t-s)y^\gamma(u_k v_\ell)(y,s)dydsd\theta \\ &\quad + \int_0^t\int_{|y|>|x|/2} F_{\ell,jk}(x,t-s)(u_k v_\ell)(y,s)dyds \\ &\quad - \int_0^t\int_{|y|>|x|/2}(\partial_m F_{\ell,jk})(x,t-s)y_m(u_k v_\ell)(y,s)dyds \\ &\equiv I_{11}' + I_{12}' + I_{13}'. \end{aligned}$$

It is easy to see that

$$\begin{aligned} |I_{11}'| &\le c|x|^{-n-3}\int_0^\infty\int |y|^2|u(y,s)||v(y,s)|dyds = c|x|^{-n-3}, \\ |I_{12}'| &\le c|x|^{-n-1}\int_0^t\int_{|y|>|x|/2}(1+|y|)^{-2n-3}(1+s)^{-\frac{3}{2}}dyds \le c|x|^{-n-3}, \\ |I_{13}'| &\le c|x|^{-n-2}\int_0^t\int_{|y|>|x|/2}(1+|y|)^{-2n-2}(1+s)^{-\frac{3}{2}}dyds \le c|x|^{-n-3}. \end{aligned}$$

This proves the first estimate of (4.7).

Due to Lemma 4.1 and the above calculation, we can now complete the proof of Theorem 3. Namely, we consider the operator

$$\Phi(u) = e^{-tA}a + \mathcal{S}[u,u]$$

on the space of divergence-free vector fields $u(y,s)$ satisfying (a), (b) and

$$\|u\|_{n+3} \equiv \sup(1+|y|)^{n+3-\alpha}(1+s)^{\frac{\alpha}{2}}|u(y,s)| < \infty,$$

where the supremum is taken over $y \in \mathbb{R}^n$, $s \ge 0$ and $0 \le \alpha \le n+3$. The foregoing calculation then shows that

$$\|\mathcal{S}[u,v]\|_{n+3} \le c\|u\|_{n+3}\cdot\|v\|_{n+3}.$$

So we can apply the contraction mapping principle, assuming that c_0 and c_1 in (1.4) be sufficiently small, to get the desired solution of the equation $u = \Phi(u)$. The details are completely parallel to the final step of the proof of Theorem 1 and so omitted here.

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