EXISTENCE OF POSITIVE SOLUTIONS TO SEMIPOSITONE FREDHOLM INTEGRAL EQUATIONS

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Abstract. New existence theorems are presented for semipositone integral equations of the form $y(t) = \mu \int_0^1 k(t,s)\, f(s,y(s))\, ds$ for $t \in [0,1]$. An application to second order boundary value problems is also discussed.

1. Introduction.

This paper presents three new existence results for semipositone Fredholm integral equations of the form

$$y(t) = \mu \int_0^1 k(t,s)\, f(s,y(s))\, ds \quad \text{for} \quad t \in [0,1], \tag{1.1}$$

where $\mu > 0$ is a constant. Existence in both $C[0,1]$ and $L^p[0,1]$ will be discussed. Throughout this paper k is nonnegative but our nonlinearity f may take negative values. Problems of this type are referred to as semipositone problems in the literature and they arise naturally in chemical reactor theory [4]. The constant μ is called the Thiele modulus and of physical interest is the existence of positive solutions to (1.1) when $\mu > 0$ is small. The literature on positive solutions to Fredholm integral equations (see [3–8] and the references therein) is almost totally devoted to (1.1) when f takes nonnegative values (i.e. positone problems). Only a few results (see [1 Chapter 4]) are available for the semipositone problem.

Existence in this paper will be established using Krasnoselskii's fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1. *Let $E = (E, \|\,.\,\|)$ be a Banach space and let $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ be continuous and completely continuous. In addition suppose either*

$$\|A\,u\| \le \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_1 \ \ \textit{and} \ \ \|A\,u\| \ge \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_2$$

or

$$\|A\,u\| \ge \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_1 \ \ \textit{and} \ \ \|A\,u\| \le \|u\| \ \ \textit{for} \ \ u \in K \cap \partial\Omega_2$$

hold. Then A has a fixed point in $K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.

2. Semipositone problems.

In this section we present three new results for the semipositone Fredholm integral equation

$$y(t) = \mu \int_0^1 k(t,s)\, f(s, y(s))\, ds \quad \text{for} \quad t \in [0,1]; \tag{2.1}$$

here $\mu > 0$ is a constant. Of physical interest is the existence of nonnegative solutions which are positive a.e. on $[0,1]$.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$\begin{cases} \text{there exists } a \in C[0,1] \text{ and } t^\star \in [0,1] \text{ with } a(t) > 0 \\ \text{for a.e. } t \in [0,1] \text{ and } a(t^\star) > 0, \text{ there exists } \kappa \in L^1[0,1] \\ \text{with } \kappa(t) \geq 0 \text{ a.e. } t \in [0,1] \text{ and } \int_0^1 \kappa(s)\, ds > 0 \text{ such} \\ \text{that } a(t)\, \kappa(s) \leq k(t,s) \text{ for all } t \in [0,1], \text{ a.e. } s \in [0,1] \end{cases} \tag{2.2}$$

$$k_t(s) = k(t,s) \leq \kappa(s) \quad \text{for all } t \in [0,1], \quad \text{a.e. } s \in [0,1] \tag{2.3}$$

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0,1] \text{ to } L^1[0,1] \tag{2.4}$$

$$\begin{cases} f : [0,1] \times [0,\infty) \to \mathbf{R} \text{ is continuous and there} \\ \text{exists a constant } M > 0 \text{ with } f(t,u) + M \geq 0 \\ \text{for } (t,u) \in [0,1] \times [0,\infty) \end{cases} \tag{2.5}$$

$$\begin{cases} f(t,u) + M \leq \psi(u) \text{ on } [0,1] \times [0,\infty) \text{ with} \\ \psi : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing} \\ \text{and } \psi(u) > 0 \text{ for } u > 0 \end{cases} \tag{2.6}$$

$$\exists\, C > 0 \text{ with } \int_0^1 k(t,s)\, ds \leq C\, a(t) \text{ for } t \in [0,1] \tag{2.7}$$

$$\exists\, r \geq \mu\, M\, C \text{ with } \frac{r}{\psi(r)} \geq \mu \sup_{t \in [0,1]} \int_0^1 k(t,s)\, ds \tag{2.8}$$

$$\begin{cases} f(t,u) + M \geq g(u) \text{ for } (t,u) \in [0,1] \times (0,\infty) \text{ with} \\ g : [0,\infty) \to [0,\infty) \text{ continuous and nondecreasing} \\ \text{and } g(u) > 0 \text{ for } u > 0 \end{cases} \tag{2.9}$$

and

$$\exists\, R > r \text{ with } R \leq \mu \int_0^1 k(t^\star, s)\, g\,(\epsilon\, R\, a(s))\, ds; \tag{2.10}$$

here $\epsilon > 0$ is any constant (choose and fix it) so that $1 - \frac{\mu\, M\, C}{R} \geq \epsilon$ (note ϵ exists since $R > r \geq \mu\, M\, C$). Then (2.1) has a nonnegative solution $y \in C[0,1]$ with $y(t) > 0$ for a.e. $t \in [0,1]$ (in fact $y(t) > 0$ at those $t's$ where $a(t) > 0$).

PROOF: To show (2.1) has a nonnegative solution we will look at

$$y(t) = \mu \int_0^1 k(t,s)\, f^\star(s, y(s) - \phi(s))\, ds \tag{2.11}$$

where $\phi(t) = \mu\, M\, \int_0^1 k(t,s)\, ds$ and

$$f^\star(t,v) = \begin{cases} f(t,v) + M, & v \geq 0 \\ f(t,0) + M, & v \leq 0. \end{cases}$$

We will show, using Theorem 1.1, that there exists a solution y_1 to (2.11) with $y_1(t) \geq \phi(t)$ for $t \in [0,1]$ (note $\phi(t) > 0$ for those $t's$ where $a(t) > 0$). If this is true then $u(t) = y_1(t) - \phi(t)$ is a nonnegative solution (positive a.e. on $[0,1]$) of (2.1) since for $t \in [0,1]$ we have

$$\begin{aligned} u(t) &= \mu \int_0^1 k(t,s)\, f^\star(s, y_1(s) - \phi(s))\, ds \,-\, \mu\, M \int_0^1 k(t,s)\, ds \\ &= \mu \int_0^1 k(t,s)\, [f(s, y_1(s) - \phi(s)) + M]\, ds \,-\, \mu\, M \int_0^1 k(t,s)\, ds \\ &= \mu \int_0^1 k(t,s)\, f(s, y_1(s) - \phi(s))\, ds = \mu \int_0^1 k(t,s)\, f(s, u(s))\, ds. \end{aligned}$$

As a result we will concentrate our study on (2.11). Let $E = (C[0,1], |\,.\,|_0)$ (here $|u|_0 = \sup_{t \in [0,1]} |u(t)|$) and

$$K = \{u \in C[0,1] : \ u(t) \geq a(t)\, |u|_0 \ \text{ for } \ t \in [0,1]\}\,.$$

Clearly K is a cone of E. Let

$$\Omega_1 = \{u \in C[0,1] : \ |u|_0 < r\} \quad \text{and} \quad \Omega_2 = \{u \in C[0,1] : \ |u|_0 < R\}.$$

Now let $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to C[0,1]$ be defined by

$$A\, y(t) = \mu \int_0^1 k(t,s)\, f^\star(s, y(s) - \phi(s))\, ds.$$

First we show $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$. If $y \in K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$ and $t \in [0,1]$ then (2.2) implies

$$A\, y(t) \geq \mu\, a(t) \int_0^1 \kappa(s)\, f^\star(s, y(s) - \phi(s))\, ds. \tag{2.12}$$

On the other hand (2.3) implies

$$|A\, y|_0 \leq \mu \int_0^1 \kappa(s)\, f^\star(s, y(s) - \phi(s))\, ds,$$

and this together with (2.12) yields

$$A\, y(t) \geq a(t)\, |A\, y|_0 \quad \text{for} \quad t \in [0,1].$$

Consequently $A\, y \in K$ so $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$. It is well known [7] that $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ is continuous and compact.

We now show

$$|A\,y|_0 \le |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_1. \tag{2.13}$$

To see this let $y \in K \cap \partial\Omega_1$. Then $|y|_0 = r$ and $y(t) \ge a(t)\,r$ for $t \in [0,1]$. Also for $t \in [0,1]$ we have

$$\begin{aligned} A\,y(t) &\le \mu \int_0^1 k(t,s)\,\psi(y(s))\,ds \le \mu\,\psi(|y|_0) \int_0^1 k(t,s)\,ds \\ &\le \mu\,\psi(r) \int_0^1 k(t,s)\,ds \end{aligned}$$

since for $s \in (0,1)$ (note $y(s) \ge 0$),

$$f^\star(s, y(s) - \phi(s)) = \begin{cases} f(s, y(s) - \phi(s)) + M \le \psi(y(s) - \phi(s)) \le \psi(y(s)) \\ \qquad\qquad\qquad\qquad \text{if} \ \ y(s) - \phi(s) \ge 0 \\ f(s,0) + M \le \psi(0) \le \psi(y(s)) \ \ \text{if} \ \ y(s) - \phi(s) < 0\,; \end{cases}$$

in fact one can show $y(s) - \phi(s) \ge 0$ for $s \in [0,1]$ (see the argument below). This together with (2.8) yields

$$|A\,y|_0 \le \mu\,\psi(r) \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds \;\le\; r = |y|_0,$$

so (2.13) holds.

Next we show

$$|A\,y|_0 \ge |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_2. \tag{2.14}$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(t) \ge a(t)\,R$ for $t \in [0,1]$. Let ϵ be as in the statement of Theorem 2.1. For $t \in [0,1]$ we have from (2.7) that

$$\begin{aligned} y(t) - \phi(t) &= y(t) - \mu\,M \int_0^1 k(t,s)\,ds \ge y(t) - \mu\,M\,C\,a(t) \\ &\ge y(t)\left[1 - \frac{\mu\,M\,C}{R}\right] \ge \epsilon\,y(t) \ge \epsilon\,a(t)\,R. \end{aligned}$$

Now with $t^\star$ as in (2.2), we have

$$\begin{aligned} A\,y(t^\star) &= \mu \int_0^1 k(t^\star,s)\,f^\star(s, y(s) - \phi(s))\,ds \\ &\ge \mu \int_0^1 k(t^\star,s)\,g\left(\epsilon\,a(s)\,R\right)\,ds, \end{aligned}$$

since for $s \in [0,1]$ we have

$$f^\star(s, y(s) - \phi(s)) = f(s, y(s) - \phi(s)) + M \ge g(y(s) - \phi(s)) \ge g(\epsilon\,a(s)\,R).$$

This together with (2.10) yields

$$A\,y(t^\star) \ge \mu \int_0^1 k(t^\star,s)\,g\left(\epsilon\,a(s)\,R\right)\,ds \;\ge\; R \;=\; |y|_0.$$

Thus $|A\,y|_0 \ge |y|_0$, so (2.14) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ i.e. $r \leq |y_1|_0 \leq R$ and $y_1(t) \geq a(t)\, r$ for $t \in [0,1]$. To finish the proof we need to show $y_1(t) \geq \phi(t)$ for $t \in [0,1]$. Now for $t \in [0,1]$ we have from (2.7) and (2.8) that

$$y_1(t) \geq a(t)\, r \geq a(t)\, \mu\, M\, C \geq \mu\, M \int_0^1 k(t,s)\, ds = \phi(t). \quad \square$$

To illustrate the applicability of Theorem 2.1 we consider the boundary value problem

$$\left\{ \begin{array}{l} y'' + \mu\, f(t,y) = 0, \ \ 0 < t < 1 \\ \alpha\, y(0) - \beta\, y'(0) = 0 \\ \gamma\, y(1) + \delta\, y'(1) = 0; \end{array} \right. \tag{2.15}$$

here $\alpha,\ \beta,\ \gamma,\ \delta \geq 0$ with $\rho = \gamma\, \beta + \alpha\, \gamma + \alpha\, \delta > 0$. Of course (2.15) is equivalent to the integral equation

$$y(t) = \mu \int_0^1 k(t,s)\, f(s, y(s))\, ds \tag{2.16}$$

where the Green's function [1] is given by

$$k(t,s) = \left\{ \begin{array}{ll} \frac{(\gamma+\delta-\gamma\, t)\,(\beta+\alpha\, s)}{\rho}, & 0 \leq s \leq t \leq 1 \\ \frac{(\beta+\alpha\, t)\,(\gamma+\delta-\gamma\, s)}{\rho}, & 0 \leq t \leq s \leq 1. \end{array} \right.$$

Let

$$\kappa(s) = k(s,s), \ \ a(t) = \frac{(\gamma+\delta-\gamma\, t)\,(\beta+\alpha\, t)}{(\gamma+\delta)\,(\beta+\alpha)} \ \ \text{and} \ \ t^\star = \frac{1}{4}. \tag{2.17}$$

It is easy to check that (2.2), (2.3) and (2.4) hold. Next we claim (2.7) holds with

$$C = \frac{(\gamma+\delta)\,(\beta+\alpha)}{\rho}. \tag{2.18}$$

To see this notice for $t \in [0,1]$ that

$$\begin{aligned} \int_0^1 k(t,s)\, ds &= \frac{1}{\rho}\,(\gamma+\delta-\gamma\, t) \int_0^t (\beta+\alpha\, s)\, ds \\ &+ \frac{1}{\rho}\,(\beta+\alpha\, t) \int_t^1 (\gamma+\delta-\gamma\, s)\, ds \\ &\leq \frac{1}{\rho}\,(\gamma+\delta-\gamma\, t)\,(\beta+\alpha\, t)\,[t+(1-t)] \\ &= \frac{1}{\rho}\,(\gamma+\delta-\gamma\, t)\,(\beta+\alpha\, t) \\ &= C\, a(t); \end{aligned}$$

here C is as in (2.18) and a is as in (2.17).

Corollary 2.2. *Suppose* (2.5), (2.6) *and* (2.9) *hold. In addition assume the following conditions are satisfied:*

$$\exists\, r \geq \frac{\mu\, M\,(\gamma+\delta)\,(\beta+\alpha)}{\rho} \ \ \textit{with} \ \ \frac{r}{\psi(r)} \geq \mu \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds \tag{2.19}$$

and

$$\exists\, R > r \quad \text{with} \quad R \le \mu \int_0^1 k\left(\frac{1}{4}, s\right) g\left(\epsilon\, R\, a(s)\right)\, ds; \tag{2.20}$$

here a *is as in* (2.17) *and* $\epsilon > 0$ *is any constant (choose and fix it) so that*

$$1 - \frac{\mu\, M\, (\gamma+\delta)\,(\beta+\alpha)}{\rho\, R} \ge \epsilon.$$

Then (2.15) *has a nonnegative solution* $y \in C[0,1]$ *with* $y(t) > 0$ *for* $t \in (0,1)$.

Example. Consider (2.15) with $f(t,u) = u^m - 1,\ m > 1$, and

$$\mu \in \left(0, \frac{\gamma\,\beta + \alpha\,\gamma + \alpha\,\delta}{(\gamma+\delta)\,(\beta+\alpha)}\right].$$

Then (2.15) has a solution y with $y(t) > 0$ for $t \in (0,1)$.

To see this we will apply Corollary 2.2 with (here $R > 1$ will be chosen later)

$$M = 1,\ \psi(u) = g(u) = u^m,\ \epsilon = \frac{1}{2}\left(1 - \frac{\mu\,(\gamma+\delta)\,(\beta+\alpha)}{\rho\, R}\right).$$

Clearly (2.5), (2.6) and (2.9) hold. In addition we know from above that

$$\sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds \le C \sup_{t\in[0,1]} a(t) \le C = \frac{(\gamma+\delta)\,(\beta+\alpha)}{\rho},$$

so (2.19) is true with $r = 1$ since

$$\frac{\mu\, M\,(\gamma+\delta)\,(\beta+\alpha)}{\rho} \le \frac{\rho}{(\gamma+\delta)\,(\beta+\alpha)}\ \frac{(\gamma+\delta)\,(\beta+\alpha)}{\rho} = 1 = r$$

and

$$\mu \sup_{t\in[0,1]} \int_0^1 k(t,s)\, ds \le \frac{\mu\,(\gamma+\delta)\,(\beta+\alpha)}{\rho} \le 1 = \frac{r}{\psi(r)}.$$

Finally notice (2.20) is satisfied for R large since

$$R \le \mu \int_0^1 k\left(\frac{1}{4}, s\right) g\left(\epsilon\, a(s)\, R\right)\, ds$$

means

$$\frac{1}{R^{m-1}} \le \mu \int_0^1 k\left(\frac{1}{4}, s\right) \epsilon^m\, [a(s)]^m\, ds,$$

and notice $\frac{1}{R^{m-1}} \to 0$ as $R \to \infty$. Thus all the conditions of Corollary 2.2 are satisfied so existence is guaranteed.

It is possible to obtain another existence result for (2.1) if we relax some of the conditions on the kernel k and we strengthen some of the conditions on the nonlinearity f. To indicate what is possible we consider for convenience the integral equation

$$y(t) = \mu \int_0^1 k(t,s)\, f(y(s))\, ds \quad \text{for} \quad t \in [0,1]; \tag{2.21}$$

here $\mu > 0$ is a constant; the obvious adjustments for (2.1) will be left to the reader.

Theorem 2.3. *Suppose the following conditions are satisfied:*

$$0 \le k_t(s) = k(t,s) \in L^1[0,1] \quad \text{for each} \quad t \in [0,1] \tag{2.22}$$

$$\text{the map} \quad t \mapsto k_t \quad \text{is continuous from} \quad [0,1] \quad \text{to} \quad L^1[0,1] \tag{2.23}$$

$$\begin{cases} K_1 \ne K_2 \quad \text{where} \quad K_1 = \sup_{t\in[0,1]} \int_0^1 k(t,s)\,ds \\ \text{and} \quad K_2 = \inf_{t\in[0,1]} \int_0^1 k(t,s)\,ds \end{cases} \tag{2.24}$$

$$\begin{cases} f : [0,\infty) \to \mathbf{R} \quad \text{is continuous and there exists a} \\ \text{constant} \quad M > 0 \quad \text{with} \quad f(u) + M \ge 0 \quad \text{for} \quad u \in [0,\infty) \end{cases} \tag{2.25}$$

$$\begin{cases} f(u) + M = \psi(u) \quad \text{on} \quad [0,\infty) \quad \text{with} \quad \psi : [0,\infty) \to [0,\infty) \\ \text{nondecreasing and} \quad \psi(u) > 0 \quad \text{for} \quad u > 0 \end{cases} \tag{2.26}$$

$$\begin{cases} \text{there exists a continuous} \quad \theta : (0,\infty) \to (0,\infty) \quad \text{such that for} \\ \text{any constant} \quad K > 0 \quad \text{we have} \quad \psi(K\,u) \ge \theta(K)\,\psi(u) \quad \text{for} \quad u > 0 \end{cases} \tag{2.27}$$

$$\begin{cases} \text{there exists} \quad 0 < M_0 < 1 \quad \text{and} \quad r > 0 \quad \text{with} \quad r \ge \frac{\mu\,M\,K_1}{M_0}, \\ \frac{r}{\psi(r)} \ge \mu\,K_1 \quad \text{and} \quad \frac{M_0}{\theta(A_0\,M_0)} \le \frac{K_2}{K_1}; \quad \text{here} \quad A_0 = \left(1 - \frac{\mu\,M\,K_1}{M_0\,r}\right) \end{cases} \tag{2.28}$$

and

$$\exists\, R > r \quad \text{with} \quad \frac{R}{\psi(\epsilon\,M_0\,R)} \le \mu\,K_2; \tag{2.29}$$

here $\epsilon > 0$ *is any constant (choose and fix it) so that* $1 - \frac{\mu\,M\,K_1}{M_0\,R} \ge \epsilon$. *Then* (2.21) *has a solution* $y \in C[0,1]$ *with* $y(t) > 0$ *for* $t \in [0,1]$.

PROOF: It is enough to show

$$y(t) = \mu \int_0^1 k(t,s)\,f^\star(y(s) - \phi(s))\,ds \tag{2.30}$$

with $\phi(t) = \mu\,M \int_0^1 k(t,s)\,ds$ and

$$f^\star(v) = \begin{cases} f(v) + M, & v \ge 0 \\ f(0) + M, & v \le 0. \end{cases}$$

has a solution y_1 with $y_1(t) \ge \phi(t)$ for $t \in [0,1]$ (note $\phi(t) > 0$ for $t \in [0,1]$). Let $E = (C[0,1], |\,.\,|_0)$ and

$$K = \{u \in C[0,1] : \ u(t) \ge M_0\,|u|_0 \ \ \text{for} \ \ t \in [0,1]\}.$$

Let Ω_1, Ω_1 be as in Theorem 2.1 and let $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to C[0,1]$ be defined by

$$A\, y(t) = \mu \int_0^1 k(t,s)\, f^\star(y(s) - \phi(s))\, ds.$$

First we show $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$. If $y \in K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$ then $y(t) \geq M_0\, |y|_0 \geq M_0\, r$ for $t \in [0,1]$. In addition notice for $t \in [0,1]$ that

$$\begin{aligned} y(t) - \phi(t) &= y(t) - \mu\, M \int_0^1 k(t,s)\, ds \geq y(t) - \mu\, M\, K_1 \\ &\geq y(t) \left[1 - \frac{\mu\, M\, K_1}{M_0\, r} \right] = A_0\; y(t), \end{aligned}$$

so $y(t) - \phi(t) \geq 0$ for $t \in [0,1]$. As a result for $t \in [0,1]$ we have

$$A\, y(t) = \mu \int_0^1 k(t,s)\, \psi(y(s) - \phi(s))\, ds \leq \mu \int_0^1 k(t,s)\, \psi(|y|_0)\, ds \leq \mu\, K_1\, \psi(|y|_0),$$

and so

$$|A\, y|_0 \leq \mu\, K_1\, \psi(|y|_0). \tag{2.31}$$

On the other hand for $t \in [0,1]$ we have

$$\begin{aligned} A\, y(t) &= \mu \int_0^1 k(t,s)\, \psi(y(s) - \phi(s))\, ds \geq \mu \int_0^1 k(t,s)\, \psi(A_0\, y(s))\, ds \\ &\geq \mu \int_0^1 k(t,s)\, \psi(A_0\, M_0\, |y|_0)\, ds \geq \mu\, \theta(A_0\, M_0)\, \psi(|y|_0) \int_0^1 k(t,s)\, ds \\ &\geq \mu\, \theta(A_0\, M_0)\, \psi(|y|_0)\, K_2 \geq \mu\, M_0\, K_1\, \psi(|y|_0) \end{aligned}$$

using (2.27) and (2.28). This together with (2.31) yields

$$A\, y(t) \geq \mu\, M_0\, K_1\, \psi(|y|_0) \geq M_0\, |A\, y|_0 \quad \text{for} \quad t \in [0,1],$$

so $A\, y \in K$. Thus $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ and also we know $A : K \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to K$ is continuous and compact. Essentially the same reasoning as in Theorem 2.1 guarantees that

$$|A\, y|_0 \leq |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_1. \tag{2.32}$$

Next we show

$$|A\, y|_0 \geq |y|_0 \quad \text{for} \quad y \in K \cap \partial\Omega_2. \tag{2.33}$$

To see this let $y \in K \cap \partial\Omega_2$ so $|y|_0 = R$ and $y(t) \geq M_0\, R$ for $t \in [0,1]$. Let ϵ be as in the statement of Theorem 2.3 and notice for $t \in [0,1]$ that

$$y(t) - \phi(t) \geq y(t) - \mu\, M\, K_1 \geq y(t) \left[1 - \frac{\mu\, M\, K_1}{M_0\, R} \right] \geq \epsilon\; y(t) \geq \epsilon\, M_0\, R.$$

Also for $t \in [0,1]$ we have

$$\begin{aligned} A\, y(t) &= \mu \int_0^1 k(t,s)\, \psi(y(s) - \phi(s))\, ds \geq \mu \int_0^1 k(t,s)\, \psi(\epsilon\, M_0\, R)\, ds \\ &\geq \mu\, \psi(\epsilon\, M_0\, R)\, K_2 \geq R = |y|_0 \end{aligned}$$

from (2.29). Thus (2.33) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$ i.e. $r \le |y_1|_0 \le R$ and $y_1(t) \ge M_0\, r$ for $t \in [0,1]$. To finish the proof we need to show $y_1(t) \ge \phi(t)$ for $t \in [0,1]$. Now for $t \in [0,1]$ we have

$$y_1(t) \ge M_0\, r \ge \mu\, M\, K_1 \ge \mu\, M \int_0^1 k(t,s)\, ds = \phi(t). \quad \square$$

Next we look for solutions to Fredholm integral equations in $L^p[0,1]$, $1 \le p < \infty$.

Theorem 2.4. *Assume that p, p_1 and p_2 are such that $1 \le p_1 \le p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Suppose the following conditions are satisfied:*

(2.34) $$\begin{cases} f : [0,1] \times [0,\infty) \to \mathbf{R} \ \text{ is a Carathéodory function and there exists} \\ a_1 \in L^{p_2}[0,1] \ \text{ and } \ a_2 > 0 \ \text{ with } \ |f(t,y)| \le a_1(t) + a_2\, |y|^{\frac{p}{p_2}} \\ \text{for a.e. } \ t \in [0,1] \end{cases}$$

(2.35) $$\exists\, M > 0 \ \text{ with } \ f(t,u) + M \ge 0 \ \text{ for a.e. } \ t \in [0,1] \ \text{ and all } \ u \in [0,\infty)$$

(2.36) $$\begin{cases} f(t,u) + M \le \psi(u) \ \text{ for a.e. } \ t \in [0,1] \ \text{ and all } \ u \in [0,\infty) \\ \text{with } \ \psi : [0,\infty) \to [0,\infty) \ \text{continuous and nondecreasing} \\ \text{and } \ \psi(u) > 0 \ \text{ for } \ u > 0 \end{cases}$$

(2.37) $$\begin{cases} \text{there exists } \ \theta \in C[0,\infty) \ \text{ with } \ \|\psi(y)\|_{p_2} \le \theta\left(\|y\|_p\right) \ \text{ for} \\ \text{all } \ y \in L^p[0,1]; \ \text{ here } \ \|u\|_p = \|u\|_{L^p} = \left(\int_0^1 |u(t)|^p\, dt\right)^{\frac{1}{p}} \end{cases}$$

(2.38) $$\begin{cases} k : [0,1] \times [0,1] \to \mathbf{R} \ \text{ is such that} \\ (t,s) \to k(t,s) \ \text{ is measurable} \end{cases}$$

(2.39) $$\begin{cases} \exists\, 0 < M_0 < 1, \ k_1 \in L^p[0,1] \ \text{ and } \ k_2 \in L^{p_1}[0,1] \\ \text{such that } \ 0 < k_1(t), \ k_2(t) \ \text{ a.e. } \ t \in [0,1] \ \text{ and} \\ M_0\, k_1(t)\, k_2(s) \le k(t,s) \le k_1(t)\, k_2(s), \ \text{ for a.e.} \\ t \in [0,1], \ \text{ a.e. } \ s \in [0,1] \end{cases}$$

(2.40) $$\begin{cases} f(t,u) + M \ge g(u) \ \text{ for a.e. } \ t \in [0,1] \ \text{ and all } \ u \in [0,\infty) \\ \text{with } \ g : [0,\infty) \to [0,\infty) \ \text{continuous and nondecreasing} \\ \text{and } \ g(u) > 0 \ \text{ for } \ u > 0 \end{cases}$$

(2.41) $$\begin{cases} \exists\, r \ge \frac{\mu\, M\, C\, \|k_1\|_p}{M_0} \ \text{ with } \ \frac{r}{\theta(r)} \ge \mu\, \|k_1\|_p\, \|k_2\|_{p_1}; \\ \\ \text{here } \ C = \int_0^1 k_2(s)\, ds \end{cases}$$

and

(2.42) $$\exists\, R > r \ \text{ with } \ R \le \mu\, M_0\, \|k_1\|_p \int_0^1 k_2(s)\, g\left(\epsilon\, R\, a(s)\right)\, ds;$$

here $\epsilon > 0$ *is any constant (choose and fix it) so that*

$$1 - \frac{\mu\, M\, C\, \|k_1\|_p}{M_0\, R} \geq \epsilon$$

and

$$a(t) = M_0\, \frac{k_1(t)}{\|k_1\|_p}.$$

Then (2.1) *has a solution* $y \in L^p[0,1]$ *with* $y(t) > 0$ *for a.e.* $t \in [0,1]$.

PROOF: It is enough to show (2.11) has a solution $y_1 \in L^p[0,1]$ with $y_1(t) \geq \phi(t)$ for a.e. $t \in [0,1]$ (note $\phi(t) > 0$ for a.e. $t \in [0,1]$). Let

$$K = \{u \in L^p[0,1] :\ u(t) \geq a(t)\, \|u\|_p \ \text{ for a.e. } \ t \in [0,1]\}\,.$$

Let

$$\Omega_1 = \{u \in L^p[0,1] :\ \|u\|_p < r\} \quad \text{and} \quad \Omega_2 = \{u \in L^p[0,1] :\ \|u\|_p < R\},$$

and let $A : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to L^p[0,1]$ be defined by

$$A\, y(t) = \mu \int_0^1 k(t,s)\, f^\star(s, y(s) - \phi(s))\, ds.$$

First we show $A : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to K$. Let $y \in K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right)$. Then

$$A\, y(t) \leq \mu\, k_1(t) \int_0^1 k_2(s)\, f^\star(s, y(s) - \phi(s))\, ds \ \text{ a.e. on } \ [0,1],$$

and so

$$\|A\, y\|_p \leq \mu\, \|k_1\|_p \int_0^1 k_2(s)\, f^\star(s, y(s) - \phi(s))\, ds. \tag{2.43}$$

On the other hand

$$A\, y(t) \geq M_0\, \mu\, k_1(t) \int_0^1 k_2(s)\, f^\star(s, y(s) - \phi(s))\, ds \ \text{ a.e. on } \ [0,1],$$

and this together with (2.43) yields

$$A\, y(t) \geq M_0\, \frac{k_1(t)}{\|k_1\|_p}\, \|A\, y\|_p = a(t)\, \|A\, y\|_p \ \text{ for a.e. } \ t \in [0,1].$$

Thus $A\, y \in K$ so $A : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to K$. It is well known [7] that $A : K \cap \left(\overline{\Omega_2}\backslash\Omega_1\right) \to K$ is continuous and compact.

We now show

$$\|A\, y\|_p \leq \|y\|_p \ \text{ for } \ y \in K \cap \partial\Omega_1. \tag{2.44}$$

To see this let $y \in K \cap \partial\Omega_1$, so $\|y\|_p = r$ and $y(t) \geq a(t)\, r$ for a.e. $t \in [0,1]$. Notice also for a.e. $t \in [0,1]$ that

$$\begin{aligned} y(t) - \phi(t) &= y(t) - \mu\, M \int_0^1 k(t,s)\, ds \geq y(t) - \mu\, M\, C\, k_1(t) \\ &\geq y(t) \left[1 - \frac{\mu\, M\, C\, \|k_1\|_p}{M_0\, r}\right] \geq 0. \end{aligned}$$

Thus for a.e. $t \in [0,1]$ we have

$$A\,y(t) \le \mu \int_0^1 k(t,s)\,\psi(y(s)-\phi(s))\,ds \le \mu\,k_1(t) \int_0^1 k_2(s)\,\psi(y(s))\,ds$$

and this together with (2.37) and (2.41) yields

$$\begin{aligned} \|A\,y\|_p &\le \mu\,\|k_1\|_p\,\|k_2\|_{p_1}\,\|\psi(y)\|_{p_2} \le \mu\,\|k_1\|_p\,\|k_2\|_{p_1}\,\theta(\|y\|_p) \\ &= \mu\,\|k_1\|_p\,\|k_2\|_{p_1}\,\theta(r) \le r = \|y\|_p, \end{aligned}$$

so (2.44) holds.

Next we show

$$\|A\,y\|_p \ge \|y\|_p \quad \text{for } \; y \in K \cap \partial\Omega_2. \tag{2.45}$$

To see this let $y \in K \cap \partial\Omega_2$ so $\|y\|_p = R$ and $y(t) \ge a(t)\,R$ for a.e. $t \in [0,1]$. Also for a.e. $t \in [0,1]$ that

$$y(t) - \phi(t) \ge y(t)\left[1 - \frac{\mu\,M\,C\,\|k_1\|_p}{M_0\,R}\right] \ge \epsilon\,y(t) \ge \epsilon\,a(t)\,R.$$

Thus for a.e. $t \in [0,1]$ we have

$$\begin{aligned} A\,y(t) &\ge \mu \int_0^1 k(t,s)\,g(y(s)-\phi(s))\,ds \\ &\ge \mu \int_0^1 k(t,s)\,g\left(\epsilon\,a(s)\,R\right)\,ds \\ &\ge \mu\,M_0\,k_1(t) \int_0^1 k_2(s)\,g\left(\epsilon\,a(s)\,R\right)\,ds. \end{aligned}$$

This together with (2.42) yields

$$\|A\,y\|_p \ge \mu\,M_0\,\|k_1\|_p \int_0^1 k_2(s)\,g\left(\epsilon\,a(s)\,R\right)\,ds \ge R = \|y\|_p,$$

so (2.45) holds.

Now Theorem 1.1 implies A has a fixed point $y_1 \in K \cap \left(\overline{\Omega_2} \setminus \Omega_1\right)$, so in particular $y_1(t) \ge a(t)\,r$ for a.e. $t \in [0,1]$. Thus for a.e. $t \in [0,1]$ we have

$$y_1(t) \ge a(t)\,r \ge a(t)\,\frac{\mu\,M\,C\,\|k_1\|_p}{M_0} = k_1(t)\,\mu\,M\,C \ge \mu\,M \int_0^1 k(t,s)\,ds = \phi(t). \quad \square$$

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