FORCED VIBRATIONS OF ABSTRACT WAVE EQUATIONS

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Abstract. Existence results of periodic solutions of certain abstract nonlinear wave equations are given when eigenvalues of linear parts of those equations are incommensurable to the time period of forcing terms.

1. Introduction

In this paper, we investigate the existence of periodic solutions for certain abstract wave equations. We are motivated by the papers of K. Ben-Naoum and J. Mawhin [1], and P.J. McKenna [8], where existence results of periodic solutions are proved for one-dimensional wave equations when the ratio between the space length and the period was irrational. Related equations are also studied by M. Yamaguchi in [10]. We proceeded in this direction in the paper [4]. We studied the equation

$$u_{tt} + Au = \varepsilon f(u,t)\,, \tag{1.1}$$

where A is a self-adjoint, unbounded linear operator with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$, ε is a small parameter and f is T-periodic in $t \in \mathbb{R}$. By a T-periodic solution of (1.1) we mean a weak solution specified below. The following results are proved in [4] under additional assumptions on A, f.

Theorem 1.1. ([4]) *Assume there exists a constant $c > 0$ such that*

$$\Big|\alpha^2 - \frac{m^2}{\lambda_i}\Big| \geq \frac{c}{\lambda_i} \qquad \forall\, m \in \mathbb{N}, \quad \forall\, \lambda_i > 0\,, \tag{1.2}$$

where $\alpha = \frac{T}{2\pi}$. Then (1.1) has a weak T-periodic solution for any ε small.

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Theorem 1.2. ([4]) *Assume*

$$\sum_{\lambda_i>0} \frac{1}{\sqrt{\lambda_i}} < \infty\,.$$

Then the Lebesque measure of the set of all positive α *not satisfying* (1.2) *is zero.*

We also studied the case when $0 < \dim\ker A < \infty$. Finally we considered the example

$$\begin{aligned} &u_{tt} - u_{xx} - n^2 u = \varepsilon f(u,t) \\ &u(t+T,\cdot) = u(t,\cdot) \quad \forall\, t \in S^T \\ &u(t,0) = u(t,\pi) = 0 \quad \forall\, t \in S^T\,, \end{aligned} \tag{1.3}$$

where $f\colon \mathbb{R}\times S^T \to \mathbb{R}$ is C^1-smooth and globally Lipschitz in u, $n \in \mathbb{N}$. Here $S^T = \mathbb{R}/[0,T]$ is the circle. The following result is proved in [4].

Theorem 1.3. ([4]) *The equation* (1.3) *has a weak* T*-periodic solution, provided that it holds*

$$\inf_{i,m\in\mathbb{N}, i>n} |i^2 - n^2 - \omega^2 m^2| > 0\,, \tag{1.4}$$

where $\omega = 1/\alpha$*, and there is a* $z \in \mathbb{R}$ *such that*

$$\begin{aligned} &\int_0^T\int_0^\pi f(z.\sin nx,t)\sin nx\,dx\,dt = 0 \\ &\int_0^T\int_0^\pi \frac{\partial f}{\partial u}(z.\sin nx,t)\sin^2 nx\,dx\,dt \neq 0\,. \end{aligned}$$

The purpose of this paper is two-fold. We firstly release the parameter ε in (1.1), so we consider the equation

$$u_{tt} + Au = f(u,t)\,. \tag{1.5}$$

We have used in [4] the Banach fixed point theorem. To get our results in this paper, we apply the Leray-Schauder fixed point theorem. For this reason, we need more precision condition than (1.2), see (2.2) below. We also study the resonant case when $0 < \dim\ker A < \infty$. We derive a Landesman-Lazer type result [7]. Finally, we present a forced beam equation as an example.

We secondly investigate more correctly and thoroughly the condition (1.4) than in [4]. By using some results of the number theory [2, 3, 6], we derive several conditions for ω when (1.4) is either satisfied or not.

2. Nonresonant Cases

Let X be a Banach space continuously embedded into a Hilbert space Y with an inner product $\langle\cdot,\cdot\rangle$ and with the corresponding norm $|\cdot|$. We assume that $A\colon X\to Y$ is bounded and the eigenvectors $\{u_j\}_1^\infty$ of A corresponding to $\{\lambda_j\}_1^\infty$ form an orthonormal basis of Y. Moreover we suppose that the linear hull of $\{u_j\}_1^\infty$ is dense in X. Furthermore, $f\colon Y\times S^T\to Y$ is continuous satisfying

$$|f(y,t)|\le c_1\big(|y|^\sigma+1\big)\quad \forall\, y\in Y,\,\forall\, t\in S^T \tag{2.1}$$

for constants $1>\sigma\ge 0$, $c_1>0$.

In this section we study the case when

$$\lambda_i\ne 0\quad \forall\, i\,. \tag{n}$$

A **weak T-periodic solution** of (1.5) is some $u\in L^2\big(S^T,Y\big)$ satisfying

$$\int_0^T\langle u(t),v_{tt}(t)+Av(t)\rangle\,dt=\int_0^T\langle f\big(u(t),t\big),v(t)\rangle\,dt \qquad \forall\, v\in C^2\big(S^T,X\big)\,. \tag{W}$$

We mean the integrability in the sense of Bochner [5]. We note that (2.1) implies the continuity of the Nemytskii operator $u\to f(u(t),t)$ from $L^2(S^T,Y)$ to $L^2(S^T,Y)$.

Lemma 2.1. *Assume the following condition*

$$\big|\alpha^2-\frac{m^2}{\lambda_i}\big|\ge\frac{c}{\lambda_i^\rho}\qquad \forall\, m\in\mathbb{N},\quad \forall\,\lambda_i>0\,, \tag{2.2}$$

where ρ, c are positive constants such that $0<\rho<1$. Then the equation

$$\int_0^T\langle u(t),v_{tt}(t)+Av(t)\rangle\,dt=\int_0^T\langle h(t),v(t)\rangle\,dt \qquad \forall\, v\in C^2\big(S^T,X\big) \tag{2.3}$$

has a unique solution $u=Lh\in L^2\big(S^T,Y\big)$ for any $h\in L^2\big(S^T,Y\big)$. Moreover, $L\colon L^2(S^T,Y)\to L^2(S^T,Y)$ is compact.

Proof. By our assumptions, the Hilbert space $L^2(S^T,Y)$ has the orthogonal basis

$$\Big\{\sin m\frac{2\pi t}{T}\cdot u_j\,,\,\cos m\frac{2\pi t}{T}\cdot u_j\Big\}\subset C^2\big(S^T,X\big) \qquad m=0,1,2,\cdots\quad j=1,2,\cdots\,. \tag{2.4}$$

We expand u (formally) and h really in the basis (2.4) to get

$$u(t)=\sum_{m,j}\Big(u_{mj}^1\sin m\frac{2\pi t}{T}+u_{mj}^2\cos m\frac{2\pi t}{T}\Big)u_j$$

$$h(t)=\sum_{m,j}\Big(h_{mj}^1\sin m\frac{2\pi t}{T}+h_{mj}^2\cos m\frac{2\pi t}{T}\Big)u_j\,.$$

Of course, we take $u^1_{0j} = 0$ and $h^1_{0j} = 0$. We have

$$\sum_j T(h^2_{0,j})^2 + \sum_{m\neq 0,j} (T/2)\big((h^1_{mj})^2 + (h^2_{mj})^2\big) < \infty\,.$$

If u is a solution of (2.3), then we take $v(t) = \sin m\frac{2\pi t}{T}\cdot u_j$ and $v(t) = \cos m\frac{2\pi t}{T}\cdot u_j$ to get

$$u^i_{mj} = \frac{\alpha^2}{\alpha^2\lambda_j - m^2} h^i_{mj}, \quad i = 1,2\,.$$

Hence if (2.3) has a solution $u \in L^2(S^T, Y)$, then it is unique and it should be given by

$$u(t) = \sum_{m,j} \frac{\alpha^2}{\alpha^2\lambda_j - m^2}\Big(h^1_{mj} \sin m\frac{2\pi t}{T} + h^2_{mj}\cos m\frac{2\pi t}{T}\Big)u_j\,. \tag{2.5}$$

The assumption (2.2) gives a constant $c_2 > 0$ such that $|u^i_{mj}| \le c_2|h^i_{mj}|$, $i = 1,2$. Hence

$$\begin{aligned}&\sum_j T(u^2_{0,j})^2 + \sum_{m\neq 0,j}(T/2)\big((u^1_{mj})^2 + (u^2_{mj})^2\big)\\ &\le c_2^2\Big(\sum_j T(h^2_{0j})^2 + \sum_{m\neq 0,j}(T/2)\big((h^1_{mj})^2 + (h^2_{mj})^2\big)\Big)\,.\end{aligned}$$

This gives $u \in L^2(S^T, Y)$ given by (2.5) and $|u| \le c_2|h|$. So L is continuous. Moreover the assumption (2.2) implies that $|\alpha^2\lambda_j - m^2| \to \infty$ as $|\lambda_j| + m \to \infty$. This gives the compactness of L.

Now we show that this u satisfies (2.3). Our assumptions give that the linear hull L_H of (2.4) is dense in $C^2(S^T, X)$: one can prove this by using the $\triangle$-approximation method like in [9], see also Fejér's theorem [9]. So for any $v \in C^2(S^T, X)$ there is a sequence $v_j \in L_H$ such that $v_j \to v$ in $C^2(S^T, X)$. This gives $v_{jtt} \to v_{tt}$ and $v_j \to v$ in $C(S^T, X)$. Hence $Av_j \to Av$ in $C(S^T, Y)$. The equality (2.3) holds for any $v_j \in L_H$, and since $X \subset Y$ continuously, we take the limit $j \to \infty$ in (2.3) for $v = v_j$ to get the validity of (2.3) for any v. The proof is finished. □

Theorem 2.2. *If the conditions* (2.1), (2.2) *and* (n) *are satisfied, then* (1.5) *has a weak T-periodic solution.*

Proof. For a $u \in L^2(S^T, Y)$, we put $F(u) = f(u(t), t) \in L^2(S^T, Y)$. Then (W) is equivalent to $u = LF(u)$. Lemma 2.1 gives the compactness of the operator $LF\colon L^2(S^T, Y) \to L^2(S^T, Y)$. Moreover, Lemma 2.1 and the condition (2.1) give

$$\begin{aligned}|LF(u)|^2 &\le c_2^2\int_0^T c_1^2\big(|u(t)|^\sigma + 1\big)^2\,dt \le 2c_1^2c_2^2\int_0^T\big(|u(t)|^{2\sigma} + 1\big)\,dt\\ &= 2c_1^2c_2^2T + 2c_1^2c_2^2\int_0^T |u(t)|^{2\sigma}\,dt\,.\end{aligned}$$

Since $0 \le \sigma < 1$, the Jensen inequality gives

$$|LF(u)|^2 \le 2c_1^2c_2^2T + 2c_1^2c_2^2T^{1-\sigma}\Big(\int_0^T |u(t)|^2\,dt\Big)^\sigma$$
$$= 2c_1^2c_2^2T^{1-\sigma}|u|^{2\sigma} + 2c_1^2c_2^2T \le 2c_1^2c_2^2\big(T^{(1-\sigma)/2}|u|^\sigma + \sqrt{T}\big)^2\,.$$

Consequently we get

$$|LF(u)| \le \sqrt{2}c_1c_2\big(T^{(1-\sigma)/2}|u|^\sigma + \sqrt{T}\big)\,.$$

By taking $K_0 > 0$ such that $\sqrt{2}c_1c_2\big(T^{(1-\sigma)/2}K_0^\sigma + \sqrt{T}\big) = K_0$, we see that the ball $B_{K_0} = \{u \in L^2(S^T, Y) \mid |u| \le K_0\}$ is mapped to itself by LF. Hence the Leray-Schauder fixed point theorem [7] finishes the proof. □

By the same way as for Theorem 1.2 of [4], we get the following result.

Theorem 2.3. *Assume*

$$\sum_{\lambda_i>0}\frac{1}{\lambda_i^{\rho-(1/2)}} < \infty\,.$$

Then the Lebesque measure of the set of all positive α not satisfying (2.2) *is zero.*

Proof. We present the proof for the reader convenience. If (2.2) is false for some $\alpha \in (K, K+1)$, $K > 0$, then for any $d > 0$ small there exist m, $\lambda_i > 0$ such that

$$\Big|\alpha^2 - \frac{m^2}{\lambda_i}\Big| \le \frac{d}{\lambda_i^\rho}\,.$$

This implies

$$\Big|\alpha - \frac{m}{\sqrt{\lambda_i}}\Big| \le \frac{d}{K\lambda_i^\rho}\,.$$

Since $\alpha \in (K, K+1)$, we have $\frac{m^2}{\lambda_i} < (K+1)^2 + 1$ for d small. Thus

$$m \le \sqrt{((K+1)^2+1)\lambda_i}\,.$$

Denote by $\mathcal{M}$ the set of all $\alpha \in (K, K+1)$ for which (2.2) does not hold. Then the Lebesque measure $\mu(\mathcal{M})$ of $\mathcal{M}$ satisfies

$$\mu(\mathcal{M}) \le \sum_{\lambda_i>0}\frac{2d}{\lambda_i^\rho K}\cdot\sqrt{((K+1)^2+1)\lambda_i}$$
$$= \frac{2d}{K}\sqrt{((K+1)^2+1)}\sum_{\lambda_i>0}\frac{1}{\lambda_i^{\rho-(1/2)}} = O(d)\,.$$

Since d is arbitrarily small, $\mu(\mathcal{M}) = 0$. The proof is finished. □

This theorem again says nothing for the case

$$Au = -u_{xx}, \quad u \in C^2\big([0,\pi], \mathbb{R}\big)$$
$$u(0) = u_{xx}(0) = u(\pi) = u_{xx}(\pi) = 0\,.$$

But taking

$$Au = u_{xxxx}, \quad u \in C^4\big([0,\pi],\mathbb{R}\big)$$
$$u(0) = u_{xx}(0) = u_{xxxx}(0) = u(\pi) = u_{xx}(\pi) = u_{xxxx}(\pi) = 0$$

we have $\lambda_i = i^4 \quad \forall\, i$, and Theorem 2.3 holds for $\rho = 4/5$. The operator $u_{tt} + u_{xxxx}$ appears in the equation of a beam.

Finally we note that equations of the following form are studied by M. Yamaguchi in the paper [10]

$$u_{tt} + Au = \varepsilon f_1(u,t) + f_2(t) \tag{2.6}$$

for a small $\varepsilon \in \mathbb{R}$, $f_1\colon Y \times S^T \to Y$, $f_2\colon S^T \to Y$ continuous. Like above, we get the following result.

Theorem 2.4. *If the conditions* (2.2), (n) *are satisfied and the following condition*

$$|f(y,t)| \le c_1\big(|y|+1\big) \quad \forall\, y \in Y,\ |y| \le \max_{t\in S^T} |f_2(t)| + 1,\ \forall\, t \in S^T$$

holds as well for a positive constant c_1, then (2.6) *has a weak T-periodic solution for any ε sufficiently small.*

3. Resonant Cases

Now we consider (1.5) under the condition $0 < \dim\ker A < \infty$. The setting of the problem (1.5) is the same as in Section 2. So we suppose (2.2), but (2.1) is assumed with $\sigma = 0$, i.e

$$|f(y,t)| \le c_1 \quad \forall\, y \in Y, \forall\, t \in S^T \tag{3.1}$$

for a positive constant c_1. By (W) any weak T-periodic solution of (1.5) satisfies

$$\int_0^T \langle f\big(u(t),t\big), u_p\rangle\, dt = 0, \quad \forall\, u_p \in \ker A\,.$$

Then Lemma 2.1 holds as well for this case but only for h such that

$$\int_0^T \langle h(t), u_p\rangle\, dt = 0, \quad \forall\, u_p \in \ker A\,.$$

Let $P\colon Y \to \ker A$ be the orthogonal projection. Then (1.5) has the form

$$\begin{aligned} w_{tt} + Aw &= Qf(w+u_p,\cdot) \\ 0 &= \tilde{P} f(w+u_p,\cdot)\,, \end{aligned} \tag{3.2}$$

where

$$Q = I - \tilde{P}, \quad \tilde{P}h = \frac{1}{T}\int_0^T Ph(t)\,dt\,,$$
$$w \in \operatorname{im} Q,\ u_p \in \ker A\,.$$

Note, these projections are defined for the following reason

$$\tilde{P}h = 0 \Longleftrightarrow \begin{cases} \int_0^T \langle h(t), u_p\rangle\, dt = 0 \\ \forall\, u_p \in \ker A\,. \end{cases}$$

Clearly the operator L of Lemma 2.1 is defined as $L\colon \operatorname{im} Q \to L^2(S^T, Y)$. Let $l_i\colon Y \to \mathbb{R}$, $i = 1, 2, \cdots, i_0 = \dim\ker A$, be continuous linear functionals such that $\tilde{P}h = 0$ if and only if $l_i(h) = 0 \quad \forall\, i = 1, 2, \cdots, i_0$. We identify $\ker A \simeq \mathbb{R}^{i_0}$.

Theorem 3.1. *Let* (2.2) *and* (3.1) *be satisfied. Suppose that*

$$\liminf_{|u_p|\to\infty} \sum_{i=1}^{i_0} \big| l_i(f(Lw + u_p, \cdot))\big| > 0 \tag{3.3}$$

uniformly for $w \in$ im$Q$ from bounded sets, i.e. $\forall\, K_2 > 0$, $\exists\, K_3 > 0$, $\exists\, \delta > 0$ such that $\sum_{i=1}^{i_0} \big|l_i(f(Lw + u_p, \cdot))\big| > \delta\ \forall\, w \in$ im$Q$, $|w| \le K_2$, $\forall\, u_p \in \ker A$, $|u_p| \ge K_3$. Then (1.5) *has a weak T-periodic solution provided that the Brouwer degree deg$\big(F(u_p), B_{K_1}, 0\big) \ne 0$, where $F\colon \mathbb{R}^{i_0} \simeq \ker A \to \mathbb{R}^{i_0}$ and B_{K_1} are defined by*

$$F(u_p) = \big(l_1(f(u_p)), l_2(f(u_p)), \cdots, l_{i_0}(f(u_p))\big)$$
$$B_{K_1} = \big\{u_p \in \ker A \simeq \mathbb{R}^{i_0} \mid |u_p| \le K_1\big\} \subset \mathbb{R}^{i_0}$$

for a sufficiently large constant $K_1 > 0$. According to (3.3)*, this degree is well-defined.*

Proof. This is a Landesman-Lazer type result [7], so we follow the standard way by using the homotopy

$$\begin{aligned} w_{tt} + Aw &= \lambda Q f(w + u_p, \cdot), \quad 0 \le \lambda \le 1\,, \\ 0 &= l_i\big(f(\lambda LQf(w + u_p, \cdot) + u_p, \cdot)\big), \quad i = 1, 2, \cdots, i_0\,. \end{aligned} \tag{3.4}$$

For $\lambda = 1$ we get (3.2). The assumptions (2.2) and (3.1) imply the existence of a constant $K_0 > 0$ such that there is no solution of (3.4) such that $|w| \ge K_0$. (3.1) and (3.3) give a constant $K_1 > 0$ such that

$$\sum_{i=1}^{i_0} \big|l_i\big(f(\lambda LQf(w + u_p, \cdot) + u_p, \cdot)\big)\big| > 0 \tag{3.5}$$

for any $w \in \operatorname{im} Q$, $|w| \le K_0$, $0 \le \lambda \le 1$ and $u_p \in \ker A$, $|u_p| \ge K_1$. Let us put

$$\Omega = \Big\{(w, u_p) \in \operatorname{im} Q \times \ker A \mid |w| \le K_0, \quad |u_p| \le K_1\Big\}\,.$$

According to the choice of the constants K_0, K_1, we can easily verify that (3.4) has no solution on $\partial\Omega$ for any $0 \le \lambda \le 1$. Consequently to compute the Leray-Schauder degree of (3.4), we take $\lambda = 0$ and we are lead to the mapping $u_p \to F(u_p)$ and to the Brouwer degree $\deg\big(F(u_p), B_{K_1}, 0\big) \ne 0$. This gives the solvability of (3.4) for $\lambda = 1$ in Ω. The proof is finished. ☐

Example. Consider

$$
\begin{aligned}
&u_{tt} + u_{xxxx} - n^4 u = f(u) + g(t)h(x) \\
&u(t+T,\cdot) = u(t,\cdot) \quad \forall\, t \in S^T \qquad (3.6)\\
&u(t,0) = u_{xx}(t,0) = u(t,\pi) = u_{xx}(t,\pi) = 0 \quad \forall\, t \in S^T \,,
\end{aligned}
$$

where $f\colon \mathbb{R} \to \mathbb{R}$, $g\colon S^T \to \mathbb{R}$, $h\colon [0,\pi] \to \mathbb{R}$ are continuous and $n \in \mathbb{N}$.

Now, we have for this case

$$
\begin{aligned}
&Au = u_{xxxx} - n^4 u, \quad \ker A = \{\sin nx\}, \quad \lambda_i = i^4 - n^4, \quad u_i = \sin ix \\
&X = \{u \in C^4\big([0,\pi],\mathbb{R}\big) \mid u(0) = u_{xx}(0) = \\
&= u_{xxxx}(0) = u(\pi) = u_{xx}(\pi) = u_{xxxx}(\pi) = 0\}, \quad Y = L^2(0,\pi)\,.
\end{aligned}
$$

Now we take $\rho = 4/5$. Then Theorem 2.3 holds and the condition (2.2) gives $\big|\alpha^2\lambda_j - m^2\big| \ge c\lambda_j^{1-\rho} = c(j^4 - n^4)^{1/5} \sim j^{4/5}$ for any $j > n$. Consequently, if $u \in \operatorname{im} Q$ then Lu is given by (2.5) and $Lu \in L^2(S^T, Z)$, where Z is defined by

$$
Z = \Big\{u(x) = \sum_{j=1}^{\infty} a_j \sin jx \mid \sum_{j=1}^{\infty} j^{8/5} a_j^2 < \infty\Big\}\,.
$$

Z is a Hilbert space with the norm $|u|_Z = \sqrt{\sum_{j=1}^{\infty} j^{8/5} a_j^2}$. Hence $\Big\{\frac{1}{j^{4/5}}\sin jx\Big\}_{j=1}^{\infty}$ is an orthonormal basis of Z. Moreover, for any $u \in Z$ we have

$$
\begin{aligned}
|u(x)| &\le \sum_{j=1}^{\infty} |a_j| \le \sqrt{\sum_{j=1}^{\infty} j^{8/5} a_j^2}\sqrt{\sum_{j=1}^{\infty} j^{-8/5}} \\
&\le |u|_Z \sqrt{1 + \int_1^{\infty} x^{-8/5}\, dx} = |u|_Z \sqrt{5/3}\,.
\end{aligned}
$$

Hence $Z \subset C[0,\pi]$ and $\max_{x\in[0,\pi]} |u(x)| \le \sqrt{5/3}|u|_Z$ $\forall\, u \in Z$. (2.5) implies the continuity of $L\colon \operatorname{im} Q \to L^2(S^T, Z) \subset L^2(S^T, C[0,\pi])$ and so there is a constant $\bar{c} > 0$ such that

$$
\int_0^T \max_{x\in[0,\pi]} \big|(Lu)(t,x)\big|^2\, dt \le \bar{c}^2 |u|^2 \quad \forall\, u \in L^2(S^T, Y)\,.
$$

We take $l_1(u) = \int_0^T\int_0^\pi u(t,x)\sin nx\, dx\, dt$. To get (3.1), we suppose

$$
\sup_{u\in\mathbb{R}} |f(u)| = \delta < \infty\,. \tag{3.7}
$$

The assumption (3.3) has the form

$$
\liminf_{|c|\to\infty} \Big| \int_0^T \int_0^\pi \tilde{f}\big((Lw)(t,x) + c\sin nx, t\big)\sin nx\, dx\, dt\Big| > 0 \tag{3.8}
$$

uniformly for

$$|w|^2 = \int_0^T \int_0^\pi w(t,x)^2 \, dx \, dt \le K_3^2 , \tag{3.9}$$

where $K_3 > 0$ is any fixed constant. Here $\tilde{f}(u,t) = f(u) + g(t)h(x)$. Since $|L| \le \bar{c}$ as $L\colon \operatorname{im} Q \to L^2(S^T, C[0,\pi])$, we see that (3.9) implies

$$\int_0^T \max_{x\in[0,\pi]} \big|(Lw)(t,x)\big|^2 \, dt \le K_3^2 \bar{c}^2 . \tag{3.10}$$

Let $\varepsilon > 0$ be small and put $I_i = \big[\frac{\pi}{n} i + \varepsilon, \frac{\pi}{n}(i+1) - \varepsilon\big]$, $i = 0, 1, \cdots, n-1$. Then

$$\begin{aligned} &\int_0^T \int_0^\pi \tilde{f}\big((Lw)(t,x) + c\sin nx, t\big) \sin nx \, dx \, dt \\ &= \sum_{i=0}^{n-1} \int_0^T \int_{I_i} \tilde{f}\big((Lw)(t,x) + c\sin nx, t\big) \sin nx \, dx \, dt + O(\varepsilon) . \end{aligned}$$

(3.10) gives a set $N_{\varepsilon,w} \subset [0,T] \simeq S^T$ such that the Lebesque measure $\mu(N_{\varepsilon,w}) \le \varepsilon$, and $\forall\, t \notin N_{\varepsilon,w}$ it holds that $\max_{x\in[0,\pi]} \big|(Lw)(t,x)\big| \le K_3 \bar{c}/\sqrt{\varepsilon}$. Hence

$$\begin{aligned} &\int_0^T \int_{I_i} \tilde{f}\big((Lw)(t,x) + c\sin nx, t\big) \sin nx \, dx \, dt \\ &= \int_{S^T \setminus N_{\varepsilon,w}} \int_{I_i} \tilde{f}\big((Lw)(t,x) + c\sin nx, t\big) \sin nx \, dx \, dt + O(\varepsilon) . \end{aligned} \tag{3.11}$$

The function $\sin nx$ does not change the sign on I_i, hence $\big|(Lw)(t,x) + c\sin nx\big| \to \infty$ as $|c| \to \infty$ for $x \in I_i$, $t \notin N_{\varepsilon,w}$. More precisely, for any $K_4 > 0$, there is a $K_5 > 0$ such that $\big|(Lw)(t,x) + c\sin nx\big| \ge K_4$ for $|c| \ge K_5$, $x \in I_i$, $t \notin N_{\varepsilon,w}$. We note that K_5 is independent of w.

Let us suppose

$$\lim_{u\to\pm\infty} f(u) = f^{\pm} \in \mathbb{R} . \tag{3.12}$$

Now we take $K_4 > 0$ sufficiently large such that for any $|c| \ge K_5$ we get from (3.11) and (3.12)

$$\begin{aligned}
&\int\limits_{S^T\setminus N_{\varepsilon,w}}\int\limits_{I_i}\tilde f\big((Lw)(t,x)+c\sin nx,t\big)\sin nx\,dx\,dt\\
&=\int\limits_{S^T\setminus N_{\varepsilon,w}}\int\limits_{I_i}\Big(f^{\operatorname{sign}(-1)^i c}+g(t)h(x)\Big)\sin nx\,dx\,dt+O(\varepsilon)\\
&=\int\limits_0^T\int\limits_{i\pi/n}^{(i+1)\pi/n}\Big(f^{\operatorname{sign}(-1)^i c}+g(t)h(x)\Big)\sin nx\,dx\,dt+O(\varepsilon)\\
&=(-1)^i\frac{2T}{n}f^{\operatorname{sign}(-1)^i c}+\int\limits_0^T g(t)\,dt\int\limits_{i\pi/n}^{(i+1)\pi/n}f(x)\sin nx\,dx+O(\varepsilon)\,.
\end{aligned}$$

Summarizing we see that (3.8) holds if

$$\frac{T}{n}\triangle_n^\pm+\int\limits_0^T g(t)\,dt\int\limits_0^\pi h(x)\sin nx\,dx\neq 0\,,$$

where

$$\triangle_n^\pm=\begin{cases}(n-1)(f^\pm-f^\mp)+2f^\pm & \text{for an odd}\quad n\\ n(f^\pm-f^\mp) & \text{for an even}\quad n\,.\end{cases}$$

The mapping $F(u_p)$ of Theorem 3.1 has now the form

$$F(c)=\int\limits_0^T\int\limits_0^\pi\big(f(c\sin nx)+g(t)h(x)\big)\sin nx\,dx\,dt\,. \tag{3.13}$$

By using (3.12) like above for (3.11), we see from (3.13) that the Brouwer degree of Theorem 3.1 given now by $\deg\big(F(c),(-K_1,K_1),0\big)$ is nonzero if it holds

(3.14)

$$\Big(\frac{T}{n}\triangle_n^+ +\int\limits_0^T g(t)\,dt\int\limits_0^\pi h(x)\sin nx\,dx\Big)\Big(\frac{T}{n}\triangle_n^- +\int\limits_0^T g(t)\,dt\int\limits_0^\pi h(x)\sin nx\,dx\Big)<0\,.$$

On the other hand, the equality below (3.1) implies that (3.6) has no solution if

$$T\pi\delta<\Big|\int\limits_0^T g(t)\,dt\int\limits_0^\pi h(x)\sin nx\,dx\Big|\,. \tag{3.15}$$

Summarizing, we get the following result.

Theorem 3.2. *There is a subset $S\subset(0,\infty)$ with a zero Lebesque measure such that for any $0<T\notin S$, if* (3.12) *and* (3.14) *are satisfied then the problem* (3.6) *has a weak T-periodic solution. On the other hand, if* (3.7) *and* (3.15) *hold for a $T>0$ then* (3.6) *has no weak T-periodic solution.*

4. Some Remarks on the condition (1.4)

This section is devoted to results concerning the condition (1.4). There is a nice characterization of such ω for $n = 0$ in terms of continued fractions [1, 6]. We intend to derive similar results for $n \in \mathbb{N}$. This situation is different from $n = 0$.

Theorem 4.1. *Let $\omega = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$. The condition (1.4) holds if and only if any $n_2 \in \mathbb{N}$, $n_2 \mid \frac{n}{(p,n)}$ satisfies*

(i) *If n_2 is odd then $p/(p, n)$ does not divide $(a^2 - b^2)/2$ for any $a > b$, $a, b \in \mathbb{N}$ such that $n_2 = ab$.*

(ii) *If n_2 is even then $p/(p, n)$ does not divide $a^2 - b^2$ for any $a > b$, $a, b \in \mathbb{N}$ such that $n_2 = 2ab$.*

Here as usually (p, n) is the largest common divisor of p and n.

Proof. The condition (1.4) does not hold if and only if there are $i, m \in \mathbb{N}$, $i > n$ such that

$$i^2 = \omega^2 m^2 + n^2$$

$$q^2 i^2 = p^2 m^2 + q^2 n^2 \,. \tag{4.1}$$

Hence $q \mid pm$ implies $q \mid m$, i.e. $m = rq$, $r \in \mathbb{N}$ and (4.1) gives

$$i^2 = p^2 r^2 + n^2 \,. \tag{4.2}$$

After dividing (4.2) by $(p, n)^2$, we get $i_1^2 = p_1^2 r^2 + n_1^2$, $p_1 = p/(p, n)$, $n_1 = n/(p, n)$, $i_1 = i/(p, n)$. If $(r, n_1) > 1$ then we have $i_2^2 = p_1^2 r_1^2 + n_2^2$ and $(n_2, p_1 r_1) = 1$.

We have two possibilities:

If n_2 is odd, then $p_1 r_1$ is even and we get

$$\frac{i_2 + n_2}{2} \frac{i_2 - n_2}{2} = \Big(\frac{p_1 r_1}{2}\Big)^2 .$$

Since $(p_1 r_1, n_2) = 1$, we get

$$i_2 + n_2 = 2A^2, \quad i_2 - n_2 = 2B^2, \quad p_1 r_1 = 2AB$$

$$i_2 = A^2 + B^2, \quad n_2 = A^2 - B^2 = (A - B)(A + B) \,.$$

So $A = (a + b)/2$, $B = (a - b)/2$, $n_2 = ab$, $a > b \in \mathbb{N}$ and $p_1 r_1 = (a^2 - b^2)/2$. This proves (i).

If n_2 is even, then $p_1 r_1$ is odd and we get

$$\frac{i_2 + p_1 r_1}{2} \frac{i_2 - p_1 r_1}{2} = \Big(\frac{n_2}{2}\Big)^2 .$$

Similarly like above we get $i_2 = A^2 + B^2$, $p_1 r_1 = A^2 - B^2$, $n_2 = 2AB$, $A > B \in \mathbb{N}$. This proves (ii). The proof is finished. ☐

Corollary 4.2. *Let $\omega = p/q$, $p, q \in \mathbb{N}$, $(p, q) = 1$ and let $n > 1$ be a prime number. The condition (1.4) holds for $n = 2$, and for $n > 2$ if and only if p does not divide $(n^2 - 1)/2$.*

Theorem 4.3. *If* $\omega = \sqrt{p/q}$ *is irrational for* $p, q \in \mathbb{N}$, $(p,q)=1$, *then* (1.4) *does not hold for any* $n \in \mathbb{N}$.

Proof. Since $\sqrt{pq}$ is irrational, the Pelle equation $i_0^2 = pqm_0^2 + 1$ has a natural number solution (infinitely many). Then $i = i_0 n$, $m = m_0 qn$ satisfy $i^2 = \frac{p}{q}m^2 + n^2$, i.e. $i^2 = \omega^2 m^2 + n^2$. The proof is finished. □

It is well-known (see [6]) that if ω has the continued fraction $\omega = [a_0, a_1, \cdots]$ such that $a_k \le M$ for some $M > 0$ and all $k \ge 1$, then it holds

$$\left|\omega - p/q\right| \ge \frac{1}{M+2}\frac{1}{q^2}$$

for any $p, q \in \mathbb{N}$. Hence

$$\left|\left(\frac{i}{m}\right)^2 - \omega^2\right| = \left|\frac{i}{m} - \omega\right|\left|\frac{i}{m} + \omega\right| \ge \frac{\omega}{M+2}\frac{1}{m^2}$$

for any $i, m \in \mathbb{N}$. Consequently, we get the following result.

Theorem 4.4. *The condition* (1.4) *holds if* $\omega = [a_0, a_1, \cdots]$ *is such that* $a_k \le M$ *for some* $M > 0$ *and all* $k \ge 1$, *and* $\omega > (M+2)n^2$.

Proof. For $i > n, m \in \mathbb{N}$, according to the above result, we have

$$|i^2 - \omega^2 m^2 - n^2| \ge |i^2 - \omega^2 m^2| - n^2$$
$$= m^2\left|\left(\frac{i}{m}\right)^2 - \omega^2\right| - n^2 \ge \frac{\omega}{M+2} - n^2 > 0\,.$$

The proof is finished. □

For instance, if $\omega = \frac{2M-1+\sqrt{5}}{2}$, $M \in \mathbb{N}$ then $\omega = [M, 1, 1, \cdots]$ and we get the condition $\frac{2M-1+\sqrt{5}}{2} > 3n^2$.

Moreover, we can get further characterizations of ω to satisfy (1.4). We follow [2, 3]. Let

$$\mu(\alpha)^{-1} = \liminf_{q\to+\infty} |q(q\alpha - p)|\,,$$

where p and q are arbitrary integers. The set of values taken by $\mu(\alpha)$ as α varies is called the Lagrange spectrum. If $\alpha = [a_0, a_1, a_2, \cdots]$ then

$$\mu(\alpha) = \limsup_{k\to+\infty}\Big([a_{k+1}, a_{k+2}, \cdots] + [0, a_k, a_{k-1}, \cdots, a_1]\Big)\,.$$

We know [6] that $\mu(\alpha) < +\infty$ if and only if α is irrational and all a_i are uniformly bounded.

According to [2], two real numbers θ and θ' are **equivalent** if there are integer numbers r, s, t, u such that

$$\theta = \frac{r\theta' + s}{t\theta' + u}, \quad ru - ts = \pm 1\,.$$

We know from [2] that $\mu(\theta) = \mu(\theta')$.

Let $\omega > 0$ be irrational such that $\mu(\omega) \ne +\infty$. Then for any $\epsilon > 0$ and for any $n < i, m \in \mathbb{N}$, expect of a finite number, we have like above

$$\left|i^2 - \omega^2 m^2 - n^2\right| \ge |i - \omega m||i + \omega m| - n^2 \ge \Big(\frac{1}{\mu(\omega)} - \epsilon\Big)\omega - n^2\,.$$

On the other hand, if $i_0^2 - \omega^2 m_0^2 - n^2 = 0$ for some $n < i_0, m_0 \in \mathbb{N}$, then the proof of Theorem 4.3 gives infinitely many such i_0, m_0.

Summarizing, we get the following result.

Theorem 4.5. *Let $\omega' > 0$ be irrational such that $\mu(\omega') \neq +\infty$. Then* (1.4) *holds for any $\omega > 0$ equivalent to ω' satisfying $\omega > \mu(\omega')n^2$.*

For instance, if we take $u = t = 1, r = s + 1, s \in \mathbb{Z}$, then $\omega = s + \frac{\omega'}{\omega'+1}$ is equivalent to ω', and Theorem 4.5 holds for any $s \in \mathbb{N}$ such that

$$s > n^2\mu(\omega') - \frac{\omega'}{\omega' + 1}\,.$$

Finally, we note [2, 3] that $\mu(\alpha) \geq \sqrt{5}$ for any α. Moreover, the Lagrange spectrum on the interval $[\sqrt{5}, 3)$ is the set $\{\sqrt{9 - 4m^{-2}}\}$, where m is a positive integer such that

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2, \quad m_{1,2} \leq m$$

holds for some positive integers m_1 and m_2. Then ω' is a root of the Markoff form F_m such that $\mu(\omega') = \sqrt{9 - 4m^{-2}}$. We also note that according to the Hall theorem [3], the Lagrange spectrum contains every number greater or equal to $\sqrt{21}$.

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