# Convergence of Formal Solutions of Singular First Order Nonlinear Partial Differential Equations of Totally Characteristic Type

By
Akira Shirai
(Nagoya University, Japan)

## 1 Introduction and Main Results

In [CT], H. Chen and H. Tahara studied the following equation:

$$\begin{cases} t\partial_t u = f(t,x,u,\partial_x u), \\ u(0,x) \equiv 0, \end{cases} \tag{1.1}$$

where $(t,x) \in \mathbf{C}^2$, and $f(t,x,u,\xi)$ is a holomorphic function in a neighbourhood of the origin of $\mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}_u \times \mathbf{C}_\xi$ satisfying

$$f(0,x,0,0) \equiv 0, \qquad \text{near} \quad x = 0. \tag{1.2}$$

By the condition (1.2), $f(t,x,u,\xi)$ is written as follows:

$$f(t,x,u,\xi) = \alpha(x)t + \beta(x)u + \gamma(x)\xi + \sum_{p+q+r\geq 2} f_{pqr}(x)t^p u^q \xi^r,$$

where $\alpha(x)$, $\beta(x)$ and $\gamma(x)$ are holomorphic functions, and $p$, $q$, $r \in \mathbf{Z}_{\geq 0} = \{0,1,2,\ldots\}$.

If $\gamma(x) \equiv 0$, equation is called of nonlinear Fuchsian type (or of Briot-Bouquet type). In this case, many mathematicians studied the various theories. For example, convergence of formal solutions ([GT (Chapters 3,5)]), the Maillet type theorem ([GT (Chapter 6)], [S]), asymptotic expansions ([O]), singular solutions ([GT (Chapters 4,5)]).

If $\gamma(x) \not\equiv 0$ and $\gamma(0) \neq 0$, we can see that the equation is solvable in $\partial u/\partial x$. Therefore, we have a unique holomorphic solution with arbitrary holomorphic initial data

$u(t,0)=\varphi(t)$ satisfying $\varphi(0)=0$ by Cauchy-Kowalevski's theorem, where $u(0,x)\equiv 0$ is automatically satisfied.

In the other case, that is, $\gamma(x)\not\equiv 0$ and $\gamma(0)=0$, the equation is called of totally characteristic type. If $\gamma(x)=xc(x)$, $c(0)\neq 0$, Chen-Tahara obtained the conditions for the formal solution to converge ([CT]). This result was generalized to several space variables by Chen-Luo ([CL]) in the case where $b_k(x)=\mu_k x_k+$ higher order (see (1.4) below), but $t$ variable is still restricted to be one dimensiomal.

If $\gamma(x)=x^j c(x),\ c(0)\neq 0,\ j\geq 2$, Chen-Luo-Tahara proved the Maillet type theorem, that is, they gave the formal Gevrey class in which the formal solution belongs ([CLT]).

In [CT], Chen-Tahara obtained the following result:

**Theorem (Chen-Tahara)** *Assume (1.2) and that $\gamma(x)=xc(x)$ with $c(0)\neq 0$. Then, if*

(1.3) $|l-\beta(0)-c(0)m|\geq \sigma(m+1)$ *for any* $(l,m)\in \mathbf{N}\times \mathbf{Z}_{\geq 0}$, $(\mathbf{N}=\{1,2,\ldots\})$

*holds for some $\sigma>0$, then the equation (1.1) has a unique holomorphic solution.*

In this paper, we consider a generalization of this Chen-Tahara's theorem to the case of several time-space variables.

Let $(t,x)=(t_1,\ldots,t_d,x_1,\ldots,x_n)\in \mathbf{C}^d\times\mathbf{C}^n$ be $(d+n)$-dimensional complex variables $(d\geq 1,\ n\geq 1)$. The following equation seems to be a natural extension of (1.1) to several time-space variables:

$$(1.4)\qquad \begin{cases} \displaystyle\sum_{i,j=1}^{d} a_{ij}(x)t_i\partial_{t_j}u+\sum_{k=1}^{n} b_k(x)\partial_{x_k}u+c(x)u \\ \displaystyle\qquad\qquad =\sum_{|l|=K} d_l(x)t^l+f_{K+1}(t,x,u,\{\partial_{t_j}u\},\{\partial_{x_k}u\}), \\ u(t,x)=O(|t|^K), \end{cases}$$

where $|t|=t_1+\cdots+t_d$, $K$ is a fixed positive integer satisfying $K\geq 2$ and $a_{ij}(x)$, $b_k(x)$, $c(x)$ and $d_l(x)$ are holomorphic in a neighbourhood of the origin, and $f_{K+1}(t,x,u,\tau,\xi)$ $(\tau=(\tau_j)\in\mathbf{C}^d,\ \xi=(\xi_k)\in\mathbf{C}^n)$ is also holomorphic in a neighbourhood of the origin with the following Taylor expansion:

$$f_{K+1}(t,x,u,\tau,\xi)=\sum_{|p|+Kq+(K-1)|r|+K|s|\geq K+1} f_{pqrs}(x)t^p u^q \tau^r \xi^s,$$

where $q\geq 0$, $p=(p_1,\ldots,p_d)\in(\mathbf{Z}_{\geq 0})^d$, $r=(r_1,\ldots,r_d)\in(\mathbf{Z}_{\geq 0})^d$, $s=(s_1,\ldots,s_n)\in(\mathbf{Z}_{\geq 0})^n$,

$$|p|=p_1+\cdots+p_d,\quad |r|=r_1+\cdots+r_d,\quad |s|=s_1+\cdots+s_n,$$

and

$$t^p=\prod_{j=1}^d {t_j}^{p_j},\quad \tau^r=\prod_{j=1}^d {\tau_j}^{r_j},\quad \xi^s=\prod_{k=1}^n {\xi_k}^{s_k}.$$

Here we remark that the assumption $K\geq 2$ implies $\partial_{t_j}u(0,0)=0$ $(j=1,2,\ldots,d)$ which assures that $(0,0,u(0,0),\{\partial_{t_j}u(0,0)\},\{\partial_{x_k}u(0,0)\})$ belongs to the domain of definition of $f_{K+1}(t,x,u,\tau,\xi)$.

Now our first theorem is stated as follows:

**Theorem 1** *Let $\{\lambda_j\}_{j=1}^d$ be the eigenvalues of the matrix $(a_{ij}(0))$. We assume that $b_k(x)\not\equiv 0$ and $b_k(0)=0$ for $k=1,2,\ldots,n$, and let $\{\mu_k\}_{k=1}^n$ be the eigenvalues of Jacobi matrix of $(b_1(x),\ldots,b_n(x))$ at $x=0$. Then the formal power series solution of (1.4) exists uniquely and converges if the following conditions are satisfied:*

*There exists a positive constant $\sigma_0>0$, such that*

$$\left|\sum_{j=1}^d \lambda_j l_j+\sum_{k=1}^n \mu_k m_k\right|\geq \sigma_0(|l|+|m|) \qquad \text{(Poincaré condition)}, \tag{1.5}$$

*and*

$$\sum_{j=1}^d \lambda_j l_j+\sum_{k=1}^n \mu_k m_k+c(0)\neq 0 \qquad \text{(Non-resonance condition)} \tag{1.6}$$

*hold for all $(l,m)\in(\mathbf{Z}_{\geq 0})^d\times(\mathbf{Z}_{\geq 0})^n$ with $|l|\geq K$ and $|m|\geq 0$.*

**Remark 1** It is easy to show the following proposition.

The conditions (1.5) and (1.6) imply that

$$\left|\sum_{j=1}^d \lambda_j l_j+\sum_{k=1}^n \mu_k m_k+c(0)\right|\geq \sigma(|l|+|m|) \tag{1.7}$$

holds by some positive constant $\sigma > 0$ for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$. In the proof of Theorem 1, this condition will be used instead of (1.5) and (1.6). ■

**Remark 2** The condition (1.7) seems to be stronger than the condition that

$$(1.8) \qquad \left| \sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{n} \mu_k m_k + c(0) \right| \geq \sigma'(|m| + 1)$$

holds by some positive constant $\sigma' > 0$ for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq K$ and $|m| \geq 0$, which is a direct generalization of Chen-Tahara's condition (1.3). However it is actually proved that (1.7) and (1.8) are equivalent. The proof can be seen in [CL]. ■

Next, we consider the following general equation:

$$(1.9) \qquad \begin{cases} f(t, x, u(t, x), \{\partial_{t_j} u(t, x)\}, \{\partial_{x_k} u(t, x)\}) = 0, \\ u(0, x) \equiv 0. \end{cases}$$

**Assumption 1** $f(t, x, u, \tau, \xi)$ $(\tau = (\tau_j) \in \mathbf{C}^d,\ \xi = (\xi_k) \in \mathbf{C}^n)$ is holomorphic in a neighbourhood of the origin, and is an entire function in $\tau$ variables for any fixed $t$, $x$, $u$ and $\xi$. Moreover we assume that

$$(1.10) \qquad f(0, x, 0, \tau, 0) \equiv 0$$

for $x \in \mathbf{C}^n$ near the origin and $\tau \in \mathbf{C}^d$, which is a generalization of the definition of singular equations defined in [MS].

For the equation (1.9), we do not know whether or not the equation has a formal solution in general. Therefore, we assume the following:

**Assumption 2** The equation (1.9) has a formal solution of the form

$$(1.11) \qquad u(t, x) = \sum_{j=1}^{d} \varphi_j(x) t_j + \sum_{|l| \geq 2, |m| \geq 0} u_{lm} t^l x^m \in \mathbf{C}[[t, x]].$$

By the existence of a formal solution, $\{\varphi_j(x)\}$ satisfy the following system formally:

$$(1.12) \qquad f(0, x, 0, \{\varphi_j(x)\}, 0) \equiv 0 \quad \text{(trivial relation)},$$

and

$$\frac{\partial}{\partial t_i} f(t,x,u(t,x),\{\partial_{t_j}u(t,x)\},\{\partial_{x_k}u(t,x)\})\Big|_{t=0} \tag{1.13}$$
$$= \frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0) + \frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x)$$
$$+ \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial \varphi_i}{\partial x_k}(x) = 0, \qquad \text{for } i = 1,2,\ldots,d.$$

The formal solution of this system is not convergent in general. Therefore, we assume

**Assumption 3** The coefficients $\{\varphi_j(x)\}$ are all holomorphic in a neighbourhood of the origin of $\mathbf{C}^n$.

**Remark 3** In the case $d = 1$ ($d$ is the dimension of $t$ variables), a sufficient condition for the formal solution of (1.13) to converge has been already obtained by Miyake-Shirai ([MS]). In the case $d \geq 2$, we give a sufficient condition for the formal solution of system (1.13) to be convergent, which will be given by Theorem 3 in Section 6, but for a while we consider the problem under Assumption 3 for simplicity of our arguments. ■

Now we put $\mathbf{a}(x) = (0,x,0,\{\varphi_j(x)\},0)$ for simplicity, and define

$$A_{ij}(x) := \frac{\partial^2 f}{\partial t_i \partial \tau_j}(\mathbf{a}(x)) + \frac{\partial^2 f}{\partial u \partial \tau_j}(\mathbf{a}(x))\varphi_i(x) + \sum_{k=1}^{n} \frac{\partial^2 f}{\partial \tau_j \partial \xi_k}(\mathbf{a}(x))\frac{\partial \varphi_i}{\partial x_k}(x), \tag{1.14}$$

for $i, j = 1,2,\ldots,d$. Moreover we define

$$B_k(x) := \frac{\partial f}{\partial \xi_k}(\mathbf{a}(x)), \quad \text{for } k = 1,2,\ldots,n. \tag{1.15}$$

**Remark 4** The functions $A_{ij}(x)$ and $B_k(x)$ correspond to $a_{ij}(x)$ and $b_k(x)$ in Theorem 1, respectively (see (1.17) below). ■

Here we assume that the equation is of totally characteristic type, that is,

**Assumption 4** $B_k(x) \not\equiv 0$ and $B_k(0) = 0$, for $k = 1,2,\ldots,n$.

Now our second theorem which is our main result is stated as follows:

**Theorem 2** *Suppose Assumptions 1, 2, 3 and 4. Let* $\{\lambda_j\}_{j=1}^d$ *be the eigenvalues of* $(A_{ij}(0))$*, and let* $\{\mu_k\}_{k=1}^n$ *be the eigenvalues of Jacobi matrix of the vector* $(B_k(x))$ *at* $x = 0$*. Then the formal solution (1.11) is convergent if the following condition is satisfied:*

*There exists a positive constant* $\sigma_0 > 0$*, such that,*

$$\left|\sum_{j=1}^d \lambda_j l_j + \sum_{k=1}^n \mu_k m_k\right| \geq \sigma_0(|l| + |m|), \qquad (\textit{Poincaré condition}) \tag{1.16}$$

*holds for all* $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ *with* $|l| \geq 2,\ |m| \geq 0$.

**Remark 5** Under the assumptions of Theorem 2, if the following non-resonance condition

$$\sum_{j=1}^d \lambda_j l_j + \sum_{k=1}^n \mu_k m_k + \frac{\partial f}{\partial u}(\mathbf{a}(0)) \neq 0$$

holds for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq 2,\ |m| \geq 0$ as an additional condition, then the formal power series solution exists uniquely, after a determination of $\varphi_j(x)$ $(j = 1, 2, \ldots, d)$.

**Remark 6** By the Poincaré condition, there exists a positive integer $K$ such that the non-resonance condition

$$\sum_{j=1}^d \lambda_j l_j + \sum_{k=1}^n \mu_k m_k + \frac{\partial f}{\partial u}(\mathbf{a}(0)) \neq 0$$

holds for all $(l, m) \in (\mathbf{Z}_{\geq 0})^d \times (\mathbf{Z}_{\geq 0})^n$ with $|l| \geq K,\ |m| \geq 0$.

We put $v(t, x) = u(t, x) - \sum_{j=1}^d \varphi_j(x) t_j - \sum_{2 \leq |l| \leq K-1} u_l(x) t^l$ as a new unknown function. By Assumptions 1, 2, 3 and 4, we can see that the coefficients $\{u_l(x)\}_{2 \leq |l| \leq K-1}$ are determined as holomorphic functions which will be proved in Appendix B. Moreover, $v(t, x)$ satisfies the equation of the following form:

$$\begin{cases} \displaystyle\sum_{i,j=1}^d A_{ij}(x) t_i \partial_{t_j} v + \sum_{k=1}^n B_k(x) \partial_{x_k} v + \frac{\partial f}{\partial u}(\mathbf{a}(x)) v \\ \displaystyle\qquad\qquad = \sum_{|l|=K} d_l(x) t^l + f_{K+1}(t, x, v, \{\partial_{t_j} v\}, \{\partial_{x_k} v\}), \\ v(t, x) = O(|t|^K). \end{cases} \tag{1.17}$$

This is an equation considered in Theorem 1. Therefore, it is sufficient to prove Theorem 1 in order to prove Theorem 2. ■

## 2 Example

In this section, we give an example.

Let us consider the following equation:

$$(2.1)\qquad \begin{cases} 9uu_t + 9xu_x - (1+x)t - xt^2 = 0, \\ u(0,x) \equiv 0, \end{cases}$$

where $(t, x) \in \mathbf{C}^2$, $u = u(t, x)$ is the unknown function, $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$.

We put $u = \varphi(x)t + v(t, x)$ $(v(t, x) = O(t^2))$ as a formal solution. We see that $\varphi(x)$ satisfies the following equation:

$$(2.2)\qquad 9\varphi(x)^2 + 9x\varphi'(x) - 1 - x = 0,$$

where $\varphi'(x) = (d\varphi/dx)(x)$.

We put $\varphi(x) = p + \psi(x)$ ($p$ is a constant and $\psi(x) = O(x)$ with $\psi(x) \in \mathbf{C}[[x]]$). By substituting this into (2.2) and by calculating the constant terms of (2.2), we have $p = \pm 1/3$.

By an easy calculation, we see that $\psi(x)$ satisfies the following equation,

$$\begin{cases} 9x\psi'(x) + 6\psi(x) = x - 9\psi(x)^2, \\ \psi(0) = 0, \end{cases} \quad \text{if } p = 1/3,$$
$$\begin{cases} 9x\psi'(x) - 6\psi(x) = x - 9\psi(x)^2, \\ \psi(0) = 0, \end{cases} \quad \text{if } p = -1/3.$$

We can easily prove that the formal power series solution $\psi(x)$ exists uniquely and converges in a neighbourhood of the origin in each case, because they are nonlinear equations of regular singular type studied by Gérard-Tahara ([GT]).

We put $f(t, x, u, \tau, \xi) = 9u\tau + 9x\xi - (1+x)t - xt^2$. In order to apply Theorem 2, we introduce functions $A(x)$ and $B(x)$ as follows:

$$\begin{aligned} A(x) &= \frac{\partial^2 f}{\partial t \partial \tau}(0, x, 0, \varphi(x), 0) \\ &\quad + \frac{\partial^2 f}{\partial u \partial \tau}(0, x, 0, \varphi(x), 0)\varphi(x) + \frac{\partial^2 f}{\partial \tau \partial \xi}(0, x, 0, \varphi(x), 0)\varphi'(x) \\ &= 9\varphi(x), \end{aligned}$$

and

$$B(x) = \frac{\partial f}{\partial \xi}(0, x, 0, \varphi(x), 0) = 9x.$$

In the case $p = 1/3$, $A(0)$ and $B'(0)$, which correspond to the eigenvalues $\{\lambda_j\}$ and $\{\mu_k\}$ of the matrices $(A_{ij}(0))$ and $\frac{\partial(B_1,\dots,B_n)}{\partial(x_1,\dots,x_n)}|_{x=0}$ respectively, are given by

$$A(0) = 3, \quad B'(0) = 9.$$

This implies that the Poincaré condition is satisfied. Moreover, since $(\partial f/\partial u)(0,0,0,\varphi(0),0) = 3$, the non-resonance condition is also satisfied. Therefore, by applying Theorem 2 we can see that the formal solution $u(t,x)$ is holomorphic in a neighbourhood of the origin.

In the case $p = -1/3$, $A(0)$ and $B'(0)$ are given by

$$A(0) = -3, \quad B'(0) = 9.$$

Moreover, $(\partial f/\partial u)(0,0,0,\varphi(0),0) = -3$ holds. In this case, the resonance does occur, and therefore, the formal power series solution does not exist.

# 3 Reduction of the Equation

As is mentioned in Remark 6, it is sufficient to study the equation (1.4).

By the assumption of Theorem 1,

$$(a_{ij}(0)) \sim \begin{pmatrix} \lambda_1 & \delta_1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \delta_{d-1} \\ & & & \lambda_d \end{pmatrix}, \quad \left.\frac{\partial(b_1,\dots,b_n)}{\partial(x_1,\dots,x_n)}\right|_{x=0} \sim \begin{pmatrix} \mu_1 & \nu_1 & & \\ & \mu_2 & \ddots & \\ & & \ddots & \nu_{n-1} \\ & & & \mu_n \end{pmatrix},$$

where $\delta_j,\ \nu_k = 0$ or $1$ $(1 \le j \le d-1, 1 \le k \le n-1)$.

Then by transforming the variables, (1.4) is reduced to the following form:

$$\text{(3.1)} \qquad \begin{aligned} (\Lambda + \Delta)v(t,x) &= \sum_{|l|=K} \alpha_l(x)t^l + \sum_{i,j=1}^{d} \beta_{ij}(x)t_i\partial_{t_j}v + \gamma(x)v \\ &\quad + \sum_{k=1}^{n} \varphi_k(x)\partial_{x_k}v + \widetilde{f}_{K+1}(t,x,v,\{\partial_{t_j}v\},\{\partial_{x_k}v\}), \end{aligned}$$

with $v(t,x) = O(|t|^K)$, where

$$\Lambda = \sum_{j=1}^{d} \lambda_j t_j \partial_{t_j} + \sum_{k=1}^{n} \mu_k x_k \partial_{x_k} + c(0),$$

$$\Delta = \sum_{j=1}^{d-1} \delta_j t_j \partial_{t_{j+1}} + \sum_{k=1}^{n-1} \nu_k x_k \partial_{x_{k+1}},$$

and $\alpha_l(x)$, $\beta_{ij}(x)$, $\gamma(x)$ and $\varphi_k(x)$ are holomorphic in a neighbourhood of the origin, and satisfy $\beta_{ij}(x) = O(|x|)$, $\gamma(x) = O(|x|)$ and $\varphi_k(x) = O(|x|^2)$, and $\widetilde{f}_{K+1}(t, x, u, \tau, \xi)$ is a holomorphic function which has a similar Taylor expansion with $f_{K+1}(t, x, u, \tau, \xi)$.

In the following sections, we shall prove the existence and convergence of the unique formal solution of (3.1).

# 4 Preparation to prove Theorem 1

Let $\mathbf{C}[t, x]_{L,M}$ be the set of homogeneous polynomials of degree $L$ in $t$ variables and of degree $M$ in $x$ variables, that is,

$$\mathbf{C}[t, x]_{L,M} = \left\{ f_{LM}(t, x) = \sum_{|l|=L,\ |m|=M} f_{lm} t^l x^m \,\middle|\, f_{lm} \in \mathbf{C} \right\}.$$

For the operator $\Lambda + \Delta$, the following lemma holds:

**Lemma 1** *For all $L \geq K$ and $M \geq 0$, the operator*

$$\Lambda + \Delta : \ \mathbf{C}[t, x]_{L,M} \longrightarrow \mathbf{C}[t, x]_{L,M}$$

*is invertible. Moreover, if the majorant relation $f_{LM}(t, x) \ll F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M$ ($f_{LM}(x) \in \mathbf{C}[t, x]_{L,M},\ F > 0$) holds, then we obtain the following majorant relation:*

$$(4.1) \qquad (\Lambda + \Delta)^{-1} f_{LM}(t, x) \ll \frac{C}{L + M} F \times (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M,$$

*where $C > 0$ is a positive constant independent of $L$ and $M$.*

*Proof.* We define a norm of $u_{LM}(t, x) \in \mathbf{C}[t, x]_{L,M}$ by

$$||u_{LM}|| := \inf\Big\{ C > 0 \mid u_{LM}(t, x) \ll C (t_1 + \cdots + t_d)^L (x_1 + \cdots + x_n)^M \Big\}.$$

We remark that $\mathbf{C}[t, x]_{L,M}$ becomes a Banach space by this norm.

First, by (1.7) it is easily checked that $\Lambda$ is invertible on $\mathbf{C}[t, x]_{L,M}$ and

$$(4.2) \qquad ||\Lambda^{-1}|| \leq \frac{1}{\sigma(L + M)}$$

holds for the operator norm of $\Lambda^{-1}$ on $\mathbf{C}[t,x]_{L,M}$.

Next, since $u_{LM}(t,x) \ll ||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M$, we have

$$\begin{aligned}\Delta u_{LM}(t,x) &\ll \sum_{j=1}^{d-1} L|\delta_j|\cdot||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M\\ &\quad+\sum_{k=1}^{n-1} M|\nu_k|\cdot||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M\\ &\ll \left\{L(d-1)\max_{j=1,\dots,d-1}|\delta_j|+M(n-1)\max_{k=1,\dots,n-1}|\nu_k|\right\}\times\\ &\quad\times||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M.\end{aligned}$$

Here we make a change of variables by $t_j=\varepsilon^{j-1}\tau_j$, $x_k=\varepsilon^{k-1}y_k$, then $\delta_j$ and $\nu_k$ (the components of nilpotent part of Jordan canonical form) turns to $\varepsilon\delta_j$ and $\varepsilon\nu_k$, respectively. Therefore, by choosing $\varepsilon$ sufficiently small, we may assume that the components of nilpotent part are small enough. Hence we may assume that

$$\max_{j=1,\dots,d-1}|\delta_j|<\frac{\sigma}{2(d-1)},\qquad \max_{k=1,\dots,n-1}|\nu_k|<\frac{\sigma}{2(n-1)}. \tag{4.3}$$

Then

$$\Delta u_{LM}(t,x)\ll\frac{\sigma(L+M)}{2}\,||u_{LM}||(t_1+\cdots+t_d)^L(x_1+\cdots+x_n)^M$$

holds, and we obtain

$$||\Delta||\le\frac{\sigma(L+M)}{2}.$$

Therefore, the operator norm of $\Delta\Lambda^{-1}$ is estimated by

$$||\Delta\Lambda^{-1}||\le\frac{1}{\sigma(L+M)}\frac{\sigma(L+M)}{2}=\frac{1}{2}<1.$$

By using the Neumann's series, we can see that $\Lambda+\Delta$ is invertible and the norm of the inverse operator is estimated by

$$||(\Lambda+\Delta)^{-1}||\le\frac{2}{\sigma}\frac{1}{L+M},$$

which we want to prove since $C=2/\sigma$ is independent of $L$ and $M$. ■

Now, we define some notations which are used in the proof of Theorem 1.

**Definition** (1) Let $(t,x)\in\mathbf{C}^d\times\mathbf{C}^n$ $(d\ge 0,\ n\ge 0)$ be complex variables. For formal power series $f(t,x)=\sum_{|l|\geq 0,\ |m|\geq 0} f_{l,m}t^l x^m$, we define

$$|f|(t,x)=\sum_{|l|\geq 0,\ |m|\geq 0} |f_{l,m}|t^l x^m.$$

(2) Let $(t,X)\in \mathbf{C}^d\times\mathbf{C}$ $(d\geq 0)$ be complex variables. For formal power series $g(t,X)=\sum_{|l|\geq 0,\ M\geq 0} g_{l,M}t^l X^M$, we define the shift operator $S$ by

$$S(g)(t,X)=\sum_{|l|\geq 0,\ M\geq 0} g_{l,M+1}t^l X^M=\frac{g(t,X)-g(t,0)}{X}.$$

**Remark 7** The following facts are easily shown:

- $f(t,x)\ll |f|(t,x)$;
- If $f(t,x)$ and $g(t,X)$ are convergent power series, then $|f|(t,x)$ and $S(g)(t,X)$ are also convergent.

■

# 5 Proof of Theorem 1

## 5.1 Existence of the Formal Solution

Let

$$u(t,x)=\sum_{|l|\geq K,|m|\geq 0} u_{lm}t^l x^m=\sum_{L\geq K} u_L(t,x)=\sum_{L\geq K,M\geq 0} u_{LM}(t,x)$$

be a formal solution of (3.1), where

$$u_{LM}(t,x)=\sum_{|l|=L,|m|=M} u_{lm}t^l x^m\in\mathbf{C}[t,x]_{L,M},$$

$$u_L(t,x)=\sum_{|l|=L} u_l(x)t^l=\sum_{M\geq 0} u_{LM}(t,x).$$

We put $P=\Lambda+\Delta$ for simplicity. We substitute $u(t,x)=\sum_{L\geq K} u_L(t,x)$ into (3.1), then we have the following recursion formula:

$$
\begin{cases}
Pu_K(t,x) = \displaystyle\sum_{|l|=K} \alpha_l(x)t^l + \sum_{i,j=1}^{d} \beta_{ij}(x)t_i\partial_{t_j}u_K(t,x) \\
\qquad\qquad +\gamma(x)u_K(t,x) + \displaystyle\sum_{k=1}^{n} \varphi_k(x)\partial_{x_k}u_K(t,x), \\
Pu_L(t,x) = \displaystyle\sum_{i,j=1}^{d} \beta_{ij}(x)t_i\partial_{t_j}u_L(t,x) + \gamma(x)u_L(t,x) + \sum_{k=1}^{n} \varphi_k(x)\partial_{x_k}u_L(t,x) \\
\qquad\qquad +G_L(t,x,\{u_p\}_{K\le p<L}, \{\partial_{t_j}u_p\}_{K\le p<L}, \{\partial_{x_k}u_p\}_{K\le p<L}), \text{ for } L > K,
\end{cases}
$$

where $G_L(t,x,\zeta,\tau,\xi)$ is a polynomial of $(t,\zeta,\tau,\xi)$.

First, we consider the case $L = K$. We substitute $u_K(t,x) = \sum_{M\ge 0} u_{KM}(t,x)$ into the above recursion formula, we have

$$
\begin{cases}
Pu_{K0}(t,x) = \displaystyle\sum_{|l|=K} \alpha_l(0)t^l, \\
Pu_{KM}(t,x) = \displaystyle\sum_{|l|=K} \alpha_l^M(x)t^l + \sum_{i,j=1}^{d}\sum_{p=1}^{M} \beta_{ij}^p(x)t_i\partial_{t_j}u_{K,M-p}(t,x) \\
\qquad\qquad + \displaystyle\sum_{p=1}^{M} \gamma^p(x)u_{K,M-p}(t,x) + \sum_{k=1}^{n}\sum_{p=2}^{M} \varphi_k^p(x)\partial_{x_k}u_{K,M-p+1}(t,x),
\end{cases}
$$

where we put

$$
\begin{aligned}
\alpha_l(x) &= \sum_{M\ge 0} \alpha_l^M(x), \quad \alpha_l^M(x) = \sum_{|m|=M} \alpha_{lm}x^m, \\
\beta_{ij}(x) &= \sum_{M\ge 1} \beta_{ij}^M(x), \quad \beta_{ij}^M(x) = \sum_{|m|=M} \beta_{ijm}x^m, \\
\gamma(x) &= \sum_{M\ge 1} \gamma^M(x), \quad \gamma^M(x) = \sum_{|m|=M} \gamma_m x^m, \\
\varphi_k(x) &= \sum_{M\ge 2} \varphi_k^M(x), \quad \varphi_k^M(x) = \sum_{|m|=M} \varphi_{km}x^m.
\end{aligned}
$$

By Lemma 1, we know that the solution sequence $\{u_{KM}(t,x)\}_{M\ge 0}$ exists uniquely. Moreover, by the same argument, we see that $\{u_{LM}(t,x)\}$ $(L > K)$ exist uniquely. These show that the formal solution exists uniquely.

## 5.2 Convergence of the Formal Solution

We put $U(t,x) = Pu(t,x)$ as a new unknown function. By Lemma 1, the equation (3.1) is reduced to the following equation:

$$(5.1) \qquad \begin{aligned} U(t,x) &= \sum_{|l|=K} \alpha_l(x)t^l + \sum_{i,j=1}^{d} \beta_{ij}(x)t_i\partial_{t_j}P^{-1}U(t,x) \\ &\quad +\gamma(x)P^{-1}U(t,x) + \sum_{k=1}^{n} \varphi_k(x)\partial_{x_k}P^{-1}U(t,x) \\ &\quad +\widetilde{f}_{K+1}(t,x,P^{-1}U(t,x),\{\partial_{t_j}P^{-1}U(t,x)\},\{\partial_{x_k}P^{-1}U(t,x)\}). \end{aligned}$$

We know that (5.1) has a unique formal solution of the form

$$U(t,x) = \sum_{|l|\geq K, |m|\geq 0} U_{lm}t^l x^m = \sum_{L\geq K} U_L(t,x) = \sum_{L\geq K, M\geq 0} U_{LM}(t,x).$$

In order to get a majorant series of $U(t,x)$, we prepare some majorant series for the coefficients of (5.1). We put $T = t_1 + \cdots + t_d$, $\quad X = x_1 + \cdots + x_n$, and choose

$$\sum_{|l|=K} \alpha_l(x)t^l \ll A(X)T^K, \quad \beta_{ij}(x) \ll |\beta_{ij}|(X,\ldots,X) =: XB_{ij}(X),$$

$$\gamma(x) \ll |\gamma|(X,\ldots,X) =: XG(X), \quad \varphi_k(x) \ll |\varphi_k|(X,\ldots,X) =: X^2\Phi_k(X),$$

$$\begin{aligned} \widetilde{f}_{K+1}(t,x,u,\tau,\xi) &\ll |\widetilde{f}_{K+1}|(T,\ldots,T,X,\ldots,X,u,\tau,\xi) \\ &=: F_{K+1}(T,X,u,\tau,\xi) \\ &= \sum_{|p|+Kq+(K-1)|r|+K|s|\geq K+1} F_{pqrs}(X)T^{|p|}u^q\tau^r\xi^s, \end{aligned}$$

where $A(X)$, $B_{ij}(X)$, $G(X)$ and $\Phi_k(X)$ are holomorphic in a neighbourhood of $X=0$, and $F_{K+1}(T,X,u,\tau,\xi)$ is also holomorphic near $(T,X,u,\tau,\xi) = (0,0,0,0,0)$.

Now, we consider the following equation:

$$(5.2) \qquad \begin{aligned} w(T,X) &= A(X)T^K + C\sum_{i.j=1}^{d} XB_{ij}(X)w(T,X) \\ &\quad +CXG(X)w(T,X) + C\sum_{k=1}^{n} X^2\Phi_k(x)(t,x)S(w)(T,X) \\ &\quad +F_{K+1}\left(T,X,Cw,\left\{\frac{Cw}{T}\right\},\{CS(w)\}\right), \end{aligned}$$

where $C$ is a positive constant appeared in Lemma 1.

Let $w(T,X) = \sum_{L\geq K, M\geq 0} w_{LM}(T,X)$ be the formal solution of (5.2). By the construction of (5.2), we can easily check that $U(t,x) \ll w(T,X)$ by the next lemma.

**Lemma 2** *For two formal power series $U(t,x)$ and $w(T,X)$ satisfying*

$$U(t,x) = \sum_{L\geq K, M\geq 0} U_{LM}(t,x) \ll w(T,X) = \sum_{L\geq K, M\geq 0} w_{LM} T^L X^M,$$

*the following majorant relations hold:*

(1) $P^{-1}U(t,x) \ll Cw(T,X)$,

(2) $t_i \partial_{t_j} P^{-1}U(t,x) \ll Cw(T,X)$,

(3) $\partial_{t_j} P^{-1}U(t,x) \ll \dfrac{Cw(T,X)}{T}$,

(4) $\partial_{x_k} P^{-1}U(t,x) \ll CS(w)(T,X)$.

*Proof.* By using Lemma 1, we can easily prove this lemma. First, (1) is proved as follows:

$$P^{-1}U(t,x) = \sum_{L\geq K, M\geq 0} P^{-1}U_{LM}(t,x) \ll \sum_{L\geq K, M\geq 0} \frac{C}{L+M}\, w_{LM} T^L X^M \ll Cw(T,X).$$

Secondly, (2) and (3) are proved as follows:

$$\begin{aligned} t_i \partial_{t_j} P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} t_i \partial_{t_j} P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 0} \frac{CL}{L+M}\, w_{LM} T^L X^M \ll Cw(T,X); \end{aligned}$$

$$\begin{aligned} \partial_{t_j} P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} \partial_{t_j} P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 0} \frac{CL}{L+M}\, w_{LM} T^{L-1} X^M \ll \frac{Cw(T,X)}{T}. \end{aligned}$$

Finally, (4) is proved as follows:

$$\begin{aligned} \partial_{x_k} P^{-1}U(t,x) &= \sum_{L\geq K, M\geq 0} \partial_{x_k} P^{-1}U_{LM}(t,x) \\ &\ll \sum_{L\geq K, M\geq 1} \frac{CM}{L+M}\, w_{LM} T^L X^{M-1} \ll CS(w)(T,X). \end{aligned}$$

This completes the proof. ■

Since $w(T,X) \gg 0$, we have

$$XS(w)(T,X) = w(T,X) - w(T,0) \ll w(T,X). \tag{5.3}$$

Let us consider the following equation:

$$\begin{aligned} v(T,X) &= A(X)T^K + CXh(X)v(T,X) \\ &\quad + F_{K+1}\left(T, X, Cv, \left\{\frac{Cv}{T}\right\}, \{CS(v)\}\right), \end{aligned} \tag{5.4}$$

with $v(T,X) = O(T^K)$, where $h(X) = \sum_{i,j=1}^{d} B_{ij}(X) + G(X) + \sum_{k=1}^{n} \Phi_k(X)$. Then the following majorant relation is obvious:

$$w(T,X) \ll v(T,X).$$

We put $y(T,X) = v(T,X)/T$ as a new unknown function. By substituting this into (5.4), we see that $y(T,X)$ satisfies

$$\begin{aligned} y(T,X) &= A(X)T^{K-1} + CXh(X)y(T,X) \\ &\quad + \frac{1}{T} F_{K+1}\left(T, X, CTy, \{Cy\}, \{CTS(y)\}\right), \end{aligned} \tag{5.5}$$

with $y(T,X) = O(T^{K-1})$.

We decompose the formal solution $y(T,X)$ as follows:

$$y(T,X) = y_1(X)T^{K-1} + y_2(X)T^K + T^K z(T,X).$$

We remark that $y_1(X)$ and $y_2(X)$ are holomorphic functions in a neighbourhood of $X = 0$. Indeed, $y_1(X)$ and $y_2(X)$ are given by

$$y_1(X) = \frac{A(X)}{1 - CXh(X)},$$

$$y_2(X) = \frac{1}{1 - CXh(X)} \sum_{|p| + Kq + (K-1)|r| + K|s| = K+1} F_{pqrs}(X) \{Cy_1(X)\}^{q+|r|} \{CS(y_1)(X)\}^{|s|}.$$

These are holomorphic functions in a neighbourhood of $X = 0$.

In this case, $z(T,X)$ satisfies the following equation:

$$\begin{cases} z(T,X) = CXh(X)z(T,X) + H(T, X, Tz(T,X), TS(z)(T,X)), \\ z(0,X) \equiv 0, \end{cases} \tag{5.6}$$

where

$$\begin{aligned} H(T,X,\eta_1,\eta_2) &= \frac{1}{T^{K+1}}\big[F_{K+1}(T,X,Cy_1(X)T^K+Cy_2(X)T^{K+1}+CT^K\eta_1, \\ &\quad \{Cy_1(X)T^{K-1}+Cy_2(X)T^K+CT^{K-1}\eta_1\}, \\ &\quad \{CS(y_1)(X)T^K+CS(y_2)(X)T^{K+1}+CT^K\eta_2\})\big] \\ &\quad - \sum_{|p|+Kq+(K-1)|r|+K|s|=K+1} F_{pqrs}(X)(Cy_1(X))^{q+|r|}(CS(y_1)(X))^{|s|}. \end{aligned}$$

**Remark 8** The order of zeros in $T$ variable of $H(T,X,CTz(T,X),CTS(z)(T,X))$ is greater than or equal to 1. ■

In order to prove the convergence of $z(T,X)$, it is sufficient to show the following:

**Lemma 3** *There exists a small positive constant $\varepsilon > 0$ such that $z_\varepsilon(\rho) = z(\varepsilon\rho,\rho)$ is convergent in a neighbourhood of $\rho = 0$.*

*Proof.* We substitute $T = \varepsilon\rho$ and $X = \rho$ into (5.6) and by using the relation (5.3), we have

$$\rho S(z)(\varepsilon\rho,\rho) \ll z_\varepsilon(\rho).$$

By this relation, the following majorant relation also holds,

$$TS(z)(T,X)|_{T=\varepsilon\rho,X=\rho} = \varepsilon\rho S(z)(\varepsilon\rho,\rho) \ll \varepsilon z_\varepsilon(\rho).$$

Here we consider

$$\psi(\rho) = C\rho h(\rho)\psi(\rho) + H(\varepsilon\rho,\rho,\varepsilon\rho\psi(\rho),\varepsilon\psi(\rho)). \tag{5.7}$$

In the right hand side of (5.7), we decompose $H(\varepsilon\rho,\rho,\varepsilon\rho\psi(\rho),\varepsilon\psi(\rho))$ into the term of $\psi(\rho)$ and otherwise as follows:

$$H(\varepsilon\rho,\rho,\varepsilon\rho\psi(\rho),\varepsilon\psi(\rho)) = \varepsilon\frac{\partial H}{\partial\eta_2}(0,0,0,0)\psi(\rho) + \widetilde{H}(\varepsilon\rho,\rho,\varepsilon\rho\psi(\rho),\varepsilon\psi(\rho)).$$

We remark that the following fact holds:

$$\left.\frac{\partial\widetilde{H}}{\partial\psi}(\varepsilon\rho,\rho,\varepsilon\rho\psi,\varepsilon\psi)\right|_{\rho=0,\psi=0} = 0.$$

We put $(\partial H/\partial\eta_2)(0,0,0,0) = K_0 \geq 0$, then (5.7) is rewritten by

$$(1-\varepsilon K_0)\psi(\rho) = C\rho h(\rho)\psi(\rho) + \widetilde{H}(\varepsilon\rho, \rho, \varepsilon\rho\psi(\rho), \varepsilon\psi(\rho)). \tag{5.8}$$

We choose $\varepsilon > 0$ with $1-\varepsilon K_0 > 0$. Then by using the implicit function theorem, we can see that (5.8) has a unique holomorphic solution $\psi(\rho)$ with $\psi(0) = 0$ in a neighbourhood of $\rho = 0$. Moreover we can see $z_\varepsilon(\rho) \ll \psi(\rho)$.

Thus we complete the proof of Lemma 3. ■

# 6 Solvability of the System (1.13)

In this section, we give a sufficient condition for the formal solution of the system (1.13) to be convergent. Recall that (1.13) is

$$\begin{aligned}\frac{\partial f}{\partial t_i}(0,x,0,\{\varphi_j(x)\},0) &+ \frac{\partial f}{\partial u}(0,x,0,\{\varphi_j(x)\},0)\varphi_i(x) \\ &+\sum_{k=1}^{n}\frac{\partial f}{\partial \xi_k}(0,x,0,\{\varphi_j(x)\},0)\frac{\partial\varphi_i(x)}{\partial x_k} = 0, \quad i=1,2,\dots,d.\end{aligned} \tag{1.13}$$

(1.13) is a system version of the equation which is considered in [MS].

By Assumption 4 of Theorem 2, the condition

$$\frac{\partial f}{\partial \xi_k}(0,0,0,\{\varphi_j(0)\},0) = 0, \qquad k = 1,2,\dots,n$$

was assumed.

Let $\varphi(x) = {}^t(\varphi_1(x),\dots,\varphi_d(x))$ be the unknown functions. We put $\varphi(0) = {}^t(\varphi_1^0,\dots,\varphi_d^0) \in \mathbf{C}^d$ as the constant term of $\varphi(x)$. We substitute $\varphi_j(x) = \varphi_j^0 + \psi_j(x)$ into the system (1.13), and by restricting at $x = 0$, we see that $\{\varphi_j^0\}$ satisfies the following system:

$$\frac{\partial f}{\partial t_i}(0,0,0,\{\varphi_j^0\},0) + \frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0 = 0, \qquad i = 1,2,\dots,d. \tag{6.1}$$

This system has some roots by the assumption of the existence of a formal solution, and we fix such $\{\varphi_j^0\}$.

For such fixed $\{\varphi_j^0\}$, we see that $\{\psi_j(x)\}$ satisfies the system of the following form:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 f}{\partial\xi_k\partial x_l}(0,0,0,\{\varphi_j^0\},0)x_l\frac{\partial\psi_i}{\partial x_k}(x) \tag{6.2}$$

$$+\sum_{k=1}^{n}\sum_{p=1}^{d}\frac{\partial^2 f}{\partial\xi_k\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\psi_p(x)\frac{\partial\psi_i}{\partial x_k}(x)$$
$$+\frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0)\psi_i(x)$$
$$+\sum_{p=1}^{d}\left\{\frac{\partial^2 f}{\partial t_i\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0\right\}\psi_p(x)$$
$$+\sum_{l=1}^{n}\left\{\frac{\partial^2 f}{\partial t_i\partial x_l}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0\right\}x_l$$
$$=(\text{degree in } x \text{ is greater than or equal to } 2),\quad i=1,2,\ldots,d.$$

This system is written as follows for simplicity,

$$(6.3)\qquad \sum_{k=1}^{n}\sum_{l=1}^{n}a_{kl}x_l\frac{\partial\psi_i}{\partial x_k}(x)+\sum_{k=1}^{n}\sum_{p=1}^{d}b_{kp}\psi_p(x)\frac{\partial\psi_i}{\partial x_k}(x)$$
$$+c\psi_i(x)+\sum_{p=1}^{d}d_{ip}\psi_p(x)+\sum_{l=1}^{n}e_{il}x_l$$
$$=(\text{degree in } x \text{ is greater than or equal to } 2),\quad i=1,2,\ldots,d,$$

where

$$a_{kl}:=\frac{\partial^2 f}{\partial\xi_k\partial x_l}(0,0,0,\{\varphi_j^0\},0),\ b_{kp}:=\frac{\partial^2 f}{\partial\xi_k\partial\tau_p}(0,0,0,\{\varphi_j^0\},0),$$
$$c:=\frac{\partial f}{\partial u}(0,0,0,\{\varphi_j^0\},0),$$
$$d_{ip}:=\frac{\partial^2 f}{\partial t_i\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial\tau_p}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0,$$
$$e_{il}:=\frac{\partial^2 f}{\partial t_i\partial x_l}(0,0,0,\{\varphi_j^0\},0)+\frac{\partial^2 f}{\partial u\partial x_l}(0,0,0,\{\varphi_j^0\},0)\varphi_i^0.$$

Here we decompose $\psi_i(x)$ into $\psi_i(x)=\widetilde{\psi}_i(x)+\eta_i(x)$ ($\widetilde{\psi}_i(x)=\sum_{k=1}^{n}\psi_{ik}x_k,\ \eta_i(x)=O(|x|^2)$). We substitute this into the system (6.3) and obtain

$$(6.4)\qquad \sum_{k=1}^{n}\sum_{l=1}^{n}a_{kl}x_l\left(\frac{\partial\widetilde{\psi}_i}{\partial x_k}(x)+\frac{\partial\eta_i}{\partial x_k}(x)\right)$$
$$+\sum_{k=1}^{n}\sum_{p=1}^{d}b_{kp}(\widetilde{\psi}_p(x)+\eta_p(x))\left(\frac{\partial\widetilde{\psi}_i}{\partial x_k}(x)+\frac{\partial\eta_i}{\partial x_k}(x)\right)$$

$$+c(\widetilde{\psi}_i(x)+\eta_i(x))+\sum_{p=1}^{d} d_{ip}(\widetilde{\psi}_p(x)+\eta_p(x))+\sum_{l=1}^{n} e_{il}x_l$$
$$= \quad \text{(degree in } x \text{ is greater than or equal to 2)}, \quad i=1,2,\ldots,d.$$

By picking up the degree 1 part on the both sides, we see that $\{\widetilde{\psi}_i(x)\}$ satisfy the following system:

$$\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}x_l\frac{\partial\widetilde{\psi}_i}{\partial x_k}(x)+\sum_{k=1}^{n}\sum_{p=1}^{d} b_{kp}\widetilde{\psi}_p(x)\frac{\partial\widetilde{\psi}_i}{\partial x_k}(x)$$
$$+c\widetilde{\psi}_i(x)+\sum_{p=1}^{d} d_{ip}\widetilde{\psi}_p(x)+\sum_{l=1}^{n} e_{il}x_l=0, \tag{6.5}$$

for $i=1,2,\ldots,d$.

By the existence of a formal solution, (6.5) has some solutions $\{\widetilde{\psi}_i(x)\}$ of linear functions, and we fix such $\{\widetilde{\psi}_i(x)\}$.

For fixed $\{\varphi_i^0\}$ and $\{\widetilde{\psi}_i(x)\}$, we see that $\{\eta_i(x)\}$ satisfy the following system:

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left(a_{kl}+\sum_{p=1}^{d} b_{kp}\psi_{pl}\right)x_l\frac{\partial\eta_i}{\partial x_k}(x)+c\eta_i(x)+\sum_{p=1}^{d}\left(d_{ip}+\sum_{k=1}^{n} b_{kp}\psi_{ik}\right)\eta_p(x) \tag{6.6}$$
$$=\text{(degree in } x \text{ is greater than or equal to 2)}, \qquad i=1,2,\ldots,d.$$

We remark that the degree 2 part in the right hand side of this system does not include $\{\eta_i(x)\}$.

The following theorem holds:

**Theorem 3** *Let $(A_{kl})_{k,l=1,2,\ldots,n}$ be a matrix defined by*

$$(A_{kl})_{k,l=1,2,\ldots,n}=\left(a_{kl}+\sum_{p=1}^{d} b_{kp}\psi_{pl}\right)_{k,l=1,2,\ldots,n}.$$

*Let $\{\kappa_k\}_{k=1}^n$ be the eigenvalues of $(A_{kl})_{k,l=1,2,\ldots,n}$. If there exists a positive constant $\sigma_0$ such that the condition*

$$\left|\sum_{k=1}^{n}\kappa_k m_k\right|\geq\sigma_0|m|, \quad \text{(Poincaré condition)}$$

*holds for all $m=(m_1,\ldots,m_n)\in(\mathbf{Z}_{\geq 0})^n$ with $|m|\geq 2$, then the formal solution of the system (1.13) is convergent in a neighbourhood of the origin.*

**Remark 9** Let $(B_{ip})_{i,p=1,2,\dots,d}$ be a matrix defined by

$$(B_{ip})_{i,p=1,2,\dots,d} = \left( d_{ip} + \sum_{k=1}^{n} b_{kp}\psi_{ik} \right)_{i,p=1,2,\dots,d},$$

and let $\{\omega_j\}_{j=1}^{d}$ be the eigenvalues of $(B_{ip})_{i,p=1,2,\dots,d}$.

By the same argument in Remarks 1 and 6, we have

$$\left| \sum_{k=1}^{n} \kappa_k m_k + c + \omega_j \right| \geq \sigma|m|, \quad \text{by some} \;\; \sigma > 0, \;\text{and}\; j = 1, 2, \dots, d, \tag{6.7}$$

for large $m$, which will be used in the proof. ■

By a linear transformation of independent variables, (6.6) is reduced to the equation (A.1) in the case $K = 2$ which will be considered in Appendix A. Therefore, by using Proposition 1 in Appendix A, we can prove the convergence of formal solution of the system (6.6).

# A Convergence of Formal Solutions of Systems

We consider the following system of first order singular equations with the same principal part for each equation:

$$\left\{ \begin{pmatrix} \Lambda + \Delta & & \\ & \ddots & \\ & & \Lambda + \Delta \end{pmatrix} + \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix} \right\} \begin{pmatrix} u_1(x) \\ \vdots \\ u_d(x) \end{pmatrix} \tag{A.1}$$
$$= \begin{pmatrix} \displaystyle\sum_{|m|=K} \alpha_k^{(1)} x_k + f_{K+1}^{(1)}(x, \{u_j(x)\}, \{\partial_{x_k} u_j(x)\}) \\ \vdots \\ \displaystyle\sum_{|m|=K} \alpha_k^{(d)} x_k + f_{K+1}^{(d)}(x, \{u_j(x)\}, \{\partial_{x_k} u_j(x)\}) \end{pmatrix}$$

with $u_j(x) = O(|x|^K)$ $(j = 1, 2, \dots, d)$, where $K \geq 2$ and

$$\Lambda = \sum_{k=1}^{n} \mu_k x_k \partial_{x_k}, \quad \Delta = \sum_{k=1}^{n-1} \nu_k x_k \partial_{x_{k+1}},$$

$a_{pq}, \alpha_k^{(j)} \in \mathbf{C}$, and $f_{K+1}^{(j)}(x, \{u_j\}, \{\xi_{jk}\})$ $(j = 1, 2, \ldots, d)$ are holomorphic functions in a neighbourhood of the origin with the following Taylor expansions:

$$f_{K+1}^{(j)}(x, \{u_j\}, \{\xi_{jk}\}) = \sum_{|\alpha|+K|\beta|+(K-1)|\gamma| \geq K+1} f_{\alpha\beta\gamma}^{(j)} \prod_k x_k{}^{\alpha_k} \prod_j u_j{}^{\beta_j} \prod_{j,k} \xi_{jk}{}^{\gamma_{jk}} \in \mathbf{C}\{x, u, \xi\}.$$

The following proposition holds:

**Proposition 1** *We assume that (A.1) has a formal power series solution.*

*If the Poincaré condition*

$$\left| \sum_{k=1}^{n} \mu_k m_k \right| \geq \sigma |m| \tag{A.2}$$

*holds by some positive constant* $\sigma > 0$ *for all* $m \in (\mathbf{Z}_{\geq 0})^n$ *with* $|m| \geq K$*, then the formal solution of (A.1) converges in a neighbourhood of the origin.*

*Proof.* Proposition 1 is a vector version of Theorem 1 when $t$ is empty and the proof is essentially same except some differences as pointed below.

**Step 1** By taking a linear transformation of the unknown functions, (6.6) is reduced to the following form:

$$(\mathbf{\Lambda} + \mathbf{\Delta} + \mathbf{B}) \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix} \tag{A.3}$$

$$:= \left\{ \begin{pmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_d \end{pmatrix} + \begin{pmatrix} \Delta & & \\ & \ddots & \\ & & \Delta \end{pmatrix} + \mathbf{B} \right\} \begin{pmatrix} w_1(x) \\ \vdots \\ w_d(x) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{|m|=K} \beta_m^{(1)} x^m + g_{K+1}^{(1)}(x, w(x), \partial_x w(x)) \\ \vdots \\ \sum_{|m|=K} \beta_m^{(d)} x^m + g_{K+1}^{(d)}(x, w(x), \partial_x w(x)) \end{pmatrix},$$

where $w_j(x)$ $(j = 1, 2, \ldots, d)$ denote new unknown functions after linear transformations and

$$\Lambda_j = \Lambda + \omega_j, \quad \mathbf{B} = \begin{pmatrix} 0 & \varepsilon_1 & & \\ & \ddots & \ddots & \\ & & \ddots & \varepsilon_{d-1} \\ & & & 0 \end{pmatrix},$$

where $\omega_j$ $(j = 1, 2, \ldots, d)$ denote the eigenvalues of the matrix $(a_{ij})$ and $\varepsilon_j$ $(j = 1, 2, \ldots, d-1)$ denote the nilpotent components of the Jordan canonical form of $(a_{ij})$, and

$$g^{(i)}_{K+1}(x,\eta,\zeta) = \sum_{|\alpha|+K|\beta|+(K-1)|\gamma|\geq K+1} g^{(i)}_{\alpha\beta\gamma} x^\alpha \eta^\beta \zeta^\gamma.$$

**Step 2** Let $\mathbf{C}[x]_M = \{\sum_{|m|=M} u_m x^m \ ; \ u_m \in \mathbf{C}\}$, and define a norm of $u(x) = {}^t(u_1(x), , \ldots, u_d(x)) \in (\mathbf{C}[x]_M)^d$ by

$$||u|| := \inf\{C > 0 \ ; \ u_i(x) \ll C(x_1 + \cdots + x_n)^M, \ i = 1, 2, \ldots, d\}.$$

By the same arguments as the proof of Lemma 1 and by Remark 9, we can prove the same results as Lemma 1 for the matrix differential operator $\mathbf{\Lambda} + \mathbf{\Delta} + \mathbf{B}$, that is, there exists a positive integer $K_0$ such that $\mathbf{\Lambda}+\mathbf{\Delta}+\mathbf{B}$ is invertible on $(\mathbf{C}[x]_M)^d$ $(M \geq K_0)$ and the operater norm of $(\mathbf{\Lambda}+\mathbf{\Delta}+\mathbf{B})^{-1}$ on $(\mathbf{C}[x]_M)^d$ has the same estimate as (4.1). ¿From this observation we can construct a majorant equation whose solution is a majorant power series of the all formal solutions of the system by the same method as Section 5.2. Thus the convergence of formal solutions of the system (A.1) will be proved.

# B Analyticity of $\{u_l(x)\}$ which appeared in Remark 6

We put $u(t,x) = \sum_{j=1}^d \varphi_j(x) t_j + \sum_{L\geq 2} u_L(t,x)$ ($u_L(t,x)$ denotes the homogeneous polynomial of degree $L$ in $t$ variables.)

By an easy calculation, $u_L(t,x)$ satisfies the equation of the following form:

(B.1)
$$\left(\sum_{i,j=1}^d A_{ij}(x) t_i \partial_{t_j} + \sum_{k=1}^n B_k(x)\partial_{x_k} + \frac{\partial f}{\partial u}(\mathbf{a}(x))\right) u_L(t,x)$$
$$= (\text{given holomorphic function whose degree in } t \text{ is } L).$$

By the same method as Section 3, (B.1) is reduced to the following form by transforming the independent variables:

(B.2)
$$(\Lambda + \Delta) u_L(t,x) = \left(\sum_{i,j=1}^d \alpha_{ij}(x) t_i \partial_{t_j} + \sum_{k=1}^n \beta_k(x)\partial_{x_k} + C(x)\right) u_L(t,x)$$
$$+ (\text{given holomorphic function whose degree in } t \text{ is } L),$$

where $\alpha_{ij}(x) = O(|x|)$, $\beta_k(x) = O(|x|^2)$, $C(x) = O(|x|)$.

Here we put $u_L(t,x) = \sum_{|l|=L} u_l(x)t^l$. Then $\{u_l(x)\}_{|l|=L}$ satisfies the following equation:

$$
\text{(B.3)} \qquad \left(\sum_{j=1}^{d} \lambda_j l_j + \sum_{k=1}^{d} \mu_k x_k \partial_{x_k} + \sum_{k=1}^{n-1} \nu_k x_k \partial_{x_{k+1}} + \frac{\partial f}{\partial u}(\mathbf{a}(0))\right) u_l(x)
$$
$$
+ \sum_{j=1}^{d-1} \delta_j (l_{j+1}+1) u_{l+e(j+1)-e(j)}(x)
$$
$$
= \sum_{i,j=1}^{d} \alpha_{ij}(x)(l_{j+1}+1)u_{l+e(j+1)-e(i)}(x) + \sum_{k=1}^{n} \beta_k(x)\partial_{x_k} u_l(x) + C(x)u_l(x)
$$
$$
+(\text{given holomorphic function}), \qquad \text{for } \ |l| = L.
$$

where $e(j) = (0,\dots,0,1,0,\dots,0)$ ($j$-th component is 1).

We put $u_l(x) = \sum_{0\le|m|\le K-1} u_{lm}x^m + v_l(x)$ ($v_l(x) = O(|x|^K)$). By the assumption of the existence of formal solution, the coefficients $\{u_{jm}\}_{0\le|m|\le K-1}$ are determined. Moreover, we can see that $v_l(x)$ satisfies the equation which is the same form as (B.3).

The system (B.3) is a special one of (A.1). Therefore, by Proposition 1 in Appendix A, we obtain the analyticity of $\{v_l(x)\}$.

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Akira Shirai
Graduate School of Mathematics
Nagoya University
Nagoya, 464-8602
Japan
E-mail : m96034q@math.nagoya-u.ac.jp