ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS IN 2 VARIABLES

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§0. Introduction

Let $\lambda = (\lambda_0, \dots, \lambda_{l-1})$ be a partition of n, sometimes called a '*Young diagram* λ' of weight n. Let $H_\lambda = J(\lambda_0) \times \cdots \times J(\lambda_{l-1}) \subset GL(n)$ be the associated maximal abelian subgroup with respect to λ, where $J(m)$ is the Jordan group of size m, i.e.,

$$J(m) = \left\{ \sum_{i=0}^{m-1} h_i \Lambda^i \mid h_0 \in \mathbb{C}^\times, h_1, \dots, h_{m-1} \in \mathbb{C} \right\},$$

where the $m \times m$ matrix Λ is defined as

$$\Lambda = \begin{pmatrix} 0 & 1 & & \text{\Large 0} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \text{\Large 0} & & & 0 \end{pmatrix}.$$

We define the biholomorphic map

$$\begin{aligned} \iota : H_\lambda &\longrightarrow \prod_i (\mathbb{C}^\times \times \mathbb{C}^{\lambda_i - 1}) \\ h &\longmapsto (h_0^{(0)}, \dots, h_{\lambda_0 - 1}^{(0)}, \dots, h_0^{(l-1)}, \dots, h_{\lambda_{l-1}-1}^{(l-1)}) \end{aligned}$$

where $h = (h^{(0)}, \dots, h^{(l-1)})$, $h^{(i)} = \sum_{0 \le k < \lambda_i} h_k^{(i)} \Lambda^k \in J(\lambda_i)$.

Let $\alpha = (\alpha^{(0)}, \ldots, \alpha^{(l-1)}), \alpha^{(i)} := (\alpha_0^{(i)}, \ldots, \alpha_{\lambda_i-1}^{(i)})$ $(0 \leq i \leq l-1)$ be an n-tuple of complex numbers satisfying $\sum_{i=0}^{l-1} \alpha_0^{(i)} = -r$. We define the character $\chi(\cdot\, ;\, \alpha) : \widetilde{H}_\lambda \longrightarrow \mathbb{C}^\times$ of the universal covering group $\widetilde{H}_\lambda = \widetilde{J}(\lambda_0) \times \cdots \times \widetilde{J}(\lambda_{l-1})$ of H_λ by $\chi(h; \alpha) = \prod_{i=0}^{l-1} \chi(h^{(i)}; \alpha^{(i)})$, where

$$\chi(h^{(i)}; \alpha^{(i)}) = h_0^{\alpha_0^{(i)}} \exp\left[\sum_{j=1}^{\lambda_i-1} \alpha_j^{(i)} \theta_j\left(\frac{h_1^{(i)}}{h_0^{(i)}}, \ldots, \frac{h_{\lambda_i-1}^{(i)}}{h_0^{(i)}}\right)\right]$$

where θ_j are defined as the coefficients of the generating series

$$\log(1 + x_1 T + x_2 T^2 + \cdots) = \sum_{j=0}^{\infty} \theta_j(x_1, \ldots, x_j) T^j.$$

Recall that the hypergeometric function $\Phi(z; \alpha)$ of type λ (see [K-H-T]) is a function defined by

$$\Phi(z; \alpha) = \int_\Delta \chi(\iota^{-1}(tz); \alpha) \cdot \omega \quad \text{for} \quad z \in Z_{r,n} \tag{0.1}$$

where $Z_{r,n}$ is the set of $r \times n$ complex matrices *in general position* (see [K-H-T]) with respect to λ, $\omega := \sum_{0 \leq i < r} (-1)^i t_i dt_0 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_{r-1}$ and Δ is a *twisted cycle* in the t-space depending on z and α. Note that for $\lambda = (1, \ldots, 1)$, the hypergeometric functions of type λ coincide with the general hypergeometric function defined in [G].

The set $Z_{r,n}$ admits an action of the group $GL(r) \times H_\lambda$:

$$\begin{aligned} GL(r) \times Z_{r,n} \times H_\lambda &\longrightarrow Z_{r,n} \\ (g, z, h) \qquad &\longmapsto gzh, \end{aligned}$$

under which Φ behaves as

$$\Phi(gz; \alpha) = (\det g)^{-1} \Phi(z; \alpha) \qquad g \in GL(r) \tag{0.2}$$

$$\Phi(zh_\lambda; \alpha) = \chi(h_\lambda) \Phi(z; \alpha) \qquad h_\lambda \in H_\lambda. \tag{0.3}$$

Furthermore, the function Φ admits another symmetry:

$$\Phi(zw_\lambda; \alpha) = \Phi(z; \alpha^t w_\lambda) \qquad w_\lambda \in W_\lambda, \tag{0.4}$$

where W_λ is an analogue of the Weyl group, see [K-K].

The hypergeometric functions Φ on $Z_{2,4}$ and $Z_{2,5}$ for various partitions λ of 4 and 5 were investigated in the papers [K-H-T],[O-K] and [K-K]. It is known that the functions Φ are generalizations of Gauss', Kummer's, Bessel's, Hermite's, Airy's functions and the classical hypergeometric functions of two variables, i.e., $F_1, \Phi_1, \Phi_2, \Phi_3, G_2, \Gamma_1, \Gamma_2$ in Horn's list ([Erd 1]). In this paper, we study the hypergeometric functions of type λ in two variables on the strata of the set $M(3,6)$ of 3×6 complex matrices.We establish a classification of the functions in terms of the orbital decomposition of the set of strata and give some transformation formulae between some systems of differential equations satisfied by the functions $F_2, F_3, H_2, \mathbf{H}_3, \mathbf{H}_{11}, \Psi_1, \Xi_2$.

In Section 1 , we introduce a group $W_{\lambda(\nu)} := R_{\lambda(\nu)} \rtimes P_{\lambda(\nu)}$ which is analogous to the (classical) Weyl group and discuss its properties in detail. In Section 2, we consider the action of $P_{\lambda(\nu)}$ on the set $S_{\lambda(\nu)}$ of strata and obtain the orbital decomposition of $S_{\lambda(\nu)}$. In Section 3, we obtain suitable normal forms of the matrices in the strata relative to the action of $GL(3) \times H_\lambda$. Then we can reduce the hypergeometric function Φ into a function of two variables. In Section 4, we give relations between the classical special functions of hypergeometric type, i.e., Appell's F_2, F_3, H_2 and their confluence in Horn's list, and the hypergeometric functions of type λ in two variables. In the last section, some transformation formulae for the systems of diffenentical equations are systematically deduced from the symmetries (0.2)-(0.4) for the functions Φ.

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§1. Construction of the group W_λ.

Let $G = GL(n)$ be the general linear group, H the Cartan subgroup of G, $N_G(H)$ the normalizer of H in G. It is known that $N_G(H)$ is expressed as the *semi-direct* product $N_G(H) \simeq H \rtimes W$, where $W = N_G(H)/H$ is the Weyl group of G, which is identified with the symmetric group $\mathfrak{S}_n$. In a similar manner, we introduce the group $W_\lambda := N_G(H_\lambda)/H_\lambda$. The group W_λ is also called the

Weyl group (with respect to H_λ).

Let $\lambda^{(0)} = (1,1,1,1,1,1)$, $\lambda^{(1)} = (2,1,1,1,1)$, $\lambda^{(2)} = (2,2,1,1)$, $\lambda^{(3)} = (2,2,2)$, $\lambda^{(4)} = (3,1,1,1)$, $\lambda^{(5)} = (3,2,1)$, $\lambda^{(6)} = (3,3)$, $\lambda^{(7)} = (4,1,1)$ and $\lambda^{(8)} = (4,2)$ be the patitions of 6. The Weyl group for the Jordan group $J(m)$ is given by

$$W(m) = \left\{ \left(\phi_{ij}(x)\right)_{0\le i,j<m} \in GL(n) \mid x = (x_1,\dots,x_{m-1}) \in \mathbb{C}^\times \times \mathbb{C}^{m-2} \right\},$$

where the polynomial $\phi_{ij}(x)$ is defined by

$$\phi_{ij}(x) = \sum_{\substack{\nu_1+\cdots+\nu_i=j\\ \nu_1,\dots,\nu_i\in\mathbb{N}}} x_{\nu_1}\cdots x_{\nu_i}.$$

Set

$$P_{\lambda^{(0)}} = \mathfrak{S}_6, \quad P_{\lambda^{(1)}} = \left\{ \begin{pmatrix} I_2 & 0 \\ 0 & \mathfrak{S}_4 \end{pmatrix} \right\}$$

$$P_{\lambda^{(2)}} = \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & \mathfrak{S}_2 \end{pmatrix} \right\}$$

$$P_{\lambda^{(3)}} = \left\{ \begin{pmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ I_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & I_2 \\ 0 & I_2 & 0 \\ I_2 & 0 & 0 \end{pmatrix} \right\}$$

$$P_{\lambda^{(4)}} = \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & \mathfrak{S}_3 \end{pmatrix} \right\}, \quad P_{\lambda^{(5)}} = \left\{ \begin{pmatrix} I_3 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

$$P_{\lambda^{(6)}} = \left\{ \begin{pmatrix} I_3 & 0 \\ 0 & I_3 \end{pmatrix}, \begin{pmatrix} 0 & I_3 \\ I_3 & 0 \end{pmatrix} \right\}$$

$$P_{\lambda^{(7)}} = \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & \mathfrak{S}_2 \end{pmatrix} \right\}, \quad P_{\lambda^8} = \left\{ \begin{pmatrix} I_4 & 0 \\ 0 & I_2 \end{pmatrix} \right\}$$

where I_i is the $i \times i$ identity matrix, $\mathfrak{S}_i$ is the group of $i \times i$ permutation matrices and

$$\left\{ \begin{pmatrix} * & 0 \\ 0 & \mathfrak{S}_i \end{pmatrix} \right\} := \left\{ \begin{pmatrix} * & 0 \\ 0 & A \end{pmatrix} \mid A \in \mathfrak{S}_i \right\}.$$

Then we have the following proposition (see [K-K]).

Proposition 1.1. *For the partitions* $\lambda^{(\nu)}$*, the Weyl groups* $W_{\lambda^{(\nu)}}$ $(\nu = 0, \ldots, 8)$ *are given by* $W_{\lambda^{(\nu)}} = R_{\lambda^{(\nu)}} \rtimes P_{\lambda^{(\nu)}}$*, where*

$$\begin{aligned} R_{\lambda^{(0)}} &= I_6 & R_{\lambda^{(1)}} &= \operatorname{diag}(W(2), I_4) \\ R_{\lambda^{(2)}} &= \operatorname{diag}(W(2), W(2), I_2) & R_{\lambda^{(3)}} &= \operatorname{diag}(W(2), W(2), W(2)) \\ R_{\lambda^{(4)}} &= \operatorname{diag}(W(3), I_3) & R_{\lambda^{(5)}} &= \operatorname{diag}(W(3), W(2), 1) \\ R_{\lambda^{(6)}} &= \operatorname{diag}(W(3), W(3)) & R_{\lambda^{(7)}} &= \operatorname{diag}(W(4), I_2) \\ R_{\lambda^{(8)}} &= \operatorname{diag}(W(4), W(2)). \end{aligned}$$

Remark 1.1. In Proposition 1.1, $W := W_{\lambda^{(0)}}$ is the Weyl group of $GL(6)$, which is a finite group of order $6! = 720$. The other groups $W_{\lambda^{(\nu)}} (\nu = 1, \ldots, 8)$ are infinite groups, while the groups $P_{\lambda^{(\nu)}} (\nu = 1, \ldots, 8)$ are finite groups of order 24, 4, 6, 6, 1, 2, 2, 1, respectively.

Remark 1.2. For any $p \in P_{\lambda^{(\nu)}}$ and $h \in H_{\lambda^{(\nu)}}$, we have $p^{-1}hp \in H_{\lambda^{(\nu)}}$, where $H_{\lambda^{(\nu)}}$ is the maximal abelian subgroup of $GL(6)$ associated to $\lambda^{(\nu)}$ (see (0,1)).

§2. Orbital decomposition of the set of strata.

Let λ be a Young diagram of weight 6. we assign a number $0, 1, \ldots, 5$ to each box of λ from left to right in the first row and the from left to right in the second row and so on. For exmaple, for $\lambda = (3, 1, 1, 1)$, the numbering of the boxes is as in Figure 2.1.

A subdiagram of λ of weight 3 is a set of distinct 3 boxes of λ taken under the rule: if a box b is taken from a row, then all boxes located to the left of b in the same row must be taken. A subdiagram consisting of 3 boxes numbered by $0 \le i, j, k \le 5$ is denoted by the unordered triple (i, j, k). The determinant $\det(z_i, z_j, z_k)$ of the submatrix (z_i, z_j, z_k) of z corresponding to a subdiagram (i, j, k) is simply denoted by $D(i, j, k)$. For example, for the Young diagram $\lambda = (3, 1, 1, 1)$, the possible subdiagrams are

$$(0,1,2),(0,1,3),(0,1,4),(0,1,5),(3,4,5),(0,4,5),(0,3,5),(0,3,4).$$

Definition 2.1. *Let λ be a Young diagram of weight 6, $(i, j, k), (i, m, n)$ two subdiagrams of λ, where i, j, k, m, n are mutually distinct. We denote by the symbol $\{(i, j, k), (i, m, n)\}$ the set*

$$\left\{ z \in M(3,6) \;\middle|\; \begin{array}{c} D(i,j,k) = D(i,m,n) = 0, \\ D(p,q,r) \neq 0 \ \textit{ for any} \\ \textit{other subdiagram } (p,q,r) \end{array} \right\}$$

and call it a stratum of type $(3, 6)$ associated to λ (for short, a stratum).

For a stratum $\{(i, j, k), (i, m, n)\}$, if $A_1 \in \mathfrak{S}_3$, then $A_1 z \in \{(i, j, k), (i, m, n)\}$ for $z \in \{(i, j, k), (i, m, n)\}$; if $A_2 = (\delta_{p\sigma(q)}) \in \mathfrak{S}_6$ with the permutation σ of $\{0, 1, 2, 3, 4, 5\}$ such that $(\sigma(i), \sigma(j), \sigma(k)), (\sigma(i), \sigma(m), \sigma(n))$ are again subdiagram of λ, then $zA_2 \in \{(\sigma(i), \sigma(j), \sigma(k)), (\sigma(i), \sigma(m), \sigma(n))\}$ for $z \in \{(i, j, k), (i, m, n)\}$.

Let S_λ denote the set of strata $\{(i, j, k), (i, m, n)\}$ associated to the Young diagram λ. We simply write S for $S_{\lambda^{(0)}}$.

First we consider the right action of $W \simeq \mathfrak{S}_6$ on S. We have the following proposition.

Proposition 2.1. *(1) The Weyl group W acts transitively on S.*
(2) $\#S = 90$.

Proof. (1) The assertion easily follows from $W \simeq \mathfrak{S}_6$.

(2) For $s = \{(i, j, k), (i, m, n)\} \in S$, let $W_s := \{w \in W \ :\ sw = s\}$ be the isotropy subgroup of W. We observe that the group W_s is generated by the

three elements of W corresponding to the permutations $j \leftrightarrow k, m \leftrightarrow n$ and $\{j,k\} \leftrightarrow \{m,n\}$. Since these elements are of order 2 and commutative with each other, the order of W_s is given by $2^3 = 8$. Hence $\#S = \frac{6!}{2^3} = 90$. $\square$

For $\nu = 1, \ldots, 8$, it is easy to see that the subgroup $R_{\lambda^{(\nu)}}$ of $W_{\lambda^{(\nu)}} = R_{\lambda^{(\nu)}} \rtimes P_{\lambda^{(\nu)}}$ acts trivially on $S_{\lambda^{(\nu)}}$. Hence it is sufficient to consider the action of $P_{\lambda^{(\nu)}}$ on $S_{\lambda^{(\nu)}}$. For any stratum $s \in S_{\lambda^{(\nu)}}$, let $P_s = \{p \in P_{\lambda^{(\nu)}} \ : \ sp = s\}$ be the isotropy subgroup of $P_{\lambda^{(\nu)}}$. We shall obtain the orbital decomposition of $S_{\lambda^{(\nu)}}$.

Proposition 2.2. *Under the action of* $P_{\lambda^{(\nu)}}$ *, the orbital decomposition of* $S_{\lambda^{(\nu)}}$ *is described as follows:*

$$S_{\lambda^{(\nu)}} = \coprod_i O_{P_{\lambda^{(\nu)}}}(s_\nu^i), \tag{2.1}$$

where $s_1^1 = \{(0,1,2),(0,4,5)\}, s_1^2 = \{(4,0,1),(4,2,3)\}, s_1^3 = \{(4,0,5),(4,2,3)\},$
$s_1^4 = \{(0,2,3),(0,4,5)\}, s_2^1 = \{(0,1,2),(0,4,5)\}, s_2^2 = \{(2,0,1),(2,3,4)\},$
$s_2^3 = \{(4,0,1),(4,2,3)\}, s_2^4 = \{(0,2,3),(0,4,5)\}, s_2^5 = \{(0,1,4),(0,2,5)\},$
$s_2^6 = \{(4,0,5),(4,2,3)\}, s_3^1 = \{(0,1,2),(0,4,5)\}, s_3^2 = \{(4,0,1),(4,2,3)\},$
$s_4^1 = \{(0,1,2),(0,3,4)\}, s_4^2 = \{(0,1,3),(0,4,5)\}, s_4^3 = \{(3,0,1),(3,4,5)\},$
$s_5^1 = \{(0,1,2),(0,3,4)\}, s_5^2 = \{(0,1,2),(0,3,5)\}, s_5^3 = \{(3,0,1),(3,4,5)\},$
$s_5^4 = \{(5,0,1),(5,3,4)\}, s_6^1 = \{(0,1,2),(0,3,4)\}, s_7^1 = \{(0,1,2),(0,4,5)\},$
$s_8^1 = \{(0,1,2),(0,4,5)\}.$

Proof. We establish the decomposition (2.1) for the case $\nu = 1$. Other cases are treated in a similar manner. By a combinatorial argument, we obtain $\#S_{\lambda^{(1)}} = 39$. The elements of $P_\lambda^{(1)}$ leaving $s_1^1 = \{(0,1,2),(0,4,5)\}$ invariant are precisely the identity matrix and

$$\begin{pmatrix} I_4 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}.$$

Thus, we have $\operatorname{ord} P_{s_1^1} = 2$. Hence,$\#O_{P_{\lambda^{(1)}}}(s_1^1) = \frac{\operatorname{ord} P_{\lambda^{(1)}}}{\operatorname{ord} P_{s_1^1}} = \frac{24}{2} = 12$. Similarly, we have $\operatorname{ord} P_{s_1^2} = 2$, $\operatorname{ord} P_{s_1^3} = 2$ and $\operatorname{ord} P_{s_1^4} = 8$. Hence, $\#O_{P_{\lambda^{(1)}}}(s_1^2) =$

12, $\#O_{P_{\lambda^{(1)}}}(s_1^3) = 12$, $\#O_{P_{\lambda^{(1)}}}(s_1^4) = 3$. Therefore, $S_{\lambda^{(1)}} = \coprod_{i=1}^4 O_{P_{\lambda^{(1)}}}(s_1^i)$.
$\square$

We call s_ν^i the representative of the orbit $O_{P_{\lambda^{(\nu)}}}(s_\nu^i)$.

§3. Normal forms of matrices.

In this section, we determine normal forms of the matrices in each stratum $s_\nu^i \in S_{\lambda^{(\nu)}}$ for $\nu = 0, 1, \ldots, 8$. We have shown that the number of the strata in $S_{\lambda^{(\nu)}}$ is given by $90, 39, 18, 8, 9, 4, 2, 1, 1$ for $\nu = 0, \ldots, 8$, respectively. For each ν, we set

$$Z^\nu = \bigcup_{s \in S_{\lambda^{(\nu)}}} s. \tag{3.1}$$

For simplicity we write Z for Z^0.

Let $H_{\lambda^{(\nu)}}$ be the maximal abelian subgroup of G associated to the Young diagram $\lambda^{(\nu)}$. Consider the action of $GL(3) \times H_{\lambda^{(\nu)}}$ on Z^ν defined by

$$\begin{aligned} GL(3) \times Z^\nu \times H_{\lambda^{(\nu)}} &\longrightarrow Z^\nu \\ (g, z, h) &\longmapsto gzh. \end{aligned}$$

Proposition 3.1. *This action is well-defined, i.e. we have $gzh \in Z^\nu$ for any $(g, z, h) \in GL(3) \times Z^\nu \times H_{\lambda^{(\nu)}}$.*

Proof. Let $z = (z_0, z_1, \ldots, z_5) \in Z^\nu$. For any $g \in GL(3)$, we set $gz = (z'_0, z'_1, \ldots, z'_5)$. Since $\det(z'_i, z'_j, z'_k) = \det g \det(z_i, z_j, z_k)$, we have $\det(z'_i, z'_j, z'_k) = 0 \iff \det(z_i, z_j, z_k) = 0$. Next, we consider the action of $H_{\lambda^{(\nu)}}$ on Z^ν. Since the group $H_{\lambda^{(\nu)}}$ is a direct product of Jordan groups: $H_{\lambda^{(\nu)}} = J(\lambda_0) \times \cdots \times J(\lambda_{l-1})$, where $\lambda^{(\nu)} = (\lambda_0, \ldots, \lambda_{l-1})$, it suffices to consider the action of $J(\lambda_m)$ $(0 \leq m \leq l-1)$ on Z^ν. Set

$$j(\lambda_m) = \begin{pmatrix} h_0 & h_1 & \cdots & h_{\lambda_m - 1} \\ & h_0 & \ddots & \vdots \\ & & \ddots & h_1 \\ & & & h_0 \end{pmatrix}, \qquad (h_0, \ldots, h_{\lambda_m - 1}) \in \mathbb{C}^\times \times \mathbb{C}^{\lambda_m - 2},$$

For $z = (z_0, \ldots, z_0^{(m)}, \ldots, z_{m-1}^{(m)}, \ldots, z_5)$, we have

$$\begin{aligned} z\mathrm{diag}\,(I_p, j(\lambda_m), I_q) = (&z_0, \ldots, z_{p-1}, z_0^{(m)}h_0, z_0^{(m)}h_1 + z_1^{(m)}h_0, \ldots, \\ &z_0^{(m)}h_{\lambda_m-1} + \cdots + z_{\lambda_m-1}^{(m)}h_0, z_{5-q+1}, \ldots, z_5) \\ =: (&z'_0, \ldots, z'_5). \end{aligned}$$

Hence, there exists an $r \in \mathbb{N}$ $(0 \leq r \leq 3)$, depending on the subdiagram (i, j, k), such that $\det(z'_i, z'_j, z'_k) = h_0^r \det(z_i, z_j, z_k)$. Hence $\det(z'_i, z'_j, z'_k) = 0 \iff \det(z_i, z_j, z_k) = 0$. $\square$

Remark 3.1. For any two strata $s_1, s_2 \in O_{P_\lambda^{(\nu)}}(s)$, if $z \in s_1$, then there exists $p \in P_{\lambda^{(\nu)}}$ such that $zp \in s_2$. Obviously, p is not necessarily unique.

We first describe how to take the elements of $GL(3) \setminus Z/H$ as the normal forms of $z \in Z$. We fix one stratum $s_0 = \{(4, 0, 1), (4, 2, 3)\} \in S$.

Proposition 3.2. *For each $i = 0, \ldots, 14$, the following assertion holds: For any $z \in s_0$ there exists a unique $(x, y) \in \mathbb{C}^2$ such that*

$$(1) f_i(x, y) \neq 0, \quad (2) z \in GL(3)\overrightarrow{z}_i(x, y)H,$$

where $\vec{z}_i = \vec{z}_i(x,y)$ *and* $f_i = f_i(x,y)$ *are given by*

$$\vec{z}_0(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{z}_1(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 1 \end{pmatrix}$$

$$\vec{z}_2(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & y & 0 & x & 1 & 1 \end{pmatrix} \qquad \vec{z}_3(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & y & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_4(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & y & 0 & 1 & 1 & x \end{pmatrix} \qquad \vec{z}_5(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & x \end{pmatrix}$$

$$\vec{z}_6(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & y \end{pmatrix} \qquad \vec{z}_7(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & y \end{pmatrix}$$

$$\vec{z}_8(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & x \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad \vec{z}_9(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & x & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{10}(x,y) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & y & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & x \end{pmatrix} \qquad \vec{z}_{11}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{12}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 1 \end{pmatrix} \qquad \vec{z}_{13}(x,y) = \begin{pmatrix} 1 & y & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & x & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\vec{z}_{14}(x,y) = \begin{pmatrix} 1 & x & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & y \end{pmatrix}$$

and

$$\begin{aligned}
f_i(x,y) =& xy(1-x)(1-y)(1-x-y) && \text{for } i = 0,\ldots,3 \\
f_i(x,y) =& xy(1-x)(x-y)(1-x+y) && \text{for } i = 4,5 \\
f_i(x,y) =& xy(y-x)(1-y)(1+x-y) && \text{for } i = 6,7 \\
f_i(x,y) =& xy(1-x)(xy-1)(1-xy+y) && \text{for } i = 8,9 \\
f_i(x,y) =& xy(1-x)(xy-1)(1-xy+y) && \text{for } i = 10 \\
f_i(x,y) =& xy(1-x)(1-y)(1+xy-y) && \text{for } i = 11,12
\end{aligned}$$

$$f_i(x,y) = xy(1-x)(1-y)(xy-x-y) \qquad \text{for } i=13$$
$$f_i(x,y) = xy(xy-1)(1-y)(1-xy+x) \qquad \text{for } i=14.$$

Proof. Let $z = (z_0, z_1, \dots, z_5) \in \{(4,0,1),(4,2,3)\}$. By $D(0,2,4) \neq 0$, we see that the submatrix (z_0, z_2, z_4) belongs to $GL(3)$. Since $D(4,0,1) = D(4,2,3) = 0$, $A := (z_0, z_2, z_4)^{-1} z$ becomes

$$A = \begin{pmatrix} 1 & a_{11} & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 1 & a_{23} & 0 & a_{25} \\ 0 & a_{31} & 0 & a_{33} & 1 & a_{35} \end{pmatrix} \tag{3.2}$$

Clearly, the a_{ij}'s in this matrix are not zero. Moreover we have

$$\mathrm{diag}\,(\frac{1}{h_0}, \frac{1}{h_2}, \frac{1}{h_4}) \cdot A \cdot h = \begin{pmatrix} 1 & \frac{a_{11}h_1}{h_0} & 0 & 0 & 0 & \frac{a_{15}h_5}{h_0} \\ 0 & 0 & 1 & \frac{a_{23}h_3}{h_2} & 0 & \frac{a_{25}h_5}{h_2} \\ 0 & \frac{a_{31}h_1}{h_4} & 0 & \frac{a_{33}h_3}{h_4} & 1 & \frac{a_{35}h_5}{h_4} \end{pmatrix} \tag{3.3}$$

where $h = \mathrm{diag}\,(h_0, h_1, \dots, h_5) \in H$. One can take h suitably so that the right-hand side of (3.3) becomes $\vec{z_i}$ for some $i = 0, \dots, 14$. For example, if we take $h = \mathrm{diag}\left(1, \frac{1}{a_{11}}, \frac{a_{23}a_{31}}{a_{11}a_{33}}, \frac{a_{31}}{a_{11}a_{33}}, \frac{a_{31}}{a_{11}}, \frac{a_{31}}{a_{35}a_{11}}\right)$ and set $\frac{a_{25}a_{33}}{a_{23}a_{35}} =: x$, $\frac{a_{15}a_{31}}{a_{35}a_{11}} =: y$, then we have $\vec{z_0}$ and $f_0(x,y) = xy(1-x)(1-y)(1-x-y)$. Let $(x', y') \in \mathbb{C}^2$ be such that $z \in GL(3)\vec{z}_0(x', y')H$. Then there exists $g \in GL(3)$ and $h \in H$ such that

$$g\vec{z}_0(x', y')h = \vec{z}_0(x, y). \tag{3.4}$$

Set

$$g = \begin{pmatrix} g_{01} & g_{02} & g_{03} \\ g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \end{pmatrix} \qquad h = \mathrm{diag}\,(h_0, \dots, h_5),$$

then, by (3.4) we easily have $g = \mathrm{diag}\,(g_{01}, g_{12}, g_{23})$ and hence

$$\begin{aligned} & g\vec{z}_0(x', y')h \\ =& \mathrm{diag}\,(g_{01}, g_{12}, g_{23})\vec{z}_0(x', y')\mathrm{diag}\,(h_0, \dots, h_5) \\ =& \begin{pmatrix} g_{01}h_0 & g_{01}h_1 & 0 & 0 & 0 & g_{01}y'h_5 \\ 0 & 0 & g_{12}h_2 & g_{12}h_3 & 0 & g_{12}x'h_5 \\ 0 & g_{23}h_1 & 0 & g_{23}h_3 & g_{23}h_4 & g_{23}h_5 \end{pmatrix}. \end{aligned}$$

By (3.4) we have $g_{01} = g_{12} = g_{23} = \frac{1}{h_0}, h_1 = h_2 = h_3 = h_4 = h_5 = h_0$. Hence $g_{01}y'h_5 = y' = y, g_{12}x'h_5 = x' = x$. Thus, (x, y) is unique. □

We set $N = \left\{\overrightarrow{z_i} \,:\, 0 \leq i \leq 14\right\}$. For the stratum s_0, we denote by N_{s_0} the set $\left\{\overrightarrow{z} = \sigma\overrightarrow{z_i} \,:\, \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N\right\}$ and call $\overrightarrow{z} \in N_{s_0}$ a *normal form* of $z \in s_0$. Proof of Proposition 3.2 provides a method of finding the normal forms of the matrices in a stratum.

Proposition 3.3. *The normal forms of the matrices in any other stratum $s \in S$ can be obtained by the action of Weyl group $W \simeq \mathfrak{S}_6$ on N_{s_0}.*

Proof. Let $s \in S$ be any stratum. For any $z \in s$, there exist $\sigma_s \in \mathfrak{S}_6$ such that $z\sigma_s \in s_0$. Thus, there exists $g \in GL(3)$ and $h \in H$ such that $gz\sigma_s h = \overrightarrow{z}, \overrightarrow{z} \in N_{s_0}$. On the other hand, we also have $h' = \sigma_s h \sigma_s^{-1} \in H$ and $gzh' = \overrightarrow{z}\sigma_s^{-1} \in N_{s_0}$. Hence we can obtain N_s, the set of normal forms of $z \in s$, as $N_s = \left\{\overrightarrow{z}\sigma_s^{-1} \,:\, \overrightarrow{z} \in N_{s_0}\right\}$. □

Using the same method as above, we can obtain the normal forms of $z \in Z^\nu$ for $\nu = 1, 2, \ldots, 8$. It is sufficient to give the normal forms for the representatives s_ν^i of each orbit $O_{P_{\lambda(\nu)}}(s_\nu^i)$ in Proposition 2.2. For that purpose we first consider the cases: $\nu = 1, 2, 3$.

Proposition 3.4. *For $s_1^2 = \{(4,0,1)(4,2,3)\} \in O_{P_{\lambda(1)}}(s_1^2)$, the normal forms of $z \in s_1^2$ can be taken as $N_{s_1^2} = \left\{\sigma\overrightarrow{z_i} \,:\, \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, 0 \leq i \leq 10\right\}$. For $s_2^3 = \{(4,0,1)(4,2,3)\} \in O_{P_{\lambda(2)}}(s_2^3)$, the normal forms of $z \in s_2^3$ can be taken as $N_{s_2^3} = \left\{\sigma\overrightarrow{z_i} \,:\, \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, 0 \leq i \leq 7\right\}$. For $s_3^2 = \{(4,0,1)(4,2,3)\} \in O_{P_{\lambda(3)}}(s_3^2)$, the normal forms of $z \in s_3^2$ can be taken as $N_{s_3^2} = \left\{\sigma\overrightarrow{z_i} \,:\, \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, 0 \leq i \leq 3\right\}$. Especially, the (1,0) entry of $\overrightarrow{z_i}$ ($0 \leq i \leq 10$) for $\nu = 1$, the (1,0) and (2,3) entries of $\overrightarrow{z}_i$ ($0 \leq i \leq 7$) for $\nu = 2$, the (1,0), (2,3) and (3,5) entries of $\overrightarrow{z}_i$ ($0 \leq i \leq 3$) for $\nu = 3$ can be taken to be zero.*

This proposition is proved in an almost similar manner as in the proof of Proposition 3.2. So the proof is omitted.

We can find a complete list of the normal forms of z in s^i_ν $(\nu = 1, 2, 3)$ in the following manner. Let

$$s^1_0 = \{(0,1,2)(0,4,5)\}, \qquad s^2_0 = \{(2,0,1)(2,3,4)\},$$
$$s^3_0 = \{(0,2,3)(0,4,5)\}, \qquad s^4_0 = \{(0,1,4)(0,2,5)\},$$
$$s^5_0 = \{(4,0,5)(4,2,3)\},$$

where $s^i_0 \in S$ $(1 \le i \le 5)$. Moreover let $p_{\sigma_k} = (a_{ij}) \in \mathfrak{S}_6$ $(1 \le k \le 5)$ be the permutation matrix defined by $a_{ij} = \delta_{i\sigma_k(j)} \qquad (0 \le i, j \le 5)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 0 & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 0 & 1 \end{pmatrix} \qquad \sigma_4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 0 & 3 \end{pmatrix}$$
$$\sigma_5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 5 & 2 & 3 & 4 & 1 \end{pmatrix}.$$

Then $zp_{\sigma_k} \in s_0$ for any $z \in s^k_0$ $(1 \le k \le 5)$. Hence the set of normal forms of matrices in s^k_0 can be taken as follows

$$N_{s^k_0} = \left\{ \sigma \vec{z}_i p_{\sigma_k} \;:\; \sigma \in \mathfrak{S}_3, \vec{z}_i \in N, 0 \le i \le 14\} \right\}.$$

Thus we have obtained the following proposition, with Proposition 3.4 gives the complete list of normal forms for $\nu = 1, 2, 3$.

Proposition 3.5. *For $\nu = 1, 2, 3$ the normal forms of $z \in s^k_0 (k = 1, 2, \ldots, 5)$ can be taken as follows.*

For $\nu = 1$, set $I = \{0, 1, \ldots, 14\}$.

$$N_{s^1_1} = \left\{ \sigma \vec{z}_i p_{\sigma_1} \;:\; \sigma \in \mathfrak{S}_3, \vec{z}_i \in N, i \in I \setminus \{1, 2, 6, 12\} \right\},$$
$$N_{s^3_1} = \left\{ \sigma \vec{z}_i p_{\sigma_5} \;:\; \sigma \in \mathfrak{S}_3, \vec{z}_i \in N, i \in I \setminus \{0, 1, 5, 8\} \right\},$$
$$N_{s^4_1} = \left\{ \sigma \vec{z}_i p_{\sigma_3} \;:\; \sigma \in \mathfrak{S}_3, \vec{z}_i \in N, i \in I \setminus \{4, 5, 6, 7, 10, 14\} \right\}.$$

For $\nu = 2$,

$$N_{s_2^1} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_1} \mid \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{4,5,8,9,10,13,14\} \right\},$$
$$N_{s_2^2} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_2} \mid \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{0,3,4,5,7,8,9,10\} \right\},$$
$$N_{s_2^4} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_3} \mid \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{0,1,2,3,11,12\} \right\},$$
$$N_{s_2^5} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_4} \mid \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{1,5,6,8,10,12,13,14\} \right\},$$
$$N_{s_2^6} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_5} \mid \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{2,3,4,6,7,11,12,14\} \right\}.$$

For $\nu = 3$, $N_{s_3^1} = \left\{ \sigma \overrightarrow{z_i} p_{\sigma_1} \,:\, \sigma \in \mathfrak{S}_3, \overrightarrow{z_i} \in N, i \in \{4,5,8,9,10\} \right\}$.

Remark 3.2. In particular, the *(1,0)*, *(2,3)* and *(3,5)* entries of the normal forms can be taken to be zero as in Proposition 3.4.

Finally, for $4 \le \nu \le 8$, we consider the normal forms of the matrices in Z^ν. In a similar manner as in the proof of Proposition 3.3, we can obtain two types of normal forms for the matrices in s_ν^i as in the following Table I:

TABLE I

Matrices	Normal forms I	Normal forms II
$z \in s_4^1$	$\begin{pmatrix} 1 & 0 & 1 & 0 & x & y \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & x & y \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$
$z \in s_4^2$	$\begin{pmatrix} 1 & 0 & 1 & x & 0 & y \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & x & 0 & y \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
$z \in s_4^3$	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 0 & 1 \\ 0 & 0 & y & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 0 & 1 \\ 0 & 0 & y & 0 & 1 & 1 \end{pmatrix}$
$z \in s_5^1$	$\begin{pmatrix} 1 & 0 & 1 & 0 & x & y \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & x & y \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$
$z \in s_5^2$	$\begin{pmatrix} 1 & 0 & 1 & 0 & x & y \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 & x & y \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$

$$z \in s_5^3 \qquad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 1 & 0 \\ 0 & 0 & y & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & x & 1 & 0 & 0 \\ 0 & 0 & y & 0 & 1 & 1 \end{pmatrix}$$

$$z \in s_5^4 \qquad \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & 1 & 0 \\ 0 & 1 & y & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 1 \end{pmatrix}$$

$$z \in s_6^1 \qquad \begin{pmatrix} 1 & 0 & 1 & 0 & x & y \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & x & y \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$z \in s_7^1 \qquad \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & y \\ 0 & 1 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & y \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$z \in s_8^1 \qquad \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & y \\ 0 & 1 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & y \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Thus, for each ν, we have found the normal forms of the matrices in Z^ν. We can reduce the function Φ on Z^ν into a function of two variables by restricting Φ to the normal forms.

§4 CHG functions and the classical CHG functions of 2 variables.

Definition 4.1. For $0 \le \nu \le 8$, the function Φ given by (0.3), i.e.,

$$\Phi(z;\alpha) = \int_\Delta \chi(\iota^{-1}(tz);\alpha)\cdot\omega \qquad \text{for} \quad z \in Z^\nu \tag{4.1}$$

is called the confluent hypergeometric function of type $\lambda^{(\nu)}$ (CHG function, for short).

Let $t = (t_0, t_1, t_2)$ be the homogeneous coordinates of the complex projective plane $\mathbb{P}^2(\mathbb{C})$, $z = (z_0, \dots, z_5) \in Z^\nu$. Set $tz = (tz_0, \dots, tz_5) = (t_0, t_1, t_2)(z_0, \dots, z_5)$. For the parameters $\alpha = (\alpha_0, \dots, \alpha_5)$ and $z \in Z^\nu$, the integrand $\chi(\iota^{-1}(tz);\alpha)$ is given by (0.2).

The CHG functions of type $\lambda^{(\nu)}$ admit the symmetries described by (0.2)-(0.4) in Introduction. The symmetry (0.4) can be used to normalize the parameters $(\alpha_0, \dots, \alpha_5)$ (see [K-K]). For example, in the case of $\lambda^{(7)}$, if the parameters $(\alpha_0, \dots, \alpha_5)$ satisfy $\alpha_2 \ne 0, \alpha_5 \ne 0$, then there exists $g \in R_{\lambda^{(\nu)}}$ such that transformation $z \mapsto zg^{-1}$ takes $\Phi(z;\alpha)$ into $\Phi(zg^{-1};\beta)$ with the parameters $\beta = (\alpha_0,\, 0,\, \pm1,\, \alpha_3,\, 0,\, \pm1)$. We call the β's the normal forms of the parameters α. Later we shall list up the functions Φ on Z^ν with normalized parameters (see Table III-V).

On the other hand, there is a list of the classical CHG functions of two variables, which is known as Horn's list (see [Erd 1]). In the list, the functions $F_2, F_3, H_2, \Psi_1, \Psi_2, \Xi_1, \Xi_2, \mathbf{H}_2, \mathbf{H}_3, \mathbf{H}_4, \mathbf{H}_5$ and $\mathbf{H}_{11}$ are given by the convergent power series. Among these functions, the first three are functions with regular singularities and the rest are obtained from the first three by the process of confluence:

$$\begin{aligned}
\Psi_1(\alpha,\beta,\gamma,\gamma';x,y) &= \lim_{\varepsilon\to0} F_2(\alpha,\beta,\frac{1}{\varepsilon},\gamma,\gamma';x,\varepsilon y)\\
\Psi_2(\alpha,\gamma,\gamma';x,y) &= \lim_{\varepsilon\to0} F_2(\alpha,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\gamma,\gamma';\varepsilon x,\varepsilon y)\\
\Xi_1(\alpha,\alpha',\beta,\gamma;x,y) &= \lim_{\varepsilon\to0} F_3(\alpha,\alpha',\beta,\frac{1}{\varepsilon},\gamma;x,\varepsilon y)\\
\Xi_2(\alpha,\beta,\gamma;x,y) &= \lim_{\varepsilon\to0} F_3(\alpha,\frac{1}{\varepsilon},\beta,\frac{1}{\varepsilon},\gamma;\varepsilon x,\varepsilon y)\\
\mathbf{H}_2(\alpha,\beta,\beta',\delta;x,y) &= \lim_{\varepsilon\to0} H_2(\alpha,\beta,\beta',\frac{1}{\varepsilon},\delta;x,\varepsilon y)\\
\mathbf{H}_3(\alpha,\beta,\delta;x,y) &= \lim_{\varepsilon\to0} H_2(\alpha,\beta,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\delta;x,\varepsilon^2 y)\\
\mathbf{H}_4(\alpha,\gamma,\delta;x,y) &= \lim_{\varepsilon\to0} H_2(\alpha,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\gamma,\delta;\varepsilon x,\varepsilon y)\\
\mathbf{H}_5(\alpha,\delta;x,y) &= \lim_{\varepsilon\to0} H_2(\alpha,\frac{1}{\varepsilon},\frac{1}{\varepsilon},\frac{1}{\varepsilon},\delta;\varepsilon x,\varepsilon^2 y)\\
\mathbf{H}_{11}(\alpha,\beta',\gamma,\delta;x,y) &= \lim_{\varepsilon\to0} H_2(\alpha,\frac{1}{\varepsilon},\beta',\gamma,\delta;\varepsilon x,y)
\end{aligned}$$

Integral representations of these functions have been investigated by M. Kita [Ki], M. Yoshida [Y], B. Dwork and F. Loeser [D-L]. In particular,the last authors have given integral representations for all functions in Horn's list. For the above functions, the integrands of the integral representations are given as follows (see [D-L], [Y]):

$$F_2: \quad u^{\alpha-\gamma'}v^{\alpha-\gamma}(u+v-uv)^{\gamma+\gamma'-\alpha-2}(1-xu)^{-\beta}(1-yv)^{-\beta'}dudv$$

$$
\begin{aligned}
\Psi_1: &\quad u^{\alpha-\gamma'}v^{\alpha-\gamma}(u+v-uv)^{\gamma+\gamma'-\alpha-2}(1-xu)^{-\beta}\exp(yv)dudv\\
\Psi_2: &\quad u^{\alpha-\gamma'}v^{\alpha-\gamma}(u+v-uv)^{\gamma+\gamma'-\alpha-2}\exp(xu-yv)dudv\\
F_3: &\quad u^{\alpha-1}v^{\alpha'-1}(1-u-v)^{\gamma+\alpha-\alpha'-1}(1-xu)^{-\beta}(1-yv)^{-\beta'}dudv\\
\Xi_1: &\quad u^{\alpha-1}v^{\alpha'-1}(1-u+v)^{\gamma-\alpha-\alpha'-1}(1-xu)^{-\beta}\exp(yv)dudv\\
\Xi_2: &\quad u^{\beta-1}v^{\beta-\gamma}(1-xu)^{-\alpha}\exp(uv-v-\frac{y}{v})dudv\\
H_2: &\quad u^{\alpha+\gamma-1}v^{\gamma-1}(uv+u-1)^{\delta-\alpha-\gamma-1}(1-xu)^{-\beta}(1-yv)^{-\beta'}dudv\\
\mathbf{H}_2: &\quad u^{\delta-2}v^{\delta-\alpha-2}(1-xu)^{-\beta}(1-yv)^{-\beta'}\exp(\frac{1}{uv}-\frac{1}{v})dudv\\
\mathbf{H}_3: &\quad u^{\delta-2}v^{\delta-\alpha-2}(1-xu)^{-\beta}\exp(\frac{1}{uv}-\frac{1}{v}+yv)dudv\\
\mathbf{H}_5: &\quad u^{\delta-2}v^{\delta-\alpha-2}\exp(\frac{1}{uv}-\frac{1}{v}+xu+yv)dudv\\
H_2: &\quad u^{\alpha+\gamma-1}(1-xu)^{-\beta}(1-yv)^{-\beta'}v^{\gamma-1}(uv+u-1)^{\delta-\alpha-\gamma-1}dudv\\
\mathbf{H}_{11}: &\quad u^{\alpha+\gamma-1}v^{\gamma-1}(1-u-uv)^{\delta-\gamma-\alpha-1}(1-yv)^{-\beta'}\exp(xu)dudv\\
\mathbf{H}_4: &\quad u^{\alpha+\gamma-1}v^{\gamma-1}(1-u-uv)^{\delta-\alpha-\gamma-1}\exp(xu+yv)dudv
\end{aligned}
$$

We reinterpret these integral representations in terms of the CHG functions. The changes of variables

$$
\begin{aligned}
&(u,v)\longrightarrow(-1/u,-1/v) &&\text{for } F_2,\ \Psi_1 \text{ and } \Psi_2,\\
&(u,v)\longrightarrow(-u,-v) &&\text{for } F_3 \text{ and } \Xi_1,\\
&(u,v)\longrightarrow(-u,1/v) &&\text{for } \Xi_2,\\
&(u,v)\longrightarrow(-1/u,v) &&\text{for } H_2 \text{ and } \mathbf{H}_k\ (k=2,3,4,5,11)
\end{aligned}
$$

transform the above integral representations into the following:

$$
\begin{aligned}
F_2: &\quad v^{\beta'-\gamma'}(v+y)^{-\beta'}u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
\Psi_1: &\quad v^{-\gamma'}\exp(-\frac{y}{v})u^{\beta-\gamma}(u+x)^{-\beta}(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
\Psi_2: &\quad v^{-\gamma'}\exp(-\frac{y}{v})u^{-\gamma}\exp(-\frac{x}{u})(1+u+v)^{\gamma+\gamma'-\alpha-2}dudv\\
F_3: &\quad (1+yv)^{-\beta'}v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv\\
\Xi_1: &\quad \exp(-yv)v^{\alpha'-1}(1+u+v)^{\gamma-\alpha-\alpha'-1}u^{\alpha-1}(1+xu)^{-\beta}dudv
\end{aligned}
$$

$$\Xi_2: \quad \exp(-yv)v^{\gamma-\beta-2}\exp(-\frac{u+1}{v})u^{\beta-1}(1+xu)^{-\alpha}dudv$$
$$H_2: \quad v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}(1-yu)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv$$
$$\mathbf{H}_2: \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})(1-yv)^{-\beta'}u^{\beta-\delta}(u+x)^{-\beta}dudv$$
$$\mathbf{H}_3: \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{\beta-\delta}(u+x)^{-\beta}dudv$$
$$\mathbf{H}_5: \quad v^{\delta-\alpha-2}\exp(-\frac{u+1}{v})\exp(yv)u^{-\delta}\exp(-\frac{x}{u})dudv$$
$$H_2: \quad u^{\beta-\delta}(u+x)^{-\beta}(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$
$$\mathbf{H}_{11}: \quad u^{-\delta}\exp(-\frac{x}{u})(1-yv)^{-\beta'}v^{\gamma-1}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$
$$\mathbf{H}_4: \quad u^{-\delta}\exp(-\frac{x}{u})\exp(yv)v^{\gamma}(1+u+v)^{\delta-\alpha-\gamma-1}dudv$$

For these functions, the corresponding partitions λ and normal forms $\vec{x}_i = \vec{x}_i(x,y)$ are tabulated in the following:

TABLE II

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x,y)$	$g_i(x,y)$
F_2 ($\lambda^{(0)}$)	$\vec{x}_1 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(1-x-y)$ $(1-x)(1-y)$
Ψ_1 ($\lambda^{(1)}$)	$\vec{x}_2 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$xy(x-1)$
Ψ_2 ($\lambda^{(2)}$)	$\vec{x}_3 = \begin{pmatrix} 0 & y & 0 & x & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	xy
F_3 ($\lambda^{(0)}$)	$\vec{x}_4 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(xy-x-y)$ $(1-x)(1-y)$
Ξ_1 ($\lambda^{(1)}$)	$\vec{x}_5 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 1 & 0 & 0 \end{pmatrix}$	$xy(x-1)$

Function (λ)	Normal form $\vec{x}_i = \vec{x}_i(x,y)$	$g_i(x,y)$
Ξ_2 ($\lambda^{(2)}$)	$\vec{x}_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & x \\ 0 & y & 1 & 0 & 0 & 0 \end{pmatrix}$	$xy(x-1)$
H_2 ($\lambda^{(0)}$)	$\vec{x}_7 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_2$ ($\lambda^{(1)}$)	$\vec{x}_8 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$\mathbf{H}_3$ ($\lambda^{(2)}$)	$\vec{x}_9 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	$xy(x-1)$
$\mathbf{H}_5$ ($\lambda^{(3)}$)	$\vec{x}_{10} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & y & 0 & 0 \end{pmatrix}$	xy
H_2 ($\lambda^{(0)}$)	$\vec{x}_{11} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(xy-y+1)$ $(1-x)(1-y)$
$\mathbf{H}_{11}$($\lambda^{(1)}$)	$\vec{x}_{12} = \begin{pmatrix} 0 & x & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	$xy(y-1)$
$\mathbf{H}_4$ ($\lambda^{(2)}$)	$\vec{x}_{13} = \begin{pmatrix} 0 & x & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & y & 1 & 1 \end{pmatrix}$	xy

For the normal forms $\vec{x}_i = \vec{x}_i(x,y)$, the variables $(x,y) \in \mathbb{C}^2$ are subject to the condition $g_i(x,y) \neq 0$.

Proposition 4.1. *Let $\lambda^{(\nu)}$ and the normal forms $\vec{x_i} = \vec{x}_i(x,y)$ be given in Table III. The CHG functions on $GL(3) \setminus Z^\nu / H_{\lambda^{(\nu)}}$ with the normalized parameters β_ν ($0 \leq \nu \leq 3$) are related with the classical hypergeometric functions* *of two variables as*

$$
\begin{aligned}
\Phi_{\lambda^{(0)}}(\overrightarrow{x}_1;\beta_0) =& \int_{\Delta_1} v^{\alpha_0}(v+y)^{\alpha_1}u^{\alpha_2}(u+x)^{\alpha_3}(1+u+v)^{\alpha_5}dudv \\
=& C_1F_2(\alpha_4+1,-\alpha_3,-\alpha_1,-\alpha_2-\alpha_3,-\alpha_0-\alpha_1;x,y) \\
\Phi_{\lambda^{(1)}}(\overrightarrow{x}_2;\beta_1) =& \int_{\Delta_2} v^{\alpha_0}\exp(-\frac{y}{v})u^{\alpha_2}(u+x)^{\alpha_3}(1+u+v)^{\alpha_5}dudv \\
=& C_2\Psi_1(\alpha_4+1,-\alpha_3,-\alpha_3-\alpha_2,-\alpha_0;x,y) \\
\Phi_{\lambda^{(2)}}(\overrightarrow{x}_3;\beta_2) =& \int_{\Delta_3} v^{\alpha_0}\exp(-\frac{y}{v})u^{\alpha_2}\exp(-\frac{x}{u})(1+u+v)^{\alpha_5}dudv \\
=& C_3\Psi_2(\alpha_4+1,-\alpha_2,-\alpha_0;x,y) \\
\Phi_{\lambda^{(0)}}(\overrightarrow{x}_4;\beta_0) =& \int_{\Delta_4} (1+yv)^{\alpha_1}v^{\alpha_2}(1+u+v)^{\alpha_3}u^{\alpha_4}(1+xu)^{\alpha_5}dudv \\
=& C_4F_3(\alpha_4+1,\alpha_2+1,-\alpha_5,-\alpha_1,-\alpha_0-\alpha_1-\alpha_5;x,y) \\
\Phi_{\lambda^{(1)}}(\overrightarrow{x}_5;\beta_1) =& \int_{\Delta_5} \exp(-yv)v^{\alpha_2}(1+u+v)^{\alpha_3}u^{\alpha_4}(1+xu)^{\alpha_5}dudv \\
=& C_5\Xi_1(\alpha_4+1,\alpha_2+1,-\alpha_5,-\alpha_0-\alpha_5;x,y) \\
\Phi_{\lambda^{(2)}}(\overrightarrow{x}_6;\beta_2) =& \int_{\Delta_6} \exp(-yv)v^{\alpha_2}\exp(-\frac{u+1}{v})u^{\alpha_4}(1+xu)^{\alpha_5}dudv \\
=& C_6\Xi_2(-\alpha_5,\alpha_4+1,-\alpha_0-\alpha_5;x,y) \\
\Phi_{\lambda^{(0)}}(\overrightarrow{x}_7;\beta_0) =& \int_{\Delta_7} v^{\alpha_0}(1+u+v)^{\alpha_1}(1-yu)^{\alpha_3}u^{\alpha_4}(u+x)^{\alpha_5}dudv \\
=& C_7H_2(\alpha_2+\alpha_3+1,-\alpha_5,-\alpha_3,\alpha_0+1,-\alpha_4-\alpha_5;x,-y) \\
\Phi_{\lambda^{(1)}}(\overrightarrow{x}_8;\beta_1) =& \int_{\Delta_8} v^{\alpha_0}\exp(-\frac{u+1}{v})(1-yv)^{\alpha_3}u^{\alpha_4}(u+x)^{\alpha_5}dudv \\
=& C_8\mathbf{H}_2(\alpha_2+\alpha_3+1,-\alpha_5,-\alpha_3,\alpha_4-\alpha_5;x,-y) \\
\Phi_{\lambda^{(2)}}(\overrightarrow{x}_9;\beta_2) =& \int_{\Delta_9} v^{\alpha_0}\exp(-\frac{u+1}{v})\exp(yu)u^{\alpha_4}(u+x)^{\alpha_5}dudv \\
=& C_9\mathbf{H}_3(\alpha_2+1,,-\alpha_5,-\alpha_4-\alpha_5;x,-y) \\
\Phi_{\lambda^{(3)}}(\overrightarrow{x}_{10};\beta_3) =& \int_{\Delta_{10}} v^{\alpha_0}\exp(-\frac{u+1}{v})\exp(yu)u^{\alpha_4}\exp(-\frac{x}{u})dudv \\
=& C_{10}\mathbf{H}_5(\alpha_2+1,-\alpha_4;x,-y) \\
\Phi_{\lambda^{(0)}}(\overrightarrow{x}_{11};\beta_0) =& \int_{\Delta_{11}} u^{\alpha_0}(u+x)^{\alpha_1}(1-yv)^{\alpha_3}v^{\alpha_4}(1+u+v)^{\alpha_5}dudv
\end{aligned}
$$

$$
\begin{aligned}
&= C_{11} H_2(\alpha_2+\alpha_3+1, -\alpha_1, -\alpha_3, -\alpha_0-\alpha_1; x, -y)\\
\Phi_{\lambda^{(1)}}(\vec{x}_{12};\beta_1) &= \int_{\Delta_{12}} u^{\alpha_0}\exp(-\frac{x}{u})(1+yv)^{\alpha_3} v^{\alpha_4}(1+u+v)^{\alpha_5} dudv\\
&= C_{12}\mathbf{H}_{11}(\alpha_2+\alpha_3+1, -\alpha_3, \alpha_4+1, -\alpha_0; x, -y)\\
\Phi_{\lambda^{(2)}}(\vec{x}_{13};\beta_2) &= \int_{\Delta_{13}} u^{\alpha_0}\exp(-\frac{x}{u})\exp(yv) v^{\alpha_4}(1+u+v)^{\alpha_5} dudv\\
&= C_{13}\mathbf{H}_4(\alpha_2+2, , \alpha_4, -\alpha_0; x, -y),
\end{aligned}
$$

where $C_0, \dots, C_{13}$ *are constants and*

$$
\begin{aligned}
\beta_0 &= (\alpha_0, \dots, \alpha_5) && \alpha_0+\cdots+\alpha_5 = -3,\\
\beta_1 &= (\alpha_0, -1, \alpha_2, \dots, \alpha_5) && \alpha_0+\alpha_2+\cdots+\alpha_5 = -3,\\
\beta_2 &= (\alpha_0, -1, \alpha_2, -1, \alpha_4, \alpha_5) && \alpha_0+\alpha_2+\alpha_4+\alpha_5 = -3,\\
\beta_3 &= (\alpha_0, -1, \alpha_2, -1, \alpha_4, -1) && \alpha_0+\alpha_2+\alpha_4 = -3.
\end{aligned}
$$

Proposition 4.1 provides a reinterpretation of the classical functions F_2, Ψ_1, $\dots, \mathbf{H}_4$ in terms of the CHG functions. The properties of those functions can be described in the Table III.

In the Table III, G_i is a multi-valued holomorphic function in the domain:

$$
X_i = \{(x,y) \in \mathbb{C}^2 \mid g_i(x,y) \neq 0\}, \tag{4.2}
$$

where $g_i(x,y)$ $(1 \leq i \leq 13)$ are given in Table II.

Note that the functions $\{F_2,\ F_3,\ H_2\}$, $\{\Xi_2,\ \mathbf{H}_3\}$, $\{\Psi_1,\ \mathbf{H}_{11}\}$ belong to the same orbits, respectively.

For $\nu = 1,2,3$, we find that the orbits not appearing in Table III are:

$$
\begin{aligned}
&O_{P_{\lambda^{(1)}}}(s_1^4) && \text{for} \quad \nu = 1,\\
&O_{P_{\lambda^{(2)}}}(s_2^4), \quad O_{P_{\lambda^{(2)}}}(s_2^5), \quad O_{P_{\lambda^{(2)}}}(s_2^6) && \text{for} \quad \nu = 2,\\
&O_{P_{\lambda^{(3)}}}(s_3^2) && \text{for} \quad \nu = 3.
\end{aligned} \tag{4.3}
$$

By the symmetry of the function Φ, we have only to consider $\Phi(z_\nu^i;\beta'_\nu)$, where $z_\nu^i \in N_{s_\nu^i}$ (see Proposition 3.5) and β'_ν are the normalized parameters given by

$$
\beta'_\nu = (\beta^{(0)}, \dots, \beta^{(l-1)}), \qquad \beta^{(i)} = (\underbrace{\alpha_0^{(i)}, 0, \dots, 0, 1}_{\lambda_i}). \tag{4.4}
$$

TABLE III

Type	Function	Stratum	Orbit
$(1,1,1,1,1,1)$	$G_1 = F_2$	$\{(4,0,1),(4,2,3)\}$	S
$(2,1,1,1,1)$	$G_2 = \Psi_1$	〃	$O_{P_{\lambda(1)}}(s_1^2)$
$(2,2,1,1)$	$G_3 = \Psi_2$	〃	$O_{P_{\lambda(2)}}(s_2^3)$
$(1,1,1,1,1,1)$	$G_4 = F_3$	$\{(0,1,2),(0,4,5)\}$	S
$(2,1,1,1,1)$	$G_5 = \Xi_1$	〃	$O_{P_{\lambda(1)}}(s_1^1)$
$(2,2,1,1)$	$G_6 = \Xi_2$	〃	$O_{P_{\lambda(2)}}(s_2^1)$
$(1,1,1,1,1,1)$	$G_7 = H_2$	$\{(2,0,3),(2,4,5)\}$	S
$(2,1,1,1,1)$	$G_8 = \mathbf{H}_2$	〃	$O_{P_{\lambda(1)}}(s_1^3)$
$(2,2,1,1)$	$G_9 = \mathbf{H}_3$	〃	$O_{P_{\lambda(2)}}(s_2^1)$
$(2,2,2)$	$G_{10} = \mathbf{H}_5$	〃	$O_{P_{\lambda(3)}}(s_3^2)$
$(1,1,1,1,1,1)$	$G_{11} = H_2$	$\{(2,0,1),(2,3,4)\}$	S
$(2,1,1,1,1)$	$G_{12} = \mathbf{H}_{11}$	〃	$O_{P_{\lambda(1)}}(s_1^2)$
$(2,2,1,1)$	$G_{13} = \mathbf{H}_4$	〃	$O_{P_{\lambda(2)}}(s_2^2)$

We consider the cases $(\nu,\, i)$=(1, 4), (2, 4). In these cases, we can take

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & y & 1 & 1 & 0 & 0 \\ 0 & x & 0 & 0 & 1 & 1 \end{pmatrix}$$

as the normal form z_ν^i. The parameters β'_ν are given by

$$\beta'_1 = (\alpha_0, 1, \alpha_2, \dots, \alpha_5), \beta'_2 = (\alpha_0, 1, \alpha_2, 1, \alpha_3, \alpha_4, \alpha_5).$$

Then the function $\Phi(z_\nu^i; \beta'_\nu)$ are expressed as

$$\Phi(z_1^4; \beta'_1) = \int u^{\alpha_2}(1+u)^{\alpha_3} \exp(yu)du \int v^{\alpha_4}(1+v)^{\alpha_5} \exp(xv)dv,$$

$$\Phi(z_2^4; \beta'_2) = \int u^{\alpha_2} \exp(yu + \frac{1}{u})du \int v^{\alpha_4}(1+v)^{\alpha_5} \exp(xv)dv.$$

Noting that $\Phi(z_1^4;\beta_1')$ is the product of two Kummer's CHG functions, $\Phi(z_2^4;\beta_2')$ is the product of Kummer's CHG function and Bessel's function.

For $(\nu,\ i)$=(2, 5), (2, 6), we take

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & y \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & x & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & y & 1 & x \end{pmatrix}$$

as the normal forms z_2^5 and z_2^6, respectively. Then we have

$$\Phi(z_2^5;\beta_2') = \int u^{\alpha_2}(y+u)^{\alpha_5} \exp\frac{1}{u} \cdot v^{\alpha_4} \exp[(1+\frac{x}{u})v]dudv,$$

$$\Phi(z_2^6;\beta_2') = \int v^{\alpha_4}(1+xv)^{\alpha_5} \exp(yv) \cdot u^{\alpha_0} \exp(\frac{1}{u}+\frac{v}{u})dudv.$$

The functions $\Phi(z_2^5;\beta_2')$ and $\Phi(z_2^6;\beta_2')$ can be decomposed into the product of a elementary function and the hypergeometric function Φ_1 in Horn's list.

For $(\nu,\ i)$= (3, 2), we take

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & y \\ 1 & 1 & 0 & 0 & 0 & x \end{pmatrix}$$

as the normal form z_3^2. The parameters $\beta_3' = (\alpha_0, 1, \alpha_2, 1, \alpha_4, 1)$. Then we have

$$\Phi(z_3^2;\beta_3') = \int v^{\alpha_0} \exp(xv+\frac{1}{v})dv \int u^{\alpha_2} \exp(yu+\frac{1}{u})du,$$

which is the product of two Bessel's functions.

We list the functions $\Phi(z_\nu^i;\beta')$ for the orbits given in (4.3) in the following Table IV:

TABLE IV

$(\nu,\ i)$	Function $\Phi(z_\nu^i;\beta_\nu')$	β_ν'
(1, 4)	Kummer × Kummer	$(\alpha_0, 1, \alpha_2, \dots, \alpha_5)$
(2, 4)	Kummer × Bessel	$(\alpha_0, 1, \alpha_2, 1, \alpha_4, \alpha_5)$
(2, 5)	Elementary $\times\Phi_1$	〃
(2, 6)	〃	〃
(3, 2)	Bessel × Bessel	$(\alpha_0, 1, \alpha_2, 1, \alpha_4, 1)$

Finally, for $\nu = 4, \ldots, 8$, we take the 'Normal forms II' in Table I as the normal forms z_ν^i, and (4.4) as the normalized parameters β'_ν. Then we obtain the functions $\Phi(z_\nu^i; \beta'_\nu)$ as in the following Table V.

TABLE V

$(\nu,\ i)$	Function $\Phi(z_\nu^i; \beta'_\nu)$	β'_ν
(4, 1)	$\Phi(z_4^1; \beta'_4)$	$(\alpha_0, 0, 1, \alpha_3, \alpha_4, \alpha_5)$
(4, 2)	Kummer $\times$ Hermite	〃
(4, 3)	$\Phi(z_4^3; \beta'_4)$	〃
(5, 1)	$\Phi(z_5^1; \beta'_5)$	$(\alpha_0, 0, 1, \alpha_3, 1, \alpha_5)$
(5, 2)	Elementary $\times$ Kummer	〃
(5, 3)	Elementary $\times \Phi_{(3,1,1)}$	〃
(5, 4)	$\Phi(z_5^4; \beta'_5)$	〃
(6, 1)	Elementary $\times$ Hermite	$(\alpha_0, 0, 1, \alpha_3, 0, 1)$
(7, 1)	Kummer $\times$ Airy	$(\alpha_0, 0, 0, 1, \alpha_4, \alpha_5)$
(8, 1)	Bessel $\times$ Airy	$(\alpha_0, 0, 0, 1, \alpha_4, 1)$

In the Table,

$$\Phi(z_4^1; \beta'_4) = \int v^{\alpha_3}(x+v)^{\alpha_4}(y+u+v)^{\alpha_5} \exp(-\frac{u^2}{2}) du dv,$$
$$\Phi(z_4^3; \beta'_4) = \int u^{\alpha_3} v^{\alpha_4}(u+v)^{\alpha_5} \exp(xu - \frac{u^2}{2}) \exp(yv) du dv,$$
$$\Phi(z_5^1; \beta'_5) = \int v^{\alpha_3}(y+u+v)^{\alpha_5} \exp(\frac{x}{u} - \frac{u^2}{2}) du dv,$$
$$\Phi(z_5^4; \beta'_5) = \int u^{\alpha_3} v^{\alpha_5} \exp(xu + yv - \frac{u^2}{2}) \exp \frac{v}{u} du dv.$$

It still remains open whether these functions are expressed as the product of classical special functions of one variable.

§5 Transformation formulae.

We deduce systematically some transformation formulae for the systems of differentical equations satisfied by some classical special functions of two variables from the symmetries (0.2)-(0.4) for the function Φ.

Let p_k be the permutation matrices defined by

$$p_k = \left(\delta_{i\sigma^k(j)}\right) \qquad k = 1, \ldots, 5 \qquad 0 \le i, j \le 5, \tag{5.1}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 0 & 3 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$
$$\sigma^3 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 0 & 1 & 4 & 5 \end{pmatrix} \qquad \sigma^4 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}$$
$$\sigma^5 = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 0 & 1 & 4 & 5 \end{pmatrix}.$$

Then $p_1, p_2, p_3 \in W$, $p_4 \in W_{\lambda^{(1)}}$, $p_5 \in W_{\lambda^{(2)}}$. Set

$$\begin{aligned} h_1(x,y) =&\mathrm{diag}\,(1, \frac{1}{y}, 1, \frac{1}{x}, 1, 1), \quad h_2(x,y) = \mathrm{diag}\,(1, \frac{1}{y}, 1, 1, 1, 1), \\ h_3(x,y) =&\mathrm{diag}\,(1, 1, 1, 1, 1, \frac{1}{x}), \quad h_4(x,y) = \mathrm{diag}\,(1, 1, 1, \frac{1}{y}, 1, 1), \\ h_5(x,y) =&\mathrm{diag}\,(1, 1, 1, 1, 1, \frac{1}{x}) \end{aligned} \tag{5.2}$$

and

$$g_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} &\vec{x}_4(x,y)p_1h_1 = \vec{x}_1(\frac{1}{x}, \frac{1}{y}), \qquad \vec{x}_7(x,y)p_2h_2 = \vec{x}_1(x, \frac{1}{y}), \\ &\vec{x}_4(x,y)p_3h_3 = \vec{x}_7(\frac{1}{x}, y), \qquad g_3\vec{x}_{12}(x,y)p_4h_4 = \vec{x}_2(\frac{1}{y}, x), \\ &\vec{x}_9(x,y)p_5h_5 = \vec{x}_6(\frac{1}{x}, y). \end{aligned} \tag{5.3}$$

By Proposition 4.1, we have

$$\begin{aligned} \Phi(\vec{x}_1(x,y); \beta_0) =&C_1F_2(\alpha_4 + 1, -\alpha_3, -\alpha_1, -\alpha_2 - \alpha_3, -\alpha_0 - \alpha_1; x, y), \\ \Phi(\vec{x}_4(x,y); \beta_0) =&C_4F_3(\alpha_4 + 1, \alpha_2 + 1, -\alpha_5, -\alpha_1, -\alpha_0 - \alpha_1 - \alpha_5; x, y). \end{aligned}$$

Using the symmetries (0.2)-(0.4), we have

$$\Phi(\vec{x}_4(x,y)p_1h_1(x,y);\beta_0) = x^{-\alpha_3}y^{-\alpha_1}\Phi(\vec{x}_4(x,y);\beta_0{}^tp_1),$$

where $\beta_0{}^tp_1 = (\alpha_4,\alpha_1,\alpha_0,\alpha_5,\alpha_2,\alpha_3)$. Hence

$$\begin{aligned}&\Phi(\vec{x}_4(x,y)p_1h_1(x,y);\beta_0)\\ =&C_4x^{-\alpha_3}y^{-\alpha_1}F_3(\alpha_2+1,\alpha_0+1,-\alpha_3,-\alpha_1,-\alpha_1-\alpha_3-\alpha_4;x,y).\end{aligned}$$

By (5.3),

$$\begin{aligned}&\Phi(\vec{x}_4(x,y)p_1h_1(x,y);\beta_0) = \Phi(\vec{x}_1(\frac{1}{x},\frac{1}{y});\beta_0)\\ =&C_1F_2(\alpha_4+1,-\alpha_3,-\alpha_1,-\alpha_2-\alpha_3,-\alpha_0-\alpha_1;\frac{1}{x},\frac{1}{y}).\end{aligned}$$

Therefore, setting $\alpha = \alpha_2+1$, $\alpha' = \alpha_0+1$, $\beta = -\alpha_3$, $\beta' = -\alpha_1$, $\gamma = -\alpha_1-\alpha_3-\alpha_4$, we obtain the following proposition.

Proposition 5.1. *The system of differential equsitions satisfied by the function* $F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)$ *is identical to that of* $x^{-\beta}y^{-\beta'}F_2(\beta+\beta'-\gamma+1,\beta,\beta',\beta-\alpha+1,\beta'-\alpha'+1;\frac{1}{x},\frac{1}{y})$.

For simplicity, we write

$$\begin{aligned}&F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)\\ =&x^{-\beta}y^{-\beta'}F_2(\beta+\beta'-\gamma+1,\beta,\beta',\beta-\alpha+1,\beta'-\alpha'+1;\frac{1}{x},\frac{1}{y})\end{aligned} \tag{5.4}$$

for the transformation relation in the proposition 5.1.

Similarly, the relations between the systems of differential equsitions satisfied by the functions F_2 and H_2, H_2 and F_3, Ψ_1 and $\mathbf{H}_{11}$, Ξ_2 and $\mathbf{H}_3$, are given as follows:

$$H_2(\alpha,\beta,\beta',\gamma,\delta;x,y) = y^{-\beta'}F_2(\alpha+\beta',\beta,\beta',\delta,\beta'-\gamma+1;x,-\frac{1}{y}), \tag{5.5}$$

where $\alpha=\alpha_1+\alpha_4+1$, $\beta=-\alpha_3$, $\beta'=-\alpha_1$, $\gamma=\alpha_0+1$, $\delta=-\alpha_2-\alpha_3$;

$$
\begin{aligned}
&F_3(\alpha,\alpha',\beta,\beta',\gamma;x,y)\\
=&x^{-\beta}H_2(\beta-\gamma+1,\beta,\beta',\alpha',\beta-\alpha+1;\frac{1}{x},-y)\\
&((\alpha,\alpha')\leftrightarrow(\beta,\beta'))\\
=&x^{-\alpha}H_2(\alpha-\gamma+1,\alpha,\alpha',\beta',\alpha-\beta+1;\frac{1}{x},-y),
\end{aligned}
\tag{5.6}
$$

where $\alpha=\alpha_4+1$, $\alpha'=\alpha_0+1$, $\beta=-\alpha_5$, $\beta'=-\alpha_3$, $\gamma=-\alpha_2-\alpha_3-\alpha_5$;

$$
\mathbf{H}_{11}(\alpha,\beta',\gamma,\delta;x,y)=y^{-\beta'}\Psi_1(\alpha+\beta',\beta',\beta'-\gamma+1,\delta;-\frac{1}{y},x),
\tag{5.7}
$$

where $\alpha=\alpha_3+\alpha_4+1$, $\beta'=-\alpha_3$, $\gamma=\alpha_2+1$, $\delta=-\alpha_0$;

$$
\mathbf{H}_3(\alpha,\beta,\delta;x,y)=x^{-\beta}\Xi_2(\beta,\beta-\delta+1,-\alpha+\beta+1;\frac{1}{x},-y),
\tag{5.8}
$$

where $\alpha=\alpha_0+1$, $\beta=-\alpha_5$, $\delta=-\alpha_4-\alpha_5$.

Remark 5.1. The transformation formulae (5.4), (5.5) and (5.6) have previously been obtained by other methods, see [Erd 2].

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