MOLECULES OF THE HARDY SPACE AND THE NAVIER–STOKES EQUATIONS

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Abstract: For the incompressible Navier–Stokes system, we consider data $\vec{u}_0 \in (\mathrm{L}^3)^3$ whose Laplacian is a molecule of the Hardy space. We prove that this property is preserved by the only solution in $C\left([0,T[,(\mathrm{L}^3(\mathbb{R}^3))^3\right)$ at least at the beginning of its evolution.

In this paper we analyze the action of the Navier-Stokes equations on a subspace of $C\left([0,T[,(\mathrm{L}^3(\mathbb{R}^3))^3\right)$ which is specially adapted to study the location and oscillation of the diffusion term $\Delta\vec{u}$. Let us recall the Navier-Stokes system describing the motion of an incompressible, homogeneous, viscous fluid filling out the whole space $\mathbb{R}^3$, without the action of external forces:

$$\begin{cases} \partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}\cdot(\vec{u}\otimes\vec{u}) - \vec{\nabla}p \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

where $\vec{u}(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the velocity vector field and $p(t,x) : \mathbb{R}^+ \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is the pressure.

Dealing with the Cauchy problem with initial data $\vec{u}(0,x) = \vec{u}_0(x)$, it is possible to rewrite the system in the following integral expression:

$$\vec{u}(t) = \mathrm{e}^{t\Delta}\vec{u}_0 - \int_0^t \mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla}\cdot(\vec{u}\otimes\vec{u})(s)ds \tag{1}$$

where $\mathrm{e}^{t\Delta}$ is the heat semigroup defined by the convolution with the Gauss kernel:

$$\mathrm{e}^{t\Delta}f(x) = \left(\frac{1}{(4\pi t)^{\frac{3}{2}}}\mathrm{e}^{-\frac{|\cdot|^2}{4t}} * f\right)(x)$$

and $\mathbb{P}$ is the projection operator on the divergence–free vector fields, defined by the matrix:

$$\begin{bmatrix} \mathrm{Id} + R_1R_1 & R_1R_2 & R_1R_3 \\ R_2R_1 & \mathrm{Id} + R_2R_2 & R_2R_3 \\ R_3R_1 & R_3R_2 & \mathrm{Id} + R_3R_3 \end{bmatrix}$$

where $\widehat{R_j f}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$ are the Riesz transforms for $j = 1, 2, 3$.

In fact the integral and the differential systems are equivalent in this framework, as it is shown in [FLT] under slight assumptions verified by a large class of spaces.

Given a Banach space X, the problem of the existence and the uniqueness of a function $\vec{u}(t) \in C([0, T[, X^3)$ solution of (1) in $\mathcal{S}'$ (a so–called mild solution) is normally approached by a fixed point argument. It is now well known ([FLT], [MA], [LE2]) that for a given $\vec{u}_0 \in (\mathrm{L}^3(\mathbb{R}^3))^3$ the only solution in $C\left([0, T[, (\mathrm{L}^3(\mathbb{R}^3))^3\right)$ belongs to every $(\mathrm{L}^p(\mathbb{R}^3))^3$ space with $p > 3$ for all its existence time. We are now interested in investigating the following problem: if the data $\vec{u}_0$ have some extra properties, does the solution keep verifying them at least on a small interval of time?

In [FLT] it was already established that if $\Delta\vec{u}_0 \in (\mathcal{H}^1)^3$, the Hardy space, then the solution $\vec{u}(t)$ satisfies $\Delta\vec{u}(t) \in (\mathcal{H}^1)^3$ for $0 < t < T' \leq T$. We will consider here a particular case of the previous one where the Laplacian of the data is a "molecule" for the Hardy space, and so well-localized in the space variables; we will prove then that the solution verifies the same property, namely its diffusion term keeps on being well–localized at the beginning of the motion.

In the following the reference space will always be $\mathbb{R}^3$.

More precisely, let $\delta \in]\frac{3}{2}, \frac{9}{2}[$, $\delta \neq \frac{5}{2}, \frac{7}{2}$ and let us introduce the following functional space:

$$X_\delta = \left\{ u \in \mathcal{S}' \ : \ \begin{array}{ll} u \ \text{ vanishes at infinity,} & \\ \Delta u \in \mathrm{L}^2((1+|x|^{2\delta})dx), & \\ \int \Delta u(x)dx = 0 & \text{for } \ \frac{3}{2} < \delta < \frac{9}{2} \\ \int x_i \Delta u(x)dx = 0 & \text{for } \ \frac{5}{2} < \delta < \frac{9}{2}, \ i = 1,2,3 \\ \int x_i x_j \Delta u(x)dx = 0 & \text{for } \ \frac{7}{2} < \delta < \frac{9}{2}, \ i,j = 1,2,3 \end{array} \right\}.$$

The space X_δ normed by $\|u\|_{X_\delta} = \|\Delta u\|_{\mathrm{L}^2((1+|x|^{2\delta})dx)}$ is a Banach space.

The theorem we are going to prove is the following one.

Theorem .
Let $\vec{u}_0 \in X_\delta^3$ and $\vec{\nabla}\cdot\vec{u}_0 = 0$. Then the mild solution of the Navier–Stokes equations $\vec{u}(t) \in C([0,T[, (\mathrm{L}^3)^3)$ is also in $C\left([0, T'], {X_\delta}^3\right)$ for $0 < T' \leq T$.

We would like to stress that the solution we obtain decays pointwise in space like $\dfrac{1}{1+|x|^{\delta-\frac{1}{2}}}$ for every $t \in [0, T']$ (see also proposition 2). In the particular case $\frac{3}{2} < \delta < \frac{7}{2}$, this decay property may also be recovered by the results of Miyakawa in [MI1], who establishes a pointwise space–time asymptotic behavior with respect to space–time variables x and t for a particular class of data.

For other results close to these topics see also [AGSS], [B2], [C], [HX], [MI2], [SS], [T].

I. GENERAL PROPERTIES OF X_δ

In the following we will write

$$\mathrm{L}^2_\delta = \mathrm{L}^2((1+|x|^{2\delta})dx)$$
$$\|u\|_{\mathrm{L}^2_\delta} = \|u(x)\|_{\mathrm{L}^2((1+|x|^{2\delta})dx)},$$
$$\|u\|_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}} = \left\|(1+|x|^{\delta-\frac{1}{2}})u(x)\right\|_{\mathrm{L}^\infty}.$$

We remark that the definition of X_δ is well posed; indeed if $\Delta u \in \mathrm{L}^2_\delta$ then $\Delta u \in \mathrm{L}^1$, for $\delta > \frac{3}{2}$, $x_i\Delta u \in \mathrm{L}^1$ for $i = 1,2,3$ and $\delta > \frac{5}{2}$, $x_i x_j \Delta u \in \mathrm{L}^1$ for $i,j = 1,2,3$ and $\delta > \frac{7}{2}$. Moreover, $X_\delta \subset \Delta\mathcal{H}^1$ and more precisely if $u \in X_\delta$ then Δu is a non–normalized molecule for the Hardy space $\mathcal{H}^1$, which is defined as follows (see [ST][CW]).

Definition 1.
Let $\varepsilon > 0$. $M(x)$ is an ε–molecule for $\mathcal{H}^1(\mathbb{R}^3)$ centered at $x = 0$ if and only if:

$$(a) \quad \left(\int_{\mathbb{R}^3} |M(x)|^2 dx\right)\left(\int_{\mathbb{R}^3} |M(x)|^2 |x|^{3+\varepsilon} dx\right)^{\frac{1}{\varepsilon}} \leq 1$$

$$(b) \quad \int_{\mathbb{R}^3} M(x) dx = 0$$

In our case, the product (a) is bounded by a constant depending on u. It can be proved that any ε–molecule is in fact in $\mathcal{H}^1 \hookrightarrow \mathrm{L}^1$ ([CW]).

We prove now some decay properties of X_δ functions.

Proposition 2. *If $u \in X_\delta$ then $\left(1 + |x|^{\delta - \frac{1}{2}}\right) u(x) \in \mathrm{L}^\infty$.*

Proof. We will only consider the case $\frac{3}{2} < \delta < \frac{5}{2}$, the other ones being analogous. It is easy to see that $u \in \mathrm{L}^\infty$; in fact, because of vanishing at infinity we can write :

$$u(x) = \int_{\mathbb{R}^3} \frac{\Delta u(y)}{|x-y|} dy$$

and, by Hölder inequality,

$$\begin{aligned} |u(x)| &\leq \left(\int_{\mathbb{R}^3} \frac{1}{1+|y|^{2\delta}} \frac{1}{|x-y|^2} dy\right)^{\frac{1}{2}} \|\Delta u\|_{\mathrm{L}^2_\delta} \\ &\leq C \|\Delta u\|_{\mathrm{L}^2_\delta}. \end{aligned}$$

Now let $|x| \geq K$. We have:

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^3} \frac{\Delta u(y)}{|x-y|} dy \\ &= \int_{|y| \geq \frac{|x|}{2}, |x-y| \leq \frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|} dy + \int_{|y| \geq \frac{|x|}{2}, |x-y| > \frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|} dy + \int_{|y| < \frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|} dy. \end{aligned}$$

For the first term we can write:

$$\begin{aligned} &\left| \int_{|y| \geq \frac{|x|}{2}, |x-y| \leq \frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|} \frac{(1+|y|^{2\delta})^{\frac{1}{2}}}{(1+|y|^{2\delta})^{\frac{1}{2}}} dy \right| \\ \leq & C \|\Delta u\|_{\mathrm{L}^2_\delta} \left(\int_{|y| \geq \frac{|x|}{2}, |x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^2} \frac{1}{(1+|y|^{2\delta})} dy \right)^{\frac{1}{2}} \\ \leq & C \frac{\|\Delta u\|_{\mathrm{L}^2_\delta}}{|x|^\delta} \left(\int_{|x-y| \leq \frac{|x|}{2}} \frac{1}{|x-y|^2} dy \right)^{\frac{1}{2}} \\ \leq & C \|\Delta u\|_{\mathrm{L}^2_\delta} |x|^{\frac{1}{2} - \delta}. \end{aligned}$$

For the second one we have:

$$\begin{aligned} &\left| \int_{|y| \geq \frac{|x|}{2}, |x-y| > \frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|} \frac{(1+|y|^{2\delta})^{\frac{1}{2}}}{(1+|y|^{2\delta})^{\frac{1}{2}}} \right| \\ \leq & C \frac{\|\Delta u\|_{\mathrm{L}^2_\delta}}{|x|} \left(\int_{|y| \geq \frac{|x|}{2}} \frac{dy}{|y|^{2\delta}} \right)^{\frac{1}{2}} \\ \leq & C \|\Delta u\|_{\mathrm{L}^2_\delta} |x|^{\frac{1}{2} - \delta}. \end{aligned}$$

In order to estimate the last integral we will use the oscillation property $\int_{\mathbb{R}^3} \Delta u(x)dx = 0$. We get:

$$\begin{aligned}
&\left|\int_{|y|<\frac{|x|}{2}} \frac{\Delta u(y)}{|x-y|}dy\right| \\
=&\left|\int_{|y|<\frac{|x|}{2}} \Delta u(y)\left(\frac{1}{|x-y|}-\frac{1}{|x|}\right)dy - \int_{|y|\geq\frac{|x|}{2}} \frac{\Delta u(y)}{|x|}dy\right| \\
\leq& C\int_{|y|<\frac{|x|}{2}} |\Delta u(y)|\frac{|y|}{|x|^2}dy + \frac{1}{|x|}\int_{|y|\geq\frac{|x|}{2}} |\Delta u(y)|\,dy \\
\leq& \frac{C}{|x|^2}\|\Delta u\|_{\mathrm{L}^2_\delta}\left(\int_{|y|<\frac{|x|}{2}} \frac{|y|^2}{1+|y|^{2\delta}}dy\right)^{\frac{1}{2}} + \frac{C}{|x|}\|\Delta u\|_{\mathrm{L}^2_\delta}\left(\int_{|y|\geq\frac{|x|}{2}} \frac{1}{1+|y|^{2\delta}}dy\right)^{\frac{1}{2}} \\
\leq& \frac{C}{|x|^2}\|\Delta u\|_{\mathrm{L}^2_\delta}|x|^{\frac{5}{2}-\delta} + \frac{C}{|x|}\|\Delta u\|_{\mathrm{L}^2_\delta}|x|^{\frac{3}{2}-\delta} \\
\leq& C\|\Delta u\|_{\mathrm{L}^2_\delta}|x|^{\frac{1}{2}-\delta}.
\end{aligned}$$

Remarks. 1) We can also prove that if $\Delta u \in \mathrm{L}^2_\delta$ and u vanishes at infinity, then $u \in C_0$. In fact, starting from the equality

$$u(x) = \int_{\mathbb{R}^3} \frac{\Delta u(y)}{|x-y|}dy$$

and defining

$$u_{\varepsilon,R}(x) = \int_{\mathbb{R}^3} \frac{\Delta u(y)\chi_{B(0,R)} * \frac{1}{\varepsilon^3}\phi(\frac{y}{\varepsilon})}{|x-y|}dy$$

with $\phi \in \mathcal{D}$ and $\operatorname{supp}\phi \subset B(0,1)$, we get $u_{\varepsilon,R}(x) \in C^\infty(\mathbb{R}^3)$ and $u_{\varepsilon,R} \to u$ uniformely over $\mathbb{R}^3$.

2) If we only assume $u \in \mathrm{L}^\infty_{\delta-\frac{1}{2}}$ and $\Delta u \in \mathrm{L}^2_\delta$, it follows that $\int_{\mathbb{R}^3} \Delta u(x)dx = 0$ for every $\delta > \frac{3}{2}$. Indeed, let $\psi \in \mathcal{D}$ with $\operatorname{supp}\psi \subset B(0,1)$ and $\psi \equiv 1$ on $B(0,\frac{1}{2})$. Being $\Delta u \in \mathrm{L}^1$, we can write:

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \Delta u(x)dx\right| &= \lim_{R\to+\infty}\left|\int_{\mathbb{R}^3} \Delta u(x)\psi(\frac{x}{R})dx\right| \\
&\leq \lim_{R\to+\infty}\int_{\mathbb{R}^3} |u(x)||\Delta(\psi(\frac{x}{R}))|dx \\
&\leq \lim_{R\to+\infty}\frac{1}{R^2}\int_{\frac{R}{2}\leq|x|\leq R} |u(x)|dx \\
&\leq \lim_{R\to+\infty}\frac{1}{R^2}\frac{1}{R^{\delta-\frac{1}{2}}}R^3 = 0.
\end{aligned}$$

II. LOCALIZATION OF ELEMENTS IN X_δ

The previous proposition 2 states that $X_\delta \subset \mathrm{L}^3$; thus the Cauchy problem with data $\vec{u}_0 \in X_\delta^3$ has a unique mild solution in $C\left([0,T[,(\mathrm{L}^3)^3\right)$ ([FLT]). Our problem is now to prove, via a fixed point algorithm, that actually it is in $C\left([0,T'],{X_\delta}^3\right)$ for $0 < T' \leq T$. In order to get it, we need three kinds of results: first, we will show some more properties of decay for u and $\vec{\nabla}u$ if $u \in X_\delta$; second, we will analyze deeper the weight $w(x) = 1+|x|^{2\delta}$ in relation with the Muckenhoupt class and third, we will study the continuity on X_δ of the operators appearing in the bilinear term of the integral equation (1).

Lemma 3.
Let $u \in X_\delta$. Then:

$$\begin{aligned}
&\text{i)}\quad \vec{\nabla} u \in \left(\mathrm{L}^4\left((1+|x|^{2\delta})dx\right)\right)^3 \qquad \delta > \frac{3}{2};\\
&\text{ii)}\quad u\vec{\nabla} u \in \left(\mathrm{L}^2\left((1+|x|^{2\delta})dx\right)\right)^3 \qquad \delta > \frac{11}{6};\\
&\text{iii)}\quad u^2 \in \mathrm{L}^2\left((1+|x|^{2\delta})dx\right) \qquad \delta > \frac{5}{2}.
\end{aligned}$$

Proof. i) We have to estimate:

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left|\vec{\nabla} u(x)\right|^4 (1+|x|^{2\delta})dx\\
&\leq \int_{|x|\leq 1} \left|\vec{\nabla} u(x)\right|^4 (1+|x|^{2\delta})dx + \sum_{j\geq 0} \int_{2^j\leq |x|\leq 2^{j+1}} \left|\vec{\nabla} u(x)\right|^4 (1+|x|^{2\delta})dx.
\end{aligned}$$

Using an interpolation result, we bound the first term by:

$$\begin{aligned}
&C\int_{|x|\leq 1} \left|\vec{\nabla} u(x)\right|^4 dx \leq C\left\|\Delta u\right\|^2_{\mathrm{L}^2} \left\|u\right\|^2_{\mathrm{L}^\infty}\\
&\leq C\left\|\Delta u\right\|^2_{\mathrm{L}^2_\delta} \left\|u\right\|^2_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}} \leq C\left\|u\right\|^4_{X_\delta}.
\end{aligned} \tag{2}$$

For the second one, let $\omega \in \mathcal{D}, \omega(x) \equiv 1$ on $1 \leq |x| \leq 2$ and supp $\omega \subseteq \{\frac{1}{2} \leq |x| \leq \frac{5}{2}\}$.

Then we can write:

$$\begin{aligned}
\int_{2^j\leq |x|\leq 2^{j+1}} \left|\vec{\nabla} u(x)\right|^4 (1+|x|^{2\delta})dx &= \int_{2^j\leq |x|\leq 2^{j+1}} \left|\vec{\nabla}\left(u(x)\omega(\frac{x}{2^j})\right)\right|^4 (1+|x|^{2\delta})dx\\
&\leq C2^{2j\delta}\int_{2^j\leq |x|\leq 2^{j+1}} \left|\vec{\nabla}\left(u(x)\omega(\frac{x}{2^j})\right)\right|^4 dx\\
&\leq C2^{2j\delta}\left\|u(x)\omega(\frac{x}{2^j})\right\|^2_{\mathrm{L}^\infty}\left\|\Delta\left(u(x)\omega(\frac{x}{2^j})\right)\right\|^2_{\mathrm{L}^2}
\end{aligned}$$

(always by interpolation),

$$\begin{aligned}
&\leq C2^{2j\delta}2^{-(2\delta-1)j}\left\|u\right\|^2_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}}\Big(\left\|\Delta u(x)\omega(\frac{x}{2^j})\right\|^2_{\mathrm{L}^2} + \left\|u(x)\Delta\left(\omega(\frac{x}{2^j})\right)\right\|^2_{\mathrm{L}^2}\\
&\qquad + \left\|\vec{\nabla} u(x)\cdot\vec{\nabla}\left(\omega(\frac{x}{2^j})\right)\right\|^2_{\mathrm{L}^2}\Big)\\
&\leq C2^j\left\|u\right\|^2_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}}\Big(2^{-2j\delta}\left\|\Delta u\right\|^2_{\mathrm{L}^2_\delta} + 2^{-(2\delta-1)j}\left\|u\right\|^2_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}}\frac{1}{2^j}\\
&\qquad + \int_{\frac{2^j}{2}\leq |x|\leq \frac{5}{2}2^j}\left|\vec{\nabla} u(x)\right|^2\left|\vec{\nabla}\left(\omega(\frac{x}{2^j})\right)\right|^2 dx\Big)\\
&\leq C2^{(1-2\delta)j}\left\|u\right\|^4_{X_\delta} + 2^j\left\|u\right\|^2_{\mathrm{L}^\infty_{\delta-\frac{1}{2}}}\left(\int_{\frac{2^j}{2}\leq |x|\leq \frac{5}{2}2^j}\left|\vec{\nabla} u(x)\right|^4 dx\right)^{\frac{1}{2}}\times\\
&\qquad \left(\int_{\frac{2^j}{2}\leq |x|\leq \frac{5}{2}2^j}\left|\vec{\nabla}\left(\omega(\frac{x}{2^j})\right)\right|^4 dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq C2^{(1-2\delta)j}\,\|u\|_{X_\delta}^4+\frac{2^j}{2^{2j\delta}}\,\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}\,\varepsilon_j\int_{\frac{2^j}{2}\leq|x|\leq\frac52 2^j}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx$$
$$+\,2^j\,\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}\,\frac{1}{\varepsilon_j}\int_{\frac{2^j}{2}\leq|x|\leq\frac52 2^j}\frac{1}{2^{4j}}\left|\vec\nabla\omega(\frac{x}{2^j})\right|^4dx$$
$$\leq C2^{(1-2\delta)j}\,\|u\|_{X_\delta}^4+\varepsilon_j\,\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}\,2^{(1-2\delta)j}\int_{\frac{2^j}{2}\leq|x|\leq\frac52 2^j}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx+\frac{1}{\varepsilon_j}\,\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}.$$

Taking $\varepsilon_j=\dfrac{\varepsilon}{2^{(1-2\delta)j}\,\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}}$ we have:

$$\int_{2^j\leq|x|\leq 2^{j+1}}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx$$
$$\leq C(1+\frac1\varepsilon)2^{(1-2\delta)j}\,\|u\|_{X_\delta}^4+C\varepsilon\int_{\frac{2^j}{2}\leq|x|\leq\frac52 2^j}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx. \tag{3}$$

Summing up all the terms and combining the two estimations (2) and (3) we obtain:

$$\int_{\mathbb{R}^3}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx$$
$$\leq C(\varepsilon)\,\|u\|_{X_\delta}^4+C'\varepsilon\int_{\mathbb{R}^3}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx.$$

Choosing $0<C'\varepsilon<1$ we get:

$$\left\|\vec\nabla u\right\|_{\mathrm{L}^4\left((1+|x|^{2\delta})dx\right)}\leq C\,\|u\|_{X_\delta}.$$

ii) As in i) we have:

$$\int_{\mathbb{R}^3}|u(x)|^2\left|\vec\nabla u(x)\right|^2(1+|x|^{2\delta})dx$$
$$=\int_{|x|\leq1}|u(x)|^2\left|\vec\nabla u(x)\right|^2(1+|x|^{2\delta})dx+\sum_{j\geq0}\int_{2^j\leq|x|\leq2^{j+1}}|u(x)|^2\left|\vec\nabla u(x)\right|^2(1+|x|^{2\delta})dx.$$

The first term is bounded by:

$$C\int_{|x|\leq1}|u(x)|^2\left|\vec\nabla u(x)\right|^2dx\leq C\,\|u\|^2_{\mathrm{L}^\infty}\left(\int_{|x|\leq1}\left|\vec\nabla u(x)\right|^4dx\right)^{\frac12}$$
$$\leq C\,\|u\|^2_{X_\delta}\,\|u\|^2_{X_\delta}.$$

Let us come to the second one. For all $j\geq0$ we have, by the Hölder inequality:

$$\int_{2^j\leq|x|\leq2^{j+1}}|u(x)|^2\left|\vec\nabla u(x)\right|^2(1+|x|^{2\delta})dx$$
$$\leq\|u\|^2_{\mathrm{L}^\infty_{\delta-\frac12}}\,2^{j(1-\delta)}2^{j\frac32}\left(\int_{2^j\leq|x|\leq2^{j+1}}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx\right)^{\frac12}$$

then, using the estimation (2) in i), we bound it by

$$C\,\|u\|^4_{X_\delta}\,2^{j(\frac52-\delta)}2^{j(\frac12-\delta)}\left(1+\frac1\varepsilon\right)^{\frac12}$$
$$+\,C\varepsilon^{\frac12}\,\|u\|^2_{X_\delta}\,2^{j(\frac52-\delta)}\left(\int_{\frac{2^j}{2}\leq|x|\leq\frac52 2^j}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx\right)^{\frac12}$$
$$\leq C\,\|u\|^4_{X_\delta}\,2^{j(3-2\delta)}\left(1+\frac1{\tilde\varepsilon}\right)+C\tilde\varepsilon\,\|u\|^2_{X_\delta}\,2^{j(\frac52-\delta)}\left(\int_{\mathbb{R}^3}\left|\vec\nabla u(x)\right|^4(1+|x|^{2\delta})dx\right)^{\frac12}.$$

Now, letting $\tilde{\varepsilon} = 2^{-\frac{2}{3}j}$ and using the estimation in point i) we get

$$\leq C\,\|u\|^4_{X_\delta}\, 2^{j(3-2\delta)} + C\,\|u\|^4_{X_\delta}\, 2^{j(\frac{11}{3}-2\delta)} + C\,\|u\|^2_{X_\delta}\, 2^{j(\frac{11}{6}-\delta)}\,\|u\|^2_{X_\delta}\,.$$

Hence, summing up $j \geq 0$ we obtain

$$\left(\int_{\mathbb{R}^3} |u(x)|^2\,|\nabla u(x)|^2\,(1+|x|^{2\delta})dx\right)^{\frac{1}{2}} \leq C\,\|u\|^2_{X_\delta}\,.$$

iii) The decay at infinity implies for $\delta > \frac{5}{2}$

$$\int_{\mathbb{R}^3} |u(x)|^4(1+|x|^{2\delta})dx \leq \|u\|^4_{X_\delta} \int_{\mathbb{R}^3} \frac{1+|x|^{2\delta}}{(1+|x|^{\delta-\frac{1}{2}})^4}dx$$
$$\leq C\,\|u\|^4_{X_\delta}\,.$$

◇

III. ALMOST MUCKENHOUPT WEIGHTS

The following two sections are strongly connected. In order to apply a fixed point scheme, we are led to investigate the bicontinuity of the bilinear term on the space $C\left([0,T],{X_\delta}^3\right)\times C\left([0,T],{X_\delta}^3\right)$ and in particular to estimate $\int_0^t \left\| e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}\cdot\Delta(\vec{u}\otimes\vec{v})(s)\right\|_{\mathrm{L}^2_\delta} ds$, where $e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}$ is a matricial Calderón–Zygmund operator with L^1-norm of the same order as $\dfrac{1}{(t-s)^{\frac{1}{2}}}$.

We have already seen via proposition 2 and lemma 3 that if $\vec{u}(t) \in \mathrm{L}^\infty\left(\,]0,T[,{X_\delta}^3\right)$, then every component of

$$\Delta(\vec{u}\otimes\vec{v})(s) = \Delta\vec{u}(s)\otimes\vec{v}(s) + \vec{u}(s)\otimes\Delta\vec{v}(s) + 2\vec{\nabla}\otimes\vec{u}(s)\cdot\vec{\nabla}\otimes\vec{v}(s)$$

is in L^2_δ for $s\in[0,T[$ a.e. It is well known that Calderón–Zygmund operators are continuous on weighted L^p spaces if and only if the weight belongs to the Muckenhoupt class A_p. Unfortunately, this is not the case for our weights $w(x) = 1+|x|^{2\delta}$, $\delta > \frac{3}{2}$, which are nevertheless in a "local" Muckenhoupt class. Using this and the decay property of the convolution kernel we will able to go over the obstruction and to conclude.

Let us recall the definition of a Muckenhoupt weight (see also [M],[ST],[LE]).

Definition 4.
A weight $w(x)$ on $\mathbb{R}^3$ (a local integrable, positive function) belongs to the Muckenhoupt class A_p, $1 \leq p < +\infty$, if there exists a constant $K \geq 1$ such that for every ball B (of volume $|B|$) we have:

$$\frac{1}{|B|}\left(\int_B w(x)dx\right)^{\frac{1}{p}}\left(\int_B w(x)^{-\frac{1}{p-1}}dx\right)^{\frac{p-1}{p}} \leq K, \qquad p>1$$
$$\frac{1}{|B|}\int_B w(x)dx \leq K\ \operatorname{essinf}_{x\in B} w(x) \qquad p=1.$$

It is also possible to give a weaker definition, involving only small balls.

Definition 5.

Let $R>0$. A weight $w(x)$ on $\mathbb{R}^3$ belongs to a local Muckenhoupt class $A_p(R)$, $1\le p<+\infty$, if there exists a constant $K=K(R)$ such that if B is a ball centered at x_B and of radius $r(B)$ we have:

$$\sup_{r(B)\le R}\frac{1}{|B|}\left(\int_B w(x)dx\right)^{\frac{1}{p}}\left(\int_B w(x)^{-\frac{1}{p-1}}dx\right)^{\frac{p-1}{p}}\le K(R),\qquad p>1$$

$$\sup_{r(B)\le R}\frac{1}{|B|}\int_B w(x)dx\ \sup_{r(B)\le R}\sup_{x\in B}\frac{1}{w(x)}\le K(R)\qquad p=1.$$

Remark. The Muckenhoupt class A_p can also be characterized by the continuity of the Calderón–Zygmund operators, as stated in [ST]:

Theorem (Muckenhoupt). *Let T a Calderón–Zygmund operator and $w\in A_p$, with $1<p<+\infty$. There exists $C>0$ such that, for every $f\in \mathrm{L}^p(w(x)dx)$, we have:*

$$\int_{\mathbb{R}^3}|Tf(x)|^p\,w(x)dx\le C\int_{\mathbb{R}^3}|f(x)|^p\,w(x)dx.$$

Conversely, let $d\mu$ a positive Borel measure. If $\int_{\mathbb{R}^3}|Tf|^p\,d\mu\le C\int_{\mathbb{R}^3}|f|^p\,d\mu$, for every $f\in\mathrm{L}^p(d\mu)$, then $d\mu$ is absolutely continuous and $d\mu=w(x)dx$ with $w\in A_p$.

The function $w(x)=1+|x|^{2\delta}$ for $\delta>\frac{3}{2}$ is not in A_2 but it is in $A_2(R)$ and we can also control the growth of the constant $K(R)$ appearing in the definition.

Proposition 6.

If $\delta>\frac{3}{2}$ then $w(x)=1+|x|^{2\delta}\in A_2(R)$ and $K(R)=C(1+R^{\delta-\frac{3}{2}})$, with C a positive constant.

Proof. Let us consider for the moment the supremum over the balls of fixed radius equal to r. If $|x_B|>3r$ then $|x-x_B|\le r$ implies $\frac{2}{3}|x_B|\le|x|\le\frac{4}{3}|x_B|$ and so

$$\left(\frac{1}{r^3}\int_{|x-x_B|\le r}(1+|x|^{2\delta})dx\right)\left(\frac{1}{r^3}\int_{|x-x_B|\le r}\frac{1}{1+|x|^{2\delta}}dx\right)\le C\frac{1+(\frac{4}{3}|x_B|)^{2\delta}}{1+(\frac{2}{3}|x_B|)^{2\delta}}\le C.$$

If $|x_B|\le 3r$ then $|x-x_B|\le r$ implies $|x|\le 4r$ and we obtain the following estimation:

$$\begin{aligned}&\left(\frac{1}{r^3}\int_{|x-x_B|\le r}(1+|x|^{2\delta})dx\right)\left(\frac{1}{r^3}\int_{|x-x_B|\le r}\frac{1}{1+|x|^{2\delta}}dx\right)\\&\le C(1+4r^{2\delta})\min(1,\frac{1}{r^3})\\&\le C(1+r^{2\delta-3})\qquad\delta>\frac{3}{2}.\end{aligned}$$

Being $f(r)=1+r^{\delta-\frac{3}{2}}$ an increasing function, we get $K(R)=C(1+R^{\delta-\frac{3}{2}})$.

It will be also essential to remark that our weights can be suitably modified to become "globally" Muckenhoupt keeping their original definition for example on a cube. The next technical proposition will show this construction, already used in [LE].

Proposition 7.

Given any cube $Q\subset\mathbb{R}^3$ there exists a Muckenhoupt weight $w_Q(x)\in A_2$ which coincides with $w(x)=1+|x|^{2\delta}$, $\delta>\frac{3}{2}$ on Q.

Proof. We are going to prove this proposition for the particular cubes Q_λ centered, for $\lambda = (n, m, k) \in \mathbb{Z}^3$, at $x_\lambda = (x_n, y_m, z_k) = (\frac{2n+1}{2}, \frac{2m+1}{2}, \frac{2k+1}{2})$ and of side 3, because this is the case we will use later. Let then Q_λ be the following cube:

$$Q_\lambda = \left\{ \begin{array}{c} (x,y,z) \in \mathbb{R}^3 : x_n - \frac{3}{2} \le x \le x_n + \frac{3}{2};\ y_m - \frac{3}{2} \le y \le y_m + \frac{3}{2} \\ z_k - \frac{3}{2} \le z \le z_k + \frac{3}{2} \end{array} \right\}.$$

We are looking for a function $w_\lambda \in A_2$ such that $w_\lambda \equiv w$ on Q_λ. The definition is easy: we begin our construction defining the function w_λ on the cube $\tilde{Q}_\lambda$ centered at $(x_n + \frac{3}{2}, y_m + \frac{3}{2}, z_k + \frac{3}{2})$ of side 6 by symmetrizing successively the function w with respect to yz, xz, xy planes and then we extend the definition of w_λ all over the space by periodicity, using the tiling τ generated by $\tilde{Q}_\lambda$.

We will show that $w_\lambda(x) \in A_2$. We prove in fact that

$$\begin{aligned} &\sup_\lambda \sup_B \frac{1}{|B|} \left(\int_B w_\lambda(x) dx \right)^{\frac{1}{2}} \left(\int_B w_\lambda(x)^{-1} dx \right)^{\frac{1}{2}} \\ &\le C \sup_{r_B \le \frac{3}{2}} \frac{1}{|B|} \left(\int_B w_\lambda(x) dx \right)^{\frac{1}{2}} \left(\int_B w_\lambda(x)^{-1} dx \right)^{\frac{1}{2}} = CK(\frac{3}{2}) \end{aligned}$$

where we remark in particular the uniform estimation with respect to the chosen family Q_λ.

At first, we suppose $3|B| \ge |\tilde{Q}_\lambda|$. Then, there exists a $q \ge 0$ such that $(2^3)^q |\tilde{Q}_\lambda| \le 3|B| < (2^3)^{q+1} |\tilde{Q}_\lambda|$. Let $\tilde{Q}_C$ be the cube circumscribed to the ball B ($|B| < |\tilde{Q}_C| < 3|B|$). As the cube $\tilde{Q}_C$ can be covered by $(2^3)^{q+1}$ cubes of the tiling τ, we get:

$$\int_B w_\lambda(x) dx \le \int_{\tilde{Q}_C} w_\lambda(x) dx \le 2^{3(q+1)} \int_{\tilde{Q}_\lambda} w_\lambda(x) dx = 8 \cdot 2^{3(q+1)} \int_{Q_\lambda} w(x) dx$$

and also:

$$\int_B w_\lambda^{-1}(x) dx \le \int_{\tilde{Q}_C} w_\lambda^{-1}(x) dx \le 2^{3(q+1)} \int_{\tilde{Q}_\lambda} w_\lambda^{-1}(x) dx = 8 \cdot 2^{3(q+1)} \int_{Q_\lambda} w^{-1}(x) dx.$$

We are therefore able to conclude:

$$\begin{aligned} &\frac{1}{|B|} \left(\int_B w_\lambda(x) dx \right)^{\frac{1}{2}} \left(\int_B w_\lambda^{-1}(x) dx \right)^{\frac{1}{2}} \\ \le &\frac{3}{2^{3q} |\tilde{Q}_\lambda|} 2^{3(q+1)} 8 \left(\int_{Q_\lambda} w(x) dx \right)^{\frac{1}{2}} \left(\int_{Q_\lambda} w^{-1}(x) dx \right)^{\frac{1}{2}} \\ = &3 \cdot 8 \frac{1}{|\tilde{Q}_\lambda|} \left(\int_{Q_\lambda} w(x) dx \right)^{\frac{1}{2}} \left(\int_{Q_\lambda} w^{-1}(x) dx \right)^{\frac{1}{2}} \\ \le &C' K(\frac{3}{2}). \end{aligned}$$

In the case $3\,|B| < |\tilde{Q}_\lambda|$ we have to deal with three different situations.

If B is contained in a traslated cube of one of the eight cubes filling $\tilde{Q}_\lambda$ then by hypothesis

$$\frac{1}{|B|} \left(\int_B w_\lambda(x) dx \right)^{\frac{1}{2}} \left(\int_B \frac{1}{w_\lambda(x)} dx \right)^{\frac{1}{2}} \le K(\frac{3}{2}).$$

If otherwise B contains itself one of the preceding cube, then

$$\left(\int_B w_\lambda(x) dx \right)^{\frac{1}{2}} \le \left(\int_{\tilde{Q}_c} w_\lambda(x) dx \right)^{\frac{1}{2}} \le \left(\int_{\tilde{Q}_\lambda} w_\lambda(x) dx \right)^{\frac{1}{2}}$$

and

$$\left(\int_B \frac{1}{w_\lambda(x)}dx\right)^{\frac{1}{2}} \le \left(\int_{\tilde{Q}_c} \frac{1}{w_\lambda(x)}dx\right)^{\frac{1}{2}} \le \left(\int_{\tilde{Q}_\lambda} \frac{1}{w_\lambda(x)}dx\right)^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned}
&\frac{1}{|B|}\left(\int_B w_\lambda(x)dx\right)^{\frac{1}{2}}\left(\int_B \frac{1}{w_\lambda(x)}dx\right)^{\frac{1}{2}} \\
&\le \frac{8}{|Q_\lambda|}\left(\int_{Q_\lambda} w(x)dx\right)^{\frac{1}{2}}\left(\int_{Q_\lambda} \frac{1}{w(x)}dx\right)^{\frac{1}{2}} \le 8C'K(\frac{3}{2}).
\end{aligned}$$

Finally, if B doesn't verify anyone of the previous conditions, then there exists at last eight cubes Q_j translated of the Q_λ, filling $\tilde{Q}_\lambda$ and intersecting B. Let $S_j = B \cap Q_j$ and let us suppose that S_1 is such that $|S_1| = \max_{j=1,\dots,8} |S_j|$. Then for the symetries of the function $w_\lambda(x)$:

$$\left(\int_B w_\lambda(x)dx\right) \le 8\int_{S_1} w_\lambda(x)dx.$$

Let B_{in}, B_c be the inscribed and circumscribed balls of S_1. Being $w(x)dx$ a doubling mesure (see [ST]) there exists an $A > 0$ such that

$$\begin{aligned}
\frac{1}{A}\left(\int_{B_c} w(x)dx\right) &\le \int_{B_{in}} w(x)dx \le \int_{S_1} w_\lambda(x)dx \\
&\le \int_{S_1} w(x)dx \le \int_{B_c} w(x)dx \le A\left(\int_{B_{in}} w(x)dx\right).
\end{aligned}$$

The same argument is true even for $w(x)^{-1}$. We can then deduce

$$\begin{aligned}
&\frac{1}{|B|}\left(\int_B w_\lambda(x)dx\right)^{\frac{1}{2}}\left(\int_B \frac{1}{w_\lambda(x)}dx\right)^{\frac{1}{2}} \\
&\le \frac{8A}{|B_{in}|}\left(\int_{B_{in}} w(x)dx\right)^{\frac{1}{2}}\left(\int_{B_{in}} \frac{1}{w(x)}dx\right)^{\frac{1}{2}} \\
&\le 8AK(\frac{3}{2}).
\end{aligned}$$

which concludes the proof.

◇

Remark. The function $w(x) = 1 + |x|^3$ is in turn a local Muckenhoupt weight but not a global one. In the same way as in propositions 6 and 7 we can prove for $w(x)$ that the growth of the constant appearing in the definition 5 is $K(R) = C(1 + \log(1 + R))$ and that it is possible to define an extention to a Muckenhoupt weight. In spite of that, we are not interested in this case because neither the Laplacian of $f \in X_{\frac{3}{2}}$ is a molecule, nor $X_{\frac{3}{2}}$ is a subspace of L^3.

IV. CONVOLUTION OPERATORS ON L^2_δ.

We have seen that $w(x)$ is not a Muckenhoupt weight. Nevertheless it is possible to prove the bicontinuity of the bilinear term of the integral equation on $C\left([0,T], X_\delta{}^3\right) \times C\left([0,T], X_\delta{}^3\right)$ by decomposing the kernel of the operator into two parts, one localized around the origin and the other vanishing there. Both of these operators are actually proved to be continuous.

These are the results we have established.

Proposition 8.

Let $T_0 f = \int_{\mathbb{R}^3} K_0(x,y) f(y) dy$ be a Calderón–Zygmund operator such that $K_0(x,y) \equiv 0$ if $|x-y| > 1$. Then $T_0 \colon \mathrm{L}^2_\delta \to \mathrm{L}^2_\delta$ and $\|T_0 f\|_{\mathrm{L}^2_\delta} \le C \|f\|_{\mathrm{L}^2_\delta}$.

Proposition 9.

Let $T_1 f = \int K_1(x-y) f(y) dy$ be a convolution operator whose kernel vanishes in a neighbourhood of the origin ($K_1(x) \equiv 0$ for $|x| \le 1$) and verifies:

$$\begin{array}{lll} i) & |K_1(x)| \le \dfrac{C}{|x|^4} & \dfrac{3}{2} < \delta < \dfrac{5}{2} \\ ii) & |K_1(x)| \le \dfrac{C}{|x|^5} & \dfrac{5}{2} < \delta < \dfrac{7}{2} \\ iii) & |K_1(x)| \le \dfrac{C}{|x|^6} & \dfrac{7}{2} < \delta < \dfrac{9}{2}. \end{array}$$

Then $T_1 : \mathrm{L}^2_\delta \to \mathrm{L}^2_\delta$ and $\|T_1 f\|_{\mathrm{L}^2_\delta} \le C \|f\|_{\mathrm{L}^2_\delta}$.

Proof of proposition 8. We have to estimate

$$\begin{aligned} \|T_0 f\|^2_{\mathrm{L}^2_\delta} &= \int_{\mathbb{R}^3} \left| \int_{|x-y|<1} K_0(x,y) f(y) dy \right|^2 w(x) dx \\ &\le \sum_{\lambda \in \mathbb{Z}^3} \int_{Q(x_\lambda,1)} \left| \int_{Q(x_\lambda,3)} K_0(x,y) f(y) dy \right|^2 w(x) dx \qquad (4) \\ &\le \sum_{\lambda \in \mathbb{Z}^3} \int_{Q(x_\lambda,1)} \left| \int_{\mathbb{R}^3} K_0(x,y) f_\lambda(y) dy \right|^2 w_\lambda(x) dx \end{aligned}$$

where $Q(x_\lambda, R)$ is the cube centered at $x_\lambda = (\frac{2n+1}{2}, \frac{2m+1}{2}, \frac{2k+1}{2})$ and of side R, $f_\lambda(y) = f(y)\chi_{Q(x_\lambda,3)}$ and $w_\lambda(x)$ is the Muckenhoupt extension of $w(x)$, constructed in the proposition 7, which equals $w(x)$ on $Q(x_\lambda, 3)$. Because of the continuity of a Calderón–Zygmund operator on the Muckenhoupt weighted spaces, (4) can be controled by

$$\begin{aligned} & C \sum_{\lambda \in \mathbb{Z}^3} \int_{\mathbb{R}^3} |f_\lambda(x)|^2 w_\lambda(x) dx \\ &= C \sum_{\lambda \in \mathbb{Z}^3} \int_{Q(x_\lambda,3)} |f(x)|^2 w(x) dx \\ &\le C \|f\|^2_{\mathrm{L}^2_\delta}. \end{aligned}$$

◇

Proof of proposition 9. Let us consider

$$\begin{aligned} \|T_1 f\|_{\mathrm{L}^2_\delta} &= \left(\int_{\mathbb{R}^3} \left| \int_{|x-y|\ge 1} K_1(x-y) f(y) dy \right|^2 w(x) dx \right)^{\frac{1}{2}} \\ &\le \sum_{j=0}^{+\infty} \left(\int_{\mathbb{R}^3} \left(\int_{2^j < |x-y| \le 2^{j+1}} |K_1(x-y) f(y)| \, dy \right)^2 w(x) dx \right)^{\frac{1}{2}}. \end{aligned}$$

We are going to estimate each term of the series, in the general case $\alpha = 4, 5, 6$. We get:

$$\int_{\mathbb{R}^3}\left(\int_{2^j<|x-y|\le 2^{j+1}} |K_1(x-y)f(y)|\,dy\right)^2 w(x)dx$$
$$\le C\sum_{\lambda\in\mathbb{Z}^3}\int_{Q(x_\lambda\frac{2^j}{2},2^j)}\frac{1}{2^{2\alpha j}}\left(\int_{2^j<|x-y|\le 2^{j+1}}|f(y)|\,dy\right)^2 w(x)dx$$
$$\le C\sum_{\lambda\in\mathbb{Z}^3}\int_{Q(x_\lambda\frac{2^j}{2},2^j)}\frac{1}{2^{2\alpha j}}\left(\int_{Q(x_\lambda\frac{2^j}{2},52^j)}|f(y)|\,w(y)^{\frac12}w(y)^{-\frac12}dy\right)^2 w(x)dx$$
$$\le C\sum_{\lambda\in\mathbb{Z}^3}\int_{Q(x_\lambda\frac{2^j}{2},2^j)}\frac{1}{2^{2\alpha j}}\left(\int_{Q(x_\lambda\frac{2^j}{2},52^j)}|f(y)|^2\,w(y)dy\right)\left(\int_{Q(x_\lambda\frac{2^j}{2},52^j)}w(y)^{-1}dy\right)w(x)dx$$
$$\le C\frac{1}{2^{j(2\alpha-6)}}\sum_{\lambda\in\mathbb{Z}^3}\left(\int_{Q(x_\lambda\frac{2^j}{2},52^j)}|f(y)|^2\,w(y)dy\right)\left(\frac{1}{2^{3j}}\int_{Q(x_\lambda\frac{2^j}{2},52^j)}w(y)dy\right)\times$$
$$\left(\frac{1}{2^{3j}}\int_{Q(x_\lambda\frac{2^j}{2},52^j)}\frac{1}{w(y)}dy\right)$$
$$\le C\frac{1}{2^{j(2\alpha-6)}}\|f\|^2_{L^2_\delta}(1+2^{j(2\delta-3)})$$

the last inequality coming from the result in proposition 6:

$$\sup_{r_B\le R}\frac{1}{|B|}\left(\int_B w(x)dx\right)^{\frac12}\left(\int_B\frac{1}{w(x)}dx\right)^{\frac12}\le C(1+R^{\delta-\frac32})$$

(the role of the family of balls and cubes can be interchanged, because of the doubling property of the mesure $w(x)dx$). By hypotheses on α and δ, we are able to sum up $j\ge 0$ and to conclude.

◇

V. TENDENCY AND FLUCTUATION IN X_δ

We dispose now of all the tools we need to make the fixed point argument work in $C\left([0,T],{X_\delta}^3\right)$. For sake of clarity we will deal separately the linear and bilinear terms.

Lemma 10.
If $\vec{u}_0\in X_\delta^3$, then for all $T\in[0,+\infty[\quad e^{t\Delta}\vec{u}_0\in C\left([0,T],{X_\delta}^3\right)$.

Proof. We prove at first that for $t\in]0,T[\ \Delta e^{t\Delta}\vec{u}_0\in L^2$. The only thing we have to show is that $\Delta e^{t\Delta}\vec{u}_0 = e^{t\Delta}\Delta\vec{u}_0\in L^\infty\left([0,T[,(L^2(|x|^{2\delta}))^3\right)$. Being $|x|^{2\delta}\le C\left(|x-y|^{2\delta}+|y|^{2\delta}\right)$ we get

$$\int_{\mathbb{R}^3}\left|\int_{\mathbb{R}^3}\frac{1}{(4\pi t)^{\frac32}}e^{-\frac{|x-y|^2}{4t}}\Delta\vec{u}_0(y)dy\right|^2|x|^{2\delta}dx$$
$$\le C\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}t^{\frac{\delta}{2}}\frac{1}{(4\pi t)^{\frac32}}e^{-\frac{|x-y|^2}{4t}}\frac{|x-y|^\delta}{t^{\frac{\delta}{2}}}|\Delta\vec{u}_0(y)|\,dy+\int_{\mathbb{R}^3}\frac{1}{(4\pi t)^{\frac32}}e^{\frac{-|x-y|^2}{4t}}|y|^\delta|\Delta\vec{u}_0(y)|\,dy\right)^2dx$$
$$\le C\left(t^\delta\|\tilde\varphi_t*|\Delta\vec{u}_0|\|^2_{L^2}+\left\|e^{t\Delta}\left(|\cdot|^\delta|\Delta\vec{u}_0|\right)\right\|^2_{L^2}\right)$$

where the uniform boundedness of $\tilde\varphi_t(x)=\frac{|x|^\delta}{t^{\frac{\delta}{2}}}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{\frac32}}$ in L^1 gives the following control:

$$\left\|e^{t\Delta}\Delta\vec{u}_0\right\|^2_{L^2(|x|^{2\delta}dx)}\le C(1+t^\delta)\|\Delta\vec{u}_0\|^2_{L^2_\delta}$$

and then

$$\left\|e^{t\Delta}\Delta\vec{u}_0\right\|_{\mathrm{L}^2_\delta} \le C(1+t^{\frac{\delta}{2}})\left\|\Delta\vec{u}_0\right\|_{\mathrm{L}^2_\delta}.$$

To see that $e^{t\Delta}\vec{u}_0$ vanishes at infinity for every fixed t it is enough to remark that the convolution kernel is a test function and $\vec{u}_0$ can be approximated in the L^3–norm by test functions. Finally, the vanishing properties of the moments come from the following two remarks: $\Delta u_0 \in \mathrm{L}^1$ if $\delta > \frac{3}{2}$, $x_i\Delta u_0 \in \mathrm{L}^1$ if $\delta > \frac{5}{2}$ and $x_i x_j \Delta u_0 \in \mathrm{L}^1$ if $\delta > \frac{7}{2}$ as we stressed in part I; also, if f and g belong to L^1 and $\int_{\mathbb{R}^3} f(x)dx = 0$ then $\int_{\mathbb{R}^3}(f*g)(x)dx = 0$.

◇

Lemma 11.
The bilinear operator

$$B(\vec{u},\vec{v})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}\cdot\vec{u}\otimes\vec{v}(s)ds$$

is bicontinuous from $C\left([0,T], {X_\delta}^3\right)$ *into* $C\left([0,T], {X_\delta}^3\right)$ *for* $T \in\,]0,+\infty[$.

Proof. We begin to show that $\Delta B(\vec{u},\vec{v})(t) \in C\left([0,T], (\mathrm{L}^2_\delta)^3\right)$. Let $\varphi \in \mathcal{D}$, $\operatorname{supp}\varphi \subset B(0,1)$, $\varphi \not\equiv 0$ and consider the splitting of the kernel $K(x)$ of $\mathbb{P}$ into two parts:

$$K(x) = \varphi(x)K(x) + (1-\varphi(x))K(x) = K_0(x) + K_1(x) \quad x \neq 0$$

where $K_0(x)$ is supported near the origin and $K_1(x)$ away from it. We deal first with $\int_0^t K_0 * e^{(t-s)\Delta}\vec{\nabla}\cdot\Delta(\vec{u}\otimes\vec{v})(s)ds$. Using the bicontinuity on L^2_δ of the convolution operator $e^{(t-s)\Delta}\sqrt{t-s}\vec{\nabla}\cdot$ and the control of

$$\left\|\Delta(\vec{u}\otimes\vec{v})(s)\right\|_{\mathrm{L}^2} = \left\|\Delta\vec{u}(s)\otimes\vec{v}(s) + \vec{u}(s)\otimes\Delta\vec{v}(s) + 2\vec{\nabla}\otimes\vec{u}(s)\cdot\vec{\nabla}\otimes\vec{v}(s)\right\|_{\mathrm{L}^2_\delta}$$

ensured by proposition 2 and lemma 3, it is straightforward to show that for all $s < t$

$$\begin{aligned}
&\left\|e^{(t-s)\Delta}\vec{\nabla}\cdot\Delta(\vec{u}\otimes\vec{v})(s)ds\right\|_{\mathrm{L}^2_\delta}\\
\le&\frac{1}{\sqrt{t-s}}\left\|e^{(t-s)\Delta}\sqrt{t-s}\vec{\nabla}\cdot\Delta(\vec{u}\otimes\vec{v})(s)ds\right\|_{\mathrm{L}^2_\delta}\\
\le&C\frac{1}{\sqrt{t-s}}\left(1+(t-s)^{\frac{\delta}{2}}\right)\left\|\Delta(\vec{u}\otimes\vec{v})(s)\right\|_{\mathrm{L}^2_\delta}\\
\le&C\frac{1}{\sqrt{t-s}}\left(1+(t-s)^{\frac{\delta}{2}}\right)\left\|\Delta\vec{u}(s)\right\|_{\mathrm{L}^2_\delta}\left\|\Delta\vec{v}(s)\right\|_{\mathrm{L}^2_\delta}.
\end{aligned}$$

If we are able to prove that $K_0(x)$ is a Calderón–Zygmund kernel, proposition 8 will furnish us the control for all $t < T$, $T \in\,]0,+\infty[$:

$$\begin{aligned}
&\left\|\int_0^t K_0 * e^{(t-s)\Delta}\vec{\nabla}\cdot\Delta(\vec{u}\otimes\vec{v})(s)ds\right\|_{\mathrm{L}^2_\delta}\\
\le&C\left(\int_0^t\frac{1}{\sqrt{t-s}}\left(1+\sqrt{t-s}^{\,\delta}\right)ds\right)\sup_{0<s<T}\left\|\Delta\vec{u}(s)\right\|_{\mathrm{L}^2_\delta}\sup_{0<s<T}\left\|\Delta\vec{v}(s)\right\|_{\mathrm{L}^2_\delta}\\
\le&CT^{\frac{1}{2}}\left(1+T^{\frac{\delta}{2}}\right)\sup_{0<s<T}\left\|\Delta\vec{u}(s)\right\|_{\mathrm{L}^2_\delta}\sup_{0<s<t}\left\|\Delta\vec{v}(s)\right\|_{\mathrm{L}^2_\delta}.
\end{aligned}\tag{5}$$

Let us remind that $\mathbb{P}$ is a Calderón–Zygmund operator, whose kernel verifies the following properties:

$$\begin{aligned}
&i)\quad \hat{K}(\xi)\in\mathrm{L}^\infty;\\
&ii)\quad |K(x)| \le \frac{C}{|x|^3}\quad x\neq 0;\\
&iii)\quad \left|\vec{\nabla}K(x)\right| \le \frac{C}{|x|^4}\quad x\neq 0.
\end{aligned}$$

The same is also true for the cut function $K_0(x) = K(x)\varphi(x)$, namely

$$i) \quad \left\|\hat{K}_0\right\|_{\mathrm{L}^\infty} = \left\|\hat{\varphi} * \hat{K}\right\|_{\mathrm{L}^\infty} \le \|\hat{\varphi}\|_{\mathrm{L}^1} \left\|\hat{K}\right\|_{\mathrm{L}^\infty};$$

$$ii) \quad |K_0(x)| \le \frac{C}{|x|^3} \quad x \ne 0;$$

$$iii) \quad \left|\vec{\nabla} K_0(x)\right| \le \left|\vec{\nabla} K(x)\right| |\varphi(x)| + |K(x)| \left|\vec{\nabla}\varphi(x)\right| \le \frac{C}{|x|^4} \quad x \ne 0$$

(because φ is supported on $B(0,1)$).

Coming to $\int_0^t K_1 * \mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \Delta(\vec{u} \otimes \vec{v})(s)ds$, we cannot apply proposition 9 directly because $K_1(x)$ decreases too slowly at infinity. In spite of that, the kernel of the operator $\mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \Delta$ appearing in the integral has at least three vanishing moments and this will be crucial.

In fact there are different equivalent espressions for the aforementioned convolution, namely

$$K_1 * \mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \Delta(\vec{u} \otimes \vec{v})(s) = \begin{cases} \vec{\nabla} K_1 * \mathrm{e}^{(t-s)\Delta} \cdot \Delta(\vec{u} \otimes \vec{v})(s) \\ \Delta K_1 * \mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \vec{u} \otimes \vec{v}(s) \\ \vec{\nabla}\Delta K_1 * \mathrm{e}^{(t-s)\Delta} \cdot (\vec{u} \otimes \vec{v})(s) \end{cases}$$

We will exploit the first formulation for $\frac{3}{2} < \delta < \frac{5}{2}$, the second one for $\frac{5}{2} < \delta < \frac{7}{2}$ and the third for $\frac{7}{2} < \delta < \frac{9}{2}$. We are actually faced with $\vec{\nabla} K_1$, ΔK_1, $\vec{\nabla}\Delta K_1$, which are supported outside the origin and deacrease respectively as $\frac{1}{|x|^4}$, $\frac{1}{|x|^5}$, $\frac{1}{|x|^6}$ (we remind that the components of $\hat{K}(\xi)$ are like $\delta_{i,j} - \frac{\xi_i \xi_j}{|\xi|^2}$) and then we are allowed to apply proposition 9 to obtain after integration:

$$(6) \qquad \begin{aligned} &\left\|\int_0^t K_1 * \mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \Delta(\vec{u} \otimes \vec{v})(s)ds\right\|_{\mathrm{L}^2_\delta} \\ &\le \int_0^t \left\|K_1 * \mathrm{e}^{(t-s)\Delta}\vec{\nabla} \cdot \Delta(\vec{u} \otimes \vec{v})(s)\right\|_{\mathrm{L}^2_\delta} ds \\ &\le C \int_0^t (1 + (t-s)^{\frac{\delta}{2}})ds \sup_{0<s<T} \|\Delta\vec{u}(s)\|_{\mathrm{L}^2_\delta} \sup_{0<s<T} \|\Delta\vec{v}(s)\|_{\mathrm{L}^2_\delta} \\ &\le C(1 + T^{\frac{\delta}{2}})T \sup_{0<s<T} \|\Delta\vec{u}(s)\|_{\mathrm{L}^2_\delta} \sup_{0<s<T} \|\Delta\vec{v}(s)\|_{\mathrm{L}^2_\delta}. \end{aligned}$$

¿From the two estimations (5) and (6) we finally get:

$$\|\Delta B(\vec{u}, \vec{v})(t)\|_{\mathrm{L}^2_\delta} \le C(1 + T^{\frac{\delta}{2}})T^{\frac{1}{2}}(1 + T^{\frac{1}{2}}) \sup_{0<s<T} \|\Delta\vec{u}(s)\|_{\mathrm{L}^2_\delta} \sup_{0<s<T} \|\Delta\vec{v}(s)\|_{\mathrm{L}^2_\delta}$$

and hence the continuity at $t = 0$ of $\Delta B(\vec{u}, \vec{v})(t)$ in L^2_δ. The continuity away from the origin is classical.

Let us see the vanishing at infinity of $B(\vec{u}, \vec{v})(t)$ $\forall t \in [0, T]$. We remark that the integral converges in $\mathrm{L}^{\frac{3}{2}}$ $\left(\int_0^t \left\|\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla} \cdot \vec{u} \otimes \vec{v}(s)\right\|_{\mathrm{L}^{\frac{3}{2}}} ds < +\infty\right)$ and the subspace of $\mathrm{L}^{\frac{3}{2}}$ of functions vanishing at infinity is closed in $\mathrm{L}^{\frac{3}{2}}$. Because of this we just have to point out that $\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla} \cdot \vec{u} \otimes \vec{v}(s) \in C_0$ for all $s \in [0, T]$ (for example because $\vec{u} \otimes \vec{v}(s) \in \mathrm{L}^3$ and the convolution kernel is in $\mathrm{L}^{\frac{3}{2}}$ [PL]). A similar argument allows to establish the vanishing of the moments. Let us see that in details. Calling $a(x, s) = \Delta\mathrm{e}^{(t-s)\Delta}\mathbb{P}\vec{\nabla} \cdot \vec{u} \otimes \vec{v}(x, s)$, we know that $\Delta B(\vec{u}, \vec{v})(t) = \displaystyle\int_0^t a(s)ds$ converges in L^2_δ. If we prove that:

$$(7) \qquad \begin{cases} \displaystyle\int_{\mathbb{R}^3} a(x, s)dx = 0 & \delta > \frac{3}{2},\ 0 < s < T, \\ \displaystyle\int_{\mathbb{R}^3} x_k a(x, s)dx = 0 & \delta > \frac{5}{2},\ 0 < s < T,\ k = 1, 2, 3, \\ \displaystyle\int_{\mathbb{R}^3} x_i x_k a(x, s)dx = 0 & \delta > \frac{7}{2},\ 0 < s < T,\ i, k = 1, 2, 3, \end{cases}$$

then we are done. In fact the subspace of L^2_δ of functions verifying (7) is closed in L^2_δ. We are so left to prove the properties in (7) for $a(x,s)$, calling Θ the convolution kernel of the operator $e^{\Delta}\mathbb{P}\vec{\nabla}$. For $\delta > \frac{3}{2}$, it is enough to see that $a(x,s) = \frac{1}{(t-s)^2}\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * \Delta(\vec{u}\otimes\vec{v})(x,s)$ and, for every component θ of Θ, we have $\theta \in L^1$, $\int \theta(x)dx = 0$ and also $\Delta(\vec{u}\otimes\vec{v})(s) \in L^2_\delta \subset L^1$ for all $s \in]0,T[$. For $\delta > \frac{5}{2}$, we will write better $a(x,s) = \frac{1}{(t-s)^{\frac{5}{2}}}\vec{\nabla}\cdot\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * \vec{\nabla}\cdot(\vec{u}\otimes\vec{v})(x,s)$ where, denoting now ψ the components of $\vec{\nabla}\cdot\Theta$, we have that ψ and $x_k\psi$ belong to L^1, for $k = 1,2,3$, $\int \psi(x)dx = 0$, $\int x_k\psi(x)dx = 0$ and, moreover, $\vec{\nabla}\cdot(\vec{u}\otimes\vec{v})(s)$, $x_k\vec{\nabla}\cdot(\vec{u}\otimes\vec{v})(s)$ belong to L^1 for all $s \in]0,T[$. Finally, for $\delta > \frac{7}{2}$, we write $a(x,s) = \frac{1}{(t-s)^3}\Delta\Theta\left(\frac{\cdot}{\sqrt{t-s}}\right) * \vec{u}\otimes\vec{v}(x,s)$ and now, the components φ of $\Delta\Theta$ verify that φ, $x_k\varphi$, $x_ix_k\varphi$ belong to L^1 for $i,k = 1,2,3$, $\int\varphi(x)dx = 0$, $\int x_i\varphi(x)dx = 0$, $\int x_ix_k\varphi(x)dx = 0$ and $\vec{u}\otimes\vec{v}(s)$, $x_i\vec{u}\otimes\vec{v}(s)$, $x_ix_k\vec{u}\otimes\vec{v}(s)$ belong to L^1 for all $s \in]0,T[$.

◇

The proof of the theorem is then straightforward.

Proof of the theorem. Let $\vec{u}(s), \vec{v}(s) \in C\left([0,T'],(X_\delta)^3\right)$ where T' will be chosen in a moment. We are looking for a ball $B(0,R) \subset C\left([0,T'],(X_\delta)^3\right)$ where the operator $F(\vec{u})(t) = e^{t\Delta}\vec{u_0} - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}\cdot\vec{u}\otimes\vec{u}(s)ds$ is a contraction. The two conditions that have to be satisfied are the following ones:

$$\begin{cases} \sup\limits_{0<t<T'} \|F(\vec{u})(t)\|_{X_\delta} \le R \\ \sup\limits_{0<t<T'} \|F(\vec{u})(t) - F(\vec{v})(t)\|_{X_\delta} \le C \sup\limits_{0<t<T'} \|\vec{u}(t)-\vec{v}(t)\|_{X_\delta} \end{cases}$$

for all $\vec{u},\vec{v} \in B(0,R)$ and for a $C < 1$. Hence we have

$$\begin{aligned} \sup_{0<t<T'} \|F(\vec{u})(t)\|_{X_\delta} &\le \sup_{0<t<T'} \left\|e^{t\Delta}\vec{u}_0\right\|_{X_\delta} + C\sqrt{T'}(1+\sqrt{T'})(1+\sqrt{T'}^\delta)(\sup_{0<t<T'}\|\vec{u}(s)\|_{X_\delta})^2 \\ &\le \sup_{0<t<T'} \left\|e^{t\Delta}\vec{u}_0\right\|_{X_\delta} + C\sqrt{T'}(1+\sqrt{T'})(1+\sqrt{T'}^\delta)R^2 \end{aligned}$$

and

$$\begin{aligned} &\sup_{0<t<T'} \|F(\vec{u})(t) - F(\vec{v})(t)\|_{X_\delta} \le C\sqrt{T'}(1+\sqrt{T'})(1+\sqrt{T'}^\delta)\sup_{0<t<T'}\|\vec{u}(t)-\vec{v}(t)\|_{X_\delta} \times \\ &(\sup_{0<t<T'}\|\vec{u}(t)\|_{X_\delta} + \sup_{0<t<T'}\|\vec{v}(t)\|_{X_\delta}) \\ &\le 2RC\sqrt{T'}(1+\sqrt{T'})(1+\sqrt{T'}^\delta)\sup_{0<t<T'}\|\vec{u}(t)-\vec{v}(t)\|_{X_\delta}. \end{aligned}$$

Hence we are looking for an R and T' such that

$$\begin{cases} \sup\limits_{0<t<T'} \left\|e^{t\Delta}\vec{u_0}\right\|_{X_\delta} + \eta(T')R^2 \le R \\ 2\eta(T')R < 1, \end{cases}$$

where $\eta(T')$ stands for $C\sqrt{T'}(1+\sqrt{T'})(1+\sqrt{T'}^\delta)$. Choosing $R = \frac{1-\sqrt{1-4\eta(T')\sup_{0<t<T'}\|e^{t\Delta}\vec{u_0}\|_{X_\delta}}}{2\eta(T')}$, with T' small enough to have $\sup_{0<t<T'}\left\|e^{t\Delta}\vec{u_0}\right\|_{X_\delta} \le \frac{1}{4\eta(T')}$ (which is possible because $\sup_{0<t<T'}\left\|e^{t\Delta}\vec{u_0}\right\|_{X_\delta} \le C(\sqrt{T'}^\delta + 1)\|\vec{u}_0\|_{X_\delta}$), the two conditions are verified and F is a contraction of $B(0,R) \subset C\left([0,T'],{X_\delta}^3\right)$. Because of the uniqueness of the solutions in the L^3–framework we are done with the theorem; we point out that $[0,T'[$ was not proved to be equal to $[0,T[$, the maximal interval of existence of the solution in L^3, but it is in general smaller than it. ◇

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