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Collapsible Backward Continuation and Oscillations in Retarded Differential Equations

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1 Introduction.

We are concerned here with the oscillatory character of certain solutions of scalar differential equations of the form

$$\dot{x}(t) = f(t, x(t), x(t - r(t))), \tag{1}$$

where $t - r(t)$ is a given strictly increasing map defined for $t \geq t_o$ for some $t_o \in \mathbb{R}$. We also assume that $r(t_o) = 0$ and $0 < r(t) \leq t - t_o$ for $t \geq t_o$. Eq. (1) is a particular form of a "retarded differential equation", the delay in time being provided by the argument $t - r(t)$ and it is a special kind of a functional differential equation of the type

$$\dot{x}(t) = g(t, x_t), \tag{2}$$

where x_t is the "tail" map (as it is given, for instance, in [6]), $x_t(\theta) = x(t + \theta)$, $\theta \in [-r(t), 0]$. A particular instance of the more general situation to be investigated in this paper, was studied in [2], namely, the "scaled differential equation"

$$\dot{x}(t) = -ax(t) + ax(pt), \tag{3}$$

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where $a > 0$ and $0 < p < 1$ are given real numbers. Here, we have $t_o = 0$ and $r(t) = t - pt$. The nomenclature for Eq. (3) is due to the change of scale in time of the argument pt. Variations of this equation have been used in some mathematical models for pantograph equipment ([3, 4, 5, 9, 10, 13, 14, 15]).
Eq. (1) has an interesting feature: two different kinds of initial value problem (IVP) can be attached to it, namely, the punctual IVP "$x(t_o) = x_o$," or the functional IVP "$x(t) = \psi(t)$", where x_o is arbitrarily chosen in $\mathbb{R}$ or ψ is arbitrarily chosen in $\mathcal{C}_\tau =: \mathcal{C}([\tau', \tau], \mathbb{R})$, and $\tau > t_o$ is an arbitrary real constant, with $\tau' = \tau - r(\tau)$. Here, as usual, $\mathcal{C}([a, b], \mathbb{R})$ stands for the space of the continuous maps from $[a, b]$ into $\mathbb{R}$, equipped with the supremum norm $||\phi|| = \sup\{|\phi(t)| : t \in [a, b]\}$. One often refers to either (t_o, x_o) or (τ, ψ) as an "initial condition". This duality of IVPs directly leads to the phenomenon of collapse of backward continuation ([2]), responsible for a wild kind of oscillatory behavior of solutions of Eq. (1), as we shall see. In order to make notations simpler, we extend the above nomenclature of τ' by putting $\tau'' = (\tau')' = \tau' - r(\tau')$ and so on.
By a "solution" of the punctual IVP we mean a **differentiable** map $x(t)$, defined in an interval $[t_o, b]$ which, of course, satisfies Eq. (1) for $t \in [t_o, b]$ and $x(t_o) = x_o$. The derivative at t_o means, of course, the right hand derivative while the derivative at b is the left hand one. As usual, the notations $\dot{x}(t+)$ and $\dot{x}(t-)$ denote the right-hand and the left-hand derivative of x at t, respectively. Similarly, a "solution" of the functional IVP is a **continuous** map $x : [\tau', b] \to \mathbb{R}$, such that $b > \tau$, x is **differentiable** in $[\tau, b]$ and, of course, satisfies $x(t) = \psi(t)$, $t \in [\tau', \tau]$, $\dot{x}(t) = f(t, x(t), x(t - r(t))$, $t \in [\tau, b]$ with the derivative at τ being the right-hand one, etc.. To emphasize the dependence on τ and ψ of this solution we shall denote it by "$x(\,.\,, \tau, \psi)$." The solution of the punctual IVP will be denoted simply by "$x(\,.\,, x_o)$." We assume from now the following conditions:
Hypothesis (H): **(i)**- $f(t, x, y)$ is continuous in t and C^∞ with respect to x, y for $t \geq t_o$ (this condition implies that $f(t, \,.\,, \,.\,)$ is locally Lipschitz, i.e., for each compact rectangle $Q = [a, b] \times [c, d]$ and $t \in \mathbb{R}$ there is a constant $M_t^Q > 0$ such that

$$|f(t, x, y) - f(t, u, v)| \leq M_t^Q \max\{|x - u|, |y - v| : (x, y), (u, v) \in Q\},$$

where $|.|$ denotes the modulus or absolute value map); **(ii)**- $D_y f(t, x, y) \neq 0$ for all (t, x, y), $t \geq t_o$ ((ii) implies that $f(t, x, \,.\,)$ is strictly monotone);

(iii)- $f(t, x, .)$ is surjective; **(iv)**- $f(t, 0, 0) = 0$, $t \geq t_o$ and **(v)**- r is C^∞. Assumption (iv) implies that $x(t) \equiv 0$ is a solution of Eq. (1), which is called "the trivial (or null) solution" and it is the unique solution for both the punctual IVP "$x(t_o) = 0$" and the functional IVP "$x_\tau = 0$" as it will become clear in the sequel. This equilibrium is denoted by $x(\,.\,, 0)$.
Note that (ii), (iii) and (iv), together, imply that the sets $\mathcal{A}_t{}^+ = \{y : f(t, 0, y) > 0\}$ and $\mathcal{A}_t{}^- = \{y : f(t, 0, y) < 0\}$ are nonempty, disjoint and $A_t{}^+ \cup \mathcal{A}_t{}^- \cup \{0\} = \mathbb{R}$, $t \geq t_o$. To fix the ideas, we may think, without loss of generality, that:
Hypothesis (G): $\mathcal{A}_t{}^+ = \mathbb{R}^+ = \{y : y > 0\}$ and $\mathcal{A}_t{}^- = \mathbb{R}^- = \{y : y < 0\}$.

Note that since f is continuous, it follows that $\{y : f(t, 0, y) = 0\} = \{0\}$ for each $t \geq t_o$.

If we compare the above conditions with the simpler ones in [2], where the authors made use of the stronger semigroup property which holds for linear delay equations, we see that the present conditions are not yet sufficient to guarantee backward continuation of a *large* set of initial conditions, as it is in the linear case. In this paper we use a special property called *piling property* to be defined, as well as applications of implicit function theorems, to extend the results to a much more general class of problems.

2 Existence and Uniqueness of Solutions

Let's now prove that, under assumption (i) above, the initial value problem $x(t_o) = x_o$ has a unique solution in a sufficiently small interval $[t_o, t_o + \epsilon_{x_o}]$, $\epsilon_{x_o} > 0$. Indeed, let $\alpha > 0$ and $\delta > 0$ be given, $\alpha < \infty$, and consider the operator $T : \mathcal{C}([t_o, t_o + \alpha], \mathbb{R}) \to \mathcal{C}([t_o, t_o + \alpha], \mathbb{R})$

$$T(\phi)(t) = x_o + \int_{t_o}^{t} f(s, \phi(s), \phi(s - r(s)))ds, \;\; t \in [t_o, t_o + \alpha]. \qquad (4)$$

Note that $0 < r(t) \leq t - t_o$ implies that T is well defined. Let $L_{x_o} := \max\{|f(t, x, y)| : (t, x, y) \in Q_\alpha^\delta\}$, where

$$Q_\alpha^\delta := [t_o, t_o + \alpha] \times [x_o - \delta, x_o + \delta] \times [x_o - \delta, x_o + \delta]$$

and restrict t in Eq. (4) to vary in $[t_o, t_o + \epsilon_{x_o}]$ where $\epsilon_{x_o} = \min\{\alpha, \delta/L_{x_o}\}$. Take $\phi_o(t) \equiv x_o$, and form the sequence $\{\phi_n\}_o^\infty$ given by

$$\phi_{n+1}(t) = T(\phi_n)(t), \;\; t \in [t_o, t_o + \epsilon_{x_o}].$$

Then,

$$|\phi_n(t) - x_o| \leq \delta,$$
$$|\phi_{n+1}(t) - \phi_n(t)| \leq L_{x_o} M^n (t - t_o)^{n+1}/(n+1)!,$$

where M is an upper bound for the Lipschitz constants M_t^Q of f for $(t, x, y) \in Q^{\delta}_{\epsilon_{x_o}}$. The existence of M is guaranteed by the continuity of f with respect to t (hypothesis H(i)). Hence, the series

$$x_o + \sum_{n=0}^{\infty} [\phi_{n+1}(t) - \phi_n(t)]$$

converges uniformly in $[t_o, t_o + \epsilon_{x_o}]$ to a map $x(t)$, due to Weierstrass' comparison M-test with the series

$$|x_o| + \frac{L_{x_o}}{M} \sum_{n=o}^{\infty} \frac{(M\epsilon_{x_o})^{n+1}}{(n+1)!}$$

which clearly converges to $|x_o| + \frac{L_{x_o}}{M}(e^{M\epsilon_{x_o}} - 1)$. The partial sum of this series is $\phi_n(t)$, so that the sequence $\{\phi_n\}$ converges itself as $n \to \infty$ to $x(t)$. Clearly, x satisfies

$$x(t) = x_o + \int_{t_o}^{t} f(s, x(s), x(s - r(s)))ds, \ t \in [t_o, t_o + \epsilon_{x_o}], \qquad (5)$$

i.e., x is a solution of the punctual initial value problem $x(t_o) = x_o$ with $|x(t) - x_o| \leq \delta$, $t \in [t_o, t_o + \epsilon_{x_o}]$. Suppose now that $y(t)$ is another solution of the same initial value problem in the same interval $[t_o, t_o + \epsilon_{x_o}]$. Then,

$$y(t) = x_o + \int_{t_o}^{t} f(s, y(s), y(s - r(s))ds, \ t \in [t_o, t_o + \epsilon_{x_o}].$$

Hence,

$$||y(t) - \phi_1(t)|| \leq \frac{L_{x_o}}{M} \frac{(M\epsilon_{x_o})^2}{2}$$

and, in general,

$$||y(t) - \phi_m(t)|| \leq \frac{L_{x_o}}{M} \frac{(M\epsilon_{x_o})^{m+1}}{(m+1)!}, \ t \in [t_o, t_o + \epsilon_{x_o}].$$

Thus, $\phi_m(t) \to y(t)$, $t \in [t_o, t_o + \epsilon_{x_o}]$ which shows that x is unique. As mentioned earlier, we denote this solution by $x(\,.\,, x_o)$. Hence, we have:

(A) *The solution $x(\,.\,,x_o)$ exists and is locally unique for each given initial condition x_o.*
Clearly, from the formula (5) of $x(\,.\,,x_o)$, one sees that this solution continuously depends on x_o. Also, it is clear that the following corollary of proposition (A) holds:

(B) *No other solutions of Eq. (1) than the ones that satisfy*

$$\dot{x}(t_o+) = f(t_o, x(t_o), x(t_o))$$

may hit (or cross),from the right hand side, the plane $t = t_o$ in the (t, x, y)-space.
Let's take advantage of our exposition to introduce at this stage a new concept (the *piling property*) that will be useful in the sequel, since it will establish a powerful criterion to guarantee that certain functional initial conditions do not belong to a solution of a punctual IVP. For that matter, note that Picard-Lindelof's scheme for proving Proposition (A) yields the following general result: given a Lipschitzian $g(t, x)$ and real numbers x_o and (positive) δ_{x_o}, α_{x_o} let $\epsilon_{x_o} := \min\{\alpha_{x_o}, \delta_{x_o}/L_{x_o}(\delta_{x_o})\}$, where

$$L_{x_o}(\delta_{x_o}) := \max\{|g(t,x)| \ : \ t_o \leq t \leq t_o + \alpha_{x_o}, \ |x - x_o| \leq \delta_{x_o}\},$$

then the solution $x(t)$ of $\dot{x} = g(t, x)$, $x(t_o) = x_o$ satisfies $|x(t) - x_o| \leq \delta_{x_o}$, for $t \in [t_o, t_o + \epsilon_{x_o}]$.

We want this kind of property to be globally valid for a map $g(t, x, y)$, as stated in the definition:

Property (P)- A map $g(t, x, y) \in \mathbb{R}$, $t, x, y \in \mathbb{R}$ is said to have the *piling property at t_o* (or, more simply, "the property (P)"), if, for each $x \in \mathbb{R}$ we can find two positive numbers, δ_x, and α_x, such that $\delta_x < \infty$, and $0 \leq \delta_z - \delta_x \leq |z| - |x|$ if z is another real number with $|z| > |x|$, and such that for

$$L_x = \max\{|g(t,z,w)| : \ t_o \leq t \leq t_o + \alpha_x, \ |x - w|, |x - z| \leq \delta_x\},$$

we have $\epsilon = \inf\{\alpha_x, \ \delta_x/L_x : x \in \mathbb{R}\} > 0$.
Let's make a paragraph to show that the class of maps that satisfy (P) is "large". In fact, if $g(t, x, y)$ is a continuous map which is strictly monotone with respect to x for each fixed y, and in y for each fixed x, and such that $|g(t, x, y)| \leq K(t)(|x| + |y|) + C$, where $K : [t_o, \infty) \to [0, \infty)$ is continuous on

$[t_o, \infty)$, then g has the piling property at any t_o. Here, $C \geq 0$ is a constant. We first assume that $g(., x, y)$ is positive and increasing in x for each y and increasing in y for each x and arbitrarily choose $\alpha > 0$ and $0 < \delta < 1$,. We have $L_x(\delta) = \max\{|g(t,z,w)| : (t,z,w) \in [t_o, t_o+\alpha] \times [x-\delta, x+\delta] \times [x-\delta, x+\delta]\,\} = g(t_x, x+\delta, x+\delta)$ for some $t_x \in [t_o, t_o+\alpha]$. Hence, $L_x(\delta) \leq 2\tilde{K}|x+\delta|+C$, where $\tilde{K} :\geq \max\{K(t) : t \in [t_o, t_o+\alpha]\,\}$, $\tilde{K} < \infty$. Next, we take $\alpha_x = \alpha$ and

$$\delta_x = \begin{cases} \delta, & \text{if } |x+\delta| < 1 \\ \delta\,|x+\delta|, & \text{if } |x+\delta| \geq 1. \end{cases}$$

It is clear that if $|x+\delta| < 1$ then $\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta}{2\tilde{K}+C} = \frac{\delta}{2\tilde{K}(1+C_1)}$, where $C_1 = \frac{C}{2\tilde{K}}$. If $x + \delta > 1$, then we have

$$\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta|x+\delta|}{2\tilde{K}|x+\delta_x|+C} = \frac{\delta}{2\tilde{K}} \frac{x+\delta}{x+\delta(x+\delta)+C_1} = \frac{\delta}{2\tilde{K}} \frac{x+\delta}{(1+\delta)x+\delta^2+C_1}\ .$$

Now, we just observe that the function $h(x) = \frac{x+\delta}{x+\delta(x+\delta)+C_1}$ is decreasing provided $C_1 < \delta$ (as we may freely increase $\tilde{K}$ as needed, we may assume that $C_1 = \frac{C}{\tilde{K}} < \delta$) and $h(x) \to \frac{1}{1+\delta}$ as $x \to \infty$; hence $h(x) \geq \frac{1}{1+\delta}$. The case $x + \delta < -1$ is similar. So, $\epsilon = \inf\{\alpha_x,\ \delta_x/L_x(\delta_x) : x \in \mathbb{R}\} \geq \frac{\delta}{2\tilde{K}(1+\delta)+C} > 0$ and, of course, all the requirements of property (P) are also satisfied by g, as we wished to prove. Let's now make no assumption on the sign of g but assume that g is increasing in x for each y and decreasing in y for each x (of course, the opposite assumption works in the same way). Let α, δ, K and $L_x(\delta)$ be as in the previous case. Clearly, also in this case, the maximum absolute value $|g(t,z,w)|$ must occur at one of the corners of the rectangle $[x-\delta, x+\delta] \times [x-\delta, x+\delta]$, i.e., $L_x(\delta) = |g(t_x, x+\delta, x-\delta)|$, for some t_x. Then, $L_x(\delta) \leq K(t)(|x-\delta|+|x+\delta|)+C \leq \tilde{K}(|x-\delta|+|x+\delta|)+C$, where $\tilde{K} \geq \max\{K(t) : t \in [t_o, t_o+\alpha]\}$. There are two possibilities:

Case 1: $x \geq 0$. Here, we have that $|x+\delta| \geq |x-\delta|$ for any $\delta > 0$. We take

$$\delta_x = \begin{cases} \delta, & \text{if } |x+\delta| < 1 \\ \delta\,|x+\delta|, & \text{if } |x-\delta| \geq 1. \end{cases}$$

Then, we have $L_x(\delta_x) \leq 2\tilde{K}|x+\delta_x|+C$. If $|x+\delta| < 1$, then

$$\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta}{2\tilde{K}|x+\delta_x|+C} \geq \frac{\delta}{2\tilde{K}+C} = \frac{\delta}{2\tilde{K}} \frac{1}{1+\frac{C}{2\tilde{K}}}\ .$$

As before, we may assume that $C_1 := \frac{C}{2\tilde{K}} < \delta$. Then, in the case $|x+\delta| < 1$ we have:

$$\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta}{2\tilde{K}} \frac{1}{1+C_1} \geq \frac{\delta}{2\tilde{K}} \frac{1}{1+\delta}.$$

Now, suppose that $|x+\delta| \geq 1$. Then, since $x \geq 0$, it follows that

$$\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta(x+\delta)}{2\tilde{K}(x+\delta_x)+C} = \frac{\delta(x+\delta)}{2\tilde{K}[x+\delta(x+\delta)+C} = \frac{\delta}{2\tilde{K}} \frac{x+\delta}{(1+\delta)x+\delta^2+C_1},$$

C_1 as above. Let $h(x) = \frac{x+\delta}{(1+\delta)x+\delta^2+C_1}$. Then, $h'(x) = \frac{C_1-\delta}{[(1+\delta)x+\delta^2+C_1]^2} < 0$ with $h(0) = \frac{\delta}{\delta^2+C_1}$, $\lim_{x\to\infty} h(x) = \frac{1}{1+\delta}$. So, $h(x) \geq \frac{1}{1+\delta}$ for $x \geq 0$, and $\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta}{2\tilde{K}} \cdot \frac{1}{1+\delta}$, as wished.

Case 2: $x < 0$. This case works in a similar way, but we shall write the proof for completeness. Here, we have $|x-\delta| > |x+\delta|$ for any $x < 0$ and $\delta > 0$. We take $\alpha_x = \alpha$ and

$$\delta_x = \begin{cases} \delta, & \text{if } |x-\delta| < 1 \\ \delta\,|x-\delta|, & \text{if } |x+\delta| \geq 1. \end{cases}$$

We have $L_x(\delta_x) \leq 2\tilde{K}|x-\delta_x| + C$. If $|x-\delta| < 1$, then

$$\frac{\delta_x}{L_x(\delta_x)} = \frac{\delta}{L_x(\delta)} \geq \frac{\delta}{2\tilde{K}|x-\delta|+C} \geq \frac{\delta}{2\tilde{K}} \frac{1}{1+C_1} \geq \frac{\delta}{2\tilde{K}} \frac{1}{1+\delta},$$

where C_1 is as above. If $|x-\delta| \geq 1$ then since $|x-\delta| = -(x-\delta) = \delta - x$,

$$\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta|x-\delta|}{2\tilde{K}|x-\delta|x-\delta||+C} = \frac{\delta}{2\tilde{K}} \frac{\delta-x}{\delta^2-(1+\delta)x+C_1}.$$

We now let $h(x) := \frac{\delta-x}{\delta^2-(1+\delta)x+C_1}$ for $x < 0$. Then, $h'(x) = \frac{\delta-C_1}{[\delta^2-(1+\delta)x+C_1]^2} > 0$ and $h(0-) = \frac{\delta}{\delta^2+C_1}$, $\lim_{x\to-\infty} h(x) = \frac{1}{1+\delta}$. Hence, $h(x) \geq \frac{1}{1+\delta}$ for $x < 0$ and $\frac{\delta_x}{L_x(\delta_x)} \geq \frac{\delta}{2\tilde{K}} \cdot \frac{1}{1+\delta}$. Therefore, in both cases we have $\inf\{\frac{\delta_x}{L_x(\delta_x)} : x \in \mathbb{R}\} \geq \frac{\delta}{2\tilde{K}} \cdot \frac{1}{1+\delta}$, and so, $\epsilon = \inf\{\alpha_x, \frac{\delta_x}{L_x(\delta_x)} : x \in \mathbb{R}\} = \inf\{\alpha, \frac{\delta_x}{L_x(\delta_x)} : x \in \mathbb{R}\} > 0$.
Observe that if g is locally Lipschitzian and has the property (P) then the solution of every IVP $\dot{x}(t) = g(t, x(t))$, $x(t_o) = x_o \in \mathbb{R}$ exists, is defined in an interval $t_o \leq t \leq t_o + \epsilon$, where $\epsilon > 0$ is independent of x_o, and this solution satisfies $|x(t) - x_o| \leq \delta_{x_o}$, $t \in [t_o, t_o+\epsilon]$ as it is shown by a proof using Picard-Lindelof's scheme. Of course, a similar result holds for the punctual IVP $\dot{x}(t) = g(t, x(t), x(t-r(t)))$, $x(t_o) = x_o \in \mathbb{R}$. Moreover, we have:

Lemma 2.1 *If a locally Lipschitzian $g(t,x,y)$ has the property (P) and $y(t)$ is a map such that $y(t_1) < x_o - \delta_{x_o}$ and $y(t_2) > x_o + \delta_{x_o}$ for some $x_o \in R$ and some $t_1,\ t_2 \in [t_o, t_o+\epsilon]$, then $y(t)$ cannot be a solution of a punctual IVP $\dot{x}(t) = g(t, x(t), x(t-r(t))),\ x(t_o) = z,\ z \in \mathbb{R}$.*

Proof: First, suppose that y is a solution of the IVP with $y(t_o) = x_o$. Let $\alpha_{x_o}, \delta_{x0}$, and ϵ be as in Property (P). Suppose that $y(t)$ is such that $y(t_1) < x_o - \delta_{x_o}$ and $y(t_2) > x_o + \delta_{x_o}$ for some $t_1\,,\ t_2 \in [t_o\,, t_o + \epsilon]$. Then there will exist a first moment $t^* \in [t_o\,, t_o + \epsilon)$ such that

$$|y(t) - x_o| \leq \delta_{x_o} \quad \text{for} \;\; t \in [t_o\,, t^*]$$

and either $y(t^*) = x_o + \delta_{x_o}$ or $x_o - \delta_{x_o}$. Then

$$|g(t, y(t), y(t - r(t)))| \leq L_{x_o} \quad \text{for} \;\; t_o \leq t \leq t^*.$$

Therefore

$$|y(t^*) - x_o| \leq \int_{t_o}^{t^*} |g(s, y(s), y(s - r(s)))|\, ds \leq L_{x_o}|t^* - t_o| < L_{x_o}\epsilon.$$

Since $L_{x_o}\epsilon \leq \delta_{x_o}$, this contradicts $|y(t^*) - x_o| = \delta_{x_o}$. This establishes the result in the lemma when $z = x_o$.

Using a similar argument, we see that $y(t)$ also cannot be a solution of the IVP when we pick $z < x_o$ or when $z > x_o$. Hence the lemma is proved.

Now, regarding existence and uniqueness of the solution of a functional IVP $x_\tau = \psi$, we can use an argument that is a simple adaptation of the one used in the above proof of proposition (A). This solution $x(t, \tau, \psi)$, $t \in [\tau', \tau + \alpha]$, for all α chosen sufficiently small is the unique fixed point of the operator
$T_\psi : \mathcal{C}([\tau', \tau + \alpha], \mathbb{R}) \to \mathcal{C}([\tau', \tau + \alpha], \mathbb{R})$ given by

$$T_\psi(y)(t) \;=\; \psi(t),\ \ t \in [\tau', \tau], \tag{6}$$

$$T_\psi(y)(t) \;=\; \psi(0) + \int_\tau^t f(s, y(s), \psi(s - r(s)))ds,\ \ t \in [\tau, \tau + \alpha]. \tag{7}$$

Assumption (i) of Section I above and the computations carried out in the proof of proposition (A) clearly show that T_ψ becomes a contraction when α

is chosen sufficiently small. Hence, there is a unique fixed point $T_\psi(x) = x$ which furnishes a solution $x(t, \tau, \psi)$, $t \in [\tau', \tau + \alpha]$, given by

$$x(t, \tau, \psi) = \psi(t), \quad t \in [\tau', \tau] \tag{8}$$

$$x(t, \tau, \psi) = x(t), \quad t \in [\tau, \tau + \alpha]. \tag{9}$$

Of course, this reasoning can be repeated infinitely many times, if necessary, in order to forwardly continue the solution $x(\,.\,, \tau, \psi)$, which is, finally, found to be defined in a maximal interval $[\tau', b_\psi)$, $\tau < b_\psi \leq \infty$ (see [6]). One such solution is said to be 'forward noncontinuable". It is clear from the expression of T_ψ and the uniqueness of the solution $x(\,.\,, \tau, \psi)$, that this solution, as in the case of the punctual IVP, continuously depends on the initial condition $(\tau,\, \psi)$. Thus, we can state:

(C) *The forward noncontinuable solution of the functional initial value $x_\tau = \psi$ attached to Eq. (1) exists, is unique for each given $\psi \in \mathcal{C}_\tau$ and continuously depends on (τ, ψ).*

One can now use the above proof of (C) in order to forwardly continue the solution $x(\,.\,, x_o)$ of the punctual initial value problem from its original interval of definition $[t_o, t_o + \epsilon_{x_o}]$ (proposition (A)): Using $\tau = t_o + \epsilon_{x_o}$, and $\psi(t) = x(t, x_o)$, $t \in [\tau', \tau]$, the operator T_ψ given by expressions (6)-(7), will yield the unique continuation of $x(\,.\,, x_o)$ to an interval $[t_o, t_o + \epsilon_{x_o} + \alpha]$. As before, this process admits infinitely many iterations, so that the solution $x(\,.\,, x_o)$ is finally found to be nontinuable and defined in a forward maximal interval of the form $[t_o, b_{x_o})$, $t_o + \epsilon_{x_o} < b_{x_o} \leq \infty$. The concept of "uniqueness" in functional differential equations is of a different nature than the one in o.d.e.'s (ordinary differential equations): two distinct solutions may intersect each other at intervals of appropriate lengths; in the o.d.e. case, uniqueness means that two distinct solutions cannot intersect each other. The uniqueness referred to in proposition (A) is of the o.d.e. type at t_o, which precisely means that there exists one, and only one, solution through (t_o, x_o) for each given $x_o \in \mathbb{R}$. The solutions of punctual initial value problems with different x_o's may even coincide at intervals of the type $[c, d]$ as long as $t_o < d' < c$. So, given any $t_1 > t_o$, it is possible to have two different solutions of Eq. (1) intersecting at t_1. This is impossible at t_o.

3 Backward Continuation of Solutions

It is clear that if a functional initial condition (τ, ψ) is appropriately chosen (for instance, if $\tau' \leq \gamma < \tau$ and $\psi(t) = x(t, \gamma, \phi)$ for $t \in [\tau', \tau]$), then the solution $x(\,.\,, \tau, \psi)$ can be "backwardly continued" in the interval $[\gamma', \tau]$, i.e., there exists $\phi \in \mathcal{C}_\gamma$ such that $x(t, \gamma, \phi) = x(t, \tau, \psi)$ for $t \geq \tau'$. In other words, for an appropriately chosen initial condition τ, ψ, the solution of a "previous" functional initial value problem (namely, (γ, ϕ)), coincides with the solution $x(\,.\,, \tau, \psi)$ for $t \geq \tau'$. In particular, we shall have $x(t, \gamma, \phi) = \psi(t)$, $t \in [\tau', \tau]$. As in the case of a forwardly noncontinuable solution, we may think of a "backwardly noncontinuable" solution of Eq. (1), $x(\,.\,, \tau, \psi)$, which satisfies either one of the following: $\tau = t_o$; $\tau > t_o$ and $\dot{\psi}(\tau-)$ is not defined; $\tau > t_o$, $\dot{\psi}(\tau-)$ is defined but $\dot{\psi}(\tau-) \neq f(\tau, \psi(\tau), \psi(\tau'))$. We also have the notion of the "maximal backward interval of definition of a solution" of Eq. (1), which must be either of the form $(\gamma, \tau]$, or of the form $[\gamma, \tau]$, $\gamma \geq t_o$. And, the "maximal interval of definition" of a solution is the interval where the solution is both backwardly and forwardly noncontinuable.
We intend to show now that

(D) *For each given $\tau > t_o$ there correspond infinitely many functional initial conditions ψ which yield backwardly continuable solutions $x(\,.\,, \tau, \psi)$.*
For that matter, assume, without loss of generality, that $\tau' < \gamma < \tau$ and observe that for the backward continuation of $x(\,.\,, \tau, \psi)$ to exist in $[\gamma, \tau]$, it is necessary that ψ be differentiable in $[\gamma, \tau]$, $\dot{\psi}(\tau-) = f(\tau, \psi(\tau), \psi(\tau'))$ and $\phi \in \mathcal{C}([\gamma', \gamma], \mathbb{R})$ exists such that

$$\dot{\psi}(t) = f(t, \psi(t), \phi(t - r(t))), \; t \in [\gamma, \tau]. \tag{10}$$

Given a pair $(\tau, \psi), \psi \in C^\infty$, let $h(t, y(t))$ be formally given by

$$h(t, y(t)) = \begin{cases} \dot{x}(t, \tau, \psi) - f(t, x(t, \tau, \psi), y(t)), & \text{if } \; t \in [\tau, b_\psi) \\ \dot{\psi}(t) - f(t, \psi(t), y(t)), & \text{if } \; t \in [\gamma, \tau]. \end{cases} \tag{11}$$

Since $h(\tau, \psi(\tau')) = 0, \; D_y h(\tau, \psi(\tau')) \neq 0$, the implicit function theorem (see [1, 7]) ensures the existence of $y(t)$, t in a neighborhood of τ, such that $h(t, y(t)) = 0$, with y as smooth as ψ is in this neighborhood (with, of course, $y(t) = x(t, \tau, \psi)$ for $t \geq \tau$). This version of the IFT does not require differetiability of f with respect to t, since they use the equivalent integral equation form of the differential equation. Hence, if we let $\phi(t - r(t)) :=$

$y(t)$, $t \leq \tau$, then $x(\,.\,,\gamma,\phi)$ is a backward continuation of $x(\,.\,,\tau,\psi)$ to the interval $[\gamma,\tau]$. The condition $\dot{\psi}(t) = f(t,\psi(t),y(t))$ together with a condition of the type $\psi \in \mathcal{C}^n$, $n \geq 1$, allows a number of successive applications of the implicit function theorem, and thus yields a backward noncontinuable solution $x(\,.\,,\tau,\psi)$ defined in a maximal backward interval of the type $[\gamma, b_\psi)$ or of the type (γ, b_ψ), $\gamma < \tau$.

4 Oscillatory Solutions

Before we proceed on the question of backward oscillation of solutions, let's clarify our understanding of oscillation in the present case of Eq. (1). Recall that the classical theory of o.d.e.'s (see [8, 11]) defines a nonconstant solution as "oscillatory in an interval I" when it has at least two zeros in this interval. But, in the case of Eq. (1), where an initial condition map ψ can already be chosen satisfying this condition, it is advisable to adopt a more stringent definition: a nonequilibrium solution $y(t)$ is said to be "backwardly oscillatory", if a sequence $\{t_n\}_1^\infty$ with $y(t_n) = 0$ can be found such that $t_n \to t_o+$.
From now on, we assume that $f(t,x,x)$ has the property (P) at t_o. Then, we will prove:

(E) *To the equilibrium solution $x(t,0)$ of Eq. (1), there correspond infinitely many solutions of functional IVP's which have $(t_o,\tau]$ as their maximal backward interval of definition and are backwardly oscillatory.*

The proof of Proposition (D), specialized to the null solution of Eq. (1) (i.e., $x_o = 0$ or $\psi = 0$), shows that for any C^∞ map ϕ, the map

$$\eta(t) = [\exp(-(t-\tau')^{-2} - (\tau - t)^{-2})]\phi(t)$$

is C^∞ and satisfies $\eta^{(n)}(\tau') = \eta^{(n)}(\tau) = 0$ for $n = 0, 1, 2, ...$ This implies that all boundary conditions that arise from successive differentiation of Eq. (1) at $t = \tau'$ and at $t = \tau$ are promptly satisfied by η. The solution $x(\,.\,,\tau,\eta)$ is therefore backwardly (see [10]) continuable in the whole interval $(t_o,\tau]$ (we are using item (iii) of hypothesis **(H)** about the surjectiveness of f in order to guarantee the existence of the solution $y(t)$ of Eq. (10) for arbitrary values of $\dot{\eta}$). In order to prove our claim about the existence of infinitely many backwardly oscillatory solutions of Eq. (1), it is enough to prove that there are infinitely many η's of the category above constructed, which lead to solutions $x(\,.\,,\tau,\eta)$ of the desired oscillatory category. But, this is a nice consequence of the piling property of f at t_o (property (P)) via Lemma 2.1 as well as the above scheme for producing the perturbations η of the null equilibrium. In fact, if we choose $\tau < \epsilon$, (this is the ϵ referred to in Lemma 2.1) and $\phi(t)$ as above such that $\eta(t) = \kappa[\exp(-(t-\tau')^{-2} - (t-\tau)^{-2})]\phi(t)$, where κ is any convenient constant, satisfies $\eta(t_1) > \delta_o$ and $\eta(t_2) < -\delta_o$ for some t_1, $t_2 \in (\tau', \tau)$ (recall that we are taking $x_o = 0$), then $x(\,.\,, \tau, \eta)$ is backwardly continuable in $(t_o, \tau]$, as we have just seen. It cannot be a solution of any punctual IVP, by Lemma 2.1. Indeed, if we denote by $x(t)$ this solution, then it cannot satisfy $\lim_{t \to t_o+} x(t) = x_1$ for any $x_1 \in \mathbb{R}$, since otherwise it would also satisfy $\dot{x}(t_o+) = f(t_o, x_1, x_1)$, $x(t_o) = x_1$ and would coincide with the ordinary solution $x(\,.\,, x_1)$, a contradiction. Of course, we can choose infinitely many such ϕ's. Figure 1 below depicts one of these ϕ's and its corresponding η.

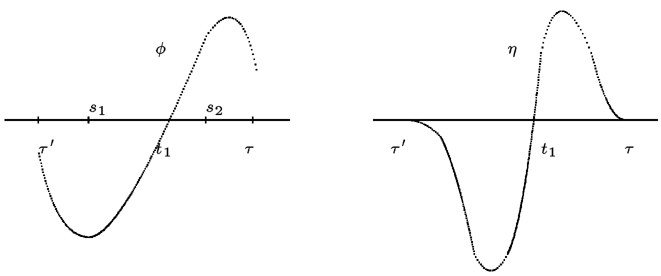

Figure 1: η - a C^∞ perturbation of $x(t) \equiv 0$

Next, we observe that for any choice of a map ϕ as above, with the further requirement that $\phi(t_1) = 0$ and $\dot{\phi}(t_1) \neq 0$ for some isolated $t_1 \in (\tau', \tau)$, we shall also have $x(t_1) = 0$ with $\dot{x}(t_1) \neq 0$. If, say, $\dot{x}(t_1) > 0$, then $f(t_1, 0, x(t_1')) > 0$, so that $x(t_1') > 0$ by hypothesis **(G)** (we recall that the symbol t' stands for $t - r(t)$). On the other hand, $\dot{x}(t_1) > 0$ means that $x(t) < 0$ for t slightly less than t_1. Combining these facts, we obtain a point t_2, $t_1' < t_2 < t_1$ such that $x(t_2) = 0$ and $\dot{x}(t_2) < 0$. So, by **(G)**, we have $x(t_2') < 0$. But, $\dot{x}(t_2) < 0$ implies that $x(t) > 0$ for t slightly less that t_2, which, as before, implies the existence of t_3, $t_2' < t_3 < t_2$ such that $x(t_2) = 0$, and so on. We thus obtain a decreasing sequence $\{t_n\}_1^\infty$, $t_n \in (t_o, \epsilon)$ such that $x(t_n) = 0$. Let us prove that $\lim_{n\to\infty} t_n = \bar{t} = t_o$. Suppose the contrary, i.e., $\bar{t} > t_o$ and there are no other zeros of $x(t)$, $t < \bar{t}$. Then, say, $x(t) > 0$ for $t < \bar{t}$. In particular, we then have $\dot{x}(\bar{t}) = f(\bar{t}, 0, x(\bar{t}')) > 0$, a contradiction. Similarly, one obtains a contradiction if $x(t) < 0$ for $t < \bar{t}$. Hence, we must have $\bar{t} = t_o$ and this finishes the proof of (E). The choice $\dot{\phi}(t_1) \neq 0$ in the above proof is in fact immaterial. If one assumes $\dot{\phi}(t_1) = 0$, then we have $\dot{x}(t_1) = f(t_1, 0, x(t_1')) = 0$, which implies (from Hypothesis (H)) $x(t_1') = 0$. If either $\dot{x}(t_1') < 0$ or $\dot{x}(t_1') > 0$, the above argument applies; if, else $\dot{x}(t_1') = 0$, we repeat this procedure: at $t_1'' < t_1'$ we must have $x(t_1'') = 0$. Continuing in this way, we get a convergent sequence $t_k := t_1^{(k)}$ such that $x(t_k) = 0$ and $\dot{x}(t_k) \to 0$. As above, we must have $t_k \to t_0$.

Before we proceed in the oscillations subject, let's make a paragraph to introduce some ideas that will be useful in the sequel. Since the null equilibrium is the solution of the punctual IVP $x(t_o) = 0$, we can extend the result of (E) to any other solution $x(\,.\,, x_o)$ as follows. Given one such solution, consider the auxiliary equation

$$\dot{y}(t) = F(t, y(t), y(t - r(t))), \tag{12}$$

where $F(t, z, w) =: \dot{x}(t, x_o) - f(t, x(t, x_o) + z, x(t - r(t), x_o) + w)$. We claim that $F(t, z, w)$ also satisfies hypothesis (H) and the piling property and that $y(t, 0) \equiv 0$ is a solution of Eq. (12). It is evident that $F(t, z, w)$ is C^∞ in z, w for $t \geq t_o$, since $f(t, u, v)$ is C^∞ in u, v for $t \geq t_o$. Consequently, it is locally Lipschitz continuous (probably with different Lipschitz constants). Regarding the boundedness of the Lipschitz constant of F, note that:

$$\begin{aligned}&|F(t, z, w) - F(t, u, v)| = \\ &\qquad |f(t, x(t, x_o) + z, x(t - r(t), x_o) + w) - f(t, x(t, x_o) + u, x(t - r(t), x_o) + v)|.\end{aligned}$$

If (z, w) and (u, v) lie in the rectangle $Q = [a, b] \times [c, d]$, then it follows that $(x(t, x_o) + z, x(t - r(t), x_o) + w), (x(t, x_o) + u, x(t - r(t), x_o) + v)$ lie in the rectangle

$$\tilde{Q}(t) := [x(t, x_o) + a, x(t, x_o) + b] \times [x(t, x_o) + c, x(t, x_o) + d]$$

If we now apply (H)(i) for f we obtain

$$|F(t, z, w) - F(t, u, v)| \leq M_t^{\tilde{Q}(t)} \max\{|u - z|, |v - w| \,:\, (z, w), (u, v) \in Q\}$$

So F satisfies a Lipschitz condition. Now suppose t lies in a compact interval; then $x(t,x_o)$ and $x(t-r(t),x_o)$ lie in a compact set, and all the rectangles $\tilde{Q}(t)$ are contained in a compact rectangle $\hat{Q}$. Then, a bound $M_t^{Q(t)} \leq M$, for t in the mentioned compact interval, follows. Consequently,

$$|F(t,z,w) - F(t,u,v)| \leq M \ \max\{|u-z|, |v-w| \, : \, (z,w),(u,v) \in Q\},$$

as wished. The other parts of (H) are clearly satisfied by F. The required boundedness of $|f(t,x(t,x_o)+z, x(t-r(t),x_o)+w)|$ in the piling property is established with the help of the Lipschitz property, as follows:

$$\begin{aligned}&|f(t,x(t,x_o)+z, x(t-r(t),x_o)+w)| \leq \\ &\quad |f(t,x(t,x_o)+z, x(t-r(t),x_o)+w) - f(t,z,w)| + |f(t,z,w)| \leq \\ &\quad M_t^Q \max\{|x(t,x_o)|, |x(t-r(t),x_o)|\} + L_{x_o}(f).\end{aligned}$$

Since $x(t,x_o)$ is a given solution, it is bounded in the interval $[t_o, t_o+\epsilon]$.

Hence, (E) is valid for Eq. (12) as well. But, to any perturbation of $y(\,.\,,0)$ there corresponds a similar perturbation of $x(\,.\,,x_o)$. Thus, there are infinitely many solutions $x(\,.\,,x_o)$ which have maximal backward interval of definition of the form $(t_o,\tau]$. Moreover, we shall now prove the following stronger result about existence of a dense class of backwardly noncontinuable solutions:

(F) *To each solution $x(t) = x(t,x_o)$ of the punctual IVP $x(t_o) = x_o$ and each $\tau > t_o$, there corresponds a dense subset V_τ of the set $\{\phi \in C^\infty \cap U_\tau, \ \dot{\phi}(\tau') = \dot{x}(\tau')\,,\, \dot{\phi}(\tau) = \dot{x}(\tau)\}$ where $U_\tau = \{\phi \in \mathcal{C}_\tau \ : \ \phi(\tau') = x(\tau')$ and $\phi(\tau) = x(\tau)\}$, V_τ consisting of initial conditions that yield backwardly noncontinuable solutions of functional IVP's which have $(t_o,\tau]$ as their maximal backward interval of definition.*

The argument used to prove (D) can also be applied when the vector field f is replaced by F, given by

$$F(t,w(t),w(t-r(t))) = f(t,x(t)+w(t),x(t-r(t))+w(t-r(t)) - f(t,x(t),x(t-r(t)),$$

where $x(t) = x(t,x_o)$. We observe that F also satisfies Property (P). Based on the ideas of the proof of (E) all we need to show now is that any continuous initial condition can be approximated by a C^∞ one with boundary conditions specified by $x(t,x_o)$. The technique to be used in this approximation scheme uses the well-known *partition of unity* constructions. We give details of the technique in order to leave it clear that the above boundary conditions are maintained if we restrict the perturbations of the initial condition to the appropriate class. Let $\phi \in U_\tau$ and $\varepsilon > 0$ be given. Weierstrass' Approximation Theorem (see, [7]) implies the existence of a polynomial $p(t)$ such that $|\phi(t) - p(t)| < \varepsilon$, $\forall t \in [\tau', \tau]$. Let $\alpha\colon \mathbb{R} \to \mathbb{R}$ be the C^∞ map defined by $\alpha(t) = 0$, if $t \leq 0$ and $\alpha(t) = e^{-1/t}$, if $t > 0$. For any δ, $0 < \delta \leq (\tau - \tau')/2$, define the maps $\beta_0(t) = \alpha(t - \tau')\alpha(\tau' + \delta - t)$, $\beta_1(t) = \alpha(t - \tau + \delta)\alpha(\tau - t)$. Clearly β_0, β_1 are C^∞, $0 \leq \beta_0(t), \beta_1(t) \leq 1$, for all $t \in \mathbb{R}$, $\beta_0(t) = 0$, if $t \notin [\tau', \tau' + \delta]$ and $\beta_1(t) = 0$, if $t \notin [\tau - \delta, \tau]$ (see [12]). Denote $c_0 = [x(\tau') - p(\tau')]/[\int_{\tau'}^{\tau'+\delta} \beta_0(u)du]$, $c_1 = [x(\tau) - p(\tau)]/[\int_{\tau-\delta}^{\tau} \beta_1(u)du]$ and define the C^∞ maps Γ_0, Γ_1 by

$$\Gamma_0(t) = c_0 \int_t^{\tau'+\delta} \beta_0(u)\, du \qquad \Gamma_1(t) = c_1 \int_{\tau-\delta}^{t} \beta_1(u)\, du.$$

It is easy to see that $\Gamma_0(\tau') = x(\tau') - p(\tau')$, $\Gamma_0(t) = 0$, for $\tau' + \delta \leq t \leq \tau$; $\Gamma_1(\tau) = x(\tau) - p(\tau)$, $\Gamma_1(t) = 0$, for $\tau' \leq t \leq \tau - \delta$; also, since Γ_0 is monotonic, we have, $|\Gamma_0(t)| \leq |x(\tau') - p(\tau')| < \varepsilon$, $\forall t \in [\tau', \tau' + \delta]$; similarly, $|\Gamma_1(t)| < \varepsilon$, $\forall t \in [\tau - \delta, \tau]$. Then the C^∞ map $q(t) = p(t) + \Gamma_0(t) + \Gamma_1(t)$ satisfies $q(\tau') = x(\tau')$, $q(\tau) = x(\tau)$, $q(t) = p(t)$, $\forall t \in [\tau' + \delta, \tau - \delta]$, and $|q(t) - \phi(t)| \leq |p(t) - \phi(t)| + |\Gamma_0(t)| + |\Gamma_1(t)| < 3\varepsilon$, $\forall t \in [\tau', \tau]$. However, $\dot{q}(\tau') = \dot{p}(\tau')$, $\dot{q}(\tau) = \dot{p}(\tau)$. We now adjust our map q in order to get the conditions on the derivatives satisfied at the endpoints τ' and τ. For any $\eta > 0$, $4\eta < \tau - \tau'$, consider the map $\beta_2(t) = \alpha(t - \tau')\alpha(\tau' + \eta - t)$. Clearly β_2 is C^∞, $0 \leq \beta_2(t) \leq 1$, $\forall t$, $\beta_2(t) = 0$, if $t \notin [\tau', \tau' + \eta]$. Denote $c_2 = [\dot{x}(\tau') - \dot{p}(\tau')]/[\int_{\tau'}^{\tau'+\eta} \beta_2(s)\, ds]$, and define the function Γ_2 by

$$\Gamma_2(t) = c_2 \int_t^{\tau'+\eta} \beta_2(u)\, du.$$

It is easy to see that $\Gamma_2(\tau') = [\dot{x}(\tau') - \dot{p}(\tau')]$, $\Gamma_2(t) = 0$, if $\tau' + \eta \leq t \leq \tau$. Now take $k_2 = [\int_{\tau'}^{\tau'+\eta} \Gamma_2(s)\, ds]/[\int_{\tau'}^{\tau'+\eta} \beta_2(s)\, ds]$, and define the map

$$\lambda_2(t) = \int_{\tau'}^{t} [\Gamma_2(s) - k_2\beta_2(s - \eta)]\, ds.$$

Clearly, $\dot{\lambda}_2(t) = \Gamma_2(t) - k_2\beta_2(t)$, so $\dot{\lambda}_2(\tau') = \Gamma_2(\tau') = [\dot{x}(\tau') - \dot{p}(\tau')]$, $\dot{\lambda}_2(t) = 0$, if $\tau' + 2\eta \leq t \leq \tau$. Also, $\lambda_2(\tau') = 0$, and by the choice of k_2, we have $\lambda_2(t) = 0$, for $\tau' + 2\eta \leq t \leq \tau$. If we choose $\eta > 0$ such that $|c_2|\eta^2 < \varepsilon$, then, since $0 \leq \beta_2(t) \leq 1$, we also have $|\lambda_2(t)| < \varepsilon$. Similarly, define $\beta_3(t) = \alpha(t - \tau + \eta)\alpha(\tau - t)$, $c_3 = [\dot{t}(\tau) - \dot{p}(\tau)]/[\int_{\tau-\eta}^{\tau} \beta_3(s)\, ds]$, $\Gamma_3(t) = c_3 \int_{\tau-\eta}^{t} \beta_3(u)\, du$, $k_3 =$

$[\int_{\tau-\eta}^{\tau} \beta_3(s)\, ds]/[\int_{\tau-\eta}^{\tau} \Gamma_3(s)\, ds]$, $\lambda_3(t) = \int_{\tau'}^{t} \Gamma_3(s) - k_3\beta_3(s+\eta)]\, ds$ and restrict η (if necessary) such that $|c_3|\eta^2 < \varepsilon$. We have $\lambda_3(\tau) = 0$, $\lambda_3(t) = 0$, if $\tau' \leq t \leq \tau - \eta$, and $\dot{\lambda}_3(\tau) = \dot{x}(\tau) - \dot{p}(\tau)$ and $|\lambda_3(t)| < \varepsilon$. Therefore, the map $\psi(t) = q(t) + \lambda_2(t) + \lambda_3(t)$ is C^∞ and satisfies $\psi(\tau) = x(\tau)$, $\psi(\tau') = x(\tau')$, $\dot{\psi}(\tau') = \dot{x}(\tau')$, $\dot{\psi}(\tau) = \dot{x}(\tau)$ and $|\psi(t) - \phi(t)| < 3\varepsilon$, $\forall t \in [\tau', \tau]$, that is $\psi \in V_\tau$. Therefore, V_τ is dense in U_τ.

Note that solutions of the functional IVP associated to F that are backwardly oscillatory correspond, in a natural way, to solutions of the functional IVP belonging to the vector field f that oscillate around $x(., x_o)$ (in the sense that there exists a sequence $t_n \to t_o+$ at which these solutions intersect).

5 Two examples

In this section we give two computable, continuous examples of the backward continuation and oscillation property established in the previous sections. Due to the involved computations that are necessary, it, it is easier (but, by no means, less illustrative) to work with a linear equation (the same one in the two examples). So, consider the equation:

$$\dot{x}(t) = -x(t) + x(t/2) \tag{13}$$

subjected to the IVP $x_2(t) = \phi(t)$. We first take $\phi(t) = a_1 t + a_2 t^2 + \ldots + a_n t^n$, $t \in [2, 4]$. We worked out a situation where the solution $x(., 4, \phi)$ can be backwardly continuable in the interval $[0.125, 4]$ (but not any further), and we provide the appropriate initial condition $\kappa : [0.125, 0.25] \to \mathbb{R}$, as well as the intermediate steps: $x_t(., 0.25, \kappa) = \xi$ (defined on $[0.25, 0, 5]$), $x_t(., 0.5, \xi) = \eta$ (defined on $[0.5, 1]$), $x_t(., 1, \eta) = \psi$ (defined on $[1, 2]$), $x_t(., 2, \psi) = \phi$. The expressions of these consecutive initial conditions are:

$$\kappa(t) = \tfrac{9604784}{677} - \tfrac{28826624}{677}t - \tfrac{487337984}{677}t^2 + \tfrac{3501375488}{2031}t^3 + \tfrac{7512522752}{2031}t^4 + \tfrac{709885952}{677}t^5 ,$$

$$\xi(t) = -\tfrac{805712}{677} + \tfrac{10410496}{677}t - \tfrac{12411904}{677}t^2 - \tfrac{109422592}{2031}t^3 + \tfrac{136773632}{2031}t^4 + \tfrac{22183936}{677}t^5 ,$$

$$\eta(t) = \tfrac{335792}{677} - \tfrac{1141504}{677}t + \tfrac{3173376}{677}t^2 - \tfrac{6276352}{2031}t^3 - \tfrac{1850368}{2031}t^4 + \tfrac{693248}{677}t^5 ,$$

$$\psi(t) = \tfrac{537456}{677} - \tfrac{201664}{677}t - \tfrac{184544}{677}t^2 + \tfrac{977888}{2031}t^3 - \tfrac{440608}{2031}t^4 + \tfrac{64992}{2031}t^5 ,$$

$$\phi(t) = \tfrac{537456}{677}t - \tfrac{319144}{677}t^2 + \tfrac{273008}{2031}t^3 - \tfrac{37693}{2031}t^4 + t^5 .$$

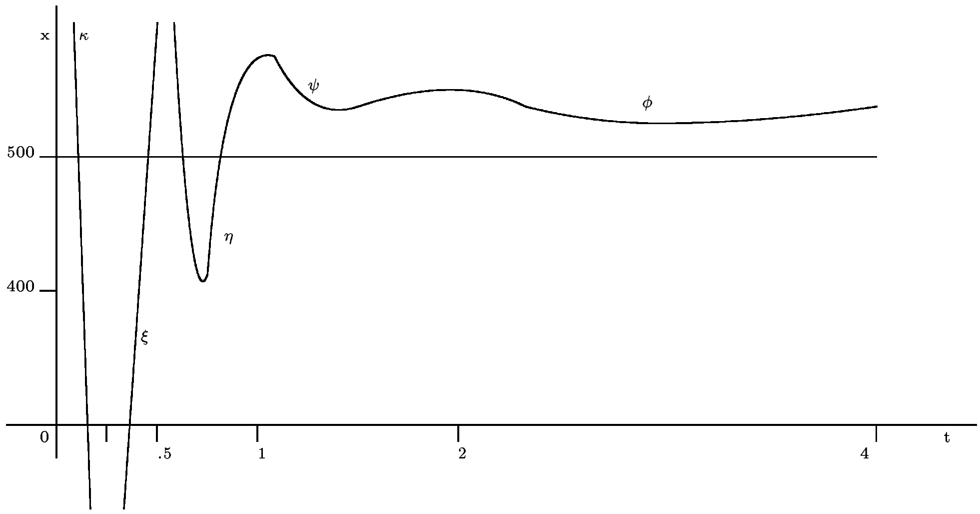

Figure 2: A four-step backward continuable solution $x(.,4,\phi)$.

It is worth realizing that, although this solution is not backwardly continuable on the interval $(0,4]$, it already depicts a tendency to the wild oscillatory behavior antecipated by the above theory.

In the next example we start with the initial condition $\phi(t) = 2[\exp(-(t-\tau')^{-2} - (\tau - t)^{-2})]$ which, as we know from Section 4, provides a backwardly continuable solution in the interval $(0,\tau]$. If we take $\tau = 20$, a simple program in the Mathematica software, e.g.,

```
Clear[f1]
f1[t_]=2*Exp[-(t-10)^(-2)-(20-t)^(-2)]
m1[t_]=f1'[t]
m2[t_]=m1'[2t]+m1[2t]
m3[t_]=m2'[2t]+m2[2t]
m4[t_]=m3'[2t]+m3[2t]
m5[t_]=m4'[2t]+m4[2t]
mr1=Plot[m1[t],{t,10,20}]
mr2=Plot[m2[t],{t,5,10}]
mr3=Plot[m3[t],{t,2.5,5}]
```

```
mr4=Plot[m4[t],{t,1.25,2.5}]
mr5=Plot[m5[t],{t,.625,1.25}]
Show[mr1,mr2,mr3,mr4,mr5]
```

furnishes the following graphs:

Fig. 3: Initial condition on [10,20]

Fig. 4: Backward continuation on [5,10]

Fig. 5: Backward continuation on [2.5,5]

Fig. 6: Backward continuation on [1.25,2.5]

Fig. 7: Backward continuation on [.625,1.25]

Fig. 8: The whole backward continuation on [.625,20]

The wild oscillatory behavior of the solution as $t \to 0^+$, is clearly depicted by the graphs. Due to the condensation of the graphs, it is useless to increase the number of steps depicted to show the phenomenon.

The above examples nicely illustrate the oscillatory properties described in [2], for the case of the linear scaled equation $\dot{x}(t) = -ax(t) + ax(pt)$. Note that the solution in the first example oscillates around the (constant) solution $x(., 500) \equiv 500$, and that one in the second example oscillates around the trivial solution.

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