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Domains of Attraction of Competition-Diffusion Systems

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1. Introduction

In this paper we study the global behavior of solutions to the *reaction-diffusion system* :

$$(1.1)\quad \begin{cases} u_t = d_1 \Delta u + u f(u,v) & \text{in } \Omega \times (0,\infty), \\ v_t = d_2 \Delta v + v g(u,v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$ and ν is the outward unit normal vector to $\partial\Omega$. The initial functions $u_0(x)$, $v_0(x)$ are not identically zero, and the functions $f(u,v)$ and $g(u,v)$ are of class $C^1(Q)$ where

$$Q = \{(u,v) \in \mathbb{R}^2 | u \geq 0, v \geq 0\}.$$

For $f(u,v)$ and $g(u,v)$ we will consider the following two cases of *competition type* which make (1.1) a *competition-diffusion system* :

(α) $f(u,v) = a_1 - b_1 u - c_1 v, \qquad g(u,v) = a_2 - b_2 u - c_2 v,$

(β) $f(u,v) = a_1 - b_1 u^2 - c_1 v^2, \quad g(u,v) = a_2 - b_2 u^2 - c_2 v^2.$

In the system (1.1) u and v are nonnegative functions which represent the population densities of two competing species. d_1 and d_2 are the *diffusion* rates of the two species, respectively. a_1 and a_2 denote the intrinsic growth rates, b_1 and c_2 account for intra-specific competitions, b_2 and c_1 are the coefficients for inter-specific competitions. For details on the backgrounds of this model, we refer the reader to [6].

Remark. The linear functions for $f(u,v)$ and $g(u,v)$ as in (α) are often used in the classical *competition-diffusion systems.* Though the quadratic functions for $f(u,v)$ and $g(u,v)$ as in (β) may not be used commonly, they make the system (1.1) the gradient system of an energy functional (after simple scalings) which helps one to analyze the system (1.1) more clearly. And, in the course of this paper it will be shown that the system (1.1) with (β) has similar properties as the system with (α).

The global behavior of solutions to the system (1.1) is related to that of its *kinetic system* which is the following system of ordinary differential equations :

$$\text{(1.2)} \qquad \begin{cases} u_t = uf(u,v) & \text{in } (0,\infty), \\ v_t = vg(u,v) & \text{in } (0,\infty). \end{cases}$$

Clearly Q is positively invariant for the flow of (1.2). The equilibria of the kinetic system (1.2) in Q consist of four points $(u_A, 0)$, $(0, v_B)$, (u_C, v_C) and $(0,0)$, where u_A, v_B, u_C and v_C are positive constants in both cases (α) and (β) of the functions $f(u,v)$ and $g(u,v)$. Throughout this paper we impose the following *strong competition* conditions on the coefficients in $f(u,v)$ and $g(u,v)$:

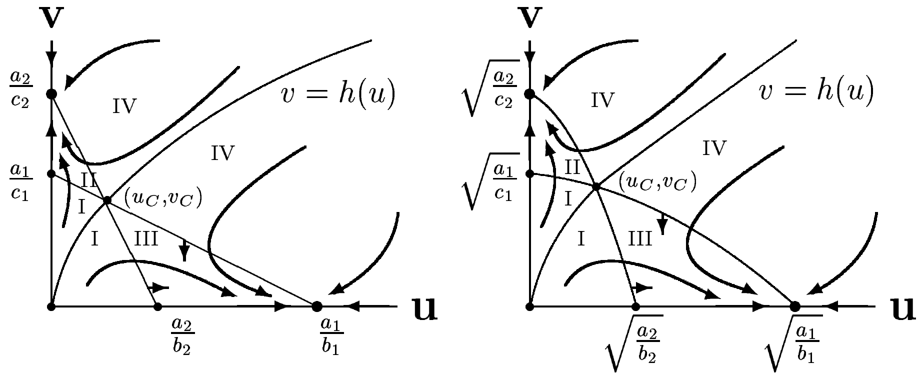
$$\text{(1.3)} \qquad \frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}.$$

We note that the condition (1.3) assures that $(0,0)$ is unstable, $(u_A, 0)$, $(0, v_B)$ are stable, and (u_C, v_C) is a saddle point for (1.2). The flows of the kinetic system (1.2) under the condition (1.3) are shown in Figure 1.

For the general properties of separatrix $h(u)$ of the kinetic system (1.2) which is illustrated in Figure 1 we refer the reader to the result due to Iida et al. [2] which is stated in Proposition 1.1 in the following. The reader may also refer to Hirsch and Smale [1], Ninomiya [5] for the properties of separatrices.

Proposition 1.1. *Suppose for the system* (1.2) *that* $f(u,v)$ *and* $g(u,v)$ *are as in either* (α) *or* (β). *Then there exists a monotone function* $h(u)$ *defined on* $[0, u_\infty)$ *with* $u_\infty \in (u_C, \infty]$ *such that*

$W_A = \{(u,v) \in Q | v < h(u)\}$ *is the basin of attraction for* $(u_A, 0)$,



I: $u_t > 0, \ v_t > 0,$ II: $u_t < 0, \ v_t > 0,$
III: $u_t > 0, \ v_t < 0,$ IV: $u_t < 0, \ v_t < 0.$

Figure 1: Flows of (1.2) with competitions as in (α) and in (β).

$W_B = \{(u,v) \in Q | v > h(u)\}$ *is the basin of attraction for* $(0, u_B)$,

$\Gamma = \{(u,v) \in Q | v = h(u)\}$ *is the separatrix.*

Moreover $h(u)$ *satisfies*

(i) $h(0) = 0$,

(ii) $h(u_C) = v_C$,

(iii) $\lim_{u \to u_\infty} h(u) = \infty$ *if* $u_\infty < \infty$,

(iv) $h'(u) > 0$ *on* $(0, u_\infty)$,

(v) $uf(u, h(u))h'(u) = h(u)g(u, h(u))$ *on* $(0, u_\infty)$,

(vi) $f(u, h(u))g(u, h(u)) > 0$ *on* $(0, u_C) \cup (u_C, u_\infty)$.

Iida et al. [2] assumed in addition to the *strong-competition* condition (1.3) that $\frac{a_1}{a_2} \geq 1$ which means that the species u is superior to the other species v in the competition sense, and they showed that the separatrix of the kinetic system (1.2) with linear competitions as in (α) is concave, i.e. $h'' \leq 0$. Also assuming that $d_1 = d_2$ they proved that the region under the concave separatrix is a domain of attraction for the equilibrium point $(u_A, 0) = (\frac{a_1}{b_1}, 0)$ in the phase plane for the *competition-diffusion* system (1.1) with linear competitions as in (α). Their result means in Figure 2 that if the initial state $(u_0(x), v_0(x))$ is chosen in the region Σ_h then the solution $(u(x,t), v(x,t))$ converges to $(u_A, 0)$, that is, only the superior species u survives and the inferior species v is wiped out eventually. In this paper we are interested in finding the initial states for which the inferior species v survives and the superior species u dies out at the end. First we consider the same situation as Iida et al. [2] and prove that the region Σ^l above the concave separatrix in Figure 2 is a domain of attraction of the equilibrium point $(0, u_B) = (0, \frac{a_2}{c_2})$. We later consider the quadratic competitions as in (β) for the *competition-diffusion* system (1.1) with $d_1 = d_2$ to show that the region Σ^l above the separatrix in Figure 3 is a domain of attraction of the equilibrium point $(0, u_B) = (0, \sqrt{\frac{a_2}{c_2}})$.

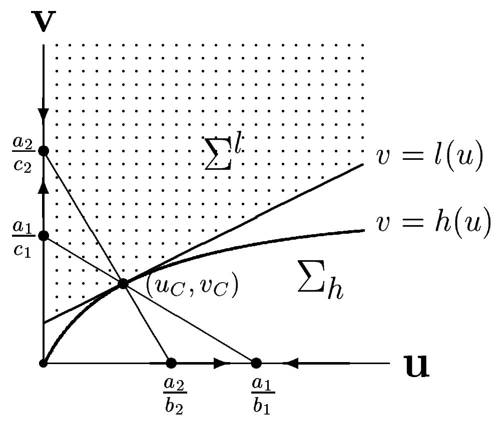


Figure 2: The set Σ^l for the system (1.1) with linear competitions.

In order to state our main results we introduce the notation which will be used throughout this paper, and reduce the system (1.1) with $d_1 = d_2$ and

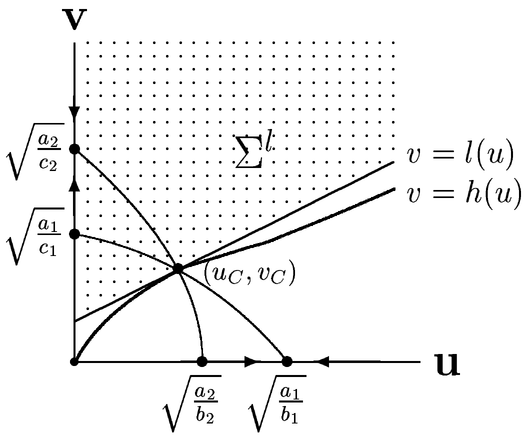


Figure 3: The set Σ^l for the system (1.1) with quadratic competitions.

the *strong competition* condition (1.3) to a simpler form.

Notation. We set $\Sigma^k := \{(u,v) \in Q |\ u \geq 0, \quad v > k(u)\}$, where $k(u)$ is a strictly increasing C^2-function with $k(0) \geq 0$. We denote $v = l(u)$ the tangent line of $v = h(u)$ at the unstable constant equilibrium (u_C, v_C) in the phase plane.

The *competition-diffusion system* (1.1) with $d_1 = d_2 = d$ and linear competitions as in (α) is reduced to the following system :

$$(1.4) \qquad \begin{cases} u_t = d\Delta u + u(a - u - bv) & \text{in } \Omega \times (0,\infty), \\ v_t = d\Delta v + v(1 - cu - v) & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0 & \text{in } \Omega \end{cases}$$

by the change of variables $\tau = a_2 t$, $\tilde{u} = \frac{b_1}{a_2} u$ and $\tilde{v} = \frac{c_2}{a_2} v$, and then using the variables t, u and v instead of τ, $\tilde{u}$ and $\tilde{v}$. The equilibria in Q for the kinetic system of (1.4) consist of four points, $(u_A, 0) = (a, 0)$, $(0, v_B) = (0, 1)$, $(u_C, v_C) = (\frac{b-a}{bc-1}, \frac{ac-1}{bc-1})$, and $(0,0)$.

The *competition-diffusion systems* (1.1) with $d_1 = d_2 = d$ and quadratic competitions as in (β) is reduced to the following system :

$$(1.5) \qquad \begin{cases} u_t = d\Delta u + u(a - u^2 - bv^2) & \text{in } \Omega \times (0, \infty), \\ v_t = d\Delta v + v(1 - cu^2 - v^2) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) \geq 0, \;\; v(x, 0) = v_0(x) \geq 0 & \text{in } \Omega \end{cases}$$

by the change of variables $\tau = a_2 t$, $\tilde{u} = \sqrt{\frac{b_1}{a_2}} u$ and $\tilde{v} = \sqrt{\frac{c_2}{a_2}} v$, and then using the variables t, u and v instead of τ, $\tilde{u}$ and $\tilde{v}$. The equilibria in Q for the kinetic system of (1.5) consist of four points, $(u_A, 0) = (\sqrt{a}, 0)$, $(0, v_B) = (0, 1)$, $(u_C, v_C) = (\sqrt{\frac{b-a}{bc-1}}, \sqrt{\frac{ac-1}{bc-1}})$, and $(0, 0)$.

For both systems (1.4) and (1.5) the *strong competition* condition (1.3) is reduced to

$$(1.6) \qquad \frac{1}{c} < a < b.$$

Now we present our main results in the following theorems.

Theorem 1.1. *Let* $\frac{1}{c} < a < b$, $a \geq 1$, *and* $(u_0(x), v_0(x)) \in \Sigma^l$. *Then a solution* $(u(x,t), v(x,t))$ *of the system* (1.4) *stays in the set* Σ^l *and converges to* $(0, 1)$ *uniformly as* $t \to \infty$.

Theorem 1.2. *Let* $\frac{1}{c} < a < b$, $a \geq 1$, *and* $(u_0(x), v_0(x)) \in \Sigma^l$. *Suppose that* $h''(u) \leq 0$ *for* $u < u_C$. *Then a solution* $(u(x,t), v(x,t))$ *of the system* (1.5) *stays in the set* Σ^l *and converges to* $(0, 1)$ *uniformly as* $t \to \infty$.

Remark. In the proofs of Theorems 1.1 and 1.2 the concavity of the graph of the function $h(u)$ plays an important role. Regarding the system (1.4) the concavity of h is proved by Iida et al. [2] as stated in Proposition 2.1. We prove similar but partial result for the system (1.5) in Proposition 2.2. A sufficient condition on the coefficients a, b, c to guarantee that $h''(u) \leq 0$ for $u < u_C$ is found in Proposition 2.2 **(vi)**.

The properties of separatrix which are needed during the constructions of domains of attraction are studied in Section 2. In Section 3 we obtain the invariance of the set $\overline{\Sigma^l}$ for the flow (1.4) and (1.5). We present lemmas regarding auxiliary functions which are used in the proofs of Theorem 1.1 and Theorem 1.2 in Section 4. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 5 and 6, respectively.

2. Properties of Separatrix of Kinetic System

In this section we investigate properties of separatrices of the kinetic systems of the *competition-diffusion systems* (1.4) and (1.5). The following result in Proposition 2.1 is due to Iida et al. [2].

Proposition 2.1. *Suppose that* $\frac{1}{b} < a < c$. *Then there exists a monotone function* $h(u)$ *defined on* $[0, \infty)$ *as the separatrix of the kinetic system of* (1.4). *Moreover the function* h *satisfies*

(i) $h(0) = 0$,

(ii) $h(u_C) = v_C$, *where* $(u_C, v_C) = (\frac{b-a}{bc-1}, \frac{ac-1}{bc-1})$

(iii) $\lim_{u\to\infty} h(u) = \infty$,

(iv) $$\begin{cases} h'' < 0 & \text{if } a > 1, \\ h'' = 0 & \text{if } a = 1, \\ h'' > 0 & \text{if } a < 1. \end{cases}$$

In particular if $a = 1$ *then* $h(u) = \frac{b-1}{c-1}\, u.$

Now for the *competition-diffusion system* (1.5) with the quadratic competitions we obtain similar results.

Proposition 2.2. *Suppose that* $\frac{1}{b} < a < c$. *Then there exists a monotone function* $h(u)$ *defined on* $[0, \infty)$ *as the separatrix of the kinetic system of* (1.5). *Moreover the function* h *satisfies*

(i) $h(0) = 0$,

(ii) $h(u_C) = v_C$, *where* $(u_C, v_C) = (\sqrt{\frac{b-a}{bc-1}}, \sqrt{\frac{ac-1}{bc-1}})$

(iii) $\lim_{u \to \infty} h(u) = \infty$,

(iv) $$\begin{cases} h''(u_C) < 0 & \text{if } a > 1, \\ h''(u_C) = 0 & \text{if } a = 1, \\ h''(u_C) > 0 & \text{if } a < 1, \end{cases}$$

(v) *If* $a = 1$ *then* $h(u) = \sqrt{\frac{b-1}{c-1}}\, u$ *for every* $u > 0$.

(vi) *Set* $D = -6\{a^2(1-ac)^2 + (b-a)^2\} + (ac-1)(b-a)\{(b-6)ca^2 + (bc - 3b - 3c + 3)a + (c-6)b\}$. *If* $a > 1$ *and* $D \le 0$ *then* $h''(u) < 0$ *for every* $u \in (0, u_C + \epsilon)$, *where* ϵ *is a small number.*

Proof. The existence of h and properties **(i)** and **(ii)** of the kinetic system of (1.5) follow from the general results in Proposition 1.1. In the case that

$a=1$ it is easy to check that $(u(t),v(t))=\left(\frac{u_C}{\sqrt{1+ke^{-2t}}},\frac{v_C}{\sqrt{1+ke^{-2t}}}\right)$ is a solution to the kinetic system of (1.5) for every $k>-1$. The solution moves on the straight line $v=\sqrt{\frac{c-1}{b-1}}\,u$ and converges to (u_C,v_C) as $t\to\infty$. This implies $h(u)\equiv\sqrt{\frac{c-1}{b-1}}\,u$, and hence **(v)** is proved.

Now we consider only the case $a>1$ in the following (the proof in the case $a<1$ is obtained in the same manner). In order to prove **(iii)**, **(iv)**, and **(vi)** we prepare the following lemmas.

Lemma 2.1. *$v=h(u)$ is defined on (u_C,∞) and $\lim_{u\to\infty}h(u)=\infty$.*

Proof. The separatrix $h(u)$ satisfies the equation $h'(u)=\frac{h(u)g(u,h(u))}{uf(u,h(u))}$. And so, $h(u)$ is strictly increasing. Let us denote thst $\xi=u-u_C$, $\eta=h(u)-v_C$, $\rho=\frac{\eta}{\xi}$, and $\rho_a=\frac{v_C}{u_C}=\sqrt{\frac{ac-1}{b-a}}$. We rewrite $f(u,h(u))$ and $g(u,h(u))$ using the notation above :

$$f(u,h(u))=a-u^2-bh(u)^2=-\xi^2-b\eta^2-2u_C\xi-2bv_C\eta,$$

$$g(u,h(u))=1-cu^2-h(u)^2=-c\xi^2-\eta^2-2cu_C\xi-2v_C\eta.$$

Then $h'(u) = \frac{(v_C+\eta)(c\xi^2+\eta^2+2cu_C\xi+2v_C\eta)}{(u_C+\xi)(\xi^2+b\eta^2+2u_C\xi+2bv_C\eta)}$. If $\rho = \frac{\eta}{\xi} = \rho_a$

$$\begin{aligned} h'(u) &= \tfrac{(\rho_a u_C+\xi\rho_a)}{(u_C+\xi)} \cdot \tfrac{(c\xi^2+\xi^2\rho_a^2+2cu_C\xi+2u_C\rho_a^2\xi)}{(\xi^2+b\rho_a^2\xi^2+2u_C\xi+2bu_C\rho_a^2\xi)} \\ &= \rho_a \cdot \tfrac{\left((c+\rho_a^2)\xi+2u_C(c+\rho_a^2)\right)}{((1+b\rho_a^2)\xi+2u_C(1+b\rho_a^2))} \\ &= \tfrac{\rho_a}{a} \\ &< \rho_a \qquad \text{for } \ a>1. \end{aligned}$$

So $v = h(u)$ can not touch the straight line $\frac{v-v_C}{u-u_C} = \rho_a$. Thus $v = h(u)$ is defined for $u \in (0,\infty)$. Let us suppose that $\lim_{u\to\infty} h(u) = h_\infty < \infty$. Then

$$\begin{aligned} h(u) &= h(2u_C) + \textstyle\int_{2u_C}^{u} h'(w)dw \\ &= h(2u_C) + \textstyle\int_{2u_C}^{u} \tfrac{h(w)\{-1+cw^2+h^2(w)\}}{w\{-a+w^2+bh^2(w)\}}dw \\ &\geq h(2u_C) + \textstyle\int_{2u_C}^{u} \tfrac{v_C\{-1+cw^2+v_C^2\}}{w\{-a+w^2+bh_\infty^2\}}dw. \end{aligned}$$

This implies $\lim_{u\to\infty} h(u) = \infty$. It is a contradiction to the hypothesis. Hence we conclude $\lim_{u\to\infty} h(u) = \infty$. □

Lemma 2.2. *Expand $h(u)$, the separatrix of the kinetic system of* (1.5), *as a series, $h(u) = v_C + h_1(u-u_C) + h_2(u-u_C)^2 + O(|u-u_C|^3)$ for u close to u_C. Then we have the following equalities :*

(2.1) $$u_C h_1(u_C + bh_1 v_C) = v_C(cu_C + h_1 v_C),$$

(2.2) $$h_2 = \frac{-h_1 u_C(3 + bh_1^2 - 2c) + v_C(c + h_1^2(3-2b))}{2(3bu_C v_C h_1 - v_C^2 + 2u_C^2)}.$$

Proof. We substitute the series expansion into the following equation satisfied by the separatrix of the kinetic system (see **(v)** in Proposition 1.1) :

$$uh'(u)f(u,h(u)) = h(u)g(u,h(u)),$$

and compare terms in both sides.

$$\begin{aligned}
&\{(u-u_C)+u_C\}\{h_1+2h_2(u-u_C)+\cdots\}\cdot \\
&\qquad \{a-(u-u_C)^2-2u_C(u-u_C)-u_C^2 \\
&\qquad\qquad -b(v_C+h_1(u-u_C)+h_2(u-u_C)^2+\cdots)^2\} \\
&=\{v_C+h_1(u-u_C)+h_2(u-u_C)^2+\cdots\}\cdot \\
&\qquad \{1-c(u-u_C)^2-2c(u-u_C)-cu_C^2 \\
&\qquad\qquad -(v_C+h_1(u-u_C)+h_2(u-u_C)^2+\cdots)^2\}.
\end{aligned}$$

Now we use the equalities $a-u_C^2-bv_C^2=0$ and $1-cu_C^2-v_C^2=0$ to have

$$\begin{aligned}
&\{(u-u_C)+u_C\}\{h_1+2h_2(u-u_C)+\cdots\}\cdot \\
&\quad \{-2(u_C+bh_1v_C)(u-u_C)-(1+bh_1^2+2v_Ch_2b)(u-u_C)^2+\cdots\} \\
&=\{v_C+h_1(u-u_C)+h_2(u-u_C)^2+\cdots\}\cdot \\
&\quad \{-2(cu_C+v_Ch_1)(u-u_C)-(c+h_1^2+2v_Ch_2)(u-u_C)^2+\cdots\}.
\end{aligned}$$

By observing the coefficients of $(u-u_C)$ on both sides we have the first equality (2.1). Next we solve the following equality between the coefficients of $(u-u_C)^2$ on both sides for h_2 to obtain the equality (2.2) :

$$-h_1 u_C(1+bh_1^2+2v_C h_2 b) - 4u_C h_2(u_C+bh_1 v_C) - 2h_1(u_C+bh_1 v_C)$$
$$= -v_C(c+h_1^2+2v_C h_2) - 2h_1(cu_C+v_C h_1). \qquad \square$$

Now we prove the local concavity of h near the unstable equilibrium (u_C, v_C).

Lemma 2.3. *Let $\frac{1}{c} < a < b$ and $a > 1$. Then $h''(u_C) < 0$.*

Proof. Let us denote

$$A := -bh_1^3 + (3-2b)\rho_a h_1^2 - (3-2c)h_1 + c\rho_a, \quad B := h_1 - \frac{1}{3b}(\rho_a - \frac{2}{\rho_a}).$$

Then from the equation (2.2) in Lemma 2.2, $h_2 = \frac{1}{6\,b\,v_C}\frac{A}{B}$. Let us define a quadratic function $F(x)$ as $F(x) := bx^2 + (\frac{1}{\rho_a} - \rho_a)x - c$ for $x \geq 0$. The equation (2.1) in Lemma 2.2 gives that $F(h_1) = 0$. By computations we observe that $F(\frac{1}{3b}(\rho_a - \frac{2}{\rho_a})) = \frac{1}{9b}(-2(\rho_a - \frac{1}{\rho_a})^2 + 1 - 9bc) < 0$.
Thus we have that $h_1 > \frac{1}{3b}(\rho_a - \frac{2}{\rho_a})$ and $B > 0$.

Now we just need to show that $A < 0$. By noticing that

$$F(\rho_a) = b\rho_a^2 + 1 - \rho_a^2 - c = (b-1)\rho_a^2 + (1-c) > 0,$$
$$F(\tfrac{\rho_a}{a}) = \tfrac{(ac-1)(1-a)}{a^2} < 0$$

we have the inequalities

$$\frac{\rho_a}{a} < h_1 < \rho_a. \tag{2.3}$$

Using the fact $F(h_1) = 0$ we can rewrite A as following by long divisions :

$$A = h_1(-1 - \tfrac{1}{b} + c - \tfrac{1}{b\rho_a^2} - 2\rho_a^2 + \tfrac{2}{b}\rho_a^2) + c(\tfrac{1}{b\rho_a} - \rho_a + \tfrac{2\rho_a}{b}).$$

If $(-1 - \frac{1}{b} + c - \frac{1}{b\rho_a^2} - 2\rho_a^2 + \frac{2}{b}\rho_a^2) \geq 0$ then from the inequality $h_1 \leq \rho_a$ we have

$$\begin{aligned} A &\leq \rho_a(-1 - \tfrac{1}{b} + c - \tfrac{1}{b\rho_a^2} - 2\rho_a^2 + \tfrac{1}{b}\rho_a^2) + c(\tfrac{1}{b\rho_a} - \rho_a + \tfrac{2\rho_a}{b}) \\ &= -\tfrac{(bc-1)(a-1)}{b(b-a)\rho_a}(2\rho_a^2 + 1) \qquad (2.4) \\ &< 0. \end{aligned}$$

If $(-1 - \frac{1}{b} + c - \frac{1}{b\rho_a^2} - 2\rho_a^2 + \frac{2}{b}\rho_a^2) \leq 0$ then we use the inequality $h_1 \geq \frac{\rho_a}{a}$ to obtain the following :

$$\begin{aligned} A &\leq \tfrac{\rho_a}{a}(-1 - \tfrac{1}{b} + c - \tfrac{1}{b\rho_a^2} - 2\rho_a^2 + \tfrac{1}{b}\rho_a^2) + c(\tfrac{1}{b\rho_a} - \rho_a + \tfrac{2\rho_a}{b}) \\ &= \tfrac{1-a}{ab\rho_a}\left(\tfrac{(ac-1)(2\rho_a^2+1)}{b-a} + bc\rho_a^2\right) \qquad (2.5) \\ &< 0. \end{aligned}$$

□

Lemma 2.4. *Let* $\frac{1}{c} < a < b$ *and* $a > 1$. *Then* $h_1 > \sqrt{\frac{c-1}{b-1}}$ *for* $c \geq 1$.

Proof. Define a quadratic function

$$Q(h) := b\sqrt{b-a}\sqrt{ac-1}\,h^2 + (b - a - ac + 1)\,h - c\sqrt{b-a}\sqrt{ac-1}.$$

Then $Q(h_1) = 0$, by the first equation proved in Lemma 2.2 and the fact that $(u_C, v_C) = \left(\sqrt{\frac{b-a}{bc-1}}, \sqrt{\frac{ac-1}{bc-1}}\right)$. We notice that the equation $Q(h) = 0$ has one negative and one positive solution. Hence in order to have the desired conclusion we just need to show that $Q(\sqrt{\frac{c-1}{b-1}}) \leq 0$.

$$\begin{aligned} Q(\sqrt{\tfrac{c-1}{b-1}}) &= b\tfrac{c-1}{b-1}\sqrt{b-a}\sqrt{ac-1} + (b-a-ac+1)\sqrt{\tfrac{c-1}{b-1}} \\ &\quad -c\sqrt{b-a}\sqrt{ac-1} \\ &= \tfrac{c-b}{b-1}\sqrt{b-a}\sqrt{ac-1} + (b-a-ac+1)\sqrt{\tfrac{c-1}{b-1}}. \end{aligned}$$

In the case $b \leq c$ we have $b - a - ac + 1 \leq 0$, since $a > 1$. It is easy to check $(c-b)^2(b-a)(ac-1) \leq (-b+a+ac-1)^2(c-1)(b-1)$, and so we have $Q(\sqrt{\frac{c-1}{b-1}}) \leq 0$. Now if $b > c$ and $b-a-ac+1 \leq 0$ then clearly $Q(\sqrt{\frac{c-1}{b-1}}) \leq 0$. So, $b > c$ and $b - a - ac + 1 > 0$ is the only case remaining, and in this case we have $(c-b)^2(b-a)(ac-1) > (-b+a+ac-1)^2(c-1)(b-1)$. Thus in the last case, we have $Q(\sqrt{\frac{c-1}{b-1}}) \leq 0$. □

Now we will observe the separatrix $v = h(u)$ away from the unstable equilibrium point (u_C, v_C) for its concavity. The separatrix h satisfies the following differential equation (see **(v)** in Proposition 1.1) :

$$uf(u,h(u))h'(u) = h(u)g(u,h(u)). \tag{2.6}$$

In order to observe $h''(u)$ we take derivatives of (2.6) :

$$\begin{aligned} uf(u,h(u))h''(u) &= h'(u)\{-f(u,h(u))+g(u,h(u))\} \\ &\quad -uh'(u)\{f_u(u,h(u))+h'(u)f_v(u,h(u))\} \\ &\quad +h(u)\{g_u(u,h(u))+h'(u)g_v(u,h(u))\}, \\ uf(u,h(u))h'''(u) &= \gamma(u)h''(u)+4h'(u)\{(1-c)u+(b-1)h(u)h'(u)\} \\ &\quad +2uh'(u)\{1+b(h'(u))^2\}-2h(u)\{c+(h'(u))^2\}. \end{aligned}$$

Let us denote

$$\begin{aligned} C &= 4h'(u)\{(1-c)u+(b-1)h(u)h'(u)\}+2uh'(u)\{1+b(h'(u))^2\} \\ &\quad -2h(u)\{c+(h'(u))^2\}. \end{aligned}$$

Then we have

$$uf(u,h(u))h'''(u) = \gamma(u)h''(u)+C. \tag{2.7}$$

Now we will investigate the sign of C. Here we remind ourselves the inequality (2.3) which implies that

$$h'(u_C) < \frac{h(u_C)}{u_C} = \rho_a, \quad \text{where } \rho_a = \sqrt{\frac{ac-1}{b-a}}.$$

We will show that $C > 0$ for $u \in (0, u_C)$ as long as

$$h_1 \le h'(u) < \frac{h(u)}{u}, \quad \rho_a < \frac{h(u)}{u}. \tag{2.8}$$

The inequalities in (2.8) imply that

$$\frac{1}{a}\frac{h(u)}{u} < h'(u) < \frac{h(u)}{u}, \tag{2.9}$$

since from the equation (2.6) we have $h'(u) = \frac{h(u)}{u}\frac{g(u,h(u))}{f(u,h(u))}$ and

$$\frac{g(u,h(u))}{f(u,h(u))} - \frac{1}{a} = \frac{b-a}{a}\frac{u^2}{f(u,h(u))}\left(-\frac{(ac-1)}{(b-a)} + \frac{h^2(u)}{u^2}\right) > 0.$$

Using the inequalities in (2.9) we have that

$$\begin{aligned} h'(u)\{1+b(h'(u))^2\} - \tfrac{h(u)}{u}\{c+(h'(u))^2\} & \\ &> h'(u)\{1+b(h'(u))^2\} - ah'(u)\{c+(h'(u))^2\} \\ &= h'(u)\{(1-ac)+(b-a)(h'(u))^2\} \\ &> h'(u)\{(1-ac)+(\tfrac{b}{a}-1)\tfrac{h(u)}{u}h'(u)\}, \end{aligned}$$

$$\begin{aligned} \tfrac{C}{2u} &= 2h'(u)\left\{(1-c)+(b-1)\tfrac{h(u)}{u}h'(u)\right\} \\ &\quad + h'(u)\{1+b(h'(u))^2\} - \tfrac{h(u)}{u}\{c+(h'(u))^2\} \\ &> h'(u)\left\{(3-2c-ac)+(\tfrac{b}{a}+2b-3)\tfrac{h(u)}{u}h'(u)\right\} \\ &> h'(u)\left\{(3-2c-ac)+(\tfrac{b}{a}+2b-3)\rho_a h'(u)\right\}. \end{aligned}$$

Denote that $\bar{\rho} := \frac{a(ac-2c-3)}{\rho_a(b+2ab-3a)}$. We adopt the function $F(x)$ in Lemma 2.3 and evaluate to have $F(\bar{\rho}) = \frac{(a-1)}{\frac{ac-1}{b-a}(b-a)(b+2ab-3a)^2}D \le 0$, where $D \le 0$ is as defined in the statement of Proposition 2.2 **(vi)**. Thus we conclude that $h_1 \ge \bar{\rho}$, and so $C > 0$ for every $u \in (0, u_C)$ as long as the condition (2.8) is satiesfied.

When $C > 0$ we obtain from (2.7) the following differential inequality :

$$h'''(u) - \frac{\gamma(u)}{uf(u,h(u))}h''(u) > 0 \quad \text{for} \quad u < u_C.$$

So we have

$$e^{-\int \frac{\gamma}{uf}} h''(u)\Big|_{u<u_C} < e^{-\int \frac{\gamma}{uf}} h''\Big|_{u=u_C}.$$

By Lemma 2.3 $h''(u_C) < 0$ and thus $h''(u) < 0$ for $u < u_C$. Finally by noticing that the concavity of h in return implies the inequalities in (2.8) we conclude the proof of **(vi)** in Proposition 2.2. $\square$

3. The Positive Invariance of the Set $\overline{\Sigma^l}$

In this section, we first consider $k(u)$, a strictly increasing C^2-function with $k(0) \geq 0$. Under a group of sufficient conditions on the function $k(u)$ it is proved that the set $\overline{\Sigma^k}$ is positively invariant for the flow (1.1) with $d_1 = d_2$. By using the result on the set $\overline{\Sigma^k}$ we will prove that the set $\overline{\Sigma^l}$ is positively invariant for the flow (1.4) and (1.5) in the phase plane.

The result in the following lemma is an immediate consequence of the general results by H. F. Weinberger [8].

Lemma 3.1. *Let $k(u)$ be a strictly increasing C^2-function satisfying*

$k(0) \geq 0,\ \ k''(u) \geq 0,$ *and*

$$k'(u)uf(u,k(u)) - k(u)g(u,k(u)) \leq 0. \tag{3.1}$$

If $(u_0(x), v_0(x)) \in \overline{\Sigma^k}$ in Ω then $(u(x,t), v(x,t))$, a solution of (1.1) with $d_1 = d_2 = d$, stays in the set $\overline{\Sigma^k}$ for every $x \in \Omega$ and $t > 0$. Moreover if $(u_0(x), v_0(x)) \in \Sigma^k$ for some $x_0 \in \Omega$ then $(u(x,t), v(x,t)) \in \Sigma^k$ for all $x \in \overline{\Omega}$ and $t > 0$.

Now, applying the results in Lemma 3.1 we prove the positive invariance of the set $\overline{\Sigma^l}$.

Lemma 3.2. *The set* $\overline{\Sigma^l}$ *is positively invariant with respect to the flow* (1.4).

Proof. We want to show that the tangent line $l(u)$ satisfies the inequality (3.1) in Lemma 3.1. We define a function $S(u)$ for $u \geq 0$ as follows :

$$S(u) := -l'(u)uf(u, l(u)) + l(u)g(u, l(u)).$$

To complete the proof we just need to show that $S(u) \geq 0$. By using the equation $l(u) = h_1(u - u_C) + v_C$ we have

$$\begin{aligned} S'(u_C) &= -h_1 u_C(-1 - b \cdot h_1) + v_C(-c - h_1), \\ S''(u) &= 2h_1(1 + b \cdot h_1 - c - h_1). \end{aligned}$$

From **(v)** in Proposition 1.2 applied to the kinetic system of (1.5) we have

$$\frac{v_C}{u_C} = \frac{h_1 + h_1^2 b}{c + h_1}. \tag{3.2}$$

Thus

$$S'(u_C) = 0. \tag{3.3}$$

Using the concavity of $h(u)$ and the fact $h(0) = 0$ from Proposition 2.1 we observe that $h_1 < \frac{v_C}{u_C}$, and thus $\frac{1+b \cdot h_1}{c+h_1} = \frac{1}{h_1} \cdot \frac{v_C}{u_C} > 1$. This inequality implies

$$S''(u) > 0. \tag{3.4}$$

Hence from (3.3) and (3.4) finally we obtain the desired inequality

$S(u) = \gamma(u_C - u)^2 \geq 0$, where γ is a positive constant. $\square$

Lemma 3.3. *The set* $\overline{\Sigma^l}$ *is positively invariant with respect to the flow* (1.5) *if* $h''(u) \leq 0$ *for* $u \in (0, u_C)$.

Proof. We apply Lemma 3.1 to the system (1.5), and so we just need to show that the tangent line $l(u)$ satisfies the conditions in Lemma 3.1. Trivially $l(u)$ is strictly increasing and $l''(u) \geq 0$. In order to check the condition (3.1) define a cubic function $S(u)$ for $u \geq 0$ as follows :

$$S(u) := -l'(u)uf(u, l(u)) + l(u)g(u, l(u)).$$

To complete the proof we show that $S(u) \geq 0$.

It is clear that

$$S(u_C) = 0, \quad \text{and} \quad S(0) = l(0)(1 - l(0)) \geq 0 \tag{3.5}$$

because of the concavity of $h(u)$ for $u \in (0, u_C)$. Using the equation $l(u) = h_1(u - u_C) + v_C$ we have

$$\begin{aligned} S'(u_C) &= -h_1 u_C\{f_u(u_C, v_C) + h_1 f_v(u_C, v_C)\} \\ &\qquad + v_C\{g_u(u_C, v_C) + h_1 g_v(u_C, v_C)\} \\ &= 2h_1 u_C(u_C + bh_1 v_C) + 2v_C(cu_C + h_1 v_C). \end{aligned}$$

From the equation (2.1) in Lemma 2.2 we obtain that

$$S'(u_C) = 0. \tag{3.6}$$

Now we write out $S(u)$ as a cubic polynomial of u as follows :

$$\begin{aligned} S(u) &= -h_1 u\{a - u^2 - b(h_1(u - u_C) + v_C)^2\} \\ &\quad + \{h_1(u - u_C) + v_C\}\{1 - cu^2 - (h_1(u - u_C) + v_C)^2\}. \end{aligned}$$

Notice that $h_1(1 + bh_1^2 - c - h_1^2)$, the coefficient of the cubic term of $S(u)$, is positive by Lemma 2.4. Finally combining (3.5) and (3.6) we conclude the desired inequality $S(u) \geq 0$ for $u \geq 0$. □

4. Properties of Auxiliary Functions

Throughout this section we let $f(u, v)$ and $g(u, v)$ be as in (α) or (β).

Lemma 4.1. *Let $(u(x,t), v(x,t))$ be a solution of (1.1) with $d_1 = d_2 = d$. For a C^2-function $k(u)$ let $p(t)$ and $q(t)$ be the solutions of*

$$\begin{cases} p_t = p\, f(p, k(p)), \\ p(0) = \min\limits_{x \in \overline{\Omega}} u_0(x), \end{cases} \tag{4.1}$$

and

$$\begin{cases} q_t = q\, g(k^{-1}(q), q), \\ q(0) = \max\limits_{x \in \overline{\Omega}} v_0(x), \end{cases} \tag{4.2}$$

respectively. If $(u(x,t), v(x,t)) \in \overline{\Sigma^k}$ *in* Ω *for* $t \geq 0$ *then* $u(x,t) \leq p(t)$, $v(x,t) \geq q(t)$ *in* Ω *for* $t \geq 0$.

Proof. The claims are easily proved by the maximum principle. We refer the reader to Protter and Weinberger [7] for the maximum principle. □

Now we observe the tangent line $v = l(u)$ of the separatrix $v = h(u)$ at (u_C, v_C) for the kinetic system (1.2) and the set $\overline{\Sigma^l}$. Let h_1 denote the slope of the line $v = l(u)$.

Lemma 4.2. *Assume the following condition on the kinetic system of* (1.1) *with* $d_1 = d_2 = d$:

$$h_1 u_C\, f_v(u_C, v_C) - v_C\, g_v(u_C, v_C) < 0. \tag{4.3}$$

Let δ *be a sufficiently small positive number. Then for any* ϵ *and* s *which are positive and sufficiently small there exists a* C^2*-function* $j(u)$ *defined for* $u \geq 0$ *such that* $j(0) = v_C - \delta - \epsilon$, $j'(u) > 0$, $j''(u) \geq 0$, $j'(u)uf(u, j(u)) - j(u)g(u, j(u)) < 0$, *and* $l(u) + s \leq j(u) \leq l(u) + 2s$ *for* $u_C - \delta \leq u \leq u_C + \delta$.

Proof. Define a monotone increasing function $\tilde{j}(u)$ by

$$\tilde{j}(u) = \begin{cases} l(u_C - \delta) + s - \epsilon + \frac{\epsilon}{u_C - \delta} u & \text{for } 0 \leq u < u_C - \delta, \\ l(u) + s & \text{for } u_C - \delta \leq u < u_C + \delta, \\ \frac{1}{\epsilon}(u - u_C - \delta) + l(u_C + \delta) + s & \text{for } u_C + \delta \leq u, \end{cases}$$

where $\epsilon = \epsilon(\delta)$ is a positive parameter satisfying that $\lim_{\delta\to 0}\epsilon(\delta) = 0$. Then for $u_C - \delta \le u < u_C + \delta$

$$
\begin{aligned}
&\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) \\
&\quad = l'(u)uf(u,l(u)+s) - (l(u)+s)g(u,l(u)+s) \\
&\quad = l'(u)uf(u,l(u)) - l(u)g(u,l(u)) + \eta(u)s + O(s^2) \\
&\quad \le \zeta(u)s + O(s^2) \quad \text{as } s \to 0,
\end{aligned}
$$

where $\zeta(u) = l'(u)uf_v(u,l(u)) - g(u,l(u)) - l(u)g_v(u,l(u))$, and $\lim_{u\to u_C}\zeta(u) = h_1 u_C f_v(u_C,v_C) - v_C g_v(u_C,v_C) < 0$ by the condition (4.3). Hence there exists a $\delta > 0$ such that $\zeta(u)$ is positive for $u_C - \delta \le u < u_C + \delta$. Thus if s is sufficeintly small then $\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) < 0$ for $u_C - \delta \le u < u_C + \delta$. For any δ sufficiently small there exists $\eta > 0$ such that $f(u,v) \le -\eta$ for $u \ge u_C + \delta$, $v \ge l(u)$. Then for $\epsilon < \delta$ and $u \ge u_C + \delta$, we have $\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) \le -\frac{1}{\epsilon}\eta u$. Hence choosing ϵ sufficiently small we have $\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) < 0$ for $u \ge u_C + \delta$. For any δ sufficiently small there exist positive constants η_1, η_2 and $\tilde{s}$ such that

$$0 < f(u,v) \le \eta_1, \quad g(u,v) > \eta_2$$

for $0 \le u < u_C - \delta$, $l(u) \le v \le l(u) + \tilde{s}$. Then for $\epsilon < \delta$, $s \le \tilde{s}$, and $0 \le u < u_C - \delta$ we have $\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) \le \epsilon\eta_1 u - \tilde{j}(0)\eta_2$.

Hence choosing ϵ sufficiently small we have

$$\tilde{j}'(u)uf(u,\tilde{j}(u)) - \tilde{j}(u)g(u,\tilde{j}(u)) < 0$$

for $0 \le u < u_C - \delta$. We now obtain a C^2-function $j(u)$ for $u \ge 0$ satisfying

$j'(u)uf(u,j(u)) - j(u)g(u,j(u)) < 0$ by smoothing $\tilde{j}(u)$. $\square$

5. Proof of Theorem 1.1

Now we prove that the solution $(u(x,t),v(x,t))$ of the system (1.4) in the set Σ^l is attracted to the constant stable equilibrium $(0,1)$. Using the positive invariance of Σ^l proved in Section 3 and applying the lemmas prepared in Section 4, we prove the convergence in the Theorem 1.1. First we check that the system (1.4) satisfies the condition in Lemma 4.2 in Section 4.

Lemma 5.1. *The kinetic systems of* (1.4) *satisfies the condition* (4.3)

$$h_1 u_C\, f_v(u_C,v_C) - v_C\, g_v(u_C,v_C) < 0,$$

in Lemma 4.2. *Thus there exists a C^2-function $j(u)$ satisyfying all the properties in* Lemma 4.2.

Proof. From the identity (3.2) and the strong competition condition (1.6) we have $\frac{h_1+h_1^2 b}{c+h_1} < bh_1$. By noticing that $f_v(u_C,v_C) = -b$ and $g_v(u_C,v_C) = -1$ for the kinetic systems of (1.4) we have $\frac{v_C}{u_C} < bh_1$. Hence the condition (4.3) is satisfied. $\square$

Proof of Theorem 1.1. We have established the positive invariance of the set $\overline{\Sigma^l}$ in Lemma 3.1. The necessary auxiliary function $j(u)$ has been constructed in Lemma 5.1. Now the rest of the proof of Theorem 1.1 follows similar lines of arguments as in [2] but with reversed inequalities and exchanges of x and y. It consists of three steps.

Step 1. There exists a positive time T and a positive constant δ such that

$$u(x,T) \le u_C + \delta, \qquad v(x,T) \ge v_C - \delta \quad \text{in } \Omega.$$

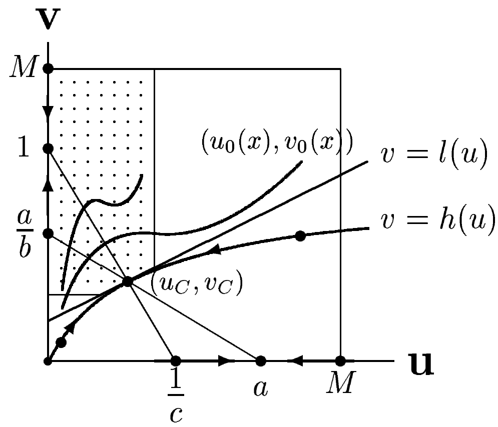


Figure 4: Figure for **Step 1.**

Proof of Step 1. By the positively invariance of the set Σ^l $(u(x,t), v(x,t)) \in \Sigma^l$ for $x \in \overline{\Omega}$ and $t \ge 0$. Since $\Sigma^l \subset \Sigma^h$, we have $(u(x,t), v(x,t)) \in \Sigma^h$ for $x \in \overline{\Omega}$ and $t \ge 0$. Let $p(t)$ and $q(t)$ be the solutions of (4.1) and (4.2), respectively in Lemma 4.1. Since $(p(t), h(p(t)))$ and $(h^{-1}(q(t)), q(t))$ are solutions of the kinetic system of (1.4) moving on the stable manifold $v = h(u)$ at (u_C, v_C),

$$\lim_{t\to\infty} p(t) = u_C, \quad \lim_{t\to\infty} q(t) = v_C.$$

By the assertion of Lemma 4.1 with $k(u) = h(u)$

$$u(x,t) \le p(t), \quad v(x,t) \ge q(t) \quad \text{in } \Omega, \quad t \ge 0.$$

Thus we conclude the proof for the first step.

Step 2. There exists $\delta' > 0$ and $T' > T$ such that

$$u(x,T') \le u_C - \delta, \quad v(x,T') \ge v_C + \delta' \quad \text{in } \Omega.$$

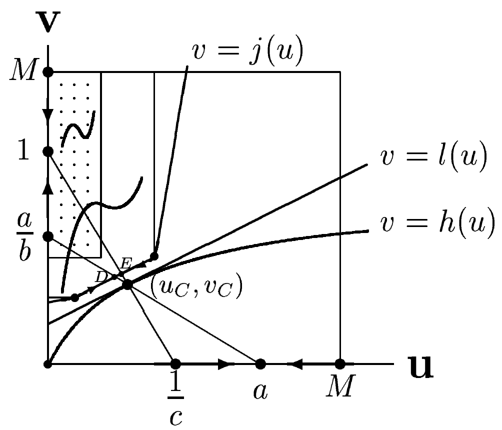


Figure 5: Figure for **Step 2.**

Proof of Step 2. By the positive invariance of Σ^l and the results in Step 1 we have

$$\begin{cases} v(x,T) > l(u(x,T)), \\ u(x,T) \le u_C + \delta, \\ v(x,T) \ge v_C - \delta \end{cases}$$

for $x \in \overline{\Omega}$. We take $s > 0$ so small that $v(x,T) \ge l(u(x,T)) + 2s$ in Ω. Then for $j(u)$ given in Lemma 4.2 we have $(u(x,T), v(x,T)) \in \overline{\Sigma^j}$ for $x \in \overline{\Omega}$. Therefore by the invariance property proved in Lemma 3.2

$$(u(x,t), v(x,t)) \in \overline{\Sigma^j} \quad \text{for } x \in \overline{\Omega}, \ \ t \ge T.$$

Let $p(t)$ and $q(t)$ be the solutions of

$$\begin{cases} p_t = p\, f(p, j(p)), \\ p(T) = u_C + \delta \end{cases}$$

and

$$\begin{cases} q_t = q\, g(j^{-1}(q), q), \\ q(T) = v_C - \delta, \end{cases}$$

respectively. Then we have

$$p_t = p\, f(p, j(p)) < 0 \quad \text{ if } \ u_D < p < u_C + \delta,$$

$$q_t = q\, g(j^{-1}(q), q) > 0 \ \text{ if } \ v_C - \delta < q < v_E,$$

where $D = (u_D, v_D)$ and $E = (u_E, v_E)$ satisfy

$$f(u_D, v_D) = 0, \quad v_D = j(u_D),$$

$$g(u_E, v_E) = 0, \quad v_E = j(u_E).$$

Since $u_D < u_C$ and $v_E > v_C$, there exists $\delta' > 0$ and $T' > 0$ such that $p(T') \leq u_C - \delta'$ and $q(T') \geq v_C + \delta'$. Hence by Lemma 4.1 with $k(u) = j(u)$

$$u(x,T') \leq u_C - \delta' \text{ and } v(x,T') \geq v_C + \delta' \text{ in } \Omega.$$

Step 3. The solution $(u(x,t), v(x,t))$ converges to $(0,1)$ uniformly in Ω.

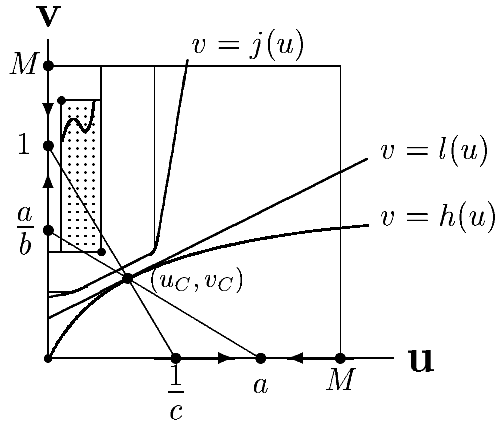


Figure 6: Figure for **Step 3.**

Proof of Step 3. Let $(\overline{u}(t), \overline{v}(t))$ and $(\underline{u}(t), \underline{v}(t))$ be the solutions of the kinetic system of (1.4) with $(\overline{u}(0), \overline{v}(0)) = (u_C - \delta', v_C + \delta')$ and $(\underline{u}(0), \underline{v}(0)) = \left(\min_{x\in\overline{\Omega}} u(x,T'), \max_{x\in\overline{\Omega}} v(x,T')\right)$, respectively. By the comparison principle for competition-diffusion systems (see, e.g., [1])

$$\underline{u}(t) \leq u(x,t) \leq \overline{u}(t),$$

$$\underline{v}(t) \geq v(x,t) \geq \overline{v}(t)$$

for $x \in \Omega$ and $t \geq T'$. Since $(\overline{u}(t), \overline{v}(t))$ and $(\underline{u}(t), \underline{v}(t))$ converge to $(0,1)$ as $t \to \infty$, $(u(x,t), v(x,t))$ converges to $(0,1)$ uniformly in Ω as $t \to \infty$. □

6. Proof of Theorem 1.2

We have examined in Proposition 2.2 a sufficient condition which guarantees the concavity of the separatrix $h(u)$ for $u \in (0, u_C + \epsilon)$. But it does not guarantee the concavity of $h(u)$ for $u \in (u_C + \epsilon, \infty)$. We just know for the separatrix $h(u)$ of the kinetic system of (1.5) that it is monotone increasing to the infinity on the interval $(u_C + \epsilon, \infty)$. In the proof of Theorem 1.2 it is not necessary to have the concavity beyond $u_C + \epsilon$, but we need to show that the tangent line $l(u)$ of the separatrix $h(u)$ at the unstable equilibrium point (u_C, v_C) actually lies above $h(u)$ in the phase plane.

Proposition 6.1. *$h(u) \leq l(u)$ for every $u > 0$, if $h''(u) \leq 0$ for $u < u_C$.*

Proof. In Lemma 3.3 we have shown the positive invariance of the set $\overline{\Sigma^l}$ for the competition-diffusion system (1.5). Now we consider the kinetic system corresponding to (1.5). $\overline{\Sigma^l}$ is also positively invariant for the kinetic system. By Proposition 1.2 the set $W_A = \{(u,v) \in Q | v < h(u)\}$ is the domain of attraction of $(\sqrt{a}, 0)$ for the kinetic system. In order to Sdraw a contradictionS suppose that $h(u_1) > l(u_1)$ for $u_1 > 0$. We take v_1 so that $h(u_1) > v_1 > l(u_1)$. Then by comparison we have that the point (u_1, v_1) on the (u,v)-phase plane is attracted to the point $(\sqrt{a}, 0)$ which is not in the set $\overline{\Sigma^l}$. It contradicts to the positively invariance of the set $\overline{\Sigma^l}$. □

Lemma 6.1. *The kinetic system of* (1.5) *satisfies the condition* (4.3)

$$h_1 u_C f_v(u_C, v_C) - v_C g_v(u_C, v_C) < 0$$

in Lemma 4.2. *Thus there exists a* C^2*-function* $j(u)$ *satisfying all the properties in* Lemma 4.2.

Proof. Using the identity (2.1) in Lemma 2.2 and that $f_v(u_C, v_C) = -2bv_C$ and $g_v(u_C, v_C) = -2v_C$ for the kinetic system of (1.5) we have that

$$bh_1^2 + (\frac{u_C}{v_C} - \frac{v_C}{u_C})h_1 - c = 0.$$

Define a quadratic function $\Gamma(h) := bh^2 + (\frac{u_C}{v_C} - \frac{v_C}{u_C})h - c$ and see that $\Gamma(\frac{v_C}{b\, u_C}) = \frac{1}{b} - c < 0$. Thus $h_1 > \frac{v_C}{b\, u_C}$, and the condition (4.3) is satisfied. □

Proof of Theorem 1.2. We have obtained the invariance of the set $\overline{\Sigma^l}$ in Theorem 3.3. Proposition 6.1 claims that the tangent line at the unstable equilibrium point (u_C, v_C) lies above the separatrix.

In Lemma 6.1 the auxiliary function $j(u)$ is constructed. Now the proof of the convergence of the solution $(u(x,t), v(x,t))$ in the set $\overline{\Sigma^l}$ is similar to the proof of Theorem 1.1. □

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