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Remarks on viscosity solutions of the Dirichlet problem for quasilinear degenerate elliptic equations

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1. Introduction.

In this paper we are concerned with the Dirichlet problem (hereafter called (**DP**)) for the quasilinear degenerate elliptic equation :

$$-g(|x|, u)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad B_R \tag{1.1}$$

$$u = \beta \quad \text{on} \quad \partial B_R, \tag{1.2}$$

where $B_R = \{x \in \mathbf{R}^N; |x| < R\}, N \geq 2, g : [0, R] \times \mathbf{R} \to \mathbf{R}^+ = [0, \infty)$ is a given continuous function, Δ is the Laplacian, and β is a real number such that $f(R, \beta) = 0$.

This investigation is a sequel of our previous work [3] where we studied the existence, uniqueness, nonuniqueness and radial property of viscosity solutions of the Dirichlet problem for the semilinear degenerate elliptic equation

$$-g(|x|)\Delta u + f(|x|, u) = 0 \quad \text{in} \quad B_R, \tag{1.3}$$

where g is a nonnegative continuous function. We refer the reader to the Monograph by Crandall, Ishii and Lions [1] for definitions, details and references of viscosity solutions.

The main purpose of the present paper is to prove existence of viscosity solutions, and to give a sufficient condition assuring the uniqueness and the radial

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symmetry of viscosity solutions of (**DP**). In what follows we consider the problem (**DP**) in case $N = 2$, since we can treat it in case $N \geq 3$ by the same arguments.

Throughout this paper we make the following assumptions:

(H1) $f(t, y) \in C([0, R] \times \mathbf{R})$ is strictly increasing in y for each fixed $t \in [0, R]$.

(H2) There exists an implicit function $\varphi(t)$ of $f(t, y) = 0$ satisfying

$$\sup_{0 \leq s \leq R, s \neq t} \left| \frac{\varphi(s) - \varphi(t)}{s - t} \right| = \Psi(t) \in L^1(0, R).$$

It is clear that $\varphi(t)$ is continuous on $[0, R]$ by (H1) and (H2).

We state our existence theorem.

Theorem 1. *Under the assumptions* (H1) *and* (H2) *there exists a radial viscosity solution of* (**DP**).

In order to establish the uniqueness of viscosity solutions for (**DP**) we introduce additional assumptions and a notion of standard viscosity solution.

(H3) $\Psi(t) \in L^\infty(0, R)$, here Ψ is the function defined in (H2).

(H4) The function g satisfies the condition : if $g(t_1, y_1) = 0$ then

$$\begin{cases} g(s, y_1) \leq \text{Const.} \mid s - t_1 \mid^2 \quad \text{for} \quad \forall s \in N(t_1), \\ g(s, y) \leq \text{Const.}(\mid s - t_1 \mid + \mid y - y_1 \mid) \quad \text{for} \quad \forall (s, y) \in N(t_1, y_1), \end{cases}$$

where $N(t_1)$ and $N(t_1, y_1)$ are small neighborhoods of t_1 and (t_1, y_1), respectively.

(H5) For f and g, we impose the following structure condition : if $0 \leq t \leq R$, $y_1 < y_2$ and $g(t, y_1) + g(t, y_2) > 0$, then

$$g(t, y_1)f(t, y_2) - g(t, y_2)f(t, y_1) > 0.$$

Example Let $a \in C^2([0, R]), b \in C^2([0, R]; [0, 1])$. Define $g \in C^2([0, R] \times \mathbf{R})$ by

$$g(t, y) = 1 - \cos[h(y - a(t); b(t))]$$

for $(t, y) \in [0, R] \times \mathbf{R}$, where $h(y; b), 0 \le b \le 1$, is a $C^2(\mathbf{R})$-function such that

(i) for every $0 < b < 1$,

(i-1) $h(y; b) = 0$ for all $y \le b - \tilde{b}$ and all $y \ge 2 - b + \tilde{b}$,

(i-2) $h(y; b) = \tilde{b}$ for all $b \le y \le 2 - b$,

(i-3) $\dfrac{dh}{dy}(y; b) \ge 0$ for all $b - \tilde{b} \le y \le b$,

(i-4) $\dfrac{dh}{dy}(y; b) \le 0$ for all $2 - b \le y \le 2 - b + \tilde{b}$;

(ii) for $b = 0, 1, h(y; b) = 0$ for all $y \in \mathbf{R}$, i.e., $h(y; 0) = h(y; 1) \equiv 0$ for $\forall y \in \mathbf{R}$.

Here, for every $b \in (0, 1), \tilde{b} := (1/2) \min\{b, 1 - b\}$. Suppose $f(t, y) = y - \varphi(t)$, and that $\varphi(t) \in C^1([0, R])$ is a given function satisfying $a(t) + b(t) \le \varphi(t) \le a(t) + 2 - b(t)$ for every $t \in [0, R]$ such that $0 < b(t) < 1$. Then $f(t, y)$ and $g(t, y)$ satisfy the assumptions (H1)-(H5).

Definition A function u is called a standard viscosity solution of (**DP**) if u is a viscosity solution and $u(x) = \varphi(x)$ for all $x \in B_R \setminus \{0\}$ satisfying $g(x, u(x)) = 0$.

By making use of a notion of standard viscosity solutions, we shall prove the uniqueness for (**DP**) :

Theorem 2 *Under the assumptions* (H1)-(H5) *there exists a unique viscosity solution u of* (**DP**). *Moreover, every viscosity solution of* (**DP**) *is standard and radially symmetric.*

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2. Approximate equations

We begin with observing the properties of solutions for the following boundary value problem :

$$\begin{cases} -g(t)(\ddot{y}(t)+\frac{1}{t}\dot{y}(t))+f(t,y(t))=0 \quad \text{in} \quad (0,R], \\ \dot{y}(0)=0 \quad \text{and} \quad y(R)=\varphi(R), \end{cases} \tag{2.1}$$

where $g(t)$ is a positive and continuous function on $[0,R]$.

LEMMA 2.1 *There exists a unique classical solution* $u \in C([0,R]) \cap C^2(0,R)$ *of* (2.1)*.*

PROOF. This lemma is a consequence of [3; Proposition 3.4].

LEMMA 2.2 *Let* $y(\cdot)$ *be the solution of* (2.1) *in Lemma 2.1. Then, for* $0 \leq t \leq R$ *we have*

$$| \, y(t) \, | \leq \max_{0\leq t\leq R} | \, \varphi(t) \, | \tag{2.2}$$

$$| \, t\dot{y}(t) \, | \leq R\Psi(t). \tag{2.3}$$

PROOF. We first remark that a C^2 function y satisfies the equation of (2.1) in $(0,R)$ if and only if y and its derivative $\dot{y}$ satisfy the integral equations

$$\begin{aligned} y(t) &= y(t_0)+t_0\dot{y}_0(t_0)\log(t/t_0) \\ &\qquad + \int_{t_0}^{t}(\log t-\log s)sg(s)^{-1}f(s,y(s))ds, \end{aligned} \tag{2.4}$$

$$t\dot{y}(t)=t_0\dot{y}(t_0)+\int_{t_0}^{t} sg(s)^{-1}f(s,y(s))ds, \tag{2.5}$$

respectively, where $0 < t_0 < R$ and $0 < t \leq R$.

To prove $y(t) \leq \max_{0 \leq t \leq R} \mid \varphi(t) \mid$ for $0 \leq t \leq R$, by contradiction we suppose $y(t_0) > \max_{0 \leq t \leq R} \mid \varphi(t) \mid$ for some $0 \leq t_0 < R$. We first assume $\dot{y}(t_0) \geq 0$. By virtue of $y(R) = \varphi(R)$, there exists a $t_1, t_0 < t_1 \leq R$ such that $y(t_1) = \varphi(t_1)$ and $y(t) > \varphi(t)$ in (t_0, t_1), hence $f(t, y(t)) > 0$ in (t_0, t_1). From (2.4), we have $y(t_1) > y(t_0) > \max_{0 \leq t \leq R} |\varphi(t)| \geq \varphi(t_1)$; this is a contradiction.
We next assume $\dot{y}(t_0) < 0$. There exists a $t_2 (0 \leq t_2 < t_0)$ such that $y(t_2) = \varphi(t_2)$ and $f(t, y(t)) > 0$ in (t_2, t_0). From (2.4) we know $y(t_2) > y(t_0) > \varphi(t_2)$; this is a contradiction. Then we get $y(t) > \varphi(t)$ for $0 < t < t_0$. Then by (2.5) we have $\dot{y}(t) < (t_0/t)\dot{y}(t_0) < 0$, hence $\lim_{t \to 0} \dot{y}(t) = -\infty$ which contradicts $\dot{y}(0) = 0$.

Similarly, we can prove $y(t) \geq -\max_{0 \leq t \leq R} \mid \varphi(t) \mid$ for $0 \leq t \leq R$. Thus (2.2) is proved.

To prove (2.3) by contradiction we suppose $\mid t_0 \dot{y}(t_0) \mid > R\Psi(t_0), 0 < t_0 \leq R$. Assume $t_0 \dot{y}(t_0) > \Psi(t_0)R$ and $y(t_0) > \varphi(t_0), 0 < t_0 < R$. Then, by (2.4), we have $y(t) > \varphi(t)$ on $[t_0, t_0 + \delta) \subset [t_0, R)$. If there exists $t_0 < t_1 < R$ such that $y(t_1) = \varphi(t_1)$ and $y(t) > \varphi(t)$ in (t_0, t_1), we get

$$\begin{aligned} y(t_1) - y(t_0) &> t_0 \dot{y}(t_0) \log(t_1/t_0) > \Psi(t_0) R \log(t_1/t_0) \\ &\geq \frac{|\varphi(t_1) - \varphi(t_0)|}{(t_1 - t_0)\eta}(t_1 - t_0)R \qquad (t_0 < \eta < t_1) \\ &\geq \varphi(t_1) - \varphi(t_0). \end{aligned}$$

Then we have $\varphi(t_1) - \varphi(t_0) > \varphi(t_1) - \varphi(t_0)$ which is impossible.
Next, assume $t_0 \dot{y}(t_0) > \Psi(t_0)R + \epsilon$ and $y(t_0) \leq \varphi(t_0), 0 < t_0 \leq R$ where ϵ is a sufficiently small positive constant. There exists a sufficiently small positive constant δ such that

$$\epsilon \log(t/t_0) + \int_{t_0}^{t} \log t/s s g(s)^{-1} f(s, y(s)) ds \leq -\epsilon(t_0 - t)/t_0 + Const(t_0 - t)^2 < 0$$

for any $t : 0 < t_0 - t < \delta$. Then we may assume $y(t_0) < \varphi(t_0)$. We use the similar method to the above. If there exists $t_2 < t_0$ such that $y(t_2) = \varphi(t_2)$ and $y(t) < \varphi(t)$ in (t_2, t_0), we have

$$y(t_2) - y(t_0) < \varphi(t_2) - \varphi(t_0).$$

This is a contradiction. Thus we get $y(t) < \varphi(t)$ on $(0, t_0)$. Hence by (2.4), we see $y(t) < y(t_0) + t_0\dot{y}(t_0)\log(t/t_0)$, which implies $\lim_{t\to 0} y(t) = -\infty$. This contradicts (2.2). Thus, we have $t\dot{y}(t) \le \Psi(t)R$ for all $0 \le t \le R$. Similarly, we have $t\dot{y}(t) \ge -R\Psi(t)$ for all $0 \le t \le R$. Consequently, (2.3) is proved. The proof of Lemma 2.2 is complete.

We shall next consider the family of approximate equations of (1.1) such that

$$-g_\varepsilon(|x|, u_\varepsilon)\Delta u_\varepsilon + f(|x|, u_\varepsilon) = 0 \quad \text{in} \quad B_R \tag{2.6}$$

with the boundary condition $u_\varepsilon(x) = \varphi(R)$ on ∂B_R, where $g_\varepsilon(t, y) = g(t, y) + \varepsilon$ and ε is a small positive parameter.

Lemma 2.3 *Let $v \in C([0, R])$. Then there exists a unique classical solution $y_{v,\epsilon}$ of*

$$\begin{cases} -\, g_{v,\epsilon}(t)(\ddot{y}_{v,\epsilon}(t) + \dfrac{1}{t}\dot{y}_{v,\epsilon}(t)) + f(t, y_{v,\epsilon}) = 0 \quad \text{in} \quad (0, R) \\ \dot{y}_{v,\epsilon}(0) = 0 \quad \text{and} \quad y_{v,\epsilon}(R) = \varphi(R), \end{cases} \tag{2.7}$$

where $g_{v,\epsilon}(t) = g(t, v(t)) + \varepsilon$. Moreover, for $y_{v,\epsilon}$ we have the same estimates as (2.2) and (2.3).

Proof. From Lemmas 2.1 and 2.2, this lemma follows.

Lemma 2.4 *There exists a classical solution y_ε of*

$$\begin{cases} -\, g_\varepsilon(t, y_\varepsilon)(\ddot{y}_\varepsilon(t) + \dfrac{1}{t}\dot{y}_\varepsilon(t)) + f(t, y_\varepsilon) = 0 \quad \text{in} \quad (0, R) \\ \dot{y}_\varepsilon(0) = 0 \quad \text{and} \quad y_\varepsilon(R) = \varphi(R), \end{cases} \tag{2.8}$$

Moreover, for y_ε we have the same estimates as (2.2) and (2.3).

Proof. The argument below is based on reduction to the Schauder fixed point theorem. From $\dot{y}_{v,\epsilon}(0) = 0$ it follows

$$\dot{y}_{v,\epsilon}(t) = \frac{1}{t}\int_0^t s g_{v,\epsilon}(s)^{-1} f(s, y_{v,\epsilon}(s))ds. \tag{2.9}$$

Since $|y_{v,\epsilon}(t)| \leq$ Const from (2.2) and $g_{v,\epsilon}(t) \geq \epsilon$ we have

$$\mid \dot{y}_{v,\epsilon}(t) \mid \leq C\frac{t}{\varepsilon}, \tag{2.10}$$

where C is a constant independent of v and ε.

We put

$$D = \left\{ v \in C([0,R]) : \max_{0\leq t\leq R} \mid v(t) \mid \leq \max_{0\leq t\leq R} \mid \varphi(t) \mid \right\}.$$

We denote a map T from D to C([0,R]) by $T(v) = y_{v,\epsilon}$, where $y_{v,\epsilon}$ is the solution in Lemma 2.3. From the estimates (2.2) and (2.10), it follows that $\{y_{v,\epsilon}\}$ are uniformly bounded and equicontinuous on $[0,R]$. Hence T is a compact map from D to D. Let $\lim_{n\to\infty} v_n = v_\infty$ in $C([0,R])$. By y_n we denote the solution y_{v_n}. Since the map T is a compact operator, there exists a subsequence $\{y_{n_j}\}$ converging to w_∞ in $C([0,T])$. Then it follows w_∞ is the solution of (2.7) replaced v by v_∞. By the uniqueness of solutions of (2.7) replaced v by v_∞, the map T is continuous in $C([0,R])$. From the Schauder fixed point theorem there exists a fixed point y_ε of the map T. Moreover, for $0 \leq t \leq R$, we have

$$\begin{cases} 1) \quad \mid y_\varepsilon(t) \mid \leq \max_{0\leq t\leq R} \mid \varphi(t) \mid \\ 2) \quad \mid t\dot{y}_\varepsilon(t) \mid \leq \max_{0\leq s\leq R, s\neq t} \left| \frac{\varphi(s)-\varphi(t)}{s-t} \right| R. \end{cases} \tag{2.11}$$

3. Existence : Proof of Theorem 1

With the help of the estimates (2.11) for approximate solutions we can prove the existence of a radial viscosity solution of **(DP)**.

Proposition 3.1 *Suppose* (H1) *and* (H2). *Then there exists a radial viscosity solution* $u \in C(\overline{B_R})$ *of the following boundary value problem :*

$$\begin{cases} \mathcal{F}[u](x) := -g(|x|,u)\Delta u + f(|x|,u) = 0 \quad \text{in} \quad B_R \setminus \{0\} \\ u(x) = \varphi(|x|) \quad \text{on} \quad \partial B_R. \end{cases} \tag{3.1}$$

Proof. Since the solutions $\{y_\varepsilon\}$ of (2.8) are locally equicontinuous on $B_R \setminus \{0\}$ and uniformly bounded by (2.11), there exists a subsequence $\{y_{\varepsilon_j}\}$ such that

$$\lim_{j\to\infty} y_{\varepsilon_j} = y \quad \text{locally uniformly} \quad \text{in} \quad (0,R].$$

Putting $u(x) = y(|\ x\ |)$ and using the stability theorem [2], we see that $u(x)$ is a radial viscosity solution of (3.1).

We next show that $y(\cdot)$ is continuous at $t=0$. For the simplicity we put $y_{\varepsilon_j} = y_j$. Since $y_j(t)$ are uniformly bounded on [0,R] it follows that $|y(t)| \le$ Const for any $t \in (0,R]$. If $\overline{\lim}_{t\to 0} y(t) = \underline{\lim}_{t\to 0} y(t)$ and putting $y(0) = \overline{\lim}_{t\to 0} y(t)$ it follows that $y(\cdot)$ is continuous at $t=0$. Suppose $\overline{\lim}_{t\to 0} y(t) > \underline{\lim}_{t\to 0} y(t)$. Then there are $\{t_j\}$ and $\{s_j\}$ such that $\lim_{j\to\infty} t_j = \lim_{j\to\infty} s_j = 0$ and $\{y(t_j)\}$ (resp. $\{y(s_j)\}$) are local maximum (resp. minimum) values. Since either $\overline{\lim}_{t\to 0} y(t) > \varphi(0)$ or $\underline{\lim}_{t\to 0} y(t) < \varphi(0)$ holds, we have a contradiction by the definition of viscosity solution.

Then we get $\overline{\lim}_{t\to 0} y(t) = \underline{\lim}_{t\to 0} y(t) = y(0)$. Thus $y(\cdot)$ is continuous at $t=0$. The proof of Proposition 3.1 is complete.

In the rest of this section we shall prove that u satisfies $\mathcal{F}[u](0) = 0$ in the viscosity sense. Let us denote y_{ε_j} by y_j, where $\{y_j\}$ are the classical solutions of (2.8) converging to y locally uniformly in $(0,R]$. Since $\{y_j(0)\}$ is bounded, we may assume that $\lim_{j\to\infty} y_j(0) = \alpha$.

Lemma 3.2 *If $g(0,u(0)) > 0$, then $u(x)$ belongs to $C^2(B_\delta)$ where δ is a sufficiently small positive number. Moreover, u satisfies $\mathcal{F}[u](0) = 0$ in the viscosity sense.*

Proof. Put $g(t) = g(t, y(t))$. Noting that $g(t) > 0$ for all $t \in [0, \delta_2)$ with small $\delta_2 > 0$, we see that there exists a unique C^2-solution w of the following two-point boundary value problem :

$$\begin{cases} -g(t)(\ddot{w}(t) + \frac{1}{t}\dot{w}(t)) + f(t,w) = 0 \quad \text{in} \quad (\delta_1, \delta_2) \\ w(\delta_1) = y(\delta_1) \quad \text{and} \quad w(\delta_2) = y(\delta_2), \end{cases} \tag{3.2}$$

where δ_1 is an arbitrary positive number such that $0 < \delta_1 < \delta_2$. By Proposition 3.1 and the standard argument in the viscosity theory (cf. [3; Lemma 3.6]), we see that $w(t) = y(t)$ on $[\delta_1, \delta_2]$, hence $y \in C^2(0, \delta_2]$. From $g(0) > 0$ we have

$$\dot{y}(t) = \frac{1}{t}\int_0^t s g(s)^{-1} f(s, y(s)) ds.$$

Hence, by L'Hôpital's rule

$$\lim_{t\to 0} \frac{1}{t}\dot{y}(t) = \frac{1}{2} g(0, y(0))^{-1} f(0, y(0)).$$

Therefore, from (2.1), we get $\lim_{t\to 0} \ddot{y}(t) = \ddot{y}(0) = \frac{1}{2} g(0, y(0))^{-1} f(0, y(0))$, hence $u \in C^2(B_{\delta_2})$. The second assertion is proved in [3]. The proof of Lemma 3.2 is complete.

We shall next study the case when $g(0, y(0)) = 0$.

Lemma 3.3 *Suppose $g(0, u(0)) = 0$ and $f(0, u(0)) = 0$. Then $u \in C(B_R)$ is a viscosity solution of* (**DP**).

Proof. It is evident that u satisfies $\mathcal{F}[u](0) = 0$ in the viscosity sense, whence u is a viscosity solution of (**DP**).

Lemma 3.4 *Suppose $g(0, u(0)) = 0$ and $f(0, u(0)) > 0$. Then u satisfies $\mathcal{F}[u](0) = 0$ in the viscosity sense.*

Proof. We first note that the assumptions $g(0,u(0))=0$ and $f(0,u(0))>0$ implies

$$\mathcal{F}[u](0) \geq 0 \quad \text{in the viscosity sense.}$$

Therefore, it suffices to prove $\mathcal{F}[u](0) \leq 0$ in the viscosity sense.
We recall that y_j in the proof of Proposition 3.1 are the classical solutions of (2.8) converging to y locally uniformly in $(0,R]$ and y is the continuous function on $[0,R]$. Let us suppose that $\lim_{j\to\infty} y_j(0)=\alpha$ exists from 1) of (2.11). For a while let j be sufficiently large numbers.
We divide our considerations into three cases such that $\alpha > y(0)$, $\alpha = y(0)$ and $\alpha < y(0)$.
Case 0: $\alpha > y(0)$. From $y(0) > \varphi(0)$ it follows $f(t, y_j(t)) > 0$ at the small neighborhood of $t=0$. By (2.9) we have

$$\dot{y}_j(t) = \frac{1}{t}\int_0^t s(g(s,y_j(s))+\epsilon_j)^{-1} f(s,y_j(s))ds. \tag{3.3}$$

Since $\dot{y}_j(t)$ is positive we see that $y_j(t)$ is increasing at the small neighborhood of $t=0$. From the continuity of y we have $y(t) \leq y(0)+(\alpha-y(0))/2$ for a small neighborhood of $t=0$. Thus, by the local uniform convergence of y_j in $(0,R]$, it follows y_j has a local maximum point in the small neighborhood of $t=0$. This is a contradiction by the maximum principle. Thus this case can not occur.
Case 1: $\alpha = y(0)$. We shall show that y_j converges y uniformly in $[0,R]$. To this end, we suppose the contrary: there is $\delta > 0$ such that $\overline{\lim_{j\to\infty}} \sup_{t\in[0,R]} \mid y_j(t)-y(t) \mid \geq \delta$. Then we will get a contradiction.
If there exists $\{t_j\}$ such that $y_j(t_j) > \alpha+\delta/2$ and $\lim_{j\to\infty} t_j = 0$ it follows y_j has a local maximum point in the small neighborhood of $t=0$ from $y(t) < y(0)+\delta/2$ for a small neighborhood of $t=0$. Then this is a contradiction. Next, we assume that $y_j(t_j) < \alpha - \delta$. From $y_j(0) > \varphi(0)$ and (3.3) we see that $y_j(t)$ is increasing at the small neighborhood of $t=0$. Then we know that y_j has a local maximum

point in the small neighborhood of $t = 0$. Thus it is a contradiction.

Therefore, noting the locally uniform convergence of $\{y_j\}$ we obtain that $\{y_j\}$ converges y uniformly on $[0, R]$.

Applying the stability theorem [2], we conclude that u is a viscosity solution of (**DP**).

Case 2: $\alpha < y(0)$. We shall show $J^{2,+}u(0) = \phi$.

From $y \in C([0, R])$ there exists a small positive number δ such that $y(t) > y(0) - \delta_0/4$ for all $t \in [0, \delta]$ where $\delta_0 = \min\{(y(0) - \alpha), (y(0) - \phi(0))\}$. Let γ_1 and γ_2 be $y(0) - 3\delta_0/4$ and $y(0) - \delta_0/2$ respectively. Then there exist $\{t_{1,j}\}$ and $\{t_{2,j}\}$ such that $\gamma_1 = y_j(t_{1,j}) < \gamma_2 = y_j(t_{2,j}), \quad t_{1,j} < t_{2,j}$ and $\lim_{j\to\infty} t_{p,j} = 0, p = 1, 2$. Then we find $\{t_j\}$ satisfying

$$\dot{y}_j(t_j) = \frac{\gamma_2 - \gamma_1}{t_{2,j} - t_{1,j}}, \; t_{1,j} < t_j < t_{2,j} \quad \text{and} \quad y_j(t) > \gamma_1 \text{ in } (t_j, \delta)$$

where $j \geq j_0$ with large j_0. Thus, we have $\dot{y}_j(\eta) > 0$ for any $t_j < \eta < \delta$. By $y(t) = \lim_{j\to\infty} y_j(t)$ and Fatou's lemma, we have

$$y(t) - y(\eta) \geq \int_\eta^t (\log t - \log s) s g(s, y(s))^{-1} f(s, y(s)) ds$$

for every $\eta < t < \delta$. Tending η to 0, we get

$$y(t) - y(0) \geq \int_0^t (\log t - \log s) s g(s, y(s))^{-1} f(s, y(s)) ds \geq 0.$$

Then by L'Hôpital's rule

$$\lim_{t\to 0} \frac{y(t) - y(0)}{t^2} = \infty;$$

from this it follows that $J^{2,+}u(0) = \phi$.

Therefore u satisfies $\mathcal{F}[u](0) \leq 0$ in the viscosity sense. The proof of Lemma 3.4 is complete.

In a similar way, we have

Lemma 3.5 *Suppose* $g(0,u(0)) = 0$ *and* $f(0,u(0)) < 0$. *Then* u *satisfies* $\mathcal{F}[u](0) = 0$ *in the viscosity sense.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. If $g(0,u(0)) > 0$, then combining Proposition 3.1 and Lemma 3.2 implies Theorem 1. On the other hand, if $g(0,u(0)) = 0$, then we apply Proposition 3.1 and Lemmas 3.3-3.5 to conclude that u is a viscosity solution (**DP**).

4. Uniqueness : Proof of Theorem 2

In this section we shall prove the uniqueness of viscosity solutions of (**DP**). Throughout this section we assume the assumptions (H1)-(H5) and that u denotes the viscosity solution of (**DP**) obtained in Theorem 1. We start with the next three lemmas which will help us to prove the uniqueness.

Lemma 4.1 *u is radial and standard.*

Proof. It suffices to prove that u is standard. Suppose $g(x_0, u(x_0)) = 0$ and $x_0 \neq 0$. We seek to show $f(|x_0|, u(x_0)) = 0$. We note that u is locally Lipschitz continuous in $B_R \setminus \{0\}$ by the assumption (H3). Denoting $t_0 = |x_0|$, we have by the assumption (H4)

$$\int^{t_0-0} g(s,y(s))^{-1} ds = \infty \quad \text{or} \quad \int_{t_0+0} g(s,y(s))^{-1} ds = \infty.$$

Thus from [3; Lemmas 4.3, 4.4 and 4.5] it follows that $f(|x_0|, u(x_0)) = 0$, that is, u is standard.

Lemma 4.2 *Suppose* $g(0,u(0)) = 0$ *and* $f(0,u(0)) < 0$ (*resp.* $f(0,u(0)) > 0$). *Then there exists a neighborhood* B_δ *of* 0 *such that* $u(0) \geq u(x)$ (*resp.* $u(0) \leq u(x)$) *for all* $x \in B_\delta$.

Proof. If there exists an $\{x_\varepsilon\}$ such that $\lim_{\varepsilon \to 0} x_\varepsilon = 0$ and $g(|x_\varepsilon|, u(x_\varepsilon)) = 0$, then $f(0,u(0)) = 0$, since u is a continuous and standard. This is a contradiction.

Thus $g(|x|, u(x)) > 0$ in $x \in B_\delta \setminus \{0\}$. Suppose $f(0, u(0)) < 0$, and that there exists a sequence $\{x_\varepsilon\}$ such that $\lim_{\varepsilon \to 0} x_\varepsilon = 0$ and $u(x_\varepsilon) > u(0)$. Remarking $y \in C^2(0, \delta)$, we get $\dot{y}(t_0) > 0$ and

$$y(t) = y(t_0) + t_0 \dot{y}_0(t_0) \log(t/t_0) \\ + \int_{t_0}^{t} (\log t - \log s) s g(s, y(s)))^{-1} f(s, y(s)) ds,$$

where $0 < t < t_0 < \delta$. Since $f(s, y(s)) < 0$ for $0 < s < \delta$, we see

$$y(t) < y(t_0) + t_0 \dot{y}_0(t_0) \log(t/t_0),$$

hence $\lim_{t \to 0} y(t) = -\infty$; this is a contradiction.

Lemma 4.3 *Under the assumption* (H5), *for each* $t \geq 0$, *the function* $g(t, \cdot)$ *is one of the following five types :*

(1) $g(t, y) > 0$ *for all* $y \in R$.

(2) $g(t, y) = 0$ *for all* $y \in R$.

(3) There exists a $\tilde{y}_1 = \tilde{y}_1(t)$ *such that* $g(t, y) > 0$ *for all* $y > \tilde{y}_1$ *and* $g(t, y) = 0$ *for all* $y \leq \tilde{y}_1$.

(4) There exists a $\tilde{y}_2 = \tilde{y}_2(t)$ *such that* $g(t, y) > 0$ *for all* $y < \tilde{y}_2$ *and* $g(t, y) = 0$ *for all* $y \geq \tilde{y}_2$.

(5) There exist two points $\tilde{y}_1 = \tilde{y}_1(t)$ *and* $\tilde{y}_2 = \tilde{y}_2(t), \tilde{y}_1 < \tilde{y}_2$, *such that* $g(t, y) > 0$ *for all* $\tilde{y}_1 < y < \tilde{y}_2$ *and* $g(t, y) = 0$ *for all* $y \leq \tilde{y}_1$ *and all* $y \geq \tilde{y}_2$.

Moreover, it follows $f(t, \tilde{y}_1) < 0$ *in the type* (3), $f(t, \tilde{y}_2) > 0$ *in the type* (4), *and* $f(t, \tilde{y}_1) < 0$ *and* $f(t, \tilde{y}_2) > 0$ *in the type* (5).

Proof. Suppose there exist three points y_1, y_2 and y_3 such that $y_1 < y_2 < y_3, g(t, y_1) > 0, g(t, y_2) = 0$ and $g(t, y_3) > 0$. Then, by the assumption (H5), we immediately see $f(t, y_2) > 0 > f(t, y_2)$ which is impossible. Thus g is one of the above five types. By the assumption (H5), we can easily verify the latter half. The proof is complete.

Let $v(x) \in C(\overline{B_R})$ be an arbitrary viscosity solution of **(DP)**. For $x \in B_R$, define

$$\overline{V}(x) = \sup\{v(Qx); Q \in O(N)\} \quad \text{and} \quad \underline{V}(x) = \inf\{v(Qx); Q \in O(N)\}, \tag{4.1}$$

where $O(N)$ denotes the set of orthogonal $N \times N$ matrices. Since $O(N)$ is compact and closed (in the matrix norm), we can replace "sup" and "inf" in (4.1) with "max" and "min", respectively.

LEMMA 4.4 ([3; Lemmas 4.1 and 4.2]) *We have the following:*
(i) $\underline{V}(x)$ *and* $\overline{V}(x)$ *are continuous on* $\overline{B_R}$.
(ii) $\overline{V}(x)$ *is a radial viscosity subsolution of* (**DP**).
(iii) $\underline{V}(x)$ *is a radial viscosity supersolution of* (**DP**).

Now we are going to show the next key proposition.

PROPOSITION 4.5 *Let u be the standard and radial viscosity solution obtained in Theorem 1, and let v be an arbitrary viscosity solution of* **(DP)**. *Then we have*

$$\overline{V}(x) \leq u(x) \quad \text{on} \quad \overline{B_R} \tag{4.2}$$

$$\underline{V}(x) \geq u(x) \quad \text{on} \quad \overline{B_R}. \tag{4.3}$$

PROOF. Let us prove (4.2). Since we seek to prove $\overline{V}(x) \leq u(x)$ on $\overline{B_R}$, we suppose to the contrary that $\overline{V}(z) > u(z)$ for some $z \in B_R$; it follows that $\overline{V} - u$ attains its positive maximum at some point $\hat{x} \in B_R$.

Case 1 : $g(|\hat{x}|, u(\hat{x})) > 0$. As in the proof of Lemma 3.2, we have $u \in C^2(B_\delta(\hat{x}))$, where $B_\delta(\hat{x})$ is a neighborhood of $\hat{x}$. Noting $(Du(\hat{x}), D^2u(\hat{x})) \in J^{2,+}\overline{V}(\hat{x})$, we have by Lemma 4.4(ii) and definition

$$\begin{cases} -g(|\hat{x}|, \overline{V}(\hat{x}))\Delta u(\hat{x}) + f(|\hat{x}|, \overline{V}(\hat{x})) \le 0 \\ -g(|\hat{x}|, u(\hat{x}))\Delta u(\hat{x}) + f(|\hat{x}|, u(\hat{x})) = 0. \end{cases}$$

This implies

$$g(|\hat{x}|, u(\hat{x}))f(|\hat{x}|, \overline{V}(\hat{x})) - g(|\hat{x}|, \overline{V}(\hat{x}))f(|\hat{x}|, u(\hat{x})) \le 0.$$

On the other hand, from $\overline{V}(\hat{x}) > u(\hat{x})$ and the assumption (H5), it follows

$$g(|\hat{x}|, u(\hat{x}))f(|\hat{x}|, \overline{V}(\hat{x})) - g(|\hat{x}|, \overline{V}(\hat{x}))f(|\hat{x}|, u(\hat{x})) > 0;$$

this is a contradiction.

Case 2 : $g(|\hat{x}|, u(\hat{x})) = 0$. For each $\varepsilon > 0$, we define

$$\Phi(x, y) := \overline{V}(x) - u(y) - \frac{1}{2\varepsilon}|x - y|^2 - \frac{1}{2\varepsilon}|x - \hat{x}|^2 - \frac{1}{2\varepsilon}|y - \hat{x}|^2$$

for every $(x, y) \in \overline{B_R \times B_R}$. Let $(x_\varepsilon, y_\varepsilon) \in B_R \times B_R$ be a maximum point of Φ over $\overline{B_R \times B_R}$. Remarking that $0 < \overline{V}(\hat{x}) - u(\hat{x}) = \Phi(\hat{x}, \hat{x}) \le \Phi(x_\varepsilon, y_\varepsilon)$, we have

$$\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 + \frac{1}{2\varepsilon}|x_\varepsilon - \hat{x}|^2 + \frac{1}{2\varepsilon}|y_\varepsilon - \hat{x}|^2 \le \overline{V}(x_\varepsilon) - \overline{V}(\hat{x}) + u(\hat{x}) - u(y_\varepsilon) \le C, \tag{4.4}$$

where C is a constant independent of ε. From this, it follows

$$|x_\varepsilon - y_\varepsilon| \to 0, \quad |x_\varepsilon - \hat{x}| \to 0 \quad \text{and} \quad |y_\varepsilon - \hat{x}| \to 0 \quad \text{as} \quad \varepsilon \downarrow 0, \tag{4.5}$$

hence by (4.4) and continuity of $\overline{V}$ and u,

$$\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0, \quad \frac{1}{\varepsilon}|x_\varepsilon - \hat{x}|^2 \to 0 \quad \text{and} \quad \frac{1}{\varepsilon}|y_\varepsilon - \hat{x}|^2 \to 0 \quad \text{as} \quad \varepsilon \downarrow 0. \tag{4.6}$$

Now applying [1; Theorem 3.2], we see that there exist $\{X_\varepsilon\}$ and $\{Y_\varepsilon\} \subset S^N$ such that $(p_\varepsilon, X_\varepsilon) \in J^{2,+}\overline{V}(x_\varepsilon)$, $\quad (q_\varepsilon, Y_\varepsilon) \in J^{2,-}u(y_\varepsilon)$ and

$$-\frac{4}{\varepsilon}\begin{pmatrix} I & O \\ O & I \end{pmatrix} \le \begin{pmatrix} X_\varepsilon & O \\ O & -Y_\varepsilon \end{pmatrix} \le \frac{1}{\varepsilon}\begin{pmatrix} 7I & -5I \\ -5I & 7I \end{pmatrix}, \tag{4.7}$$

where $p_\varepsilon = (1/\varepsilon)(x_\varepsilon - y_\varepsilon) + (1/\varepsilon)(x_\varepsilon - \hat{x})$ and $q_\varepsilon = (1/\varepsilon)(x_\varepsilon - y_\varepsilon) - (1/\varepsilon)(y_\varepsilon - \hat{x})$. By (4.7), we have

$$-\frac{4}{\varepsilon}N \le \mathrm{Tr}(X_\varepsilon) \le \frac{7}{\varepsilon}N \quad \text{and} \quad -\frac{7}{\varepsilon}N \le \mathrm{Tr}(Y_\varepsilon) \le \frac{4}{\varepsilon}N. \tag{4.8}$$

We see, by Lemma 4.4 and definition

$$-g(|x_\varepsilon|, \overline{V}(x_\varepsilon))\mathrm{Tr}(X_\varepsilon) + f(|x_\varepsilon|, \overline{V}(x_\varepsilon)) \le 0 \tag{4.9}$$

and

$$-g(|y_\varepsilon|, u(y_\varepsilon))\mathrm{Tr}(Y_\varepsilon) + f(|y_\varepsilon|, u(y_\varepsilon)) \ge 0. \tag{4.10}$$

From now on we divide our considerations into two cases.

Case 2-1: $g(|\hat{x}|, u(\hat{x})) = 0$ and $\hat{x} \neq 0$. In this case, combining that u : standard viscosity solution and the assumption (H5), we have $g(|\hat{x}|, v) = 0$ for all $v \in \mathbf{R}^N$. Hence by the assumption (H4) we have $g(|y_\varepsilon|, u(y_\varepsilon)) \le C|y_\varepsilon - \hat{x}|^2$. Combining this with (4.6) and (4.8), we have

$$-g(|y_\varepsilon|, u(y_\varepsilon))\mathrm{Tr}(Y_\varepsilon) \le \frac{C}{\varepsilon}|y_\varepsilon - \hat{x}|^2 \to 0$$

as $\varepsilon \downarrow 0$. Similarly,

$$-g(|x_\varepsilon|, \overline{V}(x_\varepsilon))\mathrm{Tr}(X_\varepsilon) \ge -\frac{C}{\varepsilon}|x_\varepsilon - \hat{x}|^2 \to 0$$

as $\varepsilon \downarrow 0$. Thus, letting $\varepsilon \downarrow 0$ in (4.9) and (4.10), we have

$$f(|\hat{x}|, \overline{V}(\hat{x})) \le 0 \le f(|\hat{x}|, u(\hat{x}));$$

this is a contradiction because $\overline{V}(\hat{x}) > u(\hat{x})$.

Case 2-2 : $g(|\hat{x}|, u(\hat{x})) = 0$ and $\hat{x} = 0$. In this case, for the function $g(0, \cdot)$, there are four types (2)-(5) in Lemma 4.3 to consider. If $g(0, \cdot)$ is the type (2), i.e., $g(0, y) \equiv 0$, then we get a contradiction by the same argument as in Case 2-1. Let us consider the case where $g(0, \cdot)$ is the type (3). In this case, from the fact $u(0) \le \tilde{y}_1$ and $f(0, \tilde{y}_1) < 0$ it follows

$$f(0, u(0)) < 0. \tag{4.11}$$

On the other hand, by Lemma 4.2 we have $u(y_\varepsilon) \le u(0) \quad (0 < \varepsilon < \varepsilon_0, \varepsilon_0 \text{ small})$, hence $g(0, u(y_\varepsilon)) = 0$. Therefore, by the same argument as in Case 2-1, we have $f(0, u(0)) \ge 0$ which contradicts (4.11). We next consider the case where $g(0, \cdot)$ is the type (4). In this case, from $\overline{V}(0) > u(0) \ge \tilde{y}_2$ and $f(0, \tilde{y}_2) > 0$ it follows

$$f(0, \overline{V}(0)) > 0. \tag{4.12}$$

Remarking $\overline{V}(x_\varepsilon) > \tilde{y}_2 \quad (0 < \varepsilon < \varepsilon_0, \varepsilon_0 \text{ small})$, we have $g(0, \overline{V}(x_\varepsilon)) = 0$. Hence, by the same argument as in Case 2-1, we have $f(0, \overline{V}(0)) \le 0$; this contradicts (4.12). It remains to consider the case where $g(0, \cdot)$ is the type (5). By proceeding the above arguments, we can easily have a contradiction. The proof of (4.2) is complete.

The proof of (4.3) is essentially the same as that of (4.2) and is therefore omitted. The proof of Proposition 4.5 is complete.

Proof of Theorem 2.

Theorem 2 is an immediate consequence of Proposition 4.5.

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