Funkcialaj Ekvacioj
Volume 45, No. 1, pp 71--88

Nonoscillatory Solutions of Fourth Order Quasilinear Differential Equations

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1. Introduction

This paper is concerned with the oscillatory and nonoscillatory behavior of solutions of fourth order quasilinear differential equations of the form

$$(p(t)|u''|^{\alpha-1}u'')'' + q(t)|u|^{\beta-1}u = 0, \tag{1.1}$$

where α, β are positive constants and $p(t)$, $q(t)$ are positive continuous functions defined on $[a, \infty)$, $a > 0$. We assume that $p(t)$ satisfies

$$\int_a^\infty \left[\frac{t}{p(t)}\right]^{1/\alpha} dt = \infty, \tag{1.2}$$

or, more strongly,

$$\int_a^\infty \frac{t}{[p(t)]^{1/\alpha}} dt = \infty \quad \text{and} \quad \int_a^\infty \left[\frac{t}{p(t)}\right]^{1/\alpha} dt = \infty. \tag{1.3}$$

By a solution of (1.1) we mean a real-valued function $u(t)$ such that $u \in C^2[b, \infty)$ and $p|u''|^{\alpha-1}u'' \in C^2[b, \infty)$ and $u(t)$ satisfies (1.1) at every point of $[b, \infty)$, where $b \geq a$ and b may depend on $u(t)$. Such a solution $u(t)$ of (1.1) is called nonoscillatory if $u(t)$ is eventually positive or eventually negative. A solution $u(t)$ of (1.1) is called oscillatory if it has an infinite sequence of zeros clustering at $t = \infty$. Equation (1.1) itself is called oscillatory if all of its solutions are oscillatory.

The main objective is to investigate the oscillatory and nonoscillatory behavior of solutions of (1.1). We first study the structure of the set of nonoscillatory solutions of (1.1). It is observed that a solution $u(t)$ which is asymptotic to a positive constant as $t \to \infty$ is "minimal" in the set of all eventually positive solutions of (1.1), and a solution $u(t)$ which is asymptotic to a positive constant multiple of the function

$$\int_a^t (t-s)\left[\frac{s}{p(s)}\right]^{1/\alpha} ds$$

as $t \to \infty$ is "maximal" in the set of all eventually positive solutions of (1.1). We establish the necessary and sufficient conditions for the existence of "minimal" and "maximal" solutions of (1.1). These necessary and sufficient conditions are given by certain integral conditions on $p(t)$ and $q(t)$. Under the assumptions $\alpha \geq 1 > \beta$ and $\alpha \leq 1 < \beta$, we can present the necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1). In the case of $\alpha \geq 1 > \beta$ [resp. $\alpha \leq 1 < \beta$], the necessary and sufficient condition is identical to the integral condition which characterizes the existence of maximal [resp. minimal] solutions.

In the case $\alpha = 1$, equation (1.1) is

$$(1.4) \qquad (p(t)u'')'' + q(t)|u|^{\beta-1}u = 0,$$

and both of conditions (1.2) and (1.3) are

$$(1.5) \qquad \int_a^\infty \frac{t}{p(t)}\, dt = \infty.$$

The oscillatory and nonoscillatory behavior of solutions of (1.4) under the condition (1.5) has been studied by Kusano and Naito [4]. The results of the present paper generalize those of [4].

Now, consider the second order quasilinear differential equation

$$(1.6) \qquad (p(t)|u'|^{\alpha-1}u')' + q(t)|u|^{\beta-1}u = 0,$$

where $\alpha > 0$, $\beta > 0$, and $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a > 0$. Suppose that

$$\int_a^\infty \frac{dt}{[p(t)]^{1/\alpha}} = \infty.$$

Then it is seen that a solution $u(t)$ of (1.6) satisfying

$$u(t) \sim c\,(>0) \quad \text{as} \quad t \to \infty \tag{1.7}$$

is minimal in the set of eventually positive solutions of (1.6), and that a solution $u(t)$ of (1.6) satisfying

$$u(t) \sim c \int_a^t \frac{ds}{[p(s)]^{1/\alpha}} \quad (c > 0) \quad \text{as} \quad t \to \infty \tag{1.8}$$

is maximal in the set of eventually positive solutions of (1.6). Moreover it is known (Elbert [1], Elbert and Kusano [2], Izyumova and Mirzov [3], Mirzov [5, 6]) that the following results hold.

(a) Equation (1.6) has a solution $u(t)$ satisfying (1.7) if and only if

$$\int_a^\infty \left[\frac{1}{p(t)} \int_t^\infty q(s)\, ds\right]^{1/\alpha} dt < \infty. \tag{1.9}$$

(b) Equation (1.6) has a solution $u(t)$ satisfying (1.8) if and only if

$$\int_a^\infty q(t) \left[\int_a^t \frac{ds}{[p(s)]^{1/\alpha}} \right]^\beta dt < \infty. \tag{1.10}$$

(c) Let $\alpha < \beta$. Equation (1.6) has a nonoscillatory solution if and only if (1.9) is satisfied.

(d) Let $\alpha > \beta$. Equation (1.6) has a nonoscillatory solution if and only if (1.10) is satisfied.

The results in the present paper for the fourth order equation (1.1) provide parallel results to the second order equation (1.6).

2. Existence of specific nonoscillatory solutions

This section is devoted to the study of the structure of the set of nonoscillatory solutions of (1.1). In particular, we establish the necessary and sufficient conditions for the existence of specific nonoscillatory solutions of (1.1). To this end, we first give the results on the signs of

$$(p(t)|u''(t)|^{\alpha-1}u''(t))', \quad u''(t) \quad \text{and} \quad u'(t) \tag{2.1}$$

for a nonoscillatory solution $u(t)$ of (1.1).

Let $u(t)$ be an eventually positive solution of (1.1). There is $T \geq a$ such that $u(t) > 0$ for $t \geq T$. By (1.1) we have

$$(p(t)|u''(t)|^{\alpha-1}u''(t))'' = -q(t)|u(t)|^{\beta-1}u(t) < 0, \quad t \geq T, \tag{2.2}$$

and so we easily find that the signs of the derivatives of $u(t)$ in (2.1) are eventually of constant signs.

Lemma 2.1. *Suppose* (1.2) *holds. If* $u(t)$ *is an eventually positive solution of* (1.1)*, then*

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0 \quad \textit{for all large } t. \tag{2.3}$$

Proof. There is $T \geq a$ such that $u(t) > 0$ for $t \geq T$. By (2.2), $(p(t)|u''(t)|^{\alpha-1}u''(t))'$ is decreasing on $[T, \infty)$. Assume that

$$(p(t)|u''(t)|^{\alpha-1}u''(t))'\Big|_{t=t_0} < 0$$

for some $t_0 \geq T$. We have

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' \leq (p(t)|u''(t)|^{\alpha-1}u''(t))'\Big|_{t=t_0} \equiv -c_0 < 0, \quad t \geq t_0,$$

and

$$p(t)|u''(t)|^{\alpha-1}u''(t) \leq p(t_0)|u''(t_0)|^{\alpha-1}u''(t_0) - c_0(t - t_0), \quad t \geq t_0.$$

Therefore there are $t_1 > t_0$ and $0 < c_1 < c_0$ such that $u''(t) < 0$ and $p(t)|u''(t)|^{\alpha-1}u''(t) \leq -c_1 t$ for $t \geq t_1$. Thus we get

$$u''(t) \leq -c_1^{1/\alpha}\left[\frac{t}{p(t)}\right]^{1/\alpha}, \quad t \geq t_1.$$

Integrating this inequality from t_1 to t, we have

$$u'(t) \leq u'(t_1) - c_1^{1/\alpha}\int_{t_1}^{t}\left[\frac{s}{p(s)}\right]^{1/\alpha} ds, \quad t \geq t_1.$$

Then condition (1.2) implies $\lim_{t\to\infty} u'(t) = -\infty$, and so $\lim_{t\to\infty} u(t) = -\infty$. This contradicts the hypothesis that $u(t) > 0$ for $t \geq T$. Thus we obtain (2.3). The proof of Lemma 2.1 is complete.

Lemma 2.2. *Suppose that* (1.3) *holds. If* $u(t)$ *is an eventually positive solution of* (1.1)*, then one of the following cases holds*:

(A) $\quad u'(t) > 0,\ u''(t) > 0,\ (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ *for all large* t;

(B) $\quad u'(t) > 0,\ u''(t) < 0,\ (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ *for all large* t.

Proof. By Lemma 2.1, $(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ for all large t, and hence $p(t)|u''(t)|^{\alpha-1}u''(t)$ is eventually increasing. We have the two possibilities: $u''(t) > 0$ for all large t or $u''(t) < 0$ for all large t. Suppose first that $u''(t) > 0$ for all large t. There is $T \geq a$ such that

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0 \quad \text{and} \quad p(t)|u''(t)|^{\alpha-1}u''(t) > 0, \quad t \geq T.$$

Then

$$p(t)|u''(t)|^{\alpha-1}u''(t) \geq p(T)|u''(T)|^{\alpha-1}u''(T) \equiv c_0 > 0, \quad t \geq T.$$

We rewrite this inequality as

$$u''(t) \geq \frac{c_0^{1/\alpha}}{[p(t)]^{1/\alpha}}, \quad t \geq T. \tag{2.4}$$

Multiplying (2.4) by t and integrating over $[T, t]$, we get

$$t\,u'(t) - T\,u'(T) - u(t) + u(T) \geq c_0^{1/\alpha} \int_T^t \frac{s}{[p(s)]^{1/\alpha}}\,ds, \quad t \geq T,$$

which yields

$$t\,u'(t) \geq \text{constant} + c_0^{1/\alpha} \int_T^t \frac{s}{[p(s)]^{1/\alpha}}\,ds, \quad t \geq T.$$

Then it follows from (1.3) that $t\,u'(t) \to \infty$ as $t \to \infty$, and consequently $u'(t) > 0$ for all large t. Thus we have the case (A).

Suppose next that $u''(t) < 0$ for all large t. We claim that $u'(t) > 0$ for all large t. Assume to the contrary that $u'(t) < 0$ for all large t. There is $T \geq a$ satisfying $u''(t) < 0$ and $u'(t) < 0$ for $t \geq T$. Then we see that

$$u(t) \leq u(T) + u'(T)(t - T), \quad t \geq T,$$

and so $u(t) \to -\infty$ as $t \to \infty$. This is a contradiction. Therefore, $u'(t) > 0$ for all large t, and we have the case (B). The proof of Lemma 2.2 is complete.

Suppose that (1.2) holds, and let $u(t)$ be an eventually positive solution of (1.1). Then we can conclude that there is $k_2 > 0$ such that

$$u(t) \le k_2 \int_a^t (t-s)\Big[\frac{s}{p(s)}\Big]^{1/\alpha} ds \quad \text{for all large } t. \tag{2.5}$$

In fact, if $u'(t) < 0$ for all large t, then $u(t)$ is a bounded function, and therefore (2.5) is clearly satisfied for some $k_2 > 0$. If $u''(t) < 0$ for all large t, then $u(t)/t$ is a bounded function. Then (2.5) is also satisfied for some $k_2 > 0$ since (1.2) implies

$$\frac{1}{t}\int_a^t (t-s)\Big[\frac{s}{p(s)}\Big]^{1/\alpha} ds \to \infty \quad \text{as } t \to \infty.$$

Suppose that both $u'(t)$ and $u''(t)$ are eventually positive. By Lemma 2.1, we have $(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ eventually. Take a number $T \ge a$ such that $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ and $(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ for $t \ge T$. By (2.2), $(p(t)|u''(t)|^{\alpha-1}u''(t))'$ is decreasing on $[T, \infty)$. Therefore

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' \le (p(t)|u''(t)|^{\alpha-1}u''(t))'\Big|_{t=T} \equiv c_0\ (>0)$$

for $t \ge T$. Integrating this inequality repeatedly, we obtain

$$u(t) \le c_3 + c_2(t-T) + \int_T^t (t-s)\Big[\frac{c_1 + c_0(s-T)}{p(s)}\Big]^{1/\alpha} ds \tag{2.6}$$

for $t \ge T$, where $c_1 = p(T)|u''(T)|^{\alpha-1}u''(T)$, $c_2 = u'(T)$ and $c_3 = u(T)$. Then it is easy to verify that (2.6) implies (2.5) with $k_2 > c_0^{1/\alpha} > 0$. Thus, if (1.2) holds and if $u(t)$ is an eventually positive solution of (1.1), then (2.5) holds for some constant $k_2 > 0$.

We observe by (2.5) that, in the set of all eventually positive solutions of (1.1), a solution $u(t)$ which satisfies

$$\lim_{t\to\infty} \frac{u(t)}{\displaystyle\int_a^t (t-s)\Big[\frac{s}{p(s)}\Big]^{1/\alpha} ds} = k, \quad 0 < k < \infty, \tag{2.7}$$

can be regarded as a "maximal" solution.

Suppose that (1.3) holds, and let $u(t)$ be an eventually positive solution of (1.1). We have $u'(t) > 0$ for all large t (see Lemma 2.2), and so there is $k_1 > 0$ such that $u(t) \geq k_1$ for all large t. Thus a solution $u(t)$ of (1.1) satisfying

$$\lim_{t\to\infty} u(t) = k, \quad 0 < k < \infty, \tag{2.8}$$

can be regarded as a "minimal" solution in the set of all eventually positive solutions of (1.1).

In the following theorems, the necessary and sufficient conditions are established for the existence of special types of nonoscillatory solutions satisfying (2.7) and (2.8).

Theorem 2.1. *Suppose* (1.2) *holds. A necessary and sufficient condition for* (1.1) *to have a nonoscillatory solution* $u(t)$ *which satisfies* (2.7) *is that*

$$\int_a^\infty q(t) \left\{ \int_a^t (t-s) \left[\frac{s}{p(s)}\right]^{1/\alpha} ds \right\}^\beta dt < \infty. \tag{2.9}$$

Proof. (Necessity) Let $u(t)$ be a nonoscillatory solution of (1.1) satisfying (2.7). Then there is $T \geq a$ such that

$$\frac{k}{2}\int_a^t (t-s)\left[\frac{s}{p(s)}\right]^{1/\alpha} ds \leq u(t) \leq 2k\int_a^t (t-s)\left[\frac{s}{p(s)}\right]^{1/\alpha} ds, \quad t \geq T.$$

Integrating (1.1) from T to t, and using Lemma 2.1, we have

$$\begin{aligned}(p(t)|u''(t)|^{\alpha-1}u''(t))'\Big|_{t=T} &\geq \int_T^t q(s)[u(s)]^\beta ds \\ &\geq \left(\frac{k}{2}\right)^\beta \int_T^t q(s)\left\{\int_a^s (s-r)\left[\frac{r}{p(r)}\right]^{1/\alpha} dr\right\}^\beta ds, \quad t \geq T.\end{aligned}$$

This yields

$$\int_T^\infty q(s)\left\{\int_a^s (s-r)\left[\frac{r}{p(r)}\right]^{1/\alpha} dr\right\}^\beta ds < \infty,$$

which implies (2.9).

(Sufficiency) We assume that (2.9) holds. Let $k > 0$ be an arbitrary number. There is $T \geq a$ such that

$$\int_T^\infty q(t)\left\{\int_a^t (t-s)\left[\frac{s}{p(s)}\right]^{1/\alpha} ds\right\}^\beta dt < \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}. \tag{2.10}$$

To prove the sufficiency part of the theorem, it is enough to show that equation (1.1) has a nonoscillatory solution $u(t)$ having the property

$$\lim_{t\to\infty} \frac{u(t)}{\displaystyle\int_T^t (t-s)\Big[\frac{s-T}{p(s)}\Big]^{1/\alpha} ds} = k, \quad 0 < k < \infty. \tag{2.11}$$

For convenience, we put

$$H(t;T) = \int_T^t (t-s)\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds, \quad t \geq T. \tag{2.12}$$

Then it follows from (2.10) that

$$\int_T^\infty q(t)\big[H(t;T)\big]^\beta dt < \frac{(2k)^\alpha - k^\alpha}{(2k)^\beta}. \tag{2.13}$$

Let $I = [T, \infty)$ and let $C[T, \infty)$ be the space of all continuous functions $u : I \to R$ with the topology of uniform convergence on compact subintervals of $[T, \infty)$. Define the subset U of $C[T, \infty)$ by

$$U = \Big\{u \in C[T,\infty) : kH(t;T) \leq u(t) \leq 2kH(t;T),\, t \geq T\Big\}.$$

The set U is a closed convex subset of $C[T, \infty)$. We also define the mapping $\Psi : U \to C[T, \infty)$ by

$$(\Psi u)(t) = \int_T^t (t-s)\left[\frac{1}{p(s)}\int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u(\xi)|^\beta d\xi\Big) dr\right]^{1/\alpha} ds, \quad t \geq T.$$

Note that Ψ is well defined on U. We will show that Ψ has a fixed point $u(t)$, i.e.,

$$u(t) = (\Psi u)(t), \quad t \geq T, \tag{2.14}$$

by using the Schauder-Tychonoff fixed point theorem.

(i) Ψ *maps* U *into* U. Let $u \in U$. By means of (2.13), we have

$$\begin{aligned}(\Psi u)(t) &\leq \int_T^t (t-s)\left[\frac{1}{p(s)}\int_T^s \Big(k^\alpha + (2k)^\beta \int_r^\infty q(\xi)\big[H(\xi;T)\big]^\beta d\xi\Big) dr\right]^{1/\alpha} ds \\ &\leq 2k \int_T^t (t-s)\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds = 2kH(t;T), \quad t \geq T,\end{aligned}$$

and

$$(\Psi u)(t) \geq k \int_T^t (t-s)\Big[\frac{s-T}{p(s)}\Big]^{1/\alpha} ds = kH(t;T), \quad t \geq T.$$

Thus $\Psi u \in U$.

(ii) Ψ *is continuous on* U. Suppose that $\{u_n\}_{n=1}^{\infty} \subset U$ and $u \in U$, and that $\lim_{n\to\infty} u_n = u$ in the topology of $C[T,\infty)$. Let $T^* > T$ be fixed. We have

$$\begin{aligned}
&\Big|(\Psi u_n)(t) - (\Psi u)(t)\Big| \\
&\quad \leq \int_T^t (t-s)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} \Big|\Big[\int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u_n(\xi)|^\beta d\xi\Big)dr\Big]^{1/\alpha} \\
&\qquad\qquad - \Big[\int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u(\xi)|^\beta d\xi\Big)dr\Big]^{1/\alpha}\Big|\, ds \\
&\quad \leq \int_T^{T^*} (T^*-s)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} F_n(s)\, ds, \quad t \in [T,T^*],
\end{aligned}$$

where

$$\begin{aligned}
F_n(s) \;=\; &\Big|\Big[\int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u_n(\xi)|^\beta d\xi\Big)dr\Big]^{1/\alpha} \\
&\qquad - \Big[\int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u(\xi)|^\beta d\xi\Big)dr\Big]^{1/\alpha}\Big|, \quad s \in [T,T^*].
\end{aligned}$$

Evidently, $\lim_{n\to\infty} F_n(s) = 0$ for each $s \in [T,T^*]$. Further we get

$$\begin{aligned}
0 \leq F_n(s) &\leq 2\Big|\int_T^{T^*} \Big\{k^\alpha + (2k)^\beta \int_r^\infty q(\xi)\big[H(\xi;T)\big]^\beta d\xi\Big\} dr\,\Big|^{1/\alpha} \\
&\leq 4k(T^*-T)^{1/\alpha}, \quad s \in [T,T^*],
\end{aligned}$$

which means that there is $M > 0$ such that $|F_n(s)| \leq M$ for $s \in [T,\, T^*]$. Then, applying the Lebesgue convergence theorem, we have

$$\lim_{n\to\infty} \int_T^{T^*} (T^*-s)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} F_n(s) ds = 0.$$

Thus we obtain

$$\Big|(\Psi u_n)(t) - (\Psi u)(t)\Big| \to 0 \;\; (n \to \infty) \;\; \text{uniformly on } [T,\, T^*] \subset I,$$

which implies $\lim_{n\to\infty} \Psi u_n = \Psi u$ in the topology of $C[T,\infty)$. This shows that Ψ is continuous on U.

(iii) ΨU *is relatively compact.* Let $T^* > T$ be fixed. In (i) we have showed that $kH(t;T) \le u(t) \le 2kH(t;T)$ for $t \ge T$. In particular, ΨU is uniformly bounded on $[T, T^*]$. Since

$$\begin{aligned} 0 \le (\Psi u)'(t) &= \int_T^t \left[\frac{1}{p(s)} \int_T^s \Big(k^\alpha + \int_r^\infty q(\xi)|u(\xi)|^\beta d\xi\Big) dr\right]^{1/\alpha} ds \\ &\le 2k \int_T^{T^*} \left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds, \quad t \in [T,\, T^*] \subset I, \end{aligned}$$

ΨU is equicontinuous on $[T,\, T^*] \subset I$. Therefore, by the Ascoli-Arzela theorem, we conclude that ΨU is a relatively compact subset of $C[T, \infty)$.

We can apply the Schauder-Tychonoff fixed point theorem to the mapping $\Psi : U \to U$. There is a point $u \in U$ such that (2.14) holds. It is easy to verify that $u(t)$ satisfies (2.11), and is a positive solution on $[T, \infty)$ of (1.1). The proof of Theorem 2.1 is complete.

Theorem 2.2. *Suppose* (1.3) *holds. A necessary and sufficient condition for* (1.1) *to have a nonoscillatory solution* $u(t)$ *which satisfies* (2.8) *is that*

$$\int_a^\infty t \left[\frac{1}{p(t)} \int_t^\infty (s-t) q(s) ds\right]^{1/\alpha} dt < \infty. \tag{2.15}$$

Proof. (Necessity) Let $u(t)$ be a nonoscillatory solution of (1.1) satisfying (2.8). There is $T \ge a$ such that $k/2 \le u(t) \le 2k$ for $t \ge T$. Evidently, the case (B) in Lemma 2.2 holds. Thus we may suppose that $u(t) > 0$, $u'(t) > 0$, $u''(t) < 0$ and $(p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$ for $t \ge T$. Integrating (1.1) from t to τ, we have

$$\int_t^\tau (p(s)|u''(s)|^{\alpha-1}u''(s))'' ds = -\int_t^\tau q(s)[u(s)]^\beta ds.$$

Letting $\tau \to \infty$, we get

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' \ge \int_t^\infty q(s)[u(s)]^\beta ds, \quad t \ge T. \tag{2.16}$$

Integrating (2.16) from t to τ, we obtain

$$\begin{aligned} &p(\tau)|u''(\tau)|^{\alpha-1}u''(\tau) \\ &\quad \ge p(t)|u''(t)|^{\alpha-1}u''(t) + \int_t^\tau \int_s^\infty q(r)[u(r)]^\beta dr\, ds, \quad \tau \ge t \ge T, \end{aligned}$$

and in the limit $\tau \to \infty$,

$$-u''(t) \geq \left[\frac{1}{p(t)} \int_t^\infty \int_s^\infty q(r)[u(r)]^\beta dr\, ds\right]^{1/\alpha}, \quad t \geq T.$$

Further, as in the above procedure we have

$$u'(t) \geq \int_t^\infty \left[\frac{1}{p(s)} \int_s^\infty \int_r^\infty q(\xi)[u(\xi)]^\beta d\xi\, dr\right]^{1/\alpha} ds, \quad t \geq T. \tag{2.17}$$

Finally, integrating (2.17) from T to t, we have

$$u(t) \geq u(T) + \int_T^t \int_s^\infty \left[\frac{1}{p(r)} \int_r^\infty \int_\xi^\infty q(\eta)[u(\eta)]^\beta d\eta\, d\xi\right]^{1/\alpha} dr\, ds, \quad t \geq T. \tag{2.18}$$

Then, since $u(t)$ has a positive finite limit as $t \to \infty$, we conclude by (2.18) that

$$\int_T^\infty (s-T)\left[\frac{1}{p(s)} \int_s^\infty (r-s)q(r)\, dr\right]^{1/\alpha} ds < \infty,$$

which gives (2.15).

(Sufficiency) We assume that (2.15) holds. Let $k > 0$ be a given constant. There is $T \geq a$ such that

$$\int_T^\infty (t-T)\left[\frac{1}{p(t)} \int_t^\infty (s-t)q(s)ds\right]^{1/\alpha} dt < \frac{k}{(2k)^{\beta/\alpha}}. \tag{2.19}$$

Let $I = [T,\infty) \subset R$ and let $C[T,\infty)$ be the set of all continuous functions $u : I \to R$ with the topology of uniform convergence on compact subintervals of $[T,\infty)$. Define the set U by

$$U = \Big\{u \in C[T,\infty) : \ |u(t)| \leq 2k,\ t \geq T\Big\},$$

which is a closed convex subset of $C[T,\infty)$. Define the mapping $\Psi : U \to C[T,\infty)$ by

$$(\Psi u)(t) = k - \int_t^\infty (s-t)\left[\frac{1}{p(s)} \int_s^\infty (r-s)q(r)|u(r)|^\beta dr\right]^{1/\alpha} ds, \quad t \geq T. \tag{2.20}$$

Obviously, Ψ is well defined on U.

(i) Ψ *maps* U *into* U. If $u \in U$, we have

$$\begin{aligned}\Big|(\Psi u)(t)\Big| &\leq k + (2k)^{\beta/\alpha}\int_t^\infty (s-t)\Big[\frac{1}{p(s)}\int_s^\infty (r-s)q(r)\,dr\Big]^{1/\alpha} ds \\ &\leq k + k = 2k, \quad t \geq T.\end{aligned}$$

This means $\Psi U \subset U$.

(ii) Ψ *is continuous on* U. Let $\{u_n\}$ be a sequence in U and let u be an element of U. Suppose that $\lim_{n\to\infty} u_n = u$ in $C[T,\infty)$. We have

$$\begin{aligned}&\Big|(\Psi u_n)(t) - (\Psi u)(t)\Big| \\ &\quad\leq \int_t^\infty (s-t)\Big[\frac{1}{p(s)}\Big]^{1/\alpha}\Big|\Big(\int_s^\infty (r-s)q(r)|u_n(r)|^\beta dr\Big)^{1/\alpha} \\ &\qquad\qquad -\Big(\int_s^\infty (r-s)q(r)|u(r)|^\beta dr\Big)^{1/\alpha}\Big| ds \\ &\quad\leq \int_T^\infty (s-T)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} F_n(s)\,ds, \quad t \geq T,\end{aligned}$$

where

$$\begin{aligned}F_n(s) = \Big|&\Big(\int_s^\infty (r-s)q(r)|u_n(r)|^\beta dr\Big)^{1/\alpha} \\ &-\Big(\int_s^\infty (r-s)q(r)|u(r)|^\beta dr\Big)^{1/\alpha}\Big|, \quad s \geq T.\end{aligned}$$

Since $\lim_{n\to\infty} F_n(s) = 0$ for each $s \geq T$ and

$$|F_n(s)| \leq 2(2k)^{\beta/\alpha}\Big[\int_s^\infty (r-s)q(r)dr\Big]^{1/\alpha}, \quad s \geq T,$$

an application of the Lebesgue convergence theorem gives

$$\lim_{n\to\infty}\int_T^\infty (s-T)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} F_n(s)ds = 0.$$

Therefore

$$\Big|(\Psi u_n)(t) - (\Psi u)(t)\Big| \to 0 \ \ (n\to\infty) \ \text{ uniformly on } \ I = [T,\,\infty),$$

and, in particular, Ψ is continuous on U.

(iii) ΨU *is relatively compact.* Since $\Psi U \subset U$ and U is uniformly bounded on I, it is evident that ΨU is also uniformly bounded on I. Differentiation of (2.20) gives

$$\begin{aligned} 0 \le (\Psi u)'(t) &= \int_t^\infty \left[\frac{1}{p(s)}\int_s^\infty (r-s)q(r)|u(r)|^\beta dr\right]^{1/\alpha} ds \\ &\le (2k)^{\beta/\alpha}\int_T^\infty \left[\frac{1}{p(s)}\int_s^\infty (r-s)q(r)\,dr\right]^{1/\alpha} ds, \quad t \ge T, \quad u \in U, \end{aligned}$$

which implies that ΨU is equicontinuous on I. Therefore ΨU is a relatively compact subset of $C[T,\infty)$.

By the Schauder-Tychonoff fixed point theorem, there is a function $u \in U$ such that

$$u(t) = (\Psi u)(t), \quad t \ge T.$$

Then it is easily seen that $u(t)$ is a solution of (1.1) on $[T,\infty)$ and satisfies (2.8). The proof of Theorem 2.2 is complete.

3. Existence of nonoscillatory solutions

In the preceding section we have established the necessary and sufficient conditions for the existence of specific nonoscillatory solutions of (1.1), that is, solutions $u(t)$ of (1.1) which satisfy the asymptotic conditions (2.7) and (2.8). This section is devoted to the study of the existence of nonoscillatory solutions of (1.1). For nonoscillatory solutions, we do not impose any asymptotic conditions, such as (2.7) or (2.8).

Theorem 3.1. *Suppose that* $\alpha \ge 1 > \beta$ *and that* (1.3) *is satisfied. Then equation* (1.1) *has a nonoscillatory solution if and only if* (2.9) *holds.*

Proof. By Theorem 2.1 we immediately see that if (2.9) holds, then (1.1) has a nonoscillatory solution $u(t)$ (satisfying (2.7)).

Conversely, assume that (1.1) has a nonoscillatory solution $u(t)$. We may suppose that $u(t)$ is eventually positive. Then, $u(t)$ satisfies either (A) or (B) in Lemma 2.2. We first discuss the case where $u(t)$ satisfies (A). Take a sufficiently large number $T \ge a$ such that

$$(3.1)\quad u(t) > 0, \quad u'(t) > 0, \quad u''(t) > 0, \text{ and } \quad (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0$$

for $t \ge T$. Then it is easy to see that

$$u''(t) \ge \left[\frac{1}{p(t)}\int_T^t (p(s)|u''(s)|^{\alpha-1}u''(s))' ds\right]^{1/\alpha}, \quad t \ge T.$$

Integrating this inequality twice, we have

$$u(t) \geq \int_T^t \int_T^s \left[\frac{1}{p(r)} \int_T^r (p(\xi)|u''(\xi)|^{\alpha-1}u''(\xi))' d\xi\right]^{1/\alpha} dr\, ds, \quad t \geq T.$$

By equation (1.1) we find that

$$\begin{aligned}
&(p(t)|u''(t)|^{\alpha-1}u''(t))'' = -q(t)[u(t)]^\beta \\
&\leq -q(t)\Big\{\int_T^t \int_T^s \left[\frac{1}{p(r)} \int_T^r (p(\xi)|u''(\xi)|^{\alpha-1}u''(\xi))' d\xi\right]^{1/\alpha} dr\, ds\Big\}^\beta \\
&\leq -\Big\{(p(t)|u''(t)|^{\alpha-1}u''(t))'\Big\}^{\beta/\alpha} q(t)\Big\{\int_T^t \int_T^s \left[\frac{1}{p(r)} \int_T^r d\xi\right]^{1/\alpha} dr\, ds\Big\}^\beta, \quad t \geq T,
\end{aligned}$$

where the fact that $(p(t)|u''(t)|^{\alpha-1}u''(t))'$ is decreasing on $[T, \infty)$ is used in the last step. We multiply this inequality by $\{(p(t)|u''(t)|^{\alpha-1}u''(t))'\}^{-(\beta/\alpha)} > 0$ and integrate the resulting inequality from T to ∞. Then, noting that $\alpha > \beta$, we obtain

$$\frac{1}{1-(\beta/\alpha)}\Big\{(p(t)|u''(t)|^{\alpha-1}u''(t))'\Big|_{t=T}\Big\}^{1-(\beta/\alpha)} \geq \int_T^\infty q(t)\Big\{\int_T^t \int_T^s \left[\frac{r-T}{p(r)}\right]^{1/\alpha} dr\, ds\Big\}^\beta dt.$$

In particular, we have

$$\int_T^\infty q(t)\Big\{\int_T^t (t-s)\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds\Big\}^\beta dt < \infty, \tag{3.2}$$

which implies (2.9).

We next discuss the case where $u(t)$ satisfies (B) in Lemma 2.2. Choose a sufficiently large number $T \geq a$ so that

$$u(t) > 0, \quad u'(t) > 0, \quad u''(t) < 0, \text{ and } \quad (p(t)|u''(t)|^{\alpha-1}u''(t))' > 0 \tag{3.3}$$

for $t \geq T$. We put

$$v(t) = p(t)|u''(t)|^{\alpha-1}u''(t), \quad t \geq T.$$

Then, $v(t) < 0, v'(t) > 0$ and $v''(t) < 0$ for $t \geq T$. Let $C_0 = -v(T)$. Evidently we have $|v(t)| \leq C_0\, (t \geq T)$. Let $H(t;T)$ be the function defined by (2.12). Then, noting that $\alpha \geq 1$, we can compute as follows:

$$\begin{aligned}
H(t;T)v'(t) &\leq \int_T^t v'(s)\int_T^s \left[\frac{r-T}{p(r)}\right]^{1/\alpha} dr\,ds \\
&= v(t)\int_T^t \left[\frac{r-T}{p(r)}\right]^{1/\alpha} dr - \int_T^t v(s)\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds \\
&\leq \int_T^t |v(s)|\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds \\
&\leq \int_T^t C_0^{1-(1/\alpha)}|v(s)|^{1/\alpha}\left[\frac{s-T}{p(s)}\right]^{1/\alpha} ds \\
&= C_0^{1-(1/\alpha)}\int_T^t \left[p(s)|u''(s)|^{\alpha}\frac{s-T}{p(s)}\right]^{1/\alpha} ds \\
&\leq C_0^{1-(1/\alpha)}\int_T^t |u''(s)|(s-T+1)ds \\
&= -C_0^{1-(1/\alpha)}\Big(u'(t)(t-T+1)-u'(T)-u(t)+u(T)\Big) \\
&\leq C_0^{1-(1/\alpha)}u'(T)+C_0^{1-(1/\alpha)}u(t) \\
&= C_0^{1-(1/\alpha)}\frac{u'(T)}{u(T)}u(T)+C_0^{1-(1/\alpha)}u(t) \\
&\leq Ku(t)
\end{aligned}$$

for $t \geq T$, where $K = C_0^{1-(1/\alpha)}\Big[\dfrac{u'(T)}{u(T)}+1\Big]$. Consequently, we obtain $u(t) \geq K^{-1}H(t;T)v'(t)$ for $t \geq T$. On the other hand, an integration of (1.1) from t to ∞ gives

$$v'(t) = (p(t)|u''(t)|^{\alpha-1}u''(t))' \geq \int_t^\infty q(s)[u(s)]^\beta\,ds, \quad t \geq T.$$

Therefore we find that

$$q(t)[u(t)]^\beta \geq K^{-\beta}q(t)\Big[H(t;T)\Big]^\beta\Big[\int_t^\infty q(s)[u(s)]^\beta ds\Big]^\beta, \quad t \geq T.$$

Hence, noting that $1 > \beta$, we obtain

$$\frac{1}{1-\beta}\Big[\int_T^\infty q(s)[u(s)]^\beta ds\Big]^{1-\beta} \geq K^{-\beta}\int_T^\infty q(t)\Big[H(t;T)\Big]^\beta dt,$$

which yields (3.2). Thus we have (2.9). This completes the proof of Theorem 3.1.

Theorem 3.2. *Suppose that $\alpha \le 1 < \beta$ and that* (1.3) *holds. Then equation* (1.1) *has a nonoscillatory solution if and only if* (2.15) *holds.*

Proof. It follows from Theorem 2.2 that if (2.15) holds, then (1.1) has a nonoscillatory solution. To prove the converse, assume that (1.1) possesses a nonoscillatory solution $u(t)$. Then, $u(t)$ satisfies either (A) or (B) in Lemma 2.2. Consider first the case (A). We take a number $T \ge a$ so that (3.1) is satisfied for $t \ge T$. Integrating (1.1) from t to ∞, we have

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' \ge \int_t^\infty q(s)[u(s)]^\beta \, ds, \quad t \ge T.$$

Further, integrating this inequality from T to t, we get

$$p(t)[u''(t)]^\alpha - p(T)[u''(T)]^\alpha \ge \int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds, \quad t \ge T,$$

from which it follows that

$$u''(t) \ge \left[\frac{1}{p(t)} \int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds\right]^{1/\alpha}, \quad t \ge T. \tag{3.4}$$

Let $T_1 > T$ be fixed. We have

$$\int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds \ge \int_T^{T_1} \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds \equiv C_0 > 0, \quad t \ge T_1.$$

Then, noting that $\alpha \le 1$, we see that

$$\left[\int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds\right]^{1/\alpha} \ge C_0^{(1/\alpha)-1} \int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds, \quad t \ge T_1.$$

By virtue of (3.4) it is seen that

$$u''(t) \ge C_0^{(1/\alpha)-1} \left[\frac{1}{p(t)}\right]^{1/\alpha} \int_T^t \int_s^\infty q(r)[u(r)]^\beta \, dr \, ds, \quad t \ge T_1. \tag{3.5}$$

Integration of (3.5) gives

$$\begin{aligned} u'(t) &\ge C_0^{(1/\alpha)-1} \int_{T_1}^t \left[\frac{1}{p(s)}\right]^{1/\alpha} \int_T^s \int_r^\infty q(\xi)[u(\xi)]^\beta \, d\xi \, dr \, ds \\ &\ge C_0^{(1/\alpha)-1} \int_t^\infty q(\xi)[u(\xi)]^\beta \, d\xi \int_{T_1}^t \left[\frac{1}{p(s)}\right]^{1/\alpha} \int_{T_1}^s dr \, ds \\ &\ge C_0^{(1/\alpha)-1} [u(t)]^\beta \int_t^\infty q(\xi) d\xi \int_{T_1}^t (s - T_1) \left[\frac{1}{p(s)}\right]^{1/\alpha} ds \end{aligned}$$

for $t \geq T_1$. Multiplying this inequality by $[u(t)]^{-\beta}$ and integrating from T_1 to ∞, we have

$$\frac{1}{\beta-1}\Big[u(T_1)\Big]^{1-\beta} \geq C_0^{(1/\alpha)-1} \int_{T_1}^{\infty} \int_s^{\infty} q(\xi)\, d\xi \int_{T_1}^{s} (r-T_1)\Big[\frac{1}{p(r)}\Big]^{1/\alpha} dr\, ds. \tag{3.6}$$

Here, we have used the condition $\beta > 1$. Then, (3.6) implies

$$\int_{T_1}^{\infty} \int_s^{\infty} q(\xi)\, d\xi\, ds < \infty,$$

and, in consequence,

$$\int_t^{\infty} (s-t) q(s)\, ds < \infty \quad \text{for all } t \geq T_1. \tag{3.7}$$

Using (3.6) again, we find that

$$\frac{1}{\beta-1}\Big[u(T_1)\Big]^{1-\beta} \geq C_0^{(1/\alpha)-1} \int_{T_1}^{\infty} (s-T_1)\Big[\frac{1}{p(s)}\Big]^{1/\alpha} \int_s^{\infty} (r-s) q(r)\, dr\, ds.$$

Since we have (3.7), it is possible to take $T_2 > T_1$ so that

$$\int_s^{\infty} (r-s) q(r)\, dr \leq 1 \quad \text{for } s \geq T_2.$$

Then, noting that $\alpha \leq 1$, we get

$$\frac{1}{\beta-1}\Big[u(T_1)\Big]^{1-\beta} \geq C_0^{(1/\alpha)-1} \int_{T_2}^{\infty} (s-T_2)\Big[\frac{1}{p(s)} \int_s^{\infty} (r-s) q(r)\, dr\Big]^{1/\alpha} ds,$$

which gives (2.15).

Next we consider the case (B). Let $T \geq a$ be a number such that (3.3) holds for $t \geq T$. We have

$$(p(t)|u''(t)|^{\alpha-1}u''(t))' \geq \int_t^{\infty} q(s)[u(s)]^{\beta}\, ds \geq [u(t)]^{\beta} \int_t^{\infty} q(s)\, ds, \quad t \geq T.$$

An integration of this inequality over $[t, \infty)$ yields

$$-p(t)|u''(t)|^{\alpha-1}u''(t) \geq [u(t)]^{\beta} \int_t^{\infty} \int_s^{\infty} q(r)\, dr\, ds = [u(t)]^{\beta} \int_t^{\infty} (s-t) q(s)\, ds, \quad t \geq T,$$

and so

$$-u''(t) \geq [u(t)]^{\beta/\alpha} \Big[\frac{1}{p(t)} \int_t^{\infty} (s-t) q(s)\, ds\Big]^{1/\alpha}, \quad t \geq T.$$

Then, integrating the above inequality from t to ∞, we get

$$u'(t) \geq [u(t)]^{\beta/\alpha} \int_t^\infty \left[\frac{1}{p(s)} \int_s^\infty (r-s) q(r)\, dr\right]^{1/\alpha} ds, \quad t \geq T. \tag{3.8}$$

Multiplying (3.8) by $[u(t)]^{-(\beta/\alpha)}$ and integrating from T to ∞, we obtain

$$\begin{aligned} \frac{1}{(\beta/\alpha)-1}\Big[u(T)\Big]^{1-(\beta/\alpha)} &\geq \int_T^\infty \int_s^\infty \left[\frac{1}{p(r)} \int_r^\infty (\xi - r) q(\xi)\, d\xi\right]^{1/\alpha} dr\, ds \\ &= \int_T^\infty (s-T)\left[\frac{1}{p(s)} \int_s^\infty (r-s) q(r)\, dr\right]^{1/\alpha} ds. \end{aligned}$$

This gives (2.15). The proof of Theorem 3.2 is complete.

By Theorems 3.1 and 3.2 we have the following corollaries, in which oscillation of equation (1.1) is characterized by integral conditions on $p(t)$ and $q(t)$.

Corollary 3.1. *Suppose that* $\alpha \geq 1 > \beta$ *and that* (1.3) *holds. Then equation* (1.1) *is oscillatory if and only if*

$$\int_a^\infty q(t) \left\{ \int_a^t (t-s) \left[\frac{s}{p(s)}\right]^{1/\alpha} ds \right\}^\beta dt = \infty.$$

Corollary 3.2. *Suppose that* $\alpha \leq 1 < \beta$ *and that* (1.3) *holds. Then equation* (1.1) *is oscillatory if and only if*

$$\begin{cases} \displaystyle\int_a^\infty t\, q(t)\, dt = \infty \quad \text{or} \\ \displaystyle\int_a^\infty t\, q(t)\, dt < \infty \quad \text{and} \quad \int_a^\infty t \left[\frac{1}{p(t)} \int_t^\infty (s-t) q(s)\, ds\right]^{1/\alpha} dt = \infty. \end{cases}$$

If $0 < \alpha \leq 1$, then the condition (1.3) is reduced to the single integral condition

$$\int_a^\infty \frac{t}{[p(t)]^{1/\alpha}}\, dt = \infty. \tag{3.9}$$

Similarly, if $\alpha \geq 1$, then (1.3) is reduced to (1.2) . Therefore we can replace the condition (1.3) in Theorem 3.2 and Corollary 3.2 [resp. Theorem 3.1 and Corollary 3.1] by (3.9) [resp. (1.2)].

Acknowledgment. The author would like to thank Professor Manabu Naito for a great deal of valuable advice.

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