Funkcialaj Ekvacioj
Volume 45, No. 1, pp 53--69

SHARP ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO NONLINEAR SCHRÖDINGER EQUATIONS IN ONE SPACE DIMENSION

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1. Introduction

We study the following nonlinear Schrödinger equation in one space dimension:

$$\begin{aligned} iu_t(t,x) &= (-1/2)\partial^2 u(t,x) + f(u(t,x)), \\ u(0,x) &= u_0(x). \end{aligned} \tag{1.1}$$

Here $(t,x) \in \mathbb{R}\times\mathbb{R}$, $\partial = \partial/\partial x$ and $f(u) = f_1(u) + f_2(u) = \lambda_1|u|^{p_1-1}u + \lambda_2|u|^{p_2-1}u$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $1 < p_1 < p_2$. The aim of this paper is to study the asymptotic behavior of the solution as $t \to \infty$, especially to obtain the second term of the asymptotic expansion of the solution in the case $p_1 = 3$.

There is a large literature on the equation (1.1); see [1, 3–9, 11, 13–23] and references therein. The well-posedness of the Cauchy Problem (1.1) has been extensively studied, and the results already obtained are satisfactory for our study of the asymptotic behavior of the solution. To put it briefly, if $1 \le p_1 < p_2 < 5$, the equation (1.1) is (conditionally) well-posed in L^2, and moreover $U(-t)u(t) \in L^{2,s}$ if $u_0 \in L^{2,s}$ with $0 < s < p_1$. Here $U(t) = \exp(it\partial^2/2)$ is the free propagator and $L^{2,s}$ denotes the weighted L^2-space of order s; more precisely, see Proposition 2.4 below. In what follows, we simply call the solution obtained by Proposition 2.4 "the solution to (1.1)."

The asymptotic behavior of the solution to (1.1) is usually explained in terms of the scattering theory. When $t \to \infty$, the solution is expected to decay by the dispersive effect of the equation. Hence we can expect

*Supported by JSPS Research Fellowships for Young Scientists.
**Supported in part by the Yukawa foundation of Osaka University.

that the nonlinearity in the equation decays rapidly enough and loses its effect as $t \to \infty$. Thus the expected profile of the solution to (1.1) is of the form $U(t)\phi$, which is a solution to the free Schrödinger equation; here ϕ is a suitable function called the scattering state of the solution. This observation is, however, correct only in case $3 < p_1 < p_2$, namely the short-range case (on the other hand, the nonlinear term with $p_1 \leq 3$ is called of long-range). Indeed, the followings are well known [1, 12, 23].

(I) If $\lambda_1, \lambda_2 \geq 0$, $3 < p_1 < p_2 < 5$ and $u_0 \in L^{2,1}$, then there exists a function $\phi \in L^2$ satisfying

$$\lim_{t\to\infty} \|u(t) - U(t)\phi\|_2 = 0, \tag{1.2}$$

where $u(t)$ is the solution to (1.1).

(II) If $3 < p_1 < p_2 < 5$, $u_0 \in L^{2,1}$ and $\|u_0\|_{L^{2,1}}$ is sufficiently small, then there exists a function $\phi \in L^{2,1}$ satisfying

$$\lim_{t\to\infty} \|U(-t)u(t) - \phi\|_{L^{2,1}} = 0, \tag{1.3}$$

where $u(t)$ is the solution to (1.1).

On the other hand, we have

(III) If $\lambda_1 \neq 0$, $p_1 \leq 3$ and $u_0 \in L^{2,1} \setminus \{0\}$, then there does not exist a function $\phi \in L^2$ satisfying (1.2) for the solution $u(t)$ to (1.1).

From the results above, the critical exponent for the existence of the scattering state is $p_1 = 3$. In this case there does not exist usual scattering state, but if we introduce the modified free dynamics of the form $U(t)\exp\bigl(-iS(t, -i\partial)\bigr)\phi$, the situation is improved, where $S(t,\xi) = \lambda_1|\hat{\phi}(\xi)|^2 \log t$ is the modifier of the Dollard type. Indeed, the following is known [11].

(IV) If $3 = p_1 < p_2 < 5$, $u_0 \in L^{2,s}$ for some $s > 1/2$, and $\|u_0\|_{L^{2,s}}$ is sufficiently small, then there exists a function $\phi \in L^2 \cap L^\infty$ satisfying

$$\lim_{t\to\infty} \|u(t) - U(t)\exp\bigl(-iS(t, -i\partial)\bigr)\phi\|_2 = 0, \tag{1.4}$$

where $u(t)$ is the solution to (1.1) and $S(t,\xi)$ is defined as above.

These results give us the asymptotic profile of the solution to (1.1). Our concern is now to know the behavior of the difference of the nonlinear solution and the asymptotic profile. In the short-range case, the following has been proved [15]:

(V) Let $3 < p_1 < p_2 < 5$, $u_0 \in L^{2,1}$ and $\|u_0\|_{L^{2,1}}$ is sufficiently small. Then there exists a function $\phi \in L^{2,1}$ satisfying

$$\begin{aligned} u(t,x) - U(t)\phi(x) = 2i^{1/2}(p_1-3)^{-1}\, t^{(2-p_1)/2} \exp(i|x|^2/2t) f_1(\hat{\phi})(x/t) \\ + o(t^{(2-p_1)/2}) \end{aligned} \tag{1.5}$$

uniformly in $\mathbb{R}$ as $t \to \infty$, where $u(t)$ is the solution to (1.1).

The nonlinear effect explicitly appears in the right-hand side. Proof of (1.5) is based on the method of stationary phase.

Remark 1.1. The scattering theory for (1.1), especially the existence and the completeness of wave operators, has been studied in various function spaces; for short-range case, see [4, 6, 8, 9, 16–18, 20, 21] and long-range case [5, 19]. In the preceding, we have mentioned only the results directly related in this paper.

The purpose of this paper is to treat the case $p_1 = 3$. The equation (1.1) with $p_1 = 3$ appears various fields of mathematical physics, and hence it is considered to be important. Since this case is of long-range, mathematical treatment is more difficult than short-range case. Now we state our theorem.

Theorem. *Let* $3 = p_1 < p_2 < 5$. *Let* $u_0 \in L^{2,s}$ *with* $s > 5/2$ *and let* $\|u_0\|_{L^{2,s}}$ *be sufficiently small. Let* $u(t)$ *be the solution to* (1.1). *Then there exists a function* $\phi \in L^{2,2} \cap \mathcal{F}^{-1}H^{2,\infty}$ *satisfying*

$$\Big\| U(-t)u(t) - \exp\big(-i\tilde{S}(t,-i\partial)\big)\big(\phi + t^{-1}\sum_{j=0}^{1}(\log t)^j \phi_{1,j}\big)\Big\|_2 = o(t^{-1}). \tag{1.6}$$

as $t \to \infty$. Here

$$\phi_{1,1} = -i\lambda_1{}^2\mathcal{F}^{-1}\big(\partial^2(|\hat{\phi}|^2)|\hat{\phi}|^2\hat{\phi}\big), \tag{1.7}$$

$$\phi_{1,0} = -\lambda_1\mathcal{F}^{-1}\big(\bar{\hat{\phi}}(\partial\hat{\phi})^2 + 2\hat{\phi}|\partial\hat{\phi}|^2 + \hat{\phi}^2\partial^2\bar{\hat{\phi}}\big) + \phi_{1,1}, \tag{1.8}$$

and

$$\begin{aligned}\tilde{S}(t,\xi) = \lambda_1|\hat{\phi}(\xi)|^2\log t - \lambda_1 t^{-1}\big(\hat{\phi}(\xi)\bar{\hat{\phi}}_{1,0}(\xi) + \bar{\hat{\phi}}(\xi)\hat{\phi}_{1,0}(\xi)\big)\\ - 2\lambda_2(p_2-3)^{-1}t^{-(p_2-3)/2}|\hat{\phi}(\xi)|^{p_2-1}.\end{aligned} \tag{1.9}$$

Furthermore,

$$\begin{aligned}u(t,x) = (it)^{-1/2}\exp\big(i|x|^2/2t - i\tilde{S}(t,x/t)\big)\big(\hat{\phi}(x/t) + t^{-1}\sum_{j=0}^{2}(\log t)^j\psi_{1,j}(x/t)\big)\\ + o(t^{-3/2})\end{aligned} \tag{1.10}$$

as $t \to \infty$ uniformly in $\mathbb{R}$. Here

$$\psi_{1,2} = (-i\lambda_1{}^2/2)(\partial|\hat{\phi}|^2)^2\,\hat{\phi}, \tag{1.11}$$

$$\psi_{1,1} = (-\lambda_1/2)(\partial^2|\hat{\phi}|^2)\,\hat{\phi} - \lambda_1(\partial|\hat{\phi}|^2)\,\partial\hat{\phi} + \hat{\phi}_{1,1}, \tag{1.12}$$

$$\psi_{1,0} = (-i/2)\partial^2\hat{\phi} + \hat{\phi}_{1,0}. \tag{1.13}$$

Remark 1.2. The restriction $p_2 < 5$ is necessary only to guarantee the existence of $L^{2,s}$-solution. If we assume $u_0 \in H^1 \cap L^{2,s}$, we can remove this restriction.

This paper is organized as follows. In section 2, we give several basic estimates and introduce a result on the well-posedness of (1.1). In section 3, we prove key Proposition 3.2. The expansion formulae given in Proposition 3.2 (and Theorem) are so complicated that we sketch the formal derivation of them at the beginning of the proof. Our main theorem easily follows from this proposition.

In the previous paper [24], one of the authors showed an analogous result for the Hartree equation which includes the nonlinearity like $f(u) = (|x|^{-1} * |u|^2)u$. The treatment of (1.1) is more difficult than that of the Hartree equation by the following reason. While the nonlinear term in the Hartree equation can be differentiated as many times as we like, the short-range nonlinear perturbation $f_2(u)$ in (1.1) is differentiable at most p_2-times because of the singularity at $u=0$. Thus, in case p_2 is close to 3, we can differentiate the phase $\tilde{S}$ only $p_2-1\approx 2$ times, which is not sufficient to prove the theorem as long as we rely on H^s-theory; indeed, we would need $s>5/2$ and hence $p_2>7/2$. To avoid this difficulty and prove the theorem for $p_2>3$, we state Proposition 3.2 in terms of the Besov space $B^{\sigma}_{r,2}$ with $\sigma\approx 2$, $r\gg 1$, which is continuously embedded in $H^{2,\infty}$.

We shall conclude this introduction by giving the notation used in this paper:
$\hat{\psi}(\xi)=(\mathcal{F}\psi)(\xi)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-ix\xi}\psi(x)dx$ is the Fourier transform of ψ, and $\mathcal{F}^{-1}\psi$ its inverse. For $1\le p\le\infty$, $\|\cdot\|_p$ means the usual L^p-norm; $L^{2,s}=\{\phi\in\mathcal{S}';\|\phi\|_{L^{2,s}}=\|(1+|x|^2)^{s/2}\phi\|_2<\infty\}$ denotes the weighted Lebesgue space; $H^{s,r}=\{\phi\in\mathcal{S}';\|\phi\|_{H^{s,r}}=\|(1-\partial^2)^{s/2}\phi\|_r<\infty\}$ denotes the Sobolev space, $H^{s,2}$ is abbreviated to H^s; $B^s_{r,q}$ denotes the Besov space (see section 2). $U(t)=\exp(it\partial^2/2)$, $M(t)=\exp(ix^2/2t)$, $[D(t)\phi](x)=(it)^{-1/2}\phi(x/t)$.

2. Preliminaries

We first summarize basic properties of the Besov space (for more details, see [2]). Let $s>0$ and $1\le r,q\le\infty$. We put

$$\|u\|_{\dot{B}^s_{r,q}}=\Big(\int_0^{\infty}t^{q(N-s)-1}\sup_{|h|\le t}\|\partial^N\delta_h^m u\|_r^q\,dt\Big)^{1/q}$$

if $q<\infty$ and put

$$\|u\|_{\dot{B}^s_{r,\infty}}=\sup_{t\in(0,\infty)}\big(t^{N-s}\sup_{|h|\le t}\|\partial^N\delta_h^m u\|_r\big).$$

Here $\delta_h^m u=\sum_{j=0}^m\binom{m}{j}(-1)^{m-j}u(\cdot+jh)$, and N,m are nonnegative integers satisfying $N<s<N+m$. The space $B^s_{r,q}=B^s_{r,q}(\mathbb{R})$ is the Banach space equipped with the norm

$$\|u\|_{B^s_{r,q}}=\|u\|_r+\|u\|_{\dot{B}^s_{r,q}}.$$

Let $0<\theta<1$ and $1/r=\theta/l_1+(1-\theta)/l_2$. Then a special case of the Gagliardo-Nirenberg type interpolation inequality $\|u\|_{\dot{B}^{\theta s}_{r,q/\theta}}\le$

$C\|u\|^{\theta}_{\dot{B}^s_{l_1,q}}\|u\|^{1-\theta}_{l_2}$ easily follows from the definition above. Using this inequality and the identity $\delta_h^m(uv) = \sum_{j=0}^m \binom{m}{j}\delta_h^j u(\cdot + jh)\, \delta_h^{m-j} v$, we have the Leibniz-rule-like inequality

$$\|uv\|_{\dot{B}^s_{r,q}} \le C\|u\|_{\dot{B}^s_{\rho_1,q}}\|v\|_{\rho_2} + C\|u\|_{\rho_3}\|v\|_{\dot{B}^s_{\rho_4,q}} \tag{2.1}$$

with $1/r = 1/\rho_1 + 1/\rho_2 = 1/\rho_3 + 1/\rho_4$.

It is well-known that $B^s_{2,2} = H^s$ as Banach spaces. Besov spaces are related to Sobolev spaces by the real interpolation as $(H^{s,r}, H^{s',r})_{\theta,q} = B^{(1-\theta)s+\theta s'}_{r,q}$, where $0 < \theta < 1$ and $s \neq s'$. We have continuous inclusions $B^s_{r,q} \subset B^{s'}_{r',q'}$ if $q \le q'$, $r \le r'$ and $1/r - s \le 1/r' - s'$; $B^s_{r,q} \subset H^{s',r'}$ if $r \le r'$ and $1/r - s < 1/r' - s'$. The inequality

$$\|\partial u\|_{B^s_{r,q}} \le C\|u\|_{B^{s+1}_{r,q}} \tag{2.2}$$

follows from the latter inclusion and the definition of Besov spaces.

We proceed to estimates of the $B^s_{r,q}$-norm of composite functions.

Definition 2.1. Let $p \ge 1$. We say that a function $g : \mathbb{C} \to \mathbb{C}$ belongs to the class $A(p)$ if $g \in C^m(\mathbb{R}^2, \mathbb{R}^2)$ for any nonnegative integer $m < p$, if $g(0) = g'(0) = \cdots = g^{(m)}(0) = 0$, and if

$$|g^{(m)}(z_1) - g^{(m)}(z_2)| \le C \begin{cases} (|z_1| + |z_2|)^{p-m-1}|z_1 - z_2|, & m < p-1, \\ |z_1 - z_2|^{p-m}, & p-1 \le m < p. \end{cases}$$

Remark 2.1. (i) $g(z) = |z|^{p-1}z \in A(p)$ for $p \ge 1$; (ii) if $g(z) \in A(p)$ with $p \ge 2$, then $g(z)/z \in A(p-1)$.

Lemma 2.1. *Let $p \ge 1$, $g \in A(p)$, $0 < s < p$, and $1 \le r, q \le \infty$. Then*

$$\|g(u)\|_{B^s_{r,q}} \le C\|u\|^{p-1}_{\rho_1}\|u\|_{B^s_{\rho_2,q}} \tag{2.3}$$

with $1 \le \rho_1, \rho_2 \le \infty$ such that $1/r = (p-1)/\rho_1 + 1/\rho_2$. Moreover, let $p \ge 2$ and $0 < s < p-1$. Then

$$\begin{aligned} \|g(u) - g(v)\|_{B^s_{r,q}} &\le C(\|u\|_{\rho_1} + \|v\|_{\rho_1})^{p-1}\|u - v\|_{B^s_{\rho_2,q}} \\ &\quad + C(\|u\|_{\rho_3} + \|v\|_{\rho_3})^{p-2}(\|u\|_{B^s_{\rho_4,q}} + \|v\|_{B^s_{\rho_4,q}})\|u - v\|_{\rho_5} \end{aligned} \tag{2.4}$$

with $1 \le \rho_3, \rho_4, \rho_5 \le \infty$ *such that* $1/r = (p-2)/\rho_3 + 1/\rho_4 + 1/\rho_5$. *Furthermore, let* k *be a positive integer. Then in case* $g(z) = |z|^{2k}z$ $(p = 2k+1)$ *or* $g(z) = |z|^{2k}$ $(p = 2k)$, *the inequalities* (2.3), (2.4) *hold valid without the restriction* $s < p$.

Proof. In case $g(z) = |z|^{2k}z$ or $g(z) = |z|^{2k}$, we can prove (2.3) directly from (2.1) and Hölder's inequality. For the proof of (2.3) for general $g \in A(p)$, see [6, 10]. So we prove only the inequality (2.4). Since

$$g(u) - g(v) = \int_0^1 g'(\theta u + (1-\theta)v)(u-v)d\theta$$

and since $g' \in A(p-1)$, we have

$$\begin{aligned}\|g(u)-g(v)\|_{B^s_{r,q}} \le C\int_0^1 &\big\{\|\theta u + (1-\theta)v\|_{\rho_1}^{p-1}\|u-v\|_{B^s_{\rho_2,q}} \\ &+ \|\theta u + (1-\theta)v\|_{\rho_3}^{p-2}\|\theta u + (1-\theta)v\|_{B^s_{\rho_4,q}}\|u-v\|_{\rho_5}\big\}d\theta.\end{aligned}$$

Here we have used the inequalities (2.1) and (2.3). Thus we obtain (2.4). □

In the proof of the main theorem, functions of type $\exp(i\phi)$ repeatedly appear. To treat these factors, we need the following lemma.

Lemma 2.2. *Let* $1 \le r, q \le \infty$, $s > 0$, *and let* ϕ *be a real-valued function. Then*

$$\|\exp(i\phi)\|_{\dot{B}^s_{r,q}} \le C(1 + \|\phi\|_\infty)^{[s]}\|\phi\|_{\dot{B}^s_{r,q}}, \tag{2.5}$$

where $[s]$ *means the integral part of* s.

Proof. See [24]. □

From this lemma and the inequality (2.1), we obtain the following inequality:

$$\|\exp(i\phi)u\|_{B^s_{r,q}} \le C(1 + \|\phi\|_\infty)^{[s]}\|\phi\|_{\dot{B}^s_{\rho_1,q}}\|u\|_{\rho_2} + C\|u\|_{B^s_{r,q}} \tag{2.6}$$

with $1/r = 1/\rho_1 + 1/\rho_2$.

Lemma 2.3. *Let* $s > 0$, $1 < r < \infty$ *and* $0 \le a \le 1$. *Then for any* $t \ne 0$ *we have the estimates*

$$\|\mathcal{F}\big(M(t)-1\big)\mathcal{F}^{-1}f\|_{B^s_{r,2}} \le C|t|^{-1}\|f\|_{B^{s+2}_{r,2}} \tag{2.7}$$

and

$$\|\mathcal{F}\big(M(t)-1-(i\xi^2/2t)\big)\mathcal{F}^{-1}f\|_{B^s_{r,2}} \le C|t|^{-1-a}\|f\|_{B^{s+2+2a}_{r,2}}. \tag{2.8}$$

Moreover, if $r=2$, (*and therefore* $B^s_{2,2}=H^s$) *we have*

$$\|\mathcal{F}\big(M(t)-1\big)\mathcal{F}^{-1}f\|_{H^s} \le C|t|^{-a}\|f\|_{H^{s+2a}} \tag{2.9}$$

for any $0\le a\le 1$.

Proof. By Mihlin's multiplier theorem (see [2]), the Fourier multiplier with symbol

$$t(1+\xi^2)^{-1}\big(M(t,\xi)-1\big)$$

is uniformly bounded on L^r. So we have

$$\|\mathcal{F}\big(M(t)-1\big)\mathcal{F}^{-1}f\|_{H^{s,r}} \le C|t|^{-1}\|f\|_{H^{s+2,r}}.$$

Thus the real interpolation method implies (2.7). The estimate (2.8) is proved similarly. The estimate (2.9) is clear. □

On the well-posedness of the equation (1.1), the following is known. Here we state the result in more general form than that we need in later section.

Proposition 2.4. *Let* $1\le p_1<p_2<5$ *and* $f(u)=f_1(u)+f_2(u)$ *with* $f_j\in A(p_j)$, $j=1,2$. *Let* f_j *be gauge invariant, namely* $f_j(e^{i\theta}u)=e^{i\theta}f_j(u)$. *Then the equation* (1.1) *has a unique solution in*

$$C(\mathbb{R};L^2)\cap\Big\{\bigcap_{j=1}^{2}L^{4(p_j+1)/(p_j-1)}_{loc}(\mathbb{R};L^{p_j+1})\Big\}$$

for any $u_0\in L^2$. *Moreover,*

(i) $u\in L^q_{loc}(\mathbb{R};L^r)$ *for any* (q,r) *satisfying* $0<2/q=1/2-1/r\le 1/2$;
(ii) *if* $u_0\in L^{2,s}$, $0<s<p_1$, *then* $U^{-1}u\in C(\mathbb{R};L^{2,s})$;
(iii) *if* $\|u_0\|_{L^{2,s}}$ *is sufficiently small, then* $\sup_{|t|\le 1}\|U(-t)u(t)\|_{L^{2,s}}\le 2\|u_0\|_{L^{2,s}}$;
(iv) $\|u(t)\|_2=\|u_0\|_2$ *for all* $t\in\mathbb{R}$.

Proof. See [3, 6, 7, 13, 14, 22] □

Remark 2.2. The uniqueness of the solution to (1.1) belonging to $C(\mathbb{R}; L^2)$ does not follows from the proposition above, unless assuming $u \in \cap_{j=1}^{2} L_{loc}^{4(p_j+1)/(p_j-1)}(\mathbb{R}; L^{p_j+1})$. So we say that the Cauchy problem (1.1) is "conditionally" well-posed in L^2. On the other hand, it is well-known that (1.1) is (unconditionally) well-posed in H^1.

3. Proof of the Theorem

This section is devoted to the proof of the main theorem in this paper. Throughout this section, we assume $p_1 = 3$. So in what follows, we abbreviate p_2 to p for simplicity. In this section we treat the case $t \geq 1$, since we consider the asymptotic behavior of the solution as $t \to \infty$. First we introduce a decay estimate of the small solution to (1.1) obtained by Hayashi-Naumkin [11].

Proposition 3.1. *Let $u_0 \in L^{2,s}$ with $s > 1/2$ and let $\|u_0\|_{L^{2,s}}$ be sufficiently small. Then the solution $u(t)$ to (1.1) satisfies*

$$\sup_{t\geq 1} t^{1/2}\|u(t)\|_\infty \leq C\|u_0\|_{L^{2,s}}. \tag{3.1}$$

The main theorem follows from the proposition below.

Proposition 3.2. *Let $5/2 < s < p$, $2 < \sigma < \min(s-1/2; p-1)$ and let $\|u_0\|_{L^{2,s}}$ be sufficiently small. Let $u(t)$ be the solution to (1.1) and put*

$$\Phi(t) = \int_1^t \left\{\lambda_1 \tau^{-1}|\mathcal{F}U(-\tau)u(\tau)|^2 + \lambda_2 \tau^{-(p-1)/2}|\mathcal{F}U(-\tau)u(\tau)|^{p-1}\right\}d\tau.$$

Then there exist a complex-valued function ϕ and a real-valued function Φ_∞ belonging to $\mathcal{F}^{-1}B^\sigma_{r,2}$ for any $2 \leq r < \infty$, and satisfying the estimates below:

$$\|\Phi(t) - \Phi_\infty - \lambda_1|\hat{\phi}|^2 \log t\|_{B^\sigma_{r,2}} \leq Ct^{-\varepsilon}, \tag{3.2}$$

$$\|\exp(i\Phi(t) - i\Phi_\infty)\mathcal{F}U(-t)u(t) - \hat{\phi}\|_{B^\sigma_{r,2}} \leq Ct^{-\varepsilon}, \tag{3.3}$$

$$\|\mathcal{F}U(-t)u(t) - \exp(-i\lambda_1|\hat{\phi}|^2 \log t)\hat{\phi}\|_{B^\sigma_{r,2}} \leq Ct^{-\varepsilon}, \tag{3.4}$$

and

$$\| \exp(i\Phi(t) - i\Phi_\infty)\mathcal{F}U(-t)u(t) - \hat{\phi} - t^{-1}\sum_{j=0}^{1}(\log t)^j\hat{\phi}_{1,j}\|_{B^{\sigma-2}_{r,2}} \le Ct^{-1-\varepsilon} \tag{3.5}$$

Here ε *is some positive number and* $\phi_{1,j}$ $(j = 0, 1)$ *are defined by* (1.7) *and* (1.8).

Remark 3.1. (i) The size of $\|u_0\|_{L^{2,s}}$ depends on p, λ_1, λ_2, s and σ. (ii) If we take r in Proposition 3.2 sufficiently large so that $B^\sigma_{r,2} \subset H^{2,\infty}$, then we find $\hat{\phi} \in H^{2,\infty}$. On the other hand, if we take $r = 2$, then we find $\hat{\phi} \in H^2$. Hence $\phi \in L^{2,2} \cap \mathcal{F}^{-1}H^{2,\infty}$ and $\phi_{1,j} \in \mathcal{F}^{-1}(L^2 \cap L^\infty)$, $j = 0, 1$. (iii) In order to prove only (3.2)-(3.4), we can relax the assumption for s as $1/2 < s < p$. The assumption $s > 5/2$ is used to prove (3.5).

Proof. We first sketch how we formally obtain the expansion formulae in the proposition above. We note that the free propagator is decomposed as $U(t) = M(t)D(t)\mathcal{F}M(t)$, and hence $\mathcal{F}M(t)U(-t) = D^{-1}(t)M^{-1}(t)$. Put $v(t) = U(-t)u(t)$. Then by the decomposition of $U(t)$ above, $\hat{v}$ satisfies

$$i\hat{v}_t = \mathcal{F}M(-t)\mathcal{F}^{-1}\big\{t^{-1}f_1(\widehat{Mv}(t)) + t^{-(p-1)/2}f_2(\widehat{Mv}(t))\big\}. \tag{3.6}$$

By the expansion $\mathcal{F}M(\pm t)\mathcal{F}^{-1} \simeq 1 \mp i\partial^2/2t$ and the Taylor expansions of f_1 and f_2, the right-hand side of (3.6) is expanded as

$$\begin{aligned}(1 + i\partial^2/2t)&\big\{t^{-1}f_1(\hat{v} - (i/2t)\partial^2\hat{v}) + t^{-(p-1)/2}f_2(\hat{v} - (i/2t)\partial^2\hat{v})\big\}\\ &\simeq t^{-1}f_1(\hat{v}) + t^{-(p-1)/2}f_2(\hat{v}) + (1/2t^2)\big\{i\partial^2 f_1(\hat{v}) - f_1'(\hat{v})i\partial^2\hat{v}\big\}.\end{aligned}$$

Therefore, by transposing $t^{-1}f_1(\hat{v}) + t^{-(p-1)/2}f_2(\hat{v})$ into the left-hand side of (3.6) and multiplying $e^{i\Phi}$, we have

$$i(e^{i\Phi}\hat{v})_t \simeq (1/2t^2)e^{i\Phi}\big\{i\partial^2 f_1(\hat{v}) - f_1'(\hat{v})i\partial^2\hat{v}\big\}. \tag{3.7}$$

Since the right-hand side is time integrable on $[1, \infty)$, we obtain (3.3). We substitute this profile of $\hat{v}$ into the definition of Φ and get (3.2). The estimate (3.4) immediately follows from (3.2) and (3.3). Substituting the profiles of $\hat{v}$ and Φ into the right-hand side of (3.7) and integrating with respect to t, we obtain the sharp asymptotics (3.5).

In our proof, $P(\eta)$ denotes various polynomials of η. We use this notation as follows: if we write $f(t) \le P(\log t)$, this inequality implies that there exists some polynomial of $\log t$ dominating $f(t)$. We let $a = \min\bigl(s - \sigma - 1/2; p - 3; 1\bigr)/2$ and $0 < \nu < a/2$. We proceed in several steps.

Step 1. We prove the estimates of $\hat{v}(t)$, $\widehat{Mv}(t)$ and $\widehat{Mv}(t) - \hat{v}(t)$ in several function spaces. Let $\|u_0\|_{L^{2,s}}$ be sufficiently small so that Proposition 3.1 holds. Then

$$\|\widehat{Mv}\|_\infty = \|D^{-1}M^{-1}u\|_\infty \le C\|u_0\|_{L^{2,s}}. \tag{3.8}$$

We estimate $\|\hat{v}(t)\|_{H^s} = \|\widehat{Mv}(t)\|_{H^s}$. By Lemma 2.1 and (3.8), the H^s-norm of the right-hand side of (3.6) is dominated by

$$\begin{aligned} &Ct^{-1}\|f_1(\widehat{Mv})\|_{H^s} + Ct^{-(p-1)/2}\|f_2(\widehat{Mv})\|_{H^s} \\ &\quad \le C\bigl\{t^{-1}\|\widehat{Mv}\|_\infty^2 + t^{-(p-1)/2}\|\widehat{Mv}\|_\infty^{p-1}\bigr\}\|\hat{v}\|_{H^s} \le Ct^{-1}\|u_0\|_{L^{2,s}}^2\|\hat{v}\|_{H^s}. \end{aligned}$$

Hence by the equation (3.6) and the estimates above,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{v}(t)\|_{H^s}^2 &= \operatorname{Im}\bigl(\mathcal{F}M(-t)\mathcal{F}^{-1}(t^{-1}f_1(\widehat{Mv}) + t^{-(p-1)/2}f_2(\widehat{Mv})), \hat{v}\bigr)_{H^s} \\ &\le Ct^{-1}\|u_0\|_{L^{2,s}}^2\|\hat{v}(t)\|_{H^s}^2. \end{aligned}$$

If $C\|u_0\|_{L^{2,s}}^2 \le \nu$, then Gronwall's inequality yields

$$\|\hat{v}(t)\|_{H^s} \le Ct^\nu. \tag{3.9}$$

Combining Lemma 2.3, estimates (3.8), (3.9) and Sobolev's inequality, we have

$$\|\widehat{Mv}(t) - \hat{v}(t)\|_{H^{s-2a}} \le Ct^{-a}\|\hat{v}(t)\|_{H^s} \le Ct^{-a+\nu}, \tag{3.10}$$

$$\|\widehat{Mv}(t) - \hat{v}(t)\|_\infty \le \|\widehat{Mv}(t) - \hat{v}(t)\|_{H^{s-2}} \le Ct^{-1}\|\hat{v}(t)\|_{H^s} \le Ct^{-1+\nu}, \tag{3.11}$$

and

$$\|\hat{v}(t)\|_\infty \le \|\widehat{Mv}(t)\|_\infty + \|\widehat{Mv}(t) - \hat{v}(t)\|_\infty \le C + Ct^{-1+\nu} \le C. \tag{3.12}$$

Step 2. In this step, we prove (3.2)-(3.4). We rewrite the equation (3.6) as

$$i\hat{v}_t = I_1 + I_2 + t^{-1} f_1(\hat{v}) + t^{-(p-1)/2} f_2(\hat{v}), \tag{3.13}$$

where

$$\begin{aligned} I_1 &= \mathcal{F}\big(M(-t) - 1\big)\mathcal{F}^{-1}\big\{t^{-1} f_1(\widehat{Mv}) + t^{-(p-1)/2} f_2(\widehat{Mv})\big\}, \\ I_2 &= t^{-1}\big(f_1(\widehat{Mv}) - f_1(\hat{v})\big) + t^{-(p-1)/2}\big(f_2(\widehat{Mv}) - f_2(\hat{v})\big). \end{aligned}$$

By multiplying $e^{i\Phi}$ to the both sides of (3.13), we have

$$i\partial_t(e^{i\Phi}\hat{v}) = e^{i\Phi}(I_1 + I_2). \tag{3.14}$$

Once we prove that the $B^\sigma_{r,2}$-norm of the right-hand side of (3.14) is integrable in time as $t \to \infty$, it follows that $\text{s-}\lim_{t\to\infty} e^{i\Phi}\hat{v}$ exists in $B^\sigma_{r,2}$. By Lemma 2.1 and the estimates obtained in Step 1,

$$\|\Phi\|_\infty \le C(1 + \log t),$$

and

$$\|\Phi\|_{B^\sigma_{r,2}} \le C \int_1^t \big(\tau^{-1}\|\hat{v}(\tau)\|_\infty \|\hat{v}(\tau)\|_{B^\sigma_{r,2}} + \tau^{-(p-1)/2}\|\hat{v}(\tau)\|_\infty^{p-2}\|\hat{v}(\tau)\|_{B^\sigma_{r,2}}\big) d\tau \le C\, t^\nu.$$

Note that, to obtain the above inequality, we used $H^{s-2a} = B^{s-2a}_{2,2} \subset B^\sigma_{r,2}$ with $1/2 - s + 2a \le 1/r - \sigma$. Therefore Lemma 2.2 yields

$$\|\exp(i\Phi)\|_{\dot{B}^\sigma_{r,2}} \le t^\nu P(\log t).$$

Hence by Lemmas 2.1 and 2.3,

$$\begin{aligned} \|I_1\|_{B^\sigma_{r,2}} &\le C\|I_1\|_{H^{s-2a}} \le Ct^{-a}\|t^{-1} f_1(\widehat{Mv}) + t^{-(p-1)/2} f_2(\widehat{Mv})\|_{H^s} \\ &\le Ct^{-a}\big(t^{-1}\|\widehat{Mv}\|_\infty^2 \|\widehat{Mv}\|_{H^s} + t^{-(p-1)/2}\|\widehat{Mv}\|_\infty^{p-1}\|\widehat{Mv}\|_{H^s}\big) \\ &\le Ct^{-1-a+\nu}. \end{aligned}$$

Similarly, we can show that

$$\begin{aligned}\|I_2\|_{B^\sigma_{r,2}} &\le Ct^{-1}\{(\|\widehat{Mv}\|_\infty + \|\hat v\|_\infty)^2\|\widehat{Mv} - \hat v\|_{H^{s-2a}} \\ &\quad + C(\|\widehat{Mv}\|_\infty + \|\hat v\|_\infty)\|\hat v\|_{H^{s-2a}}\|\widehat{Mv} - \hat v\|_\infty\} \\ &\quad + Ct^{-(p-1)/2}(\|\widehat{Mv}\|_\infty + \|\hat v\|_\infty)^{p-1}\|\hat v\|_{H^s} \\ &\le (Ct^{-1-a+\nu} + Ct^{-2+2\nu} + Ct^{-(p-1)/2+\nu}) \le Ct^{-1-a+\nu}.\end{aligned}$$

Collecting these estimates and using (2.6) together with the boundedness of embedding $B^\sigma_{r,2} \subset L^\infty$, we get

$$\|e^{i\Phi}(I_1 + I_2)\|_{B^\sigma_{r,2}} \le t^{-1-a+2\nu}P(\log t).$$

Hence, there exists a function $W \in B^\sigma_{r,2}$ satisfying

$$\|e^{i\Phi}\hat v(t) - W\|_{B^\sigma_{r,2}} \le t^{-a+2\nu}P(\log t). \tag{3.15}$$

This also yields the boundedness of $\|e^{i\Phi}\hat v(t)\|_{B^\sigma_{r,2}}$. By (3.15), we can refine the estimate of $\|\Phi\|_{B^\sigma_{r,2}}$ and get

$$\begin{aligned}\|\Phi\|_{B^\sigma_{r,2}} &\le C\int_1^t \left(\tau^{-1}\|\hat v(\tau)\|_\infty \|e^{i\Phi}\hat v(\tau)\|_{B^\sigma_{r,2}} + \tau^{-(p-1)/2}\|\hat v(\tau)\|_\infty^{p-2}\|e^{i\Phi}\hat v(\tau)\|_{B^\sigma_{r,2}}\right)d\tau \\ &\le C(1 + \log t).\end{aligned}$$

By (2.6), (3.10) and Lemma 2.2, we also get

$$\begin{aligned}\|\hat v\|_{B^\sigma_{r,2}} &\le C\|e^{-i\Phi}\|_{\dot B^\sigma_{r,2}}\|e^{i\Phi}\hat v\|_\infty + C\|e^{i\Phi}\hat v\|_{B^\sigma_{r,2}} \le P(\log t), \\ \|\widehat{Mv}\|_{B^\sigma_{r,2}} &\le C\|\widehat{Mv} - \hat v\|_{B^\sigma_{r,2}} + \|\hat v\|_{B^\sigma_{r,2}} \le P(\log t).\end{aligned}$$

We put $S(t,\xi) = \lambda_1|W(\xi)|^2\log t$. Then, there exists a function $\Phi_\infty =$ s-$\lim_{t\to\infty}(\Phi(t) - S(t))$ in $B^\sigma_{r,2}$, since we can show that

$$\begin{aligned}\int_t^\infty \tau^{-1}\big\||e^{i\Phi}\hat v|^2 - |W|^2\big\|_{B^\sigma_{r,2}}d\tau &\le C\int_t^\infty \tau^{-1}\{\|e^{i\Phi}\hat v\|_{B^\sigma_{r,2}} + \|W\|_{B^\sigma_{r,2}}\}\|e^{i\Phi}\hat v - W\|_{B^\sigma_{r,2}}d\tau \\ &\le t^{-a+2\nu}P(\log t)\end{aligned}$$

by (2.1) and the embedding $B^\sigma_{r,2} \subset L^\infty$. We put $\phi = \mathcal{F}^{-1}(e^{i\Phi_\infty}W)$, which is the function as in Proposition 3.2. We remark that $S(t,\xi) = \lambda_1|\hat\phi(\xi)|^2\log t$ since $|\hat\phi| = |W|$. Clearly we have

$$\|e^{i\Psi}\hat v - \hat\phi\|_{B^\sigma_{r,2}} \le t^{-a+2\nu}P(\log t),$$

and

$$\|\Psi(t)-S(t)\|_{B^{\sigma}_{r,2}}\le t^{-a+2\nu}P(\log t),$$

where we put $\Psi(t)=\Phi(t)-\Phi_{\infty}$. Thus we obtain (3.2) and (3.3). Again by (2.1) and (2.6), we have

$$\begin{aligned}&\|\hat{v}-e^{-iS}\hat{\phi}\|_{B^{\sigma}_{r,2}}\\ \le\ & C(1+\|e^{-i\Psi}\|_{\dot{B}^{\sigma}_{r,2}})\big\{\|e^{i\Psi}\hat{v}-\hat{\phi}\|_{B^{\sigma}_{r,2}}+(1+\|\Psi-S\|_{\infty})^{[\sigma]}\|\Psi-S\|_{B^{\sigma}_{r,2}}\|\hat{\phi}\|_{B^{\sigma}_{r,2}}\big\}\\ \le\ & t^{-a+2\nu}P(\log t),\end{aligned}$$

which means (3.4). We can also show that

$$\|\widehat{Mv}-e^{-iS}\hat{\phi}\|_{B^{\sigma}_{r,2}}\le\|\widehat{Mv}-\hat{v}\|_{B^{\sigma}_{r,2}}+\|\hat{v}-e^{-iS}\hat{\phi}\|_{B^{\sigma}_{r,2}}\le t^{-a+2\nu}P(\log t).$$

Step 3. In this step we prove (3.5). We put

$$\begin{aligned}&J_1=(i/2t^2)\partial^2 f_1(e^{-iS}\hat{\phi}),\\ &J_2=-(i\lambda_1/2t^2)\big\{2|\hat{\phi}|^2\partial^2(e^{-iS}\hat{\phi})-(e^{-iS}\hat{\phi})^2\partial^2(e^{iS}\bar{\hat{\phi}})\big\},\end{aligned}$$

and estimate the differences I_k-J_k $(k=1,2)$ in $B^{\sigma-2}_{r,2}$. We start with the estimate of I_1-J_1 by writing

$$\begin{aligned}I_1-J_1&=t^{-1}\mathcal{F}\big(M(-t)-1+(i\xi^2/2t)\big)\mathcal{F}^{-1}f_1(\widehat{Mv})\\ &\quad+(i/2t^2)\partial^2\big\{f_1(\widehat{Mv})-f_1(e^{-iS}\hat{\phi})\big\}\\ &\quad+t^{-(p-1)/2}\mathcal{F}\big(M(-t)-1\big)\mathcal{F}^{-1}f_2(\widehat{Mv})\\ &=\mathrm{I}_1+\mathrm{II}_1+\mathrm{III}_1.\end{aligned}$$

The estimates of $\mathrm{I}_1,\mathrm{II}_1$ and III_1 can be achieved as follows. By Lemmas 2.1, 2.3 and the embedding $H^s\subset B^{\sigma+2a}_{r,2}$,

$$\|\mathrm{I}_1\|_{B^{\sigma-2}_{r,2}}\le Ct^{-2-a}\|f_1(\widehat{Mv})\|_{B^{\sigma+2a}_{r,2}}\le Ct^{-2-a}\|\widehat{Mv}\|^2_{\infty}\|\widehat{Mv}\|_{H^s}\le Ct^{-2-a+\nu},$$

$$
\begin{aligned}
\|\mathrm{II}_1\|_{B^{\sigma-2}_{r,2}} &\le Ct^{-2}\|f_1(\widehat{Mv}) - f_1(e^{-iS}\hat{\phi})\|_{B^{\sigma}_{r,2}} \\
&\le Ct^{-2}(\|\widehat{Mv}\|_\infty + \|\hat{\phi}\|_\infty)^2\|\widehat{Mv} - e^{-iS}\hat{\phi}\|_{B^{\sigma}_{r,2}} \\
&\quad + Ct^{-2}(\|\widehat{Mv}\|_\infty + \|\hat{\phi}\|_\infty)(\|\widehat{Mv}\|_{B^{\sigma}_{r,2}} + \|e^{-iS}\hat{\phi}\|_{B^{\sigma}_{r,2}})\|\widehat{Mv} - e^{-iS}\hat{\phi}\|_\infty \\
&\le Ct^{-2-a+2\nu}P(\log t),
\end{aligned}
$$

$$
\begin{aligned}
\|\mathrm{III}_1\|_{B^{\sigma-2}_{r,2}} &\le Ct^{-(p+1)/2}\|f_2(\widehat{Mv})\|_{B^{\sigma}_{r,2}} \le Ct^{-(p+1)/2}\|\widehat{Mv}\|_\infty^{(p-1)}\|\widehat{Mv}\|_{B^{\sigma}_{r,2}} \\
&\le t^{-(p+1)/2}P(\log t).
\end{aligned}
$$

Collecting these estimates, we get

$$
\|I_1 - J_1\|_{B^{\sigma-2}_{r,2}} \le Ct^{-2-a+2\nu}P(\log t).
$$

We proceed with the estimate of $I_2 - J_2$. We write

$$
\begin{aligned}
I_2 - J_2 &= t^{-1}\big\{f_1(\widehat{Mv}) - f_1(\hat{v} - (i/2t)\partial^2\hat{v})\big\} \\
&\quad + t^{-1}\Big[f_1(\hat{v} - (i/2t)\partial^2\hat{v}) - f_1(\hat{v}) - \lambda_1\big\{(-i/t)|\hat{v}|^2\partial^2\hat{v} + (i/2t)\hat{v}^2\partial^2\bar{\hat{v}}\big\}\Big] \\
&\quad + (-i\lambda_1/2t^2)\big\{2|\hat{v}|^2\partial^2\hat{v} - \hat{v}^2\partial^2\bar{\hat{v}} - 2|\hat{\phi}|^2\partial^2(e^{-iS}\hat{\phi}) + (e^{-iS}\hat{\phi})^2\partial^2(e^{iS}\bar{\hat{\phi}})\big\} \\
&\quad + t^{-(p-1)/2}\big\{f_2(\widehat{Mv}) - f_2(\hat{v})\big\} \\
&= \mathrm{I}_2 + \mathrm{II}_2 + \mathrm{III}_2 + \mathrm{IV}_2. \qquad (3.16)
\end{aligned}
$$

We estimate the right-hand side of (3.16) term by term. To control I_2, we need the following estimates

$$
\|\widehat{Mv} - \hat{v} - (i/2t)\partial^2\hat{v}\|_{B^{\sigma-2}_{r,2}} \le Ct^{-1-a}\|\hat{v}\|_{H^s} \le Ct^{-1-a+\nu}, \qquad (3.17)
$$

$$
\|\widehat{Mv} - \hat{v} - (i/2t)\partial^2\hat{v}\|_\infty \le Ct^{-1-a}\|\hat{v}\|_{H^s} \le Ct^{-1-a+\nu}. \qquad (3.18)
$$

Here we have used (2.2) and Lemma 2.3 together with the Sobolev inequality $\|\partial^2\hat{v}\|_\infty \le C\|\hat{v}\|_{H^s}$. By these estimates and Lemma 2.1,

$$
\begin{aligned}
\|\mathrm{I}_2\|_{B^{\sigma-2}_{r,2}} &\le Ct^{-1}(\|\widehat{Mv}\|_\infty + \|\hat{v}\|_\infty + t^{-1}\|\partial^2\hat{v}\|_\infty)^2\|\widehat{Mv} - \hat{v} - (i/2t)\partial^2\hat{v}\|_{B^{\sigma-2}_{r,2}} \\
&\quad + Ct^{-1}(\|\widehat{Mv}\|_\infty + \|\hat{v}\|_\infty + t^{-1}\|\partial^2\hat{v}\|_\infty) \\
&\qquad \times (\|\widehat{Mv}\|_{B^{\sigma-2}_{r,2}} + \|\hat{v}\|_{B^{\sigma-2}_{r,2}} + t^{-1}\|\partial^2\hat{v}\|_{B^{\sigma-2}_{r,2}})\|\widehat{Mv} - \hat{v} - (i/2t)\partial^2\hat{v}\|_\infty \\
&\le t^{-2-a+\nu}P(\log t).
\end{aligned}
$$

By direct calculation and the estimates (2.1), (2.2),

$$\begin{aligned}\|\mathrm{II}_2\|_{B^{\sigma-2}_{r,2}} &\le Ct^{-3}(\|\hat v\|_\infty\|\partial^2\hat v\|_\infty\|\partial^2\hat v\|_{B^{\sigma-2}_{r,2}} + \|\hat v\|_{B^{\sigma-2}_{r,2}}\|\partial^2\hat v\|^2_\infty)\\ &\quad + Ct^{-4}\|\partial^2\hat v\|^2_\infty\|\partial^2\hat v\|_{B^{\sigma-2}_{r,2}}\\ &\le t^{-3+2\nu}P(\log t) + t^{-4+2\nu}P(\log t) \le t^{-3+2\nu}P(\log t).\end{aligned}$$

To estimate III_2, we remark that the inequality

$$\|w_1w_2\partial^2 w_3\|_{B^{\sigma-2}_{r,2}} \le C\prod_{j=1}^{3}\|w_j\|_{B^\sigma_{r,2}}$$

holds for any $w_1, w_2, w_3 \in B^\sigma_{r,2}$ by virtue of (2.1), (2.2) and the embedding $B^\sigma_{r,2} \subset L^\infty \cap H^{2,r}$. Applying this inequality to III_2, we have

$$\|\mathrm{III}_2\|_{B^{\sigma-2}_{r,2}} \le Ct^{-2}(\|\hat v\|_{B^\sigma_{r,2}} + \|e^{-iS}\hat\phi\|_{B^\sigma_{r,2}})^2\|\hat v - e^{-iS}\hat\phi\|_{B^\sigma_{r,2}} \le t^{-2-a+2\nu}P(\log t).$$

By Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\|\mathrm{IV}_2\|_{B^{\sigma-2}_{r,2}} &\le Ct^{-(p-1)/2}(\|\widehat{Mv}\|_\infty + \|\hat v\|_\infty)^{p-2}(\|\widehat{Mv}\|_{B^\sigma_{r,2}} + \|\hat v\|_{B^\sigma_{r,2}})\|\widehat{Mv} - \hat v\|_{B^{\sigma-2}_{r,2}}\\ &\le t^{-(p-1)/2}P(\log t)t^{-1}\|\hat v\|_{B^\sigma_{r,2}}\\ &\le t^{-(p+1)/2}P(\log t).\end{aligned}$$

Thus we obtain

$$\|I_2 - J_2\|_{B^{\sigma-2}_{r,2}} \le t^{-2-a+2\nu}P(\log t).$$

Therefore

$$\begin{aligned}&\|e^{i\Psi}(I_1+I_2) - e^{iS}(J_1+J_2)\|_{B^{\sigma-2}_{r,2}}\\ &\quad\le C(1+\|e^{i\Psi}\|_{\dot B^{\sigma-2}_{\infty,2}})\{\|I_1+I_2-J_1-J_2\|_{B^{\sigma-2}_{r,2}} + \|(e^{i(S-\Psi)}-1)(J_1+J_2)\|_{B^{\sigma-2}_{r,2}}\}\\ &\quad\le t^{-2-a+2\nu}P(\log t).\end{aligned}$$

Thus, integrating (3.14) with respect to t and using the estimate proved above, we conclude

$$\|e^{i\Psi}\hat v - \hat\phi - i\int_t^\infty e^{iS}(J_1+J_2)d\tau\|_{B^{\sigma-2}_{r,2}} \le C\,t^{-1-\varepsilon} \tag{3.19}$$

for some $\varepsilon > 0$. By directly calculating the integral in the left-hand side of (3.19), we obtain Proposition 3.2. □

Proof of Theorem. We recall that $v(t) = U(-t)u(t)$ and $\Psi = \Phi - \Phi_\infty$ as in the proof of Proposition 3.2. We also use the abbreviation $v_\infty(t) = \phi + t^{-1}\sum_{j=0}^{1}(\log t)^j \phi_{1,j}$. By Proposition 3.2, we obtain

$$t^{-1}|\hat{v}(t)|^2 = t^{-1}\big|\hat{v}_\infty(t) + O(t^{-1-\varepsilon})\big|^2 = t^{-1}|\hat{\phi}|^2 + t^{-2}(\hat{\phi}\bar{\hat{\phi}}_{1,0} + \bar{\hat{\phi}}\hat{\phi}_{1,0}) + O(t^{-2-\varepsilon})$$

and

$$t^{-(p-1)/2}|\hat{v}(t)|^{p-1} = t^{-(p-1)/2}|\hat{\phi}|^{p-1} + O(t^{-2-\varepsilon})$$

in $B^{\sigma-2}_{r,2} \cap L^\infty$ for any $2 \le r < \infty$. Therefore $\Psi(t) = \tilde{S}(t) + O(t^{-1-\varepsilon})$ in $B^{\sigma-2}_{r,2} \cap L^\infty$. Now we can immediately prove the first part of the theorem, namely the L^2 asymptotics. Indeed,

$$\|\hat{v}(t) - \exp(-i\tilde{S}(t))\hat{v}_\infty(t)\|_2 \le \|e^{i\Psi}\hat{v} - \hat{v}_\infty\|_2 + \|\Psi - \tilde{S}\|_\infty \|\hat{v}_\infty(t)\|_2 \le Ct^{-1-\varepsilon}\log t.$$

This means (1.6). To prove the second part of the theorem, namely the L^∞ asymptotics, we put

$$\mathrm{I} = \mathcal{F}\big(M - 1 - (i\xi^2/2t)\big)v, \quad \mathrm{II} = \hat{v} - e^{-i\tilde{S}}\hat{v}_\infty, \quad \mathrm{III} = -(i/2t)\partial^2\hat{v} + (i/2t)\partial^2(e^{-iS}\hat{\phi}).$$

Then

$$D^{-1}M^{-1}u = \mathrm{I} + \mathrm{II} + \mathrm{III} + e^{-i\tilde{S}}\hat{v}_\infty - (i/2t)\partial^2(e^{-iS}\hat{\phi}).$$

The estimates of I, II and III are achieved as follows. By taking r sufficiently large so that $B^{\sigma-2}_{r,2} \subset L^\infty$ and applying Lemma 2.3 together with Proposition 3.2,

$$\begin{aligned}
\|\mathrm{I}\|_\infty &\le C\|\mathcal{F}\big(M - 1 - (i\xi^2/2t)\big)\mathcal{F}^{-1}\hat{v}\|_{B^{\sigma-2}_{r,2}} \le Ct^{-2}\|\hat{v}\|_{B^{\sigma}_{r,2}} \le Ct^{-1-\varepsilon};\\
\|\mathrm{II}\|_\infty &\le \|e^{i\Psi}\hat{v} - \hat{v}_\infty\|_\infty + \|(e^{i(\Psi-\tilde{S})} - 1)\hat{v}_\infty\|_\infty\\
&\le C\|e^{i\Psi}\hat{v} - \hat{v}_\infty\|_{B^{\sigma-2}_{r,2}} + \|\Psi - \tilde{S}\|_\infty\|\hat{v}_\infty\|_\infty \le Ct^{-1-\varepsilon}\log t;\\
\|\mathrm{III}\|_\infty &\le Ct^{-1}\|\hat{v} - (e^{-iS}\hat{\phi})\|_{B^{\sigma}_{r,2}} \le Ct^{-1-\varepsilon}.
\end{aligned}$$

Therefore

$$D^{-1}M^{-1}u = e^{-i\tilde{S}}\hat{v}_\infty - (i/2t)\partial^2(e^{-iS}\hat{\phi}) + O(t^{-1-\varepsilon}\log t).$$

By direct calculation, we have

$$\partial^2(e^{-iS}\hat{\phi}) = \big\{-(\partial S)^2\hat{\phi} - i\partial^2 S\hat{\phi} - 2i\partial S\partial\hat{\phi} + \partial^2\hat{\phi}\big\}e^{-iS} = \big\{\cdots\cdots\big\}e^{-i\tilde{S}} + O(t^{-1}(\log t)^2)$$

in L^∞. Thus we obtain (1.10). $\square$

Acknowledgments

One of the authors (T. W.) would like to express his deep gratitude to Prof. Takayoshi Ogawa for his warm hospitality at Kyushu University. The authors are also grateful to him for his helpful advice. The authors also thank the referee for his comments.

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