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On L^p Regularity for Weak Derivatives of Spherically Symmetric Solutions of the Porous Media Equation

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1. Introduction. We consider the Cauchy problem for the porous media equation

$$\text{(1.1)} \qquad \begin{cases} u_t = \Delta u^m & in \quad R^N \times (0,T) \\ u(x,0) = u_0 & on \quad R^N, \end{cases}$$

where $m > 1$ and $u_0 \geq 0$. By a solution of (1.1) we mean a function $u(x,t)$ such that

$$u(x,t) \geq 0 \ \text{ in } \ R^N \times (0,T),$$

$$\int_0^T \int_{R^N} \left[u(x,t)^2 + |\nabla_x u(x,t)|^2 \right] dxdt < \infty$$

and

$$\int_0^T \int_{R^N} (u\phi_t - \nabla_x u^m \cdot \nabla_x \phi)\, dxdt + \int_{R^N} u_0(x)\phi(x,0) dx = 0$$

for any continuously differentiable function $\phi(x,t)$ with compact support in $R^N \times [0,T)$. The problem(1.1) has been studied by many authors. For a detailed account of (1.1) we refer to the survey of Kalashnikov [6]. The existence and the uniqueness of solutions of (1.1) are due to [8] and [9] under some assumption on u_0.

We are concerned to the regularity property for u. The local Hölder continuity of u was shown by Caffarelli and Friedman [3]. Aronson and

Bénilan [1] proved that Δu^m belongs to $L^1_{loc}(R^N \times (0,T))$. The method of their proof is to obtain the inequarity

$$u_t \geq -\frac{1}{t}\left(m-1+\frac{2}{N}\right)^{-1} \quad in \;\; R^N \times (0,T),$$

which is in the distribution sence. Soon after Bénilan [2] proved that

$$\Delta u^m \in L^p_{loc}(R^N \times (0,T)),$$

if $1 < p < 1 + \frac{1}{m}$. Here p needs to be less than 2 in virtue of $m > 1$. When $p = 2$, there is the following result by one of the authors [4]:

$$\partial_{x_i}\partial_{x_j} u^m \in L^2(R^N \times (0,T)), \qquad i,j = 1,\cdots,N,$$

if $1 < m < \frac{3N}{3N-2}$. When u is spherically symmetric under some assumption on u_0, this result was improved in [5] as follows:

the condition $1 < m < \frac{3N}{3N-2}$ *can be weakend with* $1 < m < 3$, *for* $N = 1,2,3$.

We consider the well-known Barenblatt solution:

$$w(x,t) = (t+\tau)^{-k}\left(\left[a^2 - \frac{k(m-1)}{2Nm}\frac{|x|^2}{(t+\tau)^{\frac{2k}{N}}}\right]_+\right)^{\frac{1}{m-1}},$$

where a, $\tau > 0$ and $k = \left(m - \frac{N-2}{N}\right)^{-1}$. We rewrite w simply with $(t+\tau)^{-k}\left([g]_+\right)^{\frac{1}{m-1}}$. Then obviously

$$\partial_{x_i}\partial_{x_j} w^m = \left([g]_+\right)^{\frac{2-m}{m-1}} P_{ij} + \left([g]_+\right)^{\frac{1}{m-1}} Q_{ij},$$

where P_{ij} and Q_{ij} are smooth functions. Hence we see that

$$\partial_{x_i}\partial_{x_j} w^m \in L^p(R^N \times (0,T)), \;\; \text{if} \;\; p\left(\frac{2-m}{m-1}\right) > -1.$$

Let u be the spherically symmetric solution of (1.1). Then from the above we conjecture that for any given $p > 2$

$$\partial_{x_i}\partial_{x_j} u^m \in L^p(R^N \times (0,T)), \quad i,j = 1,\cdots,N,$$

if m is close to 1. In this paper our first aim is to verify this conjecture (see Theorem 1). Secondly we give precisely the admissive value of m in Theorem 1, when $N=1$ (see Theorem 2). The tool is to prepare some L^p - estimates with a weight by the integreation by parts.

2. Results. Throughout this paper we suppose that $p\geq 2$. And for the case of $N\geq 2$ we set

$$q = N - 2 + p - \delta, \tag{2.1}$$

where δ is a number with

$$0 < \delta < p - 1. \tag{2.2}$$

We rewrite (2.1) as follows;

$$N - 2 + p - q = \delta > 0. \tag{2.3}$$

For the case of $N=1$ we set $q=p$. From the uniqueness we see that the solution u of (1.1) is spherically symmetric, if u_0 is so.

Our first aim is to prove

Theorem 1. *Let $N\geq 2$. Let u_0 be a spherically symmetric function such that $u_0 \in C_0^4(R^N)$ and $u_0\geq 0$ in R^N. Then there exists a constant $m_0>1$, depending only on N and p, such that if $1<m<m_0$, any solution u of (1.1) satisfies*

$$\partial_{x_i}\partial_{x_j}u^m\cdot|x|^{\frac{1}{p}(p-q-2)}\in L^p(R^N\times(0,T)) \qquad i,j=1,\cdots,N.$$

In particular, we consider the case of $N=1$, where we don't assume that u is spherically symmetric. We define

$$I(p) = 1 + \frac{4(p-1)}{p(p-1)^2+4},$$

which is used in the following theorem. For example $I(2)=\frac{5}{3}, I(3)=\frac{3}{2}, I(4)=\frac{13}{10}$.

Then we have

Theorem 2. *Let $N=1$. Let u_0 belong to $C_0^4(R^1)$ such that $u_0\geq 0$ in R^1.*

Then the solution u *of* (1.1) *satisfies*

$$\partial_x^2 u^m \in L^p(R\times(0,T)),$$

if $1<m<I(p)$.

3. Preliminaries. Throughout this paper we put $\alpha = 1-\frac{1}{m}>0$. We set $v=u^m$ in (1.1). Then (1.1) becomes

$$(1.1')\qquad \begin{cases} v^{-\alpha}v_t = m\Delta v & in \quad R^N\times(0,T) \\ v(x,0)=u_0^m(x) & on \quad R^N. \end{cases}$$

By the usual way the approximating soluion of (1.1′) is considered. Let u_0 belong to $C_0^3(R^N)$. Let $\eta>0$ and n be a natural number such that *supp* $u_0 \subset B_n(O)$,where $B_n(O)$ is the open sphere with center O and radius n. We define

$$M=\sup_{R^N}\ u_0>0,\ \ Q_T^{(n)}=B_n(O)\times(0,T).$$

The following fact is due to the famous book [7]:

Let $0<\beta<1$. Then there exists a unique $v\in H^{2+\beta,1+\frac{\beta}{2}}(\overline{Q_T^{(n)}})$ of

$$(3.1)\qquad \begin{cases} v^{-\alpha}v_t = m\Delta v & in \quad Q_T^{(n)} \\ v(x,0)=(u_0(x)+\eta)^m & on \quad B_n(O) \\ v(x,t)=\eta^m & on \quad \partial B_n(O)\times(0,T), \end{cases}$$

where the space $H^{2+\beta,1+\frac{\beta}{2}}(\overline{Q_T^{(n)}})$ with norm $|\quad|_{Q_T^{(n)}}^{(2+\beta)}$ is referred to [7]. It is known that

$$\eta^m \le v \le (M+\eta)^m \ \ in\ \ Q_T^{(n)}$$

and for some β with $0<\beta<1$

$$|v|_{Q_T^{(n)}}^{(2+\beta)} \le C \quad (<\infty),$$

where β and C are independent of n.

If $u_0\in C_0^4(R^N)$, the solution of (3.1) satisfies

$$v\in H^{3+\beta,\frac{(3+\beta)}{2}}(\overline{Q_T^{(n)}})\ \ and\ \ |v|_{Q_T^{(n)}}^{(3+\beta)} \le C \quad (<\infty),$$

where C dose not depend on n. The following is known:

Taking $n \to \infty$ and $\eta \to +0$ afterwards, we can obtain the solution v of (1.1′). If we set $u = v^{\frac{1}{m}}$, then u is the solution of (1.1).

From now on let u_0 be in $C_0^4(R^N)$. So, the solution v of (3.1) belongs to $H^{3+\beta,\frac{3+\beta}{2}}(\overline{Q_T^{(n)}})$.

If v is the spherically symmetric solution of (3.1), then v satisfies

$$(3.2) \qquad \begin{cases} v^{-\alpha}v_t = m(v_{rr} + (N-1)r^{-1}v_r) & \textit{in} \quad Q_T^{(n)} \\ v(\cdot, 0) = (u_0(\cdot) + \eta)^m & \textit{on} \quad B_n(O) \\ v(n, t) = \eta^m & \textit{on} \quad (0, T). \end{cases}$$

Now we prepare the following lemmas. The next lemma is due to [5]. But we repeat its proof by way of precaution.

Lemma 3.1([5]). *For the solution v of (3.1) it holds that*

$$|\nabla v| \leq Cn^{-1} \quad on \quad \partial B_n(O) \times (0, T),$$

where C is a positive constant independent of n and η.

Proof. Let x_0 be any fixed point on $\partial B_n(O)$ and ν be the outer normal of $\partial B_n(O)$ at x_0. We take a linear function $\pi(x)$ such that

$$\pi(x^0) = \eta^m, \ \ v(x, 0) \leq \pi(x) \ \ on \ \ B_n(O)$$

and

$$\left| \frac{\partial \pi(x^0)}{\partial \nu} \right| \leq Cn^{-1},$$

where C dose not depend on n, η and x^0. Obviously

$$\pi_t - m\Delta\pi = 0 \ \ in \ \ Q_T^{(n)},$$

$$v \leq \pi \ \ on \ \ \partial B_n(O) \times (0, T) \ \ and \ \ B_n(O) \times \{t = 0\}.$$

By the maximum principle we have

$$v \leq \pi \ \ in \ \ Q_T^{(n)}.$$

Hence

$$0 \le v(x,t) - v(x^0,t) \le \pi(x) - \pi(x^0) \ \ in \ \ Q_T^{(n)},$$

which implies immediately that

$$\left|\frac{\partial v(x^0,t)}{\partial \nu}\right| \le \left|\frac{\partial \pi(x^0)}{\partial \nu}\right| \le C n^{-1}.$$

The tangential derivative of v at x^0 equals 0. Thus we have finished the proof of Lemma 3.1.

Q.E.D.

Hereafter let $N \ge 2$, if not saying.

Lemma 3.2. *For the solution v of (3.2) it holds that*

$$|v_{rr}| \le C n^{-2} \quad on \quad \partial B_n(O) \times (0,T),$$

where C is a positive constant independent of n and η.

Proof. From (3.2)

$$v_{rr} = \frac{1}{m} v^{-\alpha} v_t - (N-1) r^{-1} v_r.$$

We see that $v_t = 0$ on $\partial B_n(0) \times (0,T)$. Hence we obtain the required inequality from Lemma 3.1.

Q.E.D.

Since v_{rr} is unifomly bounded in $B_n(O) \times (0,T)$ with respect to n, and $v_r(0,t) = 0$ we have the following lemma by using the mean value theorem:

Lemma 3.3. *For the solution v of (3.2) it holds that*

$$|v_r| \le C r \quad in \quad B_{\frac{1}{2}}(O) \times (0,T),$$

where C dose not depend on n.

From (2.1) and (2.2)

(3.3) $$N - 2 - p - q = \delta - 2p < 0$$

and

$$N+1+(p-q)-2p=3-2p+\delta<0. \tag{3.4}$$

Lemma 3.4. *It holds that*

$$1-\frac{p}{(p-1)(q+2-N)^2}(q(q+1)-(N-1)(p+q))=\frac{\delta(Np-\delta)}{(p-1)(q+2-N)^2}$$

Proof. If we set

$$H=(p-1)(q+2-N)^2-p(q(q+1)-(N-1)(p+q)),$$

then

$$H=(p-1)(p-\delta)^2-p\{(p+(N-2-\delta))(p+(N-1-\delta))-(N-1)(2p+(N-2-\delta))\}.$$

Obviously

$$(p-1)(p-\delta)^2=p^3-(1+2\delta)p^2+\delta(2+\delta)p-\delta^2,$$

$$(p+(N-2-\delta))(p+(N-1-\delta))-(N-1)(2p+(N-2-\delta))=p^2-(1+2\delta)p-\delta(N-2-\delta).$$

Hence

$$H=\delta(Np-\delta).$$

This means the required equality.

Q.E.D.

From now on we denote by (,) the inner product of $L^2(0,n)$. And we write often by $o(1)$ simply the notation of *Landau*, $o(1)$ $(n\to\infty)$. We use often the integration by parts without saying.

Lemma 3.5. *For the solution v of (3.2) it holds that*

$$\left(|v_r|^p, r^{N-3-q}\right)\le\left(\frac{p}{p-\delta}\right)^2\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)+o(1)\qquad(n\to\infty).$$

Proof. Obviously

$$\begin{aligned}\left(|v_r|^p, r^{N-3-q}\right) &= \frac{1}{N-2-q}\left(|v_r|^p, (r^{N-2-q})_r\right) \\ &= -\frac{p}{N-2-q}\left(|v_r|^{p-2}v_r, v_{rr}r^{N-2-q}\right) \\ &\quad +\frac{1}{N-2-q}\left[|v_r|^p r^{N-2-q}\right]_{r=0}^{r=n}.\end{aligned}$$

From (2.3) and Lemma 3.3

$$\lim_{r\to 0}\left(|v_r|^p r^{N-2-q}\right)(r,t) = 0.$$

And from (3.3) and Lemma 3.1

$$\lim_{r\to \infty}\left(|v_r|^p r^{N-2-q}\right)(n,t) = 0.$$

Hence using (3.3) again and Cauchy's inequality, we have

$$\begin{aligned}\left(|v_r|^p, r^{N-3-q}\right) &= \frac{p}{p-\delta}\left(|v_r|^{p-2}v_r, v_{rr}r^{N-2-q}\right) + o(1) \\ &\le \frac{p}{p-\delta}\left(\varepsilon\left(|v_r|^p, r^{N-3-q}\right) + \frac{1}{4\varepsilon}\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right) + o(1).\end{aligned}$$

If we put $\varepsilon = \frac{p-\delta}{2p}$ particularly, the required inequality is obtained.

Q.E.D.

We set $\kappa = p - q$ and $\psi_\eta = (u_0+\eta)^m$ for the initial function u_0 in our Theorems. Then we have

Lemma 3.6. *Let u_0 be the function in Theorem 1. Then*

$$\left((\psi_\eta)^{-\alpha}|(\psi_\eta)_{rr}|^p, r^{N-1+\kappa}\right) \le C(M+1)^{(p-1)(m-1)}\left[\left(|(u_0)_{rr}|^p, r^{N-1+\kappa}\right) + \left(|(u_0)_r|^p, r^{N-1+\kappa}\right)\right],$$

where C dose not depend on η. If u_0 is the function in Theorem 2, then

$$\left((\psi_\eta)^{-\alpha},|(\psi_\eta)_{rr}|^p\right)\le C(M+1)^{(p-1)(m-1)}\left(1,|(u_0)_{rr}|^p\right),$$

where (,) means the $L^2(-n,n)$ inner product.

Proof. We prove the first half. We put $l=N-1+\kappa$, $s=m-1+p(2-m)$ and $w=u_0+\eta$. Obviously

$$(\psi_\eta)^{-\alpha}|(\psi_\eta)_{rr}|^p\le C\left(w^{(p-1)(m-1)}|w_{rr}|^p+w^{-s}|w_r|^{2p}\right).$$

So, it is enough to estimate $\left(w^{-s},|w_r|^{2p}r^l\right)$ only. By integration by parts

$$\begin{aligned}\left(w^{-s},|w_r|^{2p}r^l\right) &= \left(w^{-s},|w_r|^{2p-2}w_rw_rr^l\right)\\ &= s\left(w^{-s},|w_r|^{2p}r^l\right)-(2p-1)\left(w^{1-s},|w_r|^{2p-2}w_{rr}r^l\right)\\ &\quad -l\left(w^{1-s},|w_r|^{2p-2}w_rr^{l-1}\right)+\left[w^{1-s}|w_r|^{2p-2}w_rr^l\right]_{r=0}^{r=n}.\end{aligned}$$

The last term vanishes from (2.3). Hence

$$(s-1)\left(w^{-s},|w_r|^{2p}r^l\right)\le(2p-1)\left(w^{1-s},|w_r|^{2p-2}|w_{rr}|r^l\right)+l\left(w^{1-s},|w_r|^{2p-1}r^{l-1}\right).$$

By Hölder's inequality

$$\left(w^{1-s},|w_r|^{2p-2}|w_{rr}|r^l\right)\le\varepsilon\left(w^{-s},|w_r|^{2p}r^l\right)+C(\varepsilon)\left(w^{p-s},|w_{rr}|^pr^l\right),$$

$$\left(w^{1-s},|w_r|^{2p-1}r^{l-1}\right)\le\varepsilon\left(w^{-s},|w_r|^{2p}r^l\right)+C(\varepsilon)\left(w^{p-s},|w_r|^pr^{l-p}\right).$$

Noting that $p-s=(p-1)(m-1)$, we obtain the first half from the above and the inequality

$$(|w_r|^p,r^{l-p})\le C(|(u_0)_{rr}|^p,r^{N-1+\kappa}). \tag{3.5}$$

By the way of precaution we verify (3.5) at the end of this section, see *Remark*. Secondly let u_0 be the function in Theorem 2. Since $l=0$, it follows from the above that

$$\begin{aligned}(s-1)\left(w^{-s},|w_r|^{2p}\right) &\le (2p-1)\left(w^{1-s}|w_r|^{2p-2},|w_{rr}|\right)\\ &\le \varepsilon\left(w^{-s},|w_r|^{2p}\right)+C(\varepsilon)\left(w^{p-s},|w_r|^pr^{l-p}\right).\end{aligned}$$

Thus the second estimate has been obtained. More easily we can prove that

$$\Big((\psi_\eta)^{-\alpha}|(\psi_\eta)_r|^p, r^{N-1-q}\Big) \leq C(M+1)^{(p-1)(m-1)} \Big(|(u_0)_r|^p, r^{N-1-q}\Big) \tag{3.6}$$

for the function u_0 in Theorem 1.

Q.E.D.

Remark. The function w in the proof of Lemma 3.6 satisfies $w_r = O(r)(r \to 0)$, since w is spherically symmetric.

In general, let $f(r)$ be in $C^1[0,\infty)$ such that $f(0) = 0$ and $f(r) = 0$ for sufficiently large r. Let $a > 1$, $\varepsilon > 0$ and $a \neq \varepsilon$. Then it holds that

$$\int_0^\infty |f(r)|^a r^{\varepsilon-a-1} dr \leq C \int_0^\infty |f'(r)|^a r^{\varepsilon-1} dr, \tag{3.7}$$

where C depends only on a and ε.

The inequality (3.5) is obtained immediately from (3.7). So, it is enough to prove (3.7). By the mean value theorem, $f(r) = O(r)$ $(r \to 0)$. By integration by parts

$$\int_0^\infty |f(r)|^a r^{\varepsilon-a-1} dr = \frac{1}{\varepsilon - a}\int_0^\infty |f(r)|^a (r^{\varepsilon-a})' dr = \frac{1}{\varepsilon - a}\Big[|f(r)|^a r^{\varepsilon-a}\Big]_0^\infty$$

$$-\frac{a}{\varepsilon - a}\int_0^\infty |f(r)|^{a-2} f(r) f'(r) r^{\varepsilon-a} dr.$$

Hence

$$\int_0^\infty |f|^a r^{\varepsilon-a-1} dr \leq C \int_0^\infty |f|^{a-1}|f'| r^{\varepsilon-a} dr$$

$$\text{(by Hölder's inequality)} \leq C\Big(\int_0^\infty |f|^a r^{\varepsilon-a-1} dr\Big)^{\frac{a-1}{a}} \cdot \Big(\int_0^\infty |f'|^a r^{\varepsilon-1} dr\Big)^{\frac{1}{a}},$$

which means (3.7).

4. Lemma 4.1 and its proof. From now on we denote by $((\quad,\quad))$ the integral $\int_0^T (\quad,\quad)\, dt$.

Lemma 4.1. *For the solution v of (3.2) it holds that*

$$
\begin{aligned}
(4.1)\qquad & \left(\left(|v_r|^{p-2}, {v_{rr}}^2 r^{N-1-q}\right)\right) + \frac{1}{p(p-1)}((N-1)(p+q)-q(q+1)) \left(\left(|v_r|^p, r^{N-3-q}\right)\right) \\
&= \frac{\alpha}{p(p+1)} \left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right) \\
&+ \frac{\alpha((N-1)p+q)}{p(p+1)} \left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right) \\
&- \frac{1}{p(p-1)m} \left[\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)\right]_{t=0}^{t=T} \\
&+ o(1) \quad (n\to\infty).
\end{aligned}
$$

Proof. From (3.2)

$$
\left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right) = m\left(|v_r|^{p-2}, {v_{rr}}^2 r^{N-1-q}\right) + m(N-1)\left(|v_r|^{p-2}v_r, v_{rr}r^{N-2-q}\right).
$$

Obviously

$$
\begin{aligned}
\left(|v_r|^{p-2}v_r, v_{rr}r^{N-2-q}\right) &= \frac{1}{p}\left((|v_r|^p)_r, r^{N-2-q}\right) \\
&= -\frac{N-2-q}{p}\left(|v_r|^p, r^{N-3-q}\right) \\
&\quad +\frac{1}{p}\left[|v_r|^p r^{N-2-q}\right]_{r=0}^{r=n}.
\end{aligned}
$$

As previously

$$
\lim_{r\to+0}\left(|v_r|^p r^{N-2-q}\right)(r,t) = 0, \qquad \lim_{n\to\infty}\left(|v_r|^p r^{N-2-q}\right)(n,t) = 0.
$$

Hence we obtain

$$
\begin{aligned}
(4.2)\qquad \left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right) &= m\left(|v_r|^{p-2}, {v_{rr}}^2 r^{N-1-q}\right) \\
&\quad -\frac{m}{p}(N-1)(N-2-q)\left(|v_r|^p, r^{N-3-q}\right) \\
&\quad +o(1).
\end{aligned}
$$

On the other hand

$$
\left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right) = \frac{1}{p-1}\left(v^{-\alpha}v_t, (|v_r|^{p-2}v_r)_r r^{N-1-q}\right)
$$

$$
\begin{aligned}
= & -\frac{1}{p-1}\left((v^{-\alpha}v_t)_r, |v_r|^{p-2}v_r r^{N-1-q}\right) \\
& -\frac{N-1-q}{p-1}\left(v^{-\alpha}v_t, |v_r|^{p-2}v_r r^{N-2-q}\right) \\
& +\frac{1}{p-1}\left[v^{-\alpha}v_t|v_r|^{p-2}v_r r^{N-1-q}\right]_{r=0}^{r=n}.
\end{aligned}
$$

As previously

$$
\lim_{r\to+0}\left(|v_r|^{p-2}v_r r^{N-1-q}\right)(r,t) = 0.
$$

Since $(v^{-\alpha}v_t)_r = v^{-\alpha}v_{tr} - \alpha v^{-\alpha-1}v_r v_t$, we have

$$
\begin{aligned}
\left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right) = & \frac{1}{p-1}\left(v^{-\alpha}v_t, (|v_r|^{p-2}v_r)_r r^{N-1-q}\right) \\
= & -\frac{1}{p-1}\left(v^{-\alpha}v_{rt}, |v_r|^{p-2}v_r r^{N-1-q}\right) \\
& +\frac{\alpha}{p-1}\left(v^{-\alpha-1}v_t v_r, |v_r|^{p-2}v_r r^{N-1-q}\right) \\
& -\frac{N-1-q}{p-1}\left(v^{-\alpha}v_t, |v_r|^{p-2}v_r r^{N-2-q}\right) \\
& +\frac{1}{p-1}\left(v^{-\alpha}v_t|v_r|^{p-2}v_r r^{N-1-q}\right)(n,t).
\end{aligned} \tag{4.3}
$$

Here the last term vanishes by virtue of (3.2).

We can write

$$
-\frac{1}{p-1}\left(v^{-\alpha}v_{rt}, |v_r|^{p-2}v_r r^{N-1-q}\right) = -\frac{1}{p(p-1)}\left(v^{-\alpha}, (|v_r|^p)_t r^{N-1-q}\right).
$$

Hence

$$
\begin{aligned}
-\frac{1}{p-1}\left(\left(v^{-\alpha}v_{rt}, |v_r|^{p-2}v_r r^{N-1-q}\right)\right) = & -\frac{\alpha}{p(p-1)}\left(\left(v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}\right)\right) \\
& -\frac{1}{p(p-1)}\left[\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)\right]_{t=0}^{t=T}.
\end{aligned}
$$

Combining this with (4.3), we obtain

$$(4.4)\qquad \begin{aligned}\left(\left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right)\right) &= \frac{\alpha}{p}\left(\left(v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}\right)\right)\\ &-\frac{N-1-q}{p-1}\left(\left(v^{-\alpha}v_t, |v_r|^{p-2}v_r r^{N-2-q}\right)\right)\\ &-\frac{1}{p(p-1)}\left[\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)\right]_{t=0}^{t=T}.\end{aligned}$$

First we estimate the first term on the right-hand side of (4.4). From (3.2)

$$\left(v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}\right) = m\left(v^{-1}v_{rr}, |v_r|^p r^{N-1-q}\right) + m(N-1)\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right).$$

And

$$\begin{aligned}\left(v^{-1}v_{rr}, |v_r|^p r^{N-1-q}\right) &= \frac{1}{p+1}\left(v^{-1}, (|v_r|^p v_r)_r r^{N-1-q}\right)\\ &= \frac{1}{p+1}\left(v^{-2}, |v_r|^{p+2} r^{N-1-q}\right)\\ &-\frac{N-1-q}{p+1}\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\\ &+\frac{1}{p+1}\left[v^{-1}v_r|v_r|^p r^{N-1-q}\right]_{r=0}^{r=n}.\end{aligned}$$

As previously

$$\lim_{r\to+0}\left(v_r|v_r|^p r^{N-1-q}\right)(r,t) = 0, \qquad \lim_{n\to\infty}\left(v_r|v_r|^p r^{N-1-q}\right)(n,t) = 0.$$

So from the above

$$(4.5)\qquad \begin{aligned}\left(v^{-\alpha-1}v_t, |v_r|^p r^{N-1-q}\right) &= \frac{m}{p+1}\left(v^{-2}, |v_r|^{p+2} r^{N-1-q}\right)\\ &+\frac{m((N-1)p+q)}{p+1}\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\\ &+o(1).\end{aligned}$$

Next we estimate the second term on the right-hand side of (4.4). From (3.2) again

$$\left(v^{-\alpha}v_t, |v_r|^{p-2}v_r r^{N-2-q}\right) = m\left(v_{rr}v_r|v_r|^{p-2}, r^{N-2-q}\right) + m(N-1)\left(|v_r|^p, r^{N-3-q}\right).$$

And

$$\begin{aligned}\left(v_{rr}v_r|v_r|^{p-2}, r^{N-2-q}\right) &= \frac{1}{p}\left((|v_r|^p)_r, r^{N-2-q}\right)\\ &= -\frac{N-2-q}{p}\left(|v_r|^p, r^{N-3-q}\right)\\ &\quad +\frac{1}{p}\left[|v_r|^p r^{N-2-q}\right]_{r=0}^{r=n}.\end{aligned}$$

As previously the last term equals o(1). Hence we have

$$\begin{aligned}\left(v^{-\alpha}v_t, |v_r|^{p-2}v_r r^{N-2-q}\right) &= \frac{m}{p}((p-1)(N-2)+p+q)\left(|v_r|^p, r^{N-3-q}\right)\\ &\quad + o(1).\end{aligned} \tag{4.6}$$

Combining (4.4) and (4.5) with (4.6), we obtain

$$\begin{aligned}\left(\left(v^{-\alpha}v_t, |v_r|^{p-2}v_{rr}r^{N-1-q}\right)\right) &= \frac{m\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right)\\ &+ \frac{m\alpha}{p(p+1)}((N-1)p+q)\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)\\ &- \frac{1}{p(p-1)}\left[\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)\right]_{t=0}^{t=T}\\ &- \frac{m(N-1-q)}{p(p-1)}((p-1)(N-2)+p+q)\cdot\\ &\quad\left(\left(|v_r|^p, r^{N-3-q}\right)\right)\\ &+ o(1) \quad (n\to\infty).\end{aligned} \tag{4.7}$$

Here we note that

$$(N-1-q)((p-1)(N-1)+q+1)-(p-1)(N-1)(N-2-q) = (N-1)(p+q)-q(q+1).$$

Therefore the required equality follows from (4.2) and (4.7).

Q.E.D.

5. The first L^p estimate. We investigate the inequality deduced from Lemma 4.1.

We consider the first case : $(N-1)(p+q)-q(q+1)\geq 0$. From Lemma 3.5

$$\left(\left(|v_r|^{p-2}, {v_{rr}}^2 r^{N-1-q}\right)\right)+\left(\left(|v_r|^p, r^{N-3-q}\right)\right)\leq\left(1+\left(\frac{p}{p-\delta}\right)^2\right)\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)+o(1).$$

And using Lemma 4.1, we have

$$\begin{aligned}(5.1)\qquad &\left(\left(|v_r|^{p-2}, {v_{rr}}^2 r^{N-1-q}\right)\right)+\left(\left(|v_r|^p, r^{N-3-q}\right)\right)\\ &\leq\left(1+\left(\frac{p}{p-\delta}\right)^2\right)\frac{\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right)\\ &+\left(1+\left(\frac{p}{p-\delta}\right)^2\right)\frac{\alpha((N-1)p+q)}{p(p+1)}\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)\\ &+C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}+o(1)\quad (n\to\infty),\end{aligned}$$

where C is a positive constant depending only on p.

We consider the second case : $(N-1)(p+q)-q(q+1)<0$. We multiply the both sides of the inequality in Lemma 3.5, by $\frac{1}{p(p-1)}((N-1)(p+q)-q(q+1))$.

$$\begin{aligned}(5.2)\quad &-\left(\frac{p}{p-\delta}\right)^2\frac{1}{p(p-1)}(q(q+1)-(N-1)(p+q))\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)\\ &\leq\frac{1}{p(p-1)}((N-1)(p+q)-q(q+1))\left(\left(|v_r|^p, r^{N-3-q}\right)\right).\end{aligned}$$

Next adding $\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)$ to the both sides of (5.2), we obtain from Lemmas 3.4, 4.1 and (5.2) again

$$\begin{aligned}&\left(1-\frac{p}{(p-1)(p-\delta)^2}(q(q+1)-(N-1)(p+q))\right)\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)\\ &=\frac{\delta(Np-\delta)}{(p-1)(p-\delta)^2}\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)\\ &\leq\left(\left(|v_r|^{p-2}{v_{rr}}^2 r^{N-1-q}\right)\right)+\frac{(N-1)(p+q)-q(q+1)}{p(p-1)}\left(\left(|v_r|^p, r^{N-3-q}\right)\right)\\ &\leq\frac{\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right)+\frac{\alpha((N-1)p+q)}{p(p+1)}\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)\\ &+C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}+o(1).\end{aligned}$$

Hence

$$\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right) \leq \frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)}\frac{\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right)$$
$$+\frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)}\frac{\alpha((N-1)p+q)}{p(p+1)}\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)$$
$$+C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} + o(1).$$

From the above and Lemma 3.5 it follows that

$$\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right) + \left(\left(|v_r|^p, r^{N-3-q}\right)\right)$$
$$\leq \left(1+\left(\frac{p}{p-\delta}\right)^2\right)\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)$$
$$\leq \left(1+\left(\frac{p}{p-\delta}\right)^2\right)\frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)}\frac{\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right)$$
$$+\left(1+\left(\frac{p}{p-\delta}\right)^2\right)\frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)}\frac{\alpha((N-1)p+q)}{p(p+1)}\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)$$
$$+ C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} + o(1).$$

This means that

(5.3)

$$\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)+\left(\left(|v_r|^p, r^{N-3-q}\right)\right) \leq \left(1+\left(\frac{p}{p-\delta}\right)^2\right)\frac{(p-1)(p-\delta)^2}{\delta(Np-\delta)}\times$$
$$\left\{\frac{\alpha}{p(p+1)}\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right) + \frac{\alpha((N-1)p+q)}{p(p+1)}\left(\left(v^{-1}v_r, |v_r|^p r^{N-2-q}\right)\right)\right\}$$
$$+ \ C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} + o(1),$$

where C is the positive constant as in (5.1).

Combining (5.1) and (5.3) with the above, we obtain

$$\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right) + \left(\left(|v_r|^p, r^{N-3-q}\right)\right) \tag{5.4}$$
$$\leq C\alpha\left\{\left(\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right)\right) + \left(\left(v^{-1}, |v_r|^{p+1} r^{N-2-q}\right)\right)\right\}$$
$$+ C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}$$
$$+ o(1).$$

We estimate the right-hand side of (5.4). First we have by integration by parts

$$\begin{aligned}\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) &= \left(v^{-2}|v_r|^p v_r, v_r r^{N-1-q}\right)\\ &= 2\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) - (p+1)\left(v^{-1}v_{rr}|v_r|^p, r^{N-1-q}\right)\\ &\quad -(N-1-q)\left(v^{-1}v_r|v_r|^p, r^{N-2-q}\right) + \left[v^{-1}v_r|v_r|^p r^{N-1-q}\right]_{r=0}^{r=n}.\end{aligned}$$

As previously

$$\lim_{r\to+0}\left(|v_r|^{p+1}r^{N-1-q}\right)(r,t) = 0, \qquad \lim_{n\to\infty}\left(v^{-1}v_r|v_r|^p r^{N-1-q}\right)(n,t) = 0.$$

Hence

$$\begin{aligned}\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) &= (p+1)\left(v^{-1}v_{rr}|v_r|^p, r^{N-1-q}\right)\\ &\quad +(N-1-q)\left(v^{-1}v_r|v_r|^p, r^{N-2-q}\right)\\ &\quad + o(1) \quad (n\to\infty).\end{aligned} \tag{5.5}$$

We use the following Cauchy's inequalities;

$$\left|\left(v^{-1}v_{rr}|v_r|^p, r^{N-1-q}\right)\right| \le \varepsilon\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) + \frac{1}{4\varepsilon}\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right),$$

$$\left(v^{-1}|v_r|^{p+1}, r^{N-2-q}\right) \le \varepsilon\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) + \frac{1}{4\varepsilon}\left(|v_r|^p, r^{N-3-q}\right).$$

We see that $N-1-q = 1-p+\delta < 0$ from (2.3). Since $(p+1)+|N-1-q| = 2p-\delta$, it follows from the above and (5.5) that

$$(1-\varepsilon(2p-\delta))(v^{-2}|v_r|^{p+2}, r^{N-1-q}) \le \frac{p+1}{4\varepsilon}\left(|v_r|^{p-2}{v_{rr}}^2 r^{N-1-q}\right) + \frac{2p-\delta}{4\varepsilon}\left(|v_r|^p, r^{N-3-q}\right) + o(1).$$

So setting $\varepsilon = \frac{1}{2(2p-\delta)}$, we have

$$\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) \le (p+1)(2p-\delta)\left(|v_r|^{p-2}{v_{rr}}^2 r^{N-1-q}\right) + (2p-\delta)^2\left(|v_r|^p, r^{N-3-q}\right) + o(1).$$

Hence we can estimate the first term on the right-hand side of (5.4). The second term can be also estimated from the above.

Combining these inequalities with (5.4), we conclude the following

Proposition 5.1. *There is a positive constant* α_0 *such that for the solution* v *of (3.2) it holds that if* $0<\alpha<\alpha_0$,

$$\left(\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right)\right)+\left(\left(|v_r|^p, r^{N-3-q}\right)\right)\le C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}+o(1)\quad (n\to\infty),$$

where α_0 *and* C *depend only on* N *and* p*, but not on* n *and* η*.*

6. Lemma 6.1 and its proof. We put $\kappa=p-q$ as in Lemma 3.6 for the sake of simplicity.

Lemma 6.1. *Let* v *be the solution of (3.2). Then*

$$\begin{aligned}
(6.1)\qquad & \frac{\alpha}{p-1}\left(\frac{1}{p}-\alpha\right)\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right)+\left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right)\\
&+\frac{N-1+\kappa}{p-1}\left(N-1-\frac{N-2+\kappa}{p}\right)\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right)\\
&-(N-1)\left(\left(v_r|v_{rr}|^{p-2}, v_{rrr}r^{N-3+\kappa}\right)\right)+(N-1)\left(\left(v_{rr}|v_{rr}|^{p-2}, v_{rrr}r^{N-2+\kappa}\right)\right)\\
=\ & \frac{3-p}{p-1}\alpha\left(\left(v^{-1}v_r|v_{rr}|^{p-2}, v_{rr}v_{rrr}r^{N-1+\kappa}\right)\right)-\frac{\alpha\kappa}{p}\left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right)\\
&-\alpha(N-1)\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}\right)\right)\\
+\ & \frac{\alpha^2}{p-1}(N-1)\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right)\right)\\
&-\frac{\alpha}{p-1}(N-1)(N+\kappa)\left(\left(v^{-1}{v_r}^2, |v_r|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right)\\
&+(N-1)\frac{N-1+\kappa}{p-1}\left(\left(v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa}\right)\right)\\
&-\frac{1}{p(p-1)m}\left[\left(v^{-\alpha}|v_{rr}|^p, r^{N-1+\kappa}\right)\right]_{t=0}^{t=T}\\
&+o(1)\quad (n\to\infty).
\end{aligned}$$

Proof. First by integration by parts

$$-\left(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right)\ =\ -\alpha\left(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right)$$

$$+\left(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right)$$
$$-\left[v^{-\alpha}v_t|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right]_{r=0}^{r=n}.$$

Noting that $v_t = 0$ for $r = n$. We have from (2.3)

$$(6.2)\quad -\left(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right) = -\alpha\left(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right)$$
$$+\left(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right).$$

From (3.2)

(6.3)

$$-\alpha\left(v^{-\alpha-1}v_tv_r, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right) = -\alpha m\left(v^{-1}v_rv_{rr}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right)$$
$$-\alpha m(N-1)\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}\right).$$

Combining (2.3) with (3.4), we have

$$(6.4)\quad \begin{aligned}\left(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right) &= \frac{1}{p-1}\left(v^{-\alpha}v_{rt}, (|v_{rr}|^{p-2}v_{rr})_rr^{N-1+\kappa}\right)\\ &= \frac{\alpha}{p-1}\left(v^{-\alpha-1}v_rv_{rt}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right)\\ &\quad -\frac{1}{p-1}\left(v^{-\alpha}v_{trr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right)\\ &\quad -\frac{N-1+\kappa}{p-1}\left(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right)\\ &\quad +\ o(1).\end{aligned}$$

In (6.2) we use the function $\left(|v_{rr}|^{p-2}, v_{rrr}r^{N-1+\kappa}\right)_r$ in spite of $v \in H^{3+\beta, \frac{3+\beta}{2}}$ but it is not contradictory we may take an approximating smooth sequence of v.

We estimate the second term on the right-hand side of (6.4). We can write

$$\left(v^{-\alpha}v_{trr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right) = \frac{1}{p}\left(v^{-\alpha}, (|v_{rr}|^p)_tr^{N-1+\kappa}\right).$$

Since $(v^{-\alpha}v_t)_r = v^{-\alpha}v_{rt} - \alpha v^{-\alpha-1}v_tv_r$, it follows from (3.2) that

$$v^{-\alpha}v_{tr} = mv_{rrr} + m(N-1)(v_rr^{-1})_r + \alpha mv^{-1}v_rv_{rr} + \alpha m(N-1)r^{-1}v^{-1}{v_r}^2.$$

Hence (6.4) becomes

$$
\begin{aligned}
(6.5)\qquad & \left(v^{-\alpha}v_{rt}, |v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa}\right) = \frac{\alpha m}{p-1}\left(v^{-1}v_r v_{rrr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right) \\
&+ \frac{\alpha m}{p-1}(N-1)\left(v^{-1}v_r(v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right) \\
&+ \frac{\alpha^2 m}{p-1}\left(v^{-2}{v_r}^2, |v_{rr}|^{p}r^{N-1+\kappa}\right) \\
&+ \frac{\alpha^2 m}{p-1}(N-1)\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right) \\
&- \frac{1}{p(p-1)}\left(v^{-\alpha}, (|v_{rr}|^{p})_t r^{N-1+\kappa}\right) \\
&- \frac{m}{p-1}(N-1+\kappa)\left(v_{rrr}|v_{rr}|^{p-2}, v_{rr}r^{N-2+\kappa}\right) \\
&- \frac{m}{p-1}(N-1+\kappa)(N-1)\left((v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right) \\
&- \frac{\alpha m}{p-1}(N-1+\kappa)\left(v^{-1}v_r, |v_{rr}|^{p}r^{N-2+\kappa}\right) \\
&- \frac{\alpha m}{p-1}(N-1+\kappa)(N-1)\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa}\right) \\
&+ o(1).
\end{aligned}
$$

Combining (6.2) and (6.3) with (6.5), we obtain

$$
\begin{aligned}
(6.6)\qquad & -\left(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right) = \frac{2-p}{p-1}\alpha m\left(v^{-1}v_r v_{rrr}, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right) \\
&- \alpha m(N-1)\left(v^{-1}{v_r}^2 v_{rrr}, |v_{rr}|^{p-2}r^{N-2+\kappa}\right) \\
&+ \frac{\alpha m}{p-1}(N-1)\left(v^{-1}v_r(v_r r^{-1})_r, |v_{rr}|^{p-2}v_{rr}r^{N-1+\kappa}\right) \\
&+ \frac{\alpha^2 m}{p-1}\left(v^{-2}{v_r}^2, |v_{rr}|^{p}r^{N-1+\kappa}\right) \\
&+ \frac{\alpha^2 m}{p-1}(N-1)\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right) \\
&- \frac{1}{p(p-1)}\left(v^{-\alpha}, (|v_{rr}|^{p})_t r^{N-1+\kappa}\right) \\
&- \frac{m}{p-1}(N-1+\kappa)\left(v_{rrr}|v_{rr}|^{p-2}, v_{rr}r^{N-2+\kappa}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{m}{p-1}(N-1+\kappa)(N-1)\left((v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa}\right) \\
&- \frac{\alpha m}{p-1}(N-1+\kappa)\left(v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa}\right) \\
&- \frac{\alpha m}{p-1}(N-1+\kappa)(N-1)\left(v^{-1} {v_r}^2, |v_{rr}|^{p-2} v_{rr} r^{N-3+\kappa}\right) \\
&+ o(1).
\end{aligned}
$$

On the third and eighth terms on the right-hand side of (6.6), we write $(v_r r^{-1})_r = v_{rr} r^{-1} - v_r r^{-2}$, that is

$$
\left(v^{-1} v_r (v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa}\right) = \left(v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa}\right) - \left(v^{-1} {v_r}^2 v_{rr}, |v_{rr}|^{p-2} r^{N-3+\kappa}\right),
$$

$$
\left((v_r r^{-1})_r, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa}\right) = \left(|v_{rr}|^p, r^{N-3+\kappa}\right) - \left(v_r |v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa}\right).
$$

The integration over the sixth term equals

$$
\begin{aligned}
\left(\left(v^{-\alpha}, (|v_{rr}|^p)_t r^{N-1+\kappa}\right)\right) &= \alpha\left(\left(v^{-\alpha-1} v_t, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \left[\left(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa}\right)\right]_{t=0}^{t=T} \\
&= \alpha m\left(\left(v^{-1} v_{rr}, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \alpha m(N-1)\left(\left(v^{-1} v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right) \\
&\quad + \left[\left(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa}\right)\right]_{t=0}^{t=T} \qquad (\text{from}(3.2)).
\end{aligned}
$$

The seventh term is

$$
\left(|v_{rr}|^{p-2} v_{rr}, v_{rrr} r^{N-2+\kappa}\right) = \frac{1}{p}\left(\left(|v_{rr}|^p\right)_r, r^{N-2+\kappa}\right) \tag{6.7}
$$

$$
= -\frac{1}{p}(N-2+\kappa)\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + o(1) \quad (\text{as previously}).
$$

From the above (6.6) becomes

$$
\begin{aligned}
-\left(\left(v^{-\alpha} v_t, (|v_{rr}|^{p-2} v_{rrr} r^{N-1+\kappa})_r\right)\right) &= \frac{2-p}{p-1}\alpha m\left(\left(v^{-1} v_r v_{rrr}, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa}\right)\right) \\
&\quad - \alpha m(N-1)\left(\left(v^{-1} {v_r}^2 v_{rrr}, |v_{rr}|^{p-2} r^{N-2+\kappa}\right)\right)
\end{aligned} \tag{6.8}
$$

$$
\begin{aligned}
&+\frac{\alpha^2 m}{p-1}\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right)\\
&+\frac{\alpha^2 m}{p-1}(N-1)\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa}\right)\right)\\
&-\frac{1}{p(p-1)}\left[\left(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa}\right)\right]_{t=0}^{t=T}\\
&-\frac{\alpha m}{p-1}\left(\frac{N-1}{p}+\kappa\right)\left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right)\\
&-\frac{\alpha m}{p-1}(N-1)(N+\kappa)\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2} v_{rr} r^{N-3+\kappa}\right)\right)\\
&-\frac{m}{p-1}(N-1+\kappa)(N-1-\frac{1}{p}(N-2+\kappa))\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right)\\
&+\frac{m}{p-1}(N-1)(N-1+\kappa)\left(\left(v_r|v_{rr}|^{p-2}, v_{rr} r^{N-4+\kappa}\right)\right)\\
&-\frac{\alpha m}{p(p-1)}\left(\left(v^{-1}v_{rr}, |v_{rr}|^p r^{N-1+\kappa}\right)\right)\\
&+o(1).
\end{aligned}
$$

By integration by parts the last term equals

$$
\begin{aligned}
-\left(v^{-1}v_{rr}, |v_{rr}|^p r^{N-1+\kappa}\right) &= -\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\\
&\quad + p\left(v^{-1}v_r v_{rr}, |v_{rr}|^{p-2} v_{rrr} r^{N-1+\kappa}\right)\\
&\quad + (N-1+\kappa)\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\\
&\quad + o(1).
\end{aligned}
$$

Thus (6.8) is rewritten as follows:

$$
\begin{aligned}
(6.9)\qquad -\frac{1}{m}\left(\left(v^{-\alpha}v_t, (|v_{rr}|^{p-2} v_{rrr} r^{N-1+\kappa})_r\right)\right) &= \frac{3-p}{p-1}\alpha\left(\left(v^{-1}v_r v_{rrr}, |v_{rr}|^{p-2} v_{rr} r^{N-1+\kappa}\right)\right)\\
&-\alpha(N-1)\left(\left(v^{-1}{v_r}^2 v_{rrr}, |v_{rr}|^{p-2} r^{N-2+\kappa}\right)\right)\\
&+\frac{\alpha}{p-1}\left(\alpha-\frac{1}{p}\right)\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right)\\
&+\frac{\alpha^2}{p-1}(N-1)\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2} v_{rr} r^{N-2+\kappa}\right)\right)\\
&-\frac{1}{p(p-1)m}\left[\left(v^{-\alpha}, |v_{rr}|^p r^{N-1+\kappa}\right)\right]_{t=0}^{t=T}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\alpha}{p}\kappa \left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right) \\
&- \frac{\alpha}{p-1}(N-1)(N+\kappa)\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right) \\
&- \frac{1}{p-1}(N-1+\kappa)\left(N-1-\frac{1}{p}(N-2+\kappa)\right)\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) \\
&+ \frac{1}{p-1}(N-1)(N-1+\kappa)\left(\left(v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa}\right)\right) \\
&+ o(1).
\end{aligned}
$$

On the other hand from (3.2)

$$
\begin{aligned}
-\left(v^{-\alpha}v_t, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right) &= -m\left(v_{rr}, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right) \\
&\quad -m(N-1)\left(v_r r^{-1}, (|v_{rr}|^{p-2}v_{rrr}r^{N-1+\kappa})_r\right) \\
&= m\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right) \\
&\quad +m(N-1)\left(v_{rr}|v_{rr}|^{p-2}, v_{rrr}r^{N-2+\kappa}\right) \\
&\quad -m(N-1)\left(v_r|v_{rr}|^{p-2}, v_{rrr}r^{N-3+\kappa}\right) \\
&\quad + o(1).
\end{aligned}
$$

Combining this with (6.9), we conclude Lemma 6.1

Q.E.D.

7. The second L^p estimate. We estimate the term $((|v_{rr}|^p, r^{N-3+\kappa}))$ on the left-hand of (6.1). First by integration by parts

$$
\begin{aligned}
\left(|v_{rr}|^p, r^{N-3+\kappa}\right) &= \left(|v_{rr}|^{p-2}v_{rr}, v_{rr}r^{N-3+\kappa}\right) \\
&= -(p-1)\left(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}\right) \\
&\quad -(N-3+\kappa)\left(|v_{rr}|^{p-2}v_{rr}, v_r r^{N-4+\kappa}\right) \\
&\quad + \left[|v_{rr}|^{p-2}v_{rr}v_r r^{N-3+\kappa}\right]_{r=0}^{r=n}.
\end{aligned}
$$

As previously

$$
\lim_{r\to+0}\left(|v_r|^{p-2}v_{rr}v_r r^{N-3+\kappa}\right)(r,t) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(|v_r|^{p-2}v_{rr}v_r r^{N-3+\kappa}\right)(n,t) = 0,
$$

because $N-2-q=\delta-p<0$.
Hence

$$(7.1)\qquad \left(|v_{rr}|^p, r^{N-3+\kappa}\right) = -(p-1)\left(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}\right) -(N-3+\kappa)\left(|v_{rr}|^{p-2}v_{rr}, v_r r^{N-4+\kappa}\right) +o(1).$$

By Cauchy's inequality

$$\left|\left(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}\right)\right| \le \varepsilon\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right) + C(\varepsilon^{-1})\left(|v_{rr}|^{p-2}, {v_r}^2 r^{N-5+\kappa}\right) \qquad \varepsilon>0.$$

On the other hand by Hölder's inequality

$$\left|\left(|v_{rr}|^{p-2}, {v_r}^2 r^{N-5+\kappa}\right)\right| \le \varepsilon'\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + C(\varepsilon'^{-1})\left(|v_r|^p, r^{N-3-q}\right) \qquad \varepsilon'>0.$$

The above inequalities are trivial for $p=2$. From these inequalities

$$(7.2)\qquad \left|\left(|v_{rr}|^{p-2}v_{rrr}, v_r r^{N-3+\kappa}\right)\right| \le \varepsilon\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right) + C(\varepsilon^{-1})\left\{\varepsilon'\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + C(\varepsilon'^{-1})\left(|v_r|^p, r^{N-3-q}\right)\right\}.$$

Further

$$\left|\left(|v_{rr}|^{p-2}v_{rr}, v_r r^{N-4+\kappa}\right)\right| \le \varepsilon\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + C(\varepsilon^{-1})\left(|v_r|^p, r^{N-3-q}\right).$$

From the above and (7.1) we see that

$$\left(1-C(\varepsilon^{-1})\varepsilon'(p-1)-\varepsilon\,|N-3+\kappa|\right)\left(|v_{rr}|^p, r^{N-3+\kappa}\right) \le \varepsilon(p-1)\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right) + C(\varepsilon^{-1}, \varepsilon'^{-1})\left(|v_r|^p, r^{N-3-q}\right) + o(1).$$

Taking $\varepsilon>0$ as sufficiently small, we fix it. Afterwards we take $\varepsilon'>0$ anew.
Then

$$1-C(\varepsilon^{-1})\varepsilon'(p-1)-\varepsilon\,|N-3+\kappa| > \frac{1}{2}.$$

Here we use Proposition 5.1. Then the following lemma holds:

Lemma 7.1. *For the solution v of (3.2) and any $\delta > 0$ it holds that*

$$\begin{aligned}\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) &\le \delta\left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right)\\ &+C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}\\ &+o(1) \quad (n\to\infty),\end{aligned}$$

where C depends on N, p and δ^{-1}, but not on n and η.

We use (7.2) and the inequality

$$\left|\left(v_{rr}|v_{rr}|^{p-2}, v_{rrr}r^{N-2+\kappa}\right)\right| \le \varepsilon\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right) + C(\varepsilon^{-1})\left(|v_{rr}|^p, r^{N-3+\kappa}\right).$$

Then for some $A > 0$ it follows from Lemmas 6.1 and 7.1 that

$$\begin{aligned}(7.3)\qquad &\frac{\alpha}{p-1}\left(\frac{1}{p}-\alpha\right)\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + (1-A\delta)\left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right)\\ \le\ &\frac{3-p}{p-1}\alpha\left(\left(v^{-1}v_r|v_{rr}|^{p-2}, v_{rr}v_{rrr}r^{N-1+\kappa}\right)\right)\\ -\ &\alpha(N-1)\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}\right)\right)\\ &-\frac{\alpha\kappa}{p}\left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right) + \frac{\alpha^2}{p-1}(N-1)\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right)\right)\\ &-\frac{\alpha}{p-1}(N-1)(N+\kappa)\left(\left(v^{-1}{v_r}^2, |v_r|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right)\\ &+\frac{1+\delta}{p-1}(N-1)\left(\left(v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa}\right)\right) + \frac{1}{p(p-1)m}\left(v^{-\alpha}|v_{rr}|^p, r^{N-1+\kappa}\right)_{t=0}\\ &+\ C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}\\ &+\ o(1).\end{aligned}$$

Here we put $\delta = \frac{1}{2A}$ and take $\alpha > 0$ as $\alpha < \frac{1}{2p}$. Using again Lemma 7.1, we have from (7.3)

$$\begin{aligned}(7.4)\qquad &\alpha\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right) + \left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right)\\ \le\ &C\left|\left(\left(v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa}\right)\right)\right|\end{aligned}$$

$$
\begin{aligned}
&+ \ C\alpha\Big\{ \left|\left(\left(v^{-1}v_r|v_{rr}|^{p-2}, v_{rr}v_{rrr}r^{N-1+\kappa}\right)\right)\right| + \left|\left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right)\right| \\
&+ \left|\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}\right)\right)\right| + \left|\left(\left(v^{-1}{v_r}^2, |v_r|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right)\right| \Big\} \\
&+ \ C\alpha^2 \left|\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right)\right)\right| \\
&+ \ C\Big\{ \left(v^{-\alpha}|v_{rr}|^p, r^{N-1+\kappa}\right)_{t=0} + \left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} \Big\} \\
&+ \ o(1).
\end{aligned}
$$

We shall prove

Proposition 7.2. *Let v be the solution of (3.2). Then there are two positive constants α_1 and C such that if $0 < \alpha < \alpha_1$, it holds that*

$$
\begin{aligned}
&\alpha\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right) + \left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) \\
&\le C\Big\{ \left(v^{-\alpha}|v_{rr}|^p, r^{N-1+\kappa}\right)_{t=0} + \left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} \Big\} + o(1) \quad (n\to\infty),
\end{aligned}
$$

where α_1 and C depend on N and p, but not on n and η. For the case of $N = 1$, the last term $\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}$ disappears.

Proof. We set the right-hand side of (7.4) as follows:

$$
CI_1 + C\alpha\sum_{k=2}^{5} I_k + C\alpha^2 I_6 + C\{J_1 + J_2\} + o(1).
$$

We estimate each term I_i, $i = 1, \cdots, 6$. For these we apply Hölder's inequality:

$$
\begin{aligned}
I_1 &= \left|\left(\left(v_r|v_{rr}|^{p-2}, v_{rr}r^{N-4+\kappa}\right)\right)\right| \\
&\le \varepsilon\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) + C(\varepsilon^{-1})\left(\left(|v_r|^p, r^{N-3-q}\right)\right) \\
&\le \varepsilon\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) + C\left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0} + o(1) \quad (\text{by Proposition5.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \left|\left(\left(v^{-1}v_r|v_{rr}|^{p-2}, v_{rr}v_{rrr}r^{N-1+\kappa}\right)\right)\right| \\
&\le \varepsilon\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + C(\varepsilon^{-1})\left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \left|\left(\left(v^{-1}v_r, |v_{rr}|^p r^{N-2+\kappa}\right)\right)\right| \\
&\le \varepsilon\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + C(\varepsilon^{-1})\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right),
\end{aligned}
$$

$$(7.5) \qquad \begin{aligned} 2I_4 &= 2\left|\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rrr}r^{N-2+\kappa}\right)\right)\right| \\ &\le \left(\left(|v_{rr}|^{p-2}, v^{-2}{v_r}^4 r^{N-3+\kappa}\right)\right) + \left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right). \end{aligned}$$

The first term on the right-hand side of (7.5) becomes

(7.6)

$$\left(|v_{rr}|^{p-2}, v^{-2}{v_r}^4 r^{N-3+\kappa}\right) \le \varepsilon \left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right) + C(\varepsilon^{-1}) \left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right).$$

On the other hand by integration by parts

$$\begin{aligned} \left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) &= \left(v^{-2}|v_r|^p v_r, v_r r^{N-1-q}\right) \\ &= 2\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) \\ &\quad -(p+1)\left(v^{-1}|v_r|^p v_{rr}, r^{N-1-q}\right) \\ &\quad -(N-1-q)\left(v^{-1}|v_r|^p v_r, r^{N-2-q}\right) \\ &\quad + \left[v^{-1}|v_r|^p v_r r^{N-1-q}\right]_{r=0}^{r=n}. \end{aligned}$$

As previously

$$\lim_{r\to+0}\left(v^{-1}|v_r|^p v_r r^{N-1-q}\right)(r,t) = 0, \qquad \lim_{n\to\infty}\left(v^{-1}|v_r|^p v_r r^{N-1-q}\right)(n,t) = 0.$$

Hence

$$\begin{aligned} \left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) &= (p+1)\left(v^{-1}|v_r|^p v_{rr}, r^{N-1-q}\right) \\ &\quad + (N-1-q)\left(v^{-1}|v_r|^p v_r, r^{N-2-q}\right) \\ &\quad + o(1). \end{aligned}$$

Using the inequality

$$\left|\left(v^{-1}|v_r|^p v_{rr}, r^{N-1-q}\right)\right| \le \varepsilon' \left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) + C({\varepsilon'}^{-1})\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right),$$

we have

(7.7)

$$\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) \le C\left\{\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right) + \left(v^{-1}|v_r|^{p+1}, r^{N-2-q}\right)\right\} + o(1).$$

By Hölder's inequality

$$\left(|v_r|^{p-2}{v_{rr}}^2, r^{N-1-q}\right) \le C\left\{\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + \left(|v_r|^p, r^{N-3-q}\right)\right\}$$

and

$$\left|\left(v^{-1}|v_r|^{p+1}, r^{N-2-q}\right)\right| \le \varepsilon\left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) + C(\varepsilon^{-1})\left(|v_r|^p, r^{N-3-q}\right).$$

Then it follows from the above and (7.7) that

$$(7.8)\quad \left(v^{-2}|v_r|^{p+2}, r^{N-1-q}\right) \le C\Big\{\left(|v_{rr}|^p, r^{N-3+\kappa}\right) + \left(|v_r|^p, r^{N-3-q}\right)\Big\} + o(1).$$

Combining (7.5), (7.6) and (7.8) with Proposition 5.1, we obtain

$$\begin{aligned} 2I_4 \;&\le\; \varepsilon\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \left(\left(|v_{rr}|^{p-2}, {v_{rrr}}^2 r^{N-1+\kappa}\right)\right) \\ &\quad+\; C\Big\{\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) + \left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}\Big\} + o(1). \end{aligned}$$

Next we estimate I_5. First

$$\begin{aligned} I_5 \;&=\; \left|\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right)\right| \\ &\le\; C\Big\{\left(\left(v^{-2}{v_r}^4, |v_{rr}|^{p-2}r^{N-3+\kappa}\right)\right) + \left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right)\Big\} + o(1). \end{aligned}$$

From (7.6) and (7.8)

$$\begin{aligned} \left|\left(\left(v^{-1}{v_r}^2, |v_{rr}|^{p-2}v_{rr}r^{N-3+\kappa}\right)\right)\right| &\le \varepsilon\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) \\ &\quad+\; C\Big\{\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) + \left(\left(|v_r|^p, r^{N-3-q}\right)\right)\Big\}. \end{aligned}$$

Hence by Proposition 5.1

$$I_5 \le \varepsilon\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-3+\kappa}\right)\right) + C\Big\{\left(\left(|v_{rr}|^p, r^{N-3+\kappa}\right)\right) + \left(v^{-\alpha}, |v_r|^p r^{N-1-q}\right)_{t=0}\Big\} + o(1).$$

Lastly we estimate I_6. We have

$$\begin{aligned} I_6 \;&=\; \left|\left(\left(v^{-2}{v_r}^3, |v_{rr}|^{p-2}v_{rr}r^{N-2+\kappa}\right)\right)\right| \\ &\le\; C\Big\{\left(\left(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\right)\right) + \left(\left(v^{-2}{v_r}^4, |v_{rr}|^{p-2}r^{N-3+\kappa}\right)\right)\Big\}. \end{aligned}$$

From (7.6) and(7.8) we obtain

$$I_6 \le C\Big\{ \Big(\Big(v^{-2}{v_r}^2, |v_{rr}|^p r^{N-1+\kappa}\Big)\Big) + \Big(\Big(|v_{rr}|^p, r^{N-3+\kappa}\Big)\Big) + \Big(v^{-\alpha}, |v_r|^p r^{N-1-q}\Big)_{t=0} \Big\} + o(1).$$

We have finished the proof of Proposition 7.2 from (7.4) and the above estimates of each term I_i, $i = 1, \cdots, 6$.

Q.E.D.

8. Proof of Results. Let D be a bounded domain in $R^{N+1}(= R_x^N \times R_t^1)$. For any function u on $\overline{D}$ we define its norms as follows:

$$|u|_{\overline{D}}^{(2)} = \sum_{2s+|\alpha|\le 2} \sup_{\overline{D}} |\partial_t^s \partial_x^\alpha u| \, .$$

And we define

$H^{2,1}(\overline{D}) = \{\ u \mid u$ is cotinuous in $\overline{D}$, together with all derivaties of the form $\partial_t^s \partial_x^\alpha u$ for $2s + |\alpha| \le 2\}$.

Then $H^{2,1}(\overline{D})$ is a Banach space with its norm $|\quad|_{\overline{D}}^{(2)}$.

We denote by v_n^η the solution of (3.2). For the time being let $\eta > 0$ be sufficiently small and fixed. The following fact is known:

There exist a subsequence $\{v_{n_\nu}^\eta\}$ of $\{v_n^\eta\}$ and a function v^η such that

$$v_{n_\nu}^\eta \longrightarrow v^\eta \quad in \quad H^{2,1}(\overline{D}) \quad (\nu \to \infty).$$

for any bounded subdomain D in $R^N \times (0, T)$.

Hence $\partial_t^s \partial_x^\alpha v^\eta \in C(R^N \times [0, T])$ for $2s + |\alpha| \le 2$ and v^η satisfies

$$(8.1) \qquad \begin{cases} (v^\eta)^{-\alpha}(v^\eta)_t = m\left((v^\eta)_{rr} + (N-1)r^{-1}(v^\eta)_r\right) & in \quad R^N \times (0, T) \\ v^\eta(\cdot, 0) = (\psi^\eta)^m & on \quad R^N, \end{cases}$$

where $\psi^\eta = u_0 + \eta$.

We set $u^\eta = (v^\eta)^{\frac{1}{m}}$ and $0 < \eta < \eta'$. Then it is known that

$$0 \le u^\eta(x, t) \le u^{\eta'}(x, t) \le M + 1,$$

and there exsits a function u such that

$$u^\eta(x,t) \to u(x,t) \;\; a.e \;\; R^N \times (0,T) \qquad (\eta \to +0),$$

where u is the required solution of (1.1).

In general it holds that for any function f

$$\left|\partial_{x_i}\partial_{x_j} f\right| \le C\left(|f_{rr}| + r^{-1}|f_r|\right).$$

Hence it follows from (3.6), Lemma 3.6, Propositions 5.1 and 7.2 that

$$\int_0^T \int_0^n \left|\partial_{x_i}\partial_{x_j} v_n^\eta\right|^p r^{N-3+\kappa} drdt \;\le\; C + o(1), \qquad i,j = 1,\cdots,N,$$

where C is independent of n and η. Letting $n \to \infty$, we have

$$\int_0^T \int_0^\infty \left|\partial_{x_i}\partial_{x_j} v^\eta\right|^p r^{N-3+\kappa} drdt \;\le\; C, \qquad i,j = 1,\cdots,N.$$

We denote by $L^p_q(R^N \times (0,T))$ the Banach space with its norm $\left(\int_0^T \int_0^\infty |f|^p r^{N-3+p-q} drdt\right)^{\frac{1}{p}}$. Then from the above there exist a function v on $R^N \times (0,T)$ and a subsequence $\{\eta_\nu\}$ of $\{\eta\}$ such that $\eta_\nu \to 0$ as $\nu \to \infty$. And they satisfy

$$\partial_{x_i}\partial_{x_j} v^{\eta_\nu} \, converges \;\; weakly \;\; to \;\; \partial_{x_i}\partial_{x_j} v \;\; in \;\; L^p_q(R^N \times (0,T)).$$

We easily see that $v = u^m$. Thus Theorem 1 has been proved.

Lastly we prove Theorem 2.

When $N = 1$, we easily see that the solution v dose not need to be spherically symmetric in the proof of Lemma 6.1. Let v be the solution of (3.1) and let us proceed in parallel with the previous argument. But we set $\kappa = 0$, namely $p = q$. The inner product (,) means that of $L^2(-n,n)$.

From the proof of Lemma 6.1 we have

$$\begin{aligned} &\frac{\alpha}{p-1}\left(\frac{1}{p} - \alpha\right)\left(\left(v^{-2}{v_x}^2, |v_{xx}|^p\right)\right) + \left(\left(|v_{xx}|^{p-2}, {v_{xxx}}^2\right)\right) \\ = \;& \frac{3-p}{p-1}\alpha\left(\left(v^{-1}v_x|v_{xx}|^{p-2}, v_{xx}v_{xxx}\right)\right) - \frac{1}{mp(p-1)}\left[\left(v^{-\alpha}, |v_{xx}|^p\right)\right]_{t=0}^{t=T} + o(1). \end{aligned}$$

Here we put

$$X = |v_{xx}|^{\frac{p-2}{2}}|v_{xxx}|, \qquad Y = v^{-1}|v_x||v_{xx}|^{\frac{p}{2}}.$$

Then

$$
\begin{aligned}
&\left(\left(1, X^2\right)\right) + \frac{\alpha}{p-1}\left(\frac{1}{p}-\alpha\right)\left(\left(1, Y^2\right)\right) - \frac{3-p}{p-1}\alpha\left((X,Y)\right) \\
\leq\ & -\frac{1}{mp(p-1)}\left[\left(v^{-\alpha}, |v_{xx}|^p\right)\right]_{t=0}^{t=T} + o(1).
\end{aligned}
$$

Hence it is enough to hold that

$$
X^2 + \frac{\alpha}{p-1}\left(\frac{1}{p}-\alpha\right)Y^2 - \frac{3-p}{p-1}\alpha XY \geq 0,
$$

for any two variables X and Y. For this sake the following is sufficient:

$$
\left(\frac{3-p}{p-1}\right)^2 \alpha^2 - \frac{4\alpha}{p-1}\left(\frac{1}{p}-\alpha\right) < 0, \tag{8.2}
$$

which is equivalent to

$$
\alpha < \frac{4(p-1)}{p(p^2-2p+5)}.
$$

This is the same as $m < I(p)$, where $I(p)$ is the quantity in Section 2. Therefore we have

$$
\left(v^{-\alpha}, |v_{xx}|^p\right)_{t=T} \leq \left(v^{-\alpha}, |v_{xx}|^p\right)_{t=0} + o(1).
$$

Replacing T with t on the left-hand side, we integrate the both sides on $0 \leq t \leq T$. Then

$$
\left(\left(v^{-\alpha}, |v_{xx}|^p\right)\right) \leq T\left(v^{-\alpha}, |v_{xx}|^p\right)_{t=0} + o(1),
$$

which means

$$
((1, |v_{xx}|^p)) \leq T(M+1)^{m-1}\left(v^{-\alpha}, |v_{xx}|^p\right)_{t=0} + o(1).
$$

Using Lemma 3.6, we proceed in parallel with the proof of Theorem 1. The remained part is similar. Thus we complete the proof of Theorem 2.

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