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Structure of positive radial solutions for semilinear Dirichlet problems on a ball

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1 Introduction

The structure of positive radial solutions for nonlinear elliptic equations have attracted much attention for these years. In particular, many interesting and beautiful results have been obtained concerning the structure of positive radial solutions on entire space $\mathbf{R}^n$ (see, e.g., the survey paper by Ni [8]). However, it is not straightforward to extend these results to boundary value problems on bounded domains.

In this paper, we consider the structure of solutions of the semilinear elliptic equation

$$\Delta u + Q(|x|)u^p = 0 \quad \text{in } B, \tag{1.1}$$

where $p > 1$,

$$u^p = \begin{cases} |u|^p & \text{if } u > 0, \\ 0 & \text{if } u \le 0, \end{cases}$$

$Q(|x|)$ is a given nonnegative function, and

$$B = \{x \in \mathbf{R}^n; |x| < 1\}, \quad n > 2.$$

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Our main concern is the existence and uniqueness of positive radial solutions of (1.1) under the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial B. \tag{1.2}$$

Before studying the boundary value problem, we describe the result of [11] concerning the structure of positive radial solutions of (1.1) in the entire space $\mathbf{R}^n$. Since we are concerned with positive radial solutions, we consider the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + K(r)u^p = 0, & r \in (0,\infty), \\ u(0) = \alpha > 0. \end{cases} \tag{1.3}$$

Here we assume that $p > 1$, $n > 2$, and $K(r)$ satisfies

$$\text{(K)} \quad \begin{cases} K(r) \in C((0,\infty)); \\ K(r) \geq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0,\infty); \\ rK(r) \in L^1(0,1); \\ r^{n-1-(n-2)p} K(r) \in L^1(1,\infty). \end{cases}$$

We note that $K(r)$ may be unbounded at $r = 0$. Under the first and second conditions, it is shown in [7, 9] that the initial value problem (1.3) is uniquely solvable if and only if $rK(r) \in L^1(0,1)$. We denote the unique solution by $u(r;\alpha)$. It is known [11] that the solution of (1.3) is classified as

(*i*) a crossing solution: $u(r;\alpha)$ has a zero in $(0,\infty)$,

(*ii*) a slowly decaying solution: $u(r;\alpha) > 0$ on $[0,\infty)$ and $r^{n-2}u(r;\alpha) \to \infty$ as $r \to \infty$,

(*iii*) a rapidly decaying solution: $u(r;\alpha) > 0$ on $[0,\infty)$ and $\lim_{r\to\infty} r^{n-2}u(r;\alpha)$ exists and is positive.

Finally, it is known [1, 6] that if $r^{n-1-(n-2)p}K(r) \notin L^1(1,\infty)$, then any solution of (1.3) (whether or not it satisfies the initial condition) cannot be positive near ∞ so that $u(r;\alpha)$ is a crossing solution for any $\alpha \in (0,\infty)$.

It is shown in [11] that the Pohozaev identities

$$(1.4)\qquad \frac{d}{dr}\left\{\frac{1}{2}r^{n-1}u_r\{ru_r+(n-2)u\}+\frac{1}{p+1}r^nK(r)u^{p+1}\right\} \equiv G_r(r;K)u^{p+1}\equiv H_r(r;K)\{r^{n-2}u\}^{p+1},$$

play a crucial role for the structure of solution of (1.3), where $G(r;K)$ and $H(r;K)$ are functions on $(0,\infty)$ defined by

$$(1.5)\quad G(r;K) := \frac{1}{p+1}r^nK(r)-\frac{n-2}{2}\int_0^r r^{n-1}K(r)\,dr,$$

$$(1.6)\quad H(r;K) := \frac{1}{p+1}r^{2-(n-2)p}K(r)-\frac{n-2}{2}\int_r^\infty r^{1-(n-2)p}K(r)\,dr,$$

respectively. We define

$$r_G := \inf\{r\in(0,\infty);\ G(r;K)<0\},$$
$$r_H := \sup\{r\in(0,\infty);\ H(r;K)<0\}.$$

Here we put $r_G=\infty$ if $G(r;K)\ge 0$ on $(0,\infty)$, and $r_H=0$ if $H(r;K)\ge 0$ on $(0,\infty)$.

The following result was obtained in [11].

Theorem A *Suppose that* (K) *holds. Then the structure of solutions for* (1.3) *is as follows:*

(i) *If $G(r;K)\equiv 0$ on $(0,\infty)$, then the structure is of Type R: $u(r;\alpha)$ is a rapidly decaying solution for every $\alpha>0$.*

(ii) *Suppose that $G(r;K)\not\equiv 0$ on $(0,\infty)$.*

(a) *If $r_G=\infty$, then the structure is of Type C: $u(r;\alpha)$ is a crossing solution for every $\alpha>0$.*

(b) *If $0=r_H\le r_G<\infty$, then the structure is of Type S: $u(r;\alpha)$ is a slowly decaying solution for every $\alpha>0$.*

(c) *If $0<r_H\le r_G<\infty$, then the structure is of Type M: There exists $\alpha^*\in(0,\infty)$ such that*
$u(r;\alpha)$ is a crossing solution for $\alpha\in(\alpha^*,\infty)$,
$u(r;\alpha^*)$ is a rapidly decaying solution, and
$u(r;\alpha)$ is a slowly decaying solution for $\alpha\in(0,\alpha^*)$.

(iii) *Let a and b be any given numbers with $0 \le a < b \le \infty$. Then there exists $K(r)$ satisfying* (K), *$r_G = a$ and $r_H = b$ such that the structure is of none of Types R, C, S, and M.*

Thus, the structure of solutions for (1.3) can be determined completely in the case $r_H \le r_G$, but cannot be determined from only r_G and r_H in the case $r_G < r_H$. We note that $G(r; K) \equiv 0$ on $(0, \infty)$ if and only if

$$K(r) = c \cdot r^{\frac{(n-2)p-(n+2)}{2}} \tag{1.7}$$

for some constant $c > 0$. In this case, the identity $H(r; K) \equiv 0$ holds on $(0, \infty)$ and the solution of (1.3) is explicitly written as

$$u(r; \alpha) = \alpha \left\{ 1 + \frac{2\, c\, \alpha^{p-1}}{(p+1)(n-2)^2}\, r^{(n-2)(p-1)/2} \right\}^{-\frac{2}{p-1}}. \tag{1.8}$$

The proof of Theorem A in [11] is based on the effective use of the Pohozaev identities (1.4). Theorem A is a quite powerful tool to derive precise information on the structure of solutions, but it was pointed out by Kwong and Zhang [5] that the method is not applicable directly to boundary value problems on bounded domains, mainly because the above Pohozaev identities are not related with the boundary conditions. Nonetheless, we will show by modifying the method that similar results can be obtained for the boundary value problems (see [4] about general results).

Let us return to our original problem (1.1). In order to study positive radial solutions for the Dirichlet problem (1.1) with (1.2), we consider the initial value problems

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + Q(r) u^p = 0, & r \in (0,1), \\ u(0) = \alpha > 0, \end{cases} \tag{1.9}$$

and

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + Q(r) u^p = 0, & r \in (0,1), \\ u(1) = 0, \ u_r(1) = -\beta < 0. \end{cases} \tag{1.10}$$

For $Q(r)$, we will assume

$$(\mathrm{Q}) \quad \begin{cases} Q(r) \in C((0,1)); \\ Q(r) \geq 0 \text{ and } Q(r) \not\equiv 0 \text{ on } (0,1); \\ rQ(r) \in L^1(0,1/2); \\ (1-r)^p Q(r) \in L^1(1/2,1). \end{cases}$$

We note that $Q(r)$ may be unbounded at $r = 0$ or $r = 1$. As is noted above, under these conditions, (1.9) has a unique solution $u = u(r;\alpha)$ on $(0,1)$. The last condition in (Q) is a necessary and sufficient condition on the existence of a unique solution for (1.10) (see Lemma 2.7 below). We denote the unique solution by $u(r;\beta)$.

By Lemma 2.3 given in the next section, the solution of (1.9) is classified as

(i) a crossing solution: $u(r;\alpha)$ has a zero in $(0,1)$,

(ii) a singular solution: $u(r;\alpha) > 0$ on $[0,1)$ and $\lim_{r\to1} u(r;\alpha)/(1-r) = \infty$,

(iii) a regular solution: $u(r;\alpha) > 0$ on $[0,1)$ and $\lim_{r\to1} u(r;\alpha)/(1-r)$ exists and is positive.

We will see that if the last condition in (Q) does not hold, then $u(r;\alpha)$ is a crossing solution for any $\alpha \in (0,\infty)$. We note that depending on $Q(r)$, the singular solution satisfies $u(1) > 0$ or $u(1) = 0$. In the latter case, the singular solution is not differentiable at $r = 1$. Thus, if $u(r;\alpha)$ is a regular solution, then $u = u(|x|;\alpha)$ is a solution of the Dirichlet problem (1.1) with (1.2) in the class $C(\overline{B}) \cap C^1(\overline{B} - \{0\}) \cap C^2(B - \{0\})$.

Similarly, the solution of (1.10) is classified as

(i) a crossing solution: $u(r;\beta)$ has a zero in $(0,1)$,

(ii) a singular solution: $u(r;\beta) > 0$ on $(0,1)$ and $\lim_{r\to0} u(r;\beta) = \infty$,

(iii) a regular solution: $u(r;\beta) > 0$ on $(0,1)$ and $\lim_{r\to0} u(r;\beta)$ exists and is positive.

We will see that if the third condition in (Q) does not hold, then $u(r;\beta)$ is a crossing solution for any $\beta \in (0,\infty)$. Also, if $u(r;\beta)$ is a regular solution, then $u = u(|x|;\beta)$ is a solution of the Dirichlet problem (1.1) with (1.2) in the class $C(\overline{B}) \cap C^1(\overline{B} - \{0\}) \cap C^2(B - \{0\})$.

To derive an analog of Theorem A, we need to modify the Pohozaev identities and the functions $G(r;K)$ and $H(r;K)$ by taking the boundary condition (1.2) into account. We will see that the following identities are suitable for our purpose:

$$\frac{d}{dr}\left\{\frac{1}{2}r^{n-1}u_r\{ru_r + (n-2)u - r^{n-1}u_r\} + \frac{1}{p+1}(1-r^{n-2})r^n Q(r)u^{p+1}\right\}$$
$$\equiv G_r^b(r;Q)u^{p+1} \equiv H_r^b(r;Q)\left(\frac{r^{n-2}u}{1-r^{n-2}}\right)^{p+1}, \tag{1.11}$$

where $G^b(r;Q)$ and $H^b(r;Q)$ are functions on $(0,1)$ defined by

$$G^b(r;Q) := \frac{1}{p+1}(1-r^{n-2})r^n Q(r) - \frac{n-2}{2}\int_0^r r^{n-1}Q(r)\,dr, \tag{1.12}$$

$$H^b(r;Q) := \frac{1}{p+1}(1-r^{n-2})^{p+2}r^{2-(n-2)p}Q(r) - \frac{n-2}{2}\int_r^1 (1-r^{n-2})^{p+1}r^{1-(n-2)p}Q(r)\,dr. \tag{1.13}$$

We also define

$$r_G^b := \inf\{r \in (0,1);\ G^b(r;Q) < 0\},$$
$$r_H^b := \sup\{r \in (0,1);\ H^b(r;Q) < 0\}.$$

Here we put $r_G^b = 1$ if $G^b(r;Q) \geq 0$ on $(0,1)$, and $r_H^b = 0$ if $H^b(r;Q) \geq 0$ on $(0,1)$. Thus we have $0 \leq r_G^b, r_H^b \leq 1$ by definition.

Now we state our main results on (1.9) and (1.10).

Theorem 1.1 *Suppose that* (Q) *holds. Then the structure of solutions for* (1.9) *is as follows:*

(i) *If* $G^b(r;Q) \equiv 0$ *on* $(0,1)$, *then the structure is of Type R:* $u(r;\alpha)$ *is a regular solution for every* $\alpha > 0$.

(ii) *Suppose that* $G^b(r;Q) \not\equiv 0$ *on* $(0,1)$.

(a) *If* $0 < r_H^b \le r_G^b = 1$, *then the structure is of Type C:* $u(r;\alpha)$ *is a crossing solution for every* $\alpha > 0$.

(b) *If* $0 = r_H^b \le r_G^b < 1$, *then the structure is of Type S:* $u(r;\alpha)$ *is a singular solution for every* $\alpha > 0$.

(c) *If* $0 < r_H^b \le r_G^b < 1$, *then the structure is of Type M: There exists* $\alpha^b \in (0,\infty)$ *such that*
$u(r;\alpha)$ *is a crossing solution for* $\alpha \in (\alpha^b,\infty)$,
$u(r;\alpha^b)$ *is a regular solution, and*
$u(r;\alpha)$ *is a singular solution for* $\alpha \in (0,\alpha^b)$.

(iii) *Let* a *and* b *be any given numbers with* $0 \le a < b \le 1$. *Then there exists* $Q(r)$ *satisfying* (Q), $r_G^b = a$ *and* $r_H^b = b$ *such that the structure is of none of Types R, C, S, and M.*

Theorem 1.2 *Suppose that* (Q) *holds. Then the structure of solutions for* (1.10) *is as follows:*

(i) *If* $G^b(r;Q) \equiv 0$ *on* $(0,1)$, *then the structure is of Type R:* $u(r;\beta)$ *is a regular solution for every* $\beta > 0$.

(ii) *Suppose that* $G^b(r;Q) \not\equiv 0$ *on* $(0,1)$.

(a) *If* $0 = r_H^b \le r_G^b < 1$, *then the structure is of Type C:* $u(r;\beta)$ *is a crossing solution for every* $\beta > 0$.

(b) *If* $0 < r_H^b \le r_G^b = 1$, *then the structure is of Type S:* $u(r;\beta)$ *is a singular solution for every* $\beta > 0$.

(c) *If* $0 < r_H^b \le r_G^b < 1$, *then the structure is of Type M: There exists* $\beta^b \in (0,\infty)$ *such that*
$u(r;\beta)$ *is a crossing solution for* $\beta \in (\beta^b,\infty)$,
$u(r;\beta^b)$ *is a regular solution, and*
$u(r;\beta)$ *is a singular solution for* $\beta \in (0,\beta^b)$.

(iii) *Let* a *and* b *be any given numbers with* $0 \le a < b \le 1$. *Then there exists* $Q(r)$ *satisfying* (Q), $r_G^b = a$ *and* $r_H^b = b$ *such that the structure is of none of Types R, C, S, and M.*

We note that $G^b(r;Q) \equiv 0$ if and only if

$$Q(r) = c \cdot r^{\frac{(n-2)p-(n+2)}{2}} (1 - r^{n-2})^{-\frac{p+3}{2}} \tag{1.14}$$

for some $c > 0$. In this case, we have $H^b(r;Q) \equiv 0$ on $(0,1)$ and the solutions of (1.9) and (1.10) are explicitly written as

$$u(r;\alpha) = \alpha \left\{ 1 + \frac{2\,c\,\alpha^{p-1}}{(p+1)(n-2)^2} \left(\frac{r^{n-2}}{1-r^{n-2}} \right)^{\frac{p-1}{2}} \right\}^{-\frac{2}{p-1}} \tag{1.15}$$

and

$$u(r;\beta) = \frac{\beta(1-r^{n-2})}{r^{n-2}} \left\{ 1 + \frac{2\,c\,\beta^{p-1}}{(p+1)(n-2)^2} \left(\frac{1-r^{n-2}}{r^{n-2}} \right)^{\frac{p-1}{2}} \right\}^{-\frac{2}{p-1}}, \tag{1.16}$$

respectively.

There are two ways to prove the above theorems. One is to follow the proof in [11] for (1.3) by using the modified Pohozaev identities (1.11). The other is to transform (1.9) and (1.10) into the form of (1.3) by suitable changes of variables, and apply Theorem A to the transformed systems. Then we may inversely transform the results for (1.3) to those for (1.9) and (1.10). In this paper, we adopt the latter method.

This paper is organized as follows. In section 2, we give suitable changes of variables to transform (1.9) and (1.10) into the form of (1.3). Then the proofs of Theorems 1.1 and 1.2 are obtained easily from Theorem A. In section 3, as applications of Theorems 1.1 and 1.2, we give a few corollaries concerning the structure of solutions for the problems (1.9) and (1.10) with $Q(r) = r^\sigma/(1-r^{n-2})^\tau$. We will also give an application to some exterior Dirichlet problem. Section 4 is devoted to proofs of the corollaries.

2 Proofs of Theorems

In this section, we give proofs of Theorems 1.1 and 1.2. First we introduce a change of variables which transforms (1.9) to the form of (1.3).

Lemma 2.1 *Set*

$$w(s;\alpha) := u(r;\alpha) \quad \textit{and} \quad s^{-n+2} := r^{-n+2} - 1. \tag{2.1}$$

Then (1.9) *is transformed to*

$$\begin{cases} w_{ss} + \dfrac{n-1}{s}\, w_s + K(s) w^p = 0, & s \in (0,\infty), \\ w(0;\alpha) = \alpha > 0, \end{cases} \tag{2.2}$$

where

$$s^{2(n-1)} K(s) := r^{2(n-1)} Q(r). \tag{2.3}$$

Proof. Differentiating $s^{-n+2} = r^{-n+2} - 1$, we obtain

$$r^{n-1}\frac{d}{dr} = s^{n-1}\frac{d}{ds}.$$

Hence

$$\begin{aligned} r^{2(n-1)}\left(u_{rr} + \frac{n-1}{r}\, u_r\right) &= r^{n-1}(r^{n-1}u_r)_r \\ &= s^{n-1}(s^{n-1}w_s)_s \\ &= s^{2(n-1)}\left(w_{ss} + \frac{n-1}{s}\, w_s\right). \end{aligned}$$

Clearly $w(0;\alpha) = u(0;\alpha)$. Thus we obtain (2.2) with (2.3). Q.E.D.

Lemma 2.2 *Let* $K(s)$ *be defined by* (2.3)*. Then* $K(s)$ *satisfies* (K) *if and only if* $Q(r)$ *satisfies* (Q)*.*

Proof. It is clear from (2.3) that the first and second conditions in (Q) are equivalent to those in (K). We have

$$sK(s)ds = s \cdot s^{-2(n-1)} r^{2(n-1)} Q(r) \cdot s^{n-1} r^{-n+1} dr = (1 - r^{n-2}) r Q(r) dr.$$

Hence $sK(s) \in L^1(0,1)$ if and only if $rQ(r) \in L^1(0,1/2)$. Similarly, we have

$$\begin{aligned} s^{n-1-(n-2)p} K(s) ds &= s^{n-1-(n-2)p} \cdot s^{-2(n-1)} r^{2(n-1)} Q(r) \cdot s^{n-1} r^{-n+1} dr \\ &= r^{n-1-(n-2)p} (1 - r^{n-2})^p Q(r) dr. \end{aligned}$$

Hence $s^{n-1-(n-2)p} K(s) \in L^1(1,\infty)$ if and only if $(1-r)^p Q(r) \in L^1(1/2,1)$. Thus the proof is complete. Q.E.D.

Lemma 2.3 *Let $u(r;\alpha)$ be a solution of* (1.9). *If $u(r;\alpha)>0$ on $[0,1)$, then $r^{n-2}u(r;\alpha)/(1-r^{n-2})$ is non-decreasing in $r\in(0,1)$.*

Proof. Let $w(s;\alpha)$ be the solution of (2.2). It is shown in Lemma 7.2 of [11] that if $w(s;\alpha)>0$ on $[0,\infty)$, then $s^{n-2}w(s;\alpha)$ is non-decreasing in $s\in(0,\infty)$. By the transformation (2.1), this implies that if $u(r;\alpha)>0$ on $[0,1)$, then $s^{n-2}w(s;\alpha)=r^{n-2}u(r;\alpha)/(1-r^{n-2})$ is non-decreasing in $r\in(0,1)$. Q.E.D.

We see from this lemma that any solution of (1.9) is classified as one of the crossing solution, singular solution and regular solution as in the introduction.

Lemma 2.4 *By the transformation* (2.1), *the following holds.*

(*i*) *Any crossing solution of* (1.9) *corresponds to a crossing solution of* (2.2).

(*ii*) *Any singular solution of* (1.9) *corresponds to a slowly decaying solution of* (2.2).

(*iii*) *Any regular solution of* (1.9) *corresponds to a rapidly decaying solution of* (2.2).

Proof. First, suppose that $u(y;\alpha)=0$ at some $y\in(0,1)$. Then, by (2.1), we have $w(z;\alpha)=0$ at $z:=y/(1-y^{n-2})^{1/(n-2)}\in(0,\infty)$. This proves (i). Next, suppose that $u(r;\alpha)>0$ on $[0,1)$. Then, again by (2.1), we have $s^{n-2}w(s;\alpha)=r^{n-2}u(r;\alpha)/(1-r^{n-2})$. This proves (ii) and (iii). Q.E.D.

By the transformation (2.1), we will rewrite the Pohozaev identities (1.11) and the functions $G^b(r;Q)$ and $H^b(r;Q)$ defined by (1.12) and (1.13), respectively.

Lemma 2.5 *Let $K(s)$ be defined by* (2.3). *Then*

$$G^b(r;Q)\equiv G(s;K) \quad \textit{and} \quad H^b(r;Q)\equiv H(s;K).$$

Proof. By $r^{-n+2}-1 = s^{-n+2}$, $r^{2n-2}Q(r) = s^{2n-2}K(s)$ and $r^{1-n}dr = s^{1-n}ds$, we have

$$\begin{aligned}
G^b(r;Q) &= \frac{1}{p+1}(1-r^{n-2})r^nQ(r) - \frac{n-2}{2}\int_0^r r^{n-1}Q(r)\,dr \\
&= \frac{1}{p+1}\cdot(r^{-n+2}-1)\cdot r^{2n-2}Q(r) - \frac{n-2}{2}\int_0^r r^{2n-2}Q(r)\cdot r^{-n+1}\,dr \\
&= \frac{1}{p+1}\cdot s^{-n+2}\cdot s^{2n-2}K(s) - \frac{n-2}{2}\int_0^s s^{2n-2}K(s)\cdot s^{-n+1}\,ds \\
&= \frac{1}{p+1}s^nK(s) - \frac{n-2}{2}\int_0^s s^{n-1}K(s)\,ds \\
&= G(s;K).
\end{aligned}$$

Similarly,

$$\begin{aligned}
H^b(r;Q) &= \frac{1}{p+1}(1-r^{n-2})^{p+2}r^{2-(n-2)p}Q(r) \\
&\quad -\frac{n-2}{2}\int_r^1 (1-r^{n-2})^{p+1}r^{1-(n-2)p}Q(r)\,dr \\
&= \frac{1}{p+1}\cdot(r^{-n+2}-1)^{p+2}\cdot r^{2n-2}Q(r) \\
&\quad -\frac{n-2}{2}\int_r^1 (r^{-n+2}-1)^{p+1}\cdot r^{2n-2}Q(r)\cdot r^{-n+1}\,dr \\
&= \frac{1}{p+1}\cdot(s^{-n+2})^{p+2}\cdot s^{2n-2}K(s) \\
&\quad -\frac{n-2}{2}\int_s^\infty (s^{-n+2})^{p+1}\cdot s^{2n-2}K(s)\cdot s^{-n+1}\,ds \\
&= \frac{1}{p+1}s^{2-(n-2)p}K(s) - \frac{n-2}{2}\int_s^\infty s^{1-(n-2)p}K(s)\,ds \\
&= H(s;K).
\end{aligned}$$

Thus the proof is complete. Q.E.D.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We define

$$\begin{aligned}
s_G &:= \inf\{s\in(0,\infty);\ G(s;K)<0\}, \\
s_H &:= \sup\{s\in(0,\infty);\ H(s;K)<0\}.
\end{aligned}$$

By Lemmas 2.1 and 2.5, we have

$$(s_G)^{-n+2} = (r_G)^{-n+2} - 1 \quad \text{and} \quad (s_H)^{-n+2} = (r_H)^{-n+2} - 1.$$

Then, by Lemmas 2.2, 2.4 and Theorem A, the proof is complete. Q.E.D.

Remark 2.1. By the transformation (2.1), we have

$$\begin{aligned}
&r^{n-1}u_r = s^{n-1}w_s,\\
&ru_r + (n-2)u - r^{n-1}u_r = (r^{-n+2} - 1) \cdot r^{n-1}u_r + (n-2)u\\
&\qquad\qquad = s^{-n+2} \cdot s^{n-1}w_s + (n-2)w = sw_s + (n-2)w,\\
&(1 - r^{n-2})r^n Q(r)u^{p+1} = (r^{-n+2} - 1) \cdot r^{2n-2}Q(r) \cdot u^{p+1}\\
&\qquad\qquad = s^{-n+2} \cdot s^{2n-2}K(s) \cdot w^{p+1} = s^n K(s)w^{p+1}.
\end{aligned}$$

Hence the modified Pohozaev identities (1.11) are transformed to the Pohozaev identities (1.4). Also, it is easy to see that the function $Q(r)$ given by (1.14) and the exact solution given by (1.15) correspond to the function $K(r)$ given by (1.7) and the exact solution given by (1.8), respectively.

A proof of Theorem 1.2 can be obtained in essentially the same manner. We introduce a change of variables which transforms (1.10) to the form of (1.3).

Lemma 2.6 *Set*

$$s^{n-2}w(s;\beta) := u(r;\beta) \quad \textit{and} \quad s^{n-2} := r^{-n+2} - 1. \tag{2.4}$$

Then (1.10) *is transformed to*

$$\begin{cases} w_{ss} + \dfrac{n-1}{s}\, w_s + K(s)w^p = 0, & s \in (0,\infty),\\[2mm] w(0;\beta) = \dfrac{1}{n-2}\,\beta > 0, \end{cases} \tag{2.5}$$

where

$$s^{-n+4-(n-2)p}K(s) := r^{2n-2}Q(r). \tag{2.6}$$

Proof. Differentiating $s^{n-2} = r^{-n+2} - 1$, we obtain

$$r^{n-1}\frac{d}{dr} = -s^{-n+3}\frac{d}{ds}.$$

Hence

$$\begin{aligned} r^{2n-2}\left(u_{rr} + \frac{n-1}{r}\,u_r\right) &= r^{n-1}(r^{n-1}u_r)_r \\ &= s^{-n+3}\{s^{-n+3}(s^{n-2}w)_s\}_s \\ &= s^{-n+4}\left(w_{ss} + \frac{n-1}{s}\,w_s\right). \end{aligned}$$

Moreover, we have

$$\beta = -r^{n-1}u_r(r;\beta)\Big|_{r=1} = s^{-n+3}\{s^{n-2}w(s;\beta)\}_s\Big|_{s=0} = (n-2)w(0;\beta).$$

Thus we obtain (2.5) with (2.6). Q.E.D.

By using the change of variables (2.4), we will rewrite the functions $G^b(r;Q)$ and $H^b(r;Q)$ defined by (1.12) and (1.13), respectively.

The next lemma can be proved in the same manner as in Lemma 2.2.

Lemma 2.7 *Let $K(s)$ be defined by* (2.6)*. Then $K(s)$ satisfies* (K) *if and only if $Q(r)$ satisfies* (Q)*.*

Lemma 2.8 *Let $K(s)$ be defined by* (2.6)*. Then*

$$G^b(r;Q) \equiv H(s;K) \quad \textit{and} \quad H^b(r;Q) \equiv G(s;K),$$

where G and H are defined by (1.5) *and* (1.6)*, respectively.*

Proof. By $r^{-n+2} - 1 = s^{n-2}$, $r^{2n-2}Q(r) = s^{-n+4-(n-2)p}K(s)$ and $r^{-n+1}dr = -s^{n-3}ds$, we have

$$\begin{aligned} G^b(r;Q) &= \frac{1}{p+1}\cdot(r^{-n+2}-1)\cdot r^{2n-2}Q(r) - \frac{n-2}{2}\int_0^r r^{2n-2}Q(r)\cdot r^{-n+1}\,dr \\ &= \frac{1}{p+1}\cdot s^{n-2}\cdot s^{-n+4-(n-2)p}K(s) - \frac{n-2}{2}\int_s^\infty s^{-n+4-(n-2)p}K(s)\cdot s^{n-3}\,ds \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{p+1} s^{2-(n-2)p} K(s) - \frac{n-2}{2} \int_s^\infty s^{1-(n-2)p} K(s)\, ds \\
&= H(s; K).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
H^b(r; Q) &= \frac{1}{p+1} \cdot (r^{-n+2} - 1)^{p+2} \cdot r^{2n-2} Q(r) \\
&\quad - \frac{n-2}{2} \int_r^1 (r^{-n+2} - 1)^{p+1} \cdot r^{2n-2} Q(r) \cdot r^{-n+1}\, dr \\
&= \frac{1}{p+1} \cdot s^{(n-2)(p+2)} \cdot s^{-n+4-(n-2)p} K(s) \\
&\quad - \frac{n-2}{2} \int_0^s s^{(n-2)(p+1)} \cdot s^{-n+4-(n-2)p} K(s) \cdot s^{n-3}\, ds \\
&= \frac{1}{p+1} s^n K(s) - \frac{n-2}{2} \int_0^s s^{n-1} K(s)\, ds \\
&= G(s; K).
\end{aligned}
$$

Thus the proof is complete. Q.E.D.

The remaining part of the proof of Theorems 1.2 is obtained in a similar manner to Theorem 1.1. We omit the details.

Remark 2.2. By the transformation (2.4), we have

$$
\begin{aligned}
& r^{n-1} u_r = -s^{-n+3} (s^{n-2} w)_s = -s w_s - (n-2) w, \\
& r u_r + (n-2) u - r^{n-1} u_r = (r^{-n+2} - 1) \cdot r^{n-1} u_r + (n-2) u \\
& \qquad = -s^{n-2} \cdot s^{-n+3} (s^{n-2} w)_s + (n-2) s^{n-2} w = -s^{n-1} w_s, \\
& (1 - r^{n-2}) r^n Q(r) u^{p+1} = (r^{-n+2} - 1) \cdot r^{2n-2} Q(r) \cdot u^{p+1} \\
& \qquad = s^{n-2} \cdot s^{-n+4-(n-2)p} K(s) \cdot (s^{n-2} w)^{p+1} = s^n K(s) w^{p+1}.
\end{aligned}
$$

Hence the modified Pohozaev identities (1.11) are transformed to the Pohozaev identities (1.4). Also, it is easy to see that the function $Q(r)$ given by (1.14) and the exact solution given by (1.16) correspond to the function $K(r)$ given by (1.7) and the exact solution given by (1.8), respectively.

3 Applications

In this section, as applications of Theorems 1.1 and 1.2, we investigate the structure of solutions to (1.9) and (1.10) with $Q(r) = r^\sigma/(1-r^{n-2})^\tau$, where σ and τ are real parameters. We note that $Q(r)$ is unbounded at $r=0$ for $\sigma < 0$ and unbounded at $r=1$ for $\tau > 0$. We will also give an application to some exterior Dirichlet problem.

First we consider the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \dfrac{r^\sigma}{(1-r^{n-2})^\tau} u^p = 0, \quad r \in (0,1), \\ u(0) = \alpha > 0. \end{cases} \tag{3.1}$$

If $\sigma \le -2$, then $rQ(r) \notin L^1(0,1/2)$ so that (3.1) has no solution. Hence we consider the case $\sigma > -2$ only.

Corollary 3.1 *The structure of solutions of* (3.1) *with* $\sigma > -2$ *is as follows:*

(*i*) *Let* $1 < p < (n+2+2\sigma)/(n-2)$.

- (a) *If* $\tau \ge (p+3)/2$, *then the structure is of Type C.*
- (b) *If* $\tau < (p+3)/2$, *then the structure is of Type M.*

(*ii*) *Let* $p = (n+2+2\sigma)/(n-2)$.

- (a) *If* $\tau > (p+3)/2$, *then the structure is of Type C.*
- (b) *If* $\tau = (p+3)/2$, *then the structure is of Type R. Moreover the solution is written as*

$$u(r;\alpha) = \alpha \left\{ 1 + \frac{\alpha^{\frac{2(\sigma+2)}{n-2}}}{(n+\sigma)(n-2)} \left(\frac{r^{n-2}}{1-r^{n-2}} \right)^{\frac{\sigma+2}{n-2}} \right\}^{-\frac{n-2}{\sigma+2}}. \tag{3.2}$$

- (c) *If* $\tau < (p+3)/2$, *then the structure is of Type S.*

(*iii*) *Let* $p > (n+2+2\sigma)/(n-2)$.

- (a) *If* $\tau \ge p+1$, *then the structure is of Type C.*
- (b) *If* $\tau \le (p+3)/2$, *then the structure is of Type S.*

xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
Figure 1: Structure of solutions for (3.1) with $\sigma > -2$.
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx

Remark 3.1. If $\sigma > -2$, $p > (n+2+2\sigma)/(n-2)$ and $(p+3)/2 < \tau < p+1$, then it is hard to determine the structure of solutions. Numerical computations suggest that the structure of solutions would be very complicated depending on the dimension n.

For $n = 3$, Hashimoto and Ôtani [3] obtained some existence and non-existence results for the above equation. We can derive more precise information from Corollary 3.1. Especially, we can show the uniqueness of regular solutions.

Next we investigate the structure of solutions for

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \dfrac{r^\sigma}{(1-r^{n-2})^\tau} u^p = 0, & r \in (0,1), \\ u(1) = 0, \quad u_r(1) = -\beta < 0. \end{cases} \tag{3.3}$$

If $\tau \geq p+1$, then $(1-r)^p Q(r) \notin L^1(1/2, 1)$ so that (3.3) has no solution. Hence we will consider the case $\tau < p+1$ only.

Corollary 3.2 *The structure of solutions of* (3.3) *with* $\tau < p+1$ *is as follows:*

(*i*) *Let* $\sigma \leq -2$. *Then the structure is of Type C.*

(*ii*) *Let* $\sigma > -2$ *and* $1 < p < (n+2+2\sigma)/(n-2)$.

- (a) *If* $\tau \geq (p+3)/2$, *then the structure is of Type S.*
- (b) *If* $\tau < (p+3)/2$, *then the structure is of Type M.*

(*iii*) *Let* $\sigma > -2$ *and* $p = (n+2+2\sigma)/(n-2)$.

- (a) *If* $\tau > (p+3)/2$, *then the structure is of Type S.*

(b) *If* $\tau = (p+3)/2$, *then the structure is of Type R. Moreover the solution is written as*

$$u(r;\beta) = \frac{\beta(1-r^{n-2})}{r^{n-2}} \left\{ 1 + \frac{\beta^{\frac{2(\sigma+2)}{n-2}}}{(n+\sigma)(n-2)} \left(\frac{1-r^{n-2}}{r^{n-2}} \right)^{\frac{\sigma+2}{n-2}} \right\}^{-\frac{n-2}{\sigma+2}} . \tag{3.4}$$

(c) *If* $\tau < (p+3)/2$, *then the structure is of Type C.*

(*iv*) *Let* $\sigma > -2$ *and* $p > (n+2+2\sigma)/(n-2)$. *If* $\tau \leq (p+3)/2$, *then the structure is of Type C.*

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Figure 2: Structure of solutions for (3.3) with $\sigma \leq -2$.
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xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
Figure 3: Structure of solutions for (3.3) with $\sigma > -2$.
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Remark 3.2. If $\sigma > -2$, $p > (n+2+2\sigma)/(n-2)$ and $(p+3)/2 < \tau < p+1$, then it is hard to determine the structure of solutions. Numerical computations also suggest that the structure of solutions would be very complicated depending on the dimension n.

For $n = 3$, Senba, Ebihara and Furusho [10] obtained some existence results for the above equation. We can derive more precise information from Corollary 3.2.

Finally we treat the structure of solutions for the initial value problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r} u_r + \dfrac{r^\sigma}{(r^{n-2}-1)^\tau} u^p = 0, & r \in (1,\infty), \\ u(1) = 0, \quad u_r(1) = \gamma > 0, \end{cases} \tag{3.5}$$

which corresponds to the exterior Dirichlet problem. We note that this initial value problem is uniquely solvable if and only if $\tau < p+1$. We denote the unique solution by $u(r;\gamma)$. Similar to (1.3), any solution of (3.5) is classified as

(*i*) a crossing solution: $u(r;\gamma)$ has a zero in $(1,\infty)$,

(*ii*) a slowly decaying solution: $u(r;\gamma) > 0$ on $(1,\infty)$ and $r^{n-2}u(r;\gamma) \to \infty$ as $r \to \infty$,

(*iii*) a rapidly decaying solution: $u(r;\gamma) > 0$ on $(1,\infty)$ and $\lim_{r\to\infty} r^{n-2}u(r;\gamma)$ exists and is positive.

The structures of Type R, C, S and M are defined in the same way as in Theorem A.

Corollary 3.3 *The structure of solutions of* (3.5) *with* $\tau < p+1$ *is as follows:*

(*i*) *Let* $p \le (n+\sigma)/(n-2) - \tau$. *Then the structure is of Type C.*

(*ii*) *Let* $p > (n+\sigma)/(n-2) - \tau$ *and* $p > (n+2+2\sigma)/(n-2) - 2\tau$.

 (a) *If* $\tau \ge (p+3)/2$, *then the structure is of Type S.*

 (b) *If* $\tau < (p+3)/2$, *then the structure is of Type M.*

(*iii*) *Let* $p > (n+\sigma)/(n-2) - \tau$ *and* $p = (n+2+2\sigma)/(n-2) - 2\tau$.

 (a) *If* $\tau > (p+3)/2$, *then the structure is of Type S.*

 (b) *If* $\tau = (p+3)/2$, *then the structure is of Type R. Moreover the solution is written as*

$$u(r;\gamma) = \frac{\gamma(r^{n-2}-1)}{r^{n-2}} \left\{ 1 + \frac{2\gamma^{\frac{\sigma-2(n-3)}{n-2}}}{(\sigma+2)(n-2)} \left(r^{n-2}-1\right)^{\frac{\sigma-2(n-3)}{2(n-2)}} \right\}^{-\frac{2(n-2)}{\sigma-2(n-3)}}. \tag{3.6}$$

 (c) *If* $\tau < (p+3)/2$, *then the structure is of Type C.*

(*iv*) *Let* $p > (n+\sigma)/(n-2) - \tau$ *and* $p < (n+2+2\sigma)/(n-2) - 2\tau$. *If* $\tau \le (p+3)/2$, *then the structure is of Type C.*

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Figure 4: Structure of solutions for (3.5) with $\sigma \le 2(n-3)$.
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xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
Figure 5: Structure of solutions for (3.5) with $\sigma > 2(n-3)$.
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Remark 3.3. If $p > (n+\sigma)/(n-2) - \tau$, $p < (n+2+2\sigma)/(n-2) - 2\tau$ and $(p+3)/2 < \tau < p+1$, then it is hard to determine the structure of solutions. Numerical computations suggest that the structure of solutions would be very complicated depending on the dimension n.

For $n = 3$, Ebihara and Furusho [2] obtained partial results for the above equation. Hashimoto and Ôtani also have shown some results for a special case by modifying the method in [3]. We can obtain more precise information from Corollary 3.3.

4 Proofs of Corollaries

In this section, we give proofs of Crollaries 3.1, 3.2 and 3.3. Throughout this section, we set $Q(r) = r^\sigma/(1-r^{n-2})^\tau$ and simply write $G^b(r)$ and $H^b(r)$ instead of $G^b(r;Q)$ and $H^b(r;Q)$, respectively.

We first prove the following results.

Lemma 4.1 *If $\sigma > -2$ and $\tau \ge p+1$, then the structure of solutions for* (3.1) *is of Type C.*

Proof. In the proof of Lemma 2.2, we showed that $s^{n-1-(n-2)p}K(s) \in L^1(1,\infty)$ if and only if $(1-r)^pQ(r) \in L^1(1/2,1)$. It is easy to see that $(1-r)^pQ(r) \notin L^1(1/2,1)$ if $\tau \ge p+1$. Thus the structure of solutions of (3.1) is of Type C by [1, 6]. Q.E.D.

Lemma 4.2 *If $\sigma \le -2$ and $\tau < p+1$, then the structure of solutions for* (3.3) *is of Type C.*

Proof. Since $rQ(r) \notin L^1(0, 1/2)$ if $\sigma \le -2$, the structure of solutions of (3.3) is of Type C by [1, 6]. Q.E.D.

In what follows, we consider the case where $\sigma > -2$ and $\tau < p+1$. For $Q(r) = r^\sigma/(1-r^{n-2})^\tau$, the functions $G^b(r)$ and $H^b(r)$ defined by (1.12) and (1.13) are computed as

$$(4.1)\, G^b(r) = \frac{1}{p+1}\frac{r^{n+\sigma}}{(1-r^{n-2})^{\tau-1}} - \frac{n-2}{2}\int_0^r \frac{r^{n-1+\sigma}}{(1-r^{n-2})^\tau}dr,$$

$$(4.2)\, H^b(r) = \frac{1}{p+1}\frac{r^{\sigma+2-(n-2)p}}{(1-r^{n-2})^{\tau-p-2}} - \frac{n-2}{2}\int_r^1 \frac{r^{\sigma+1-(n-2)p}}{(1-r^{n-2})^{\tau-p-1}}dr.$$

Since $n+\sigma > 0$ and $\tau - p - 2 < 0$, we have

$$G^b(0) = 0, \quad H^b(1) = 0.$$

Moreover, we have

$$(4.3) \qquad G_r^b(r) = \frac{1}{p+1}\cdot\frac{r^{n-1+\sigma}}{(1-r^{n-2})^\tau}\cdot S(r),$$

$$(4.4) \qquad H_r^b(r) = \frac{1}{p+1}\cdot\frac{r^{\sigma+1-(n-2)p}}{(1-r^{n-2})^{\tau-p-1}}\cdot S(r),$$

where

$$(4.5)\;\; S(r) := \{(n-2)\tau - 2(n-1) - \sigma\}r^{n-2} + \frac{n+2+2\sigma-(n-2)p}{2}.$$

We note that

$$S(0) = \frac{n+2+2\sigma-(n-2)p}{2}, \quad S(1) = (n-2)\left(\tau - \frac{p+3}{2}\right).$$

Lemma 4.3 *Let $1 < p < (n+2+2\sigma)/(n-2)$.*

(i) *If $(p+3)/2 \le \tau < p+1$, then $r_G^b = r_H^b = 1$.*

(ii) *If $\tau < (p+3)/2$, then $0 < r_H^b < r_G^b < 1$.*

Proof. For $1 < p < (n+2+2\sigma)/(n-2)$, we have $S(0) > 0$ and

$$S(1) \begin{cases} > 0 & \text{if } \tau > (p+3)/2, \\ = 0 & \text{if } \tau = (p+3)/2, \\ < 0 & \text{if } \tau < (p+3)/2. \end{cases}$$

Hence, if $(p+3)/2 \le \tau < p+1$, then $0 \le \min\{S(0), S(1)\} < S(r)$ on $(0,1)$, which implies $G_r^b(r) > 0$ on $(0,1)$ by (4.3) and $H_r^b(r) > 0$ on $(0,1)$ by (4.4). Thus we obtain $r_G^b = 1 = r_H^b$, which proves (i).

Next, if $\tau < (p+3)/2$, then $S(0) > 0$, $S(1) < 0$, and the function $S(r)$ defined by (4.5) has a unique zero in $(0,1)$. Hence, by (4.3) and (4.4), there exists $r_* \in (0,1)$ such that

$$G_r^b(r) > 0 \text{ on } (0, r_*) \quad \text{and} \quad G_r^b(r) < 0 \text{ on } (r_*, 1)$$

and

$$H_r^b(r) > 0 \text{ on } (0, r_*) \quad \text{and} \quad H_r^b(r) < 0 \text{ on } (r_*, 1).$$

Since $G^b(0) = H^b(1) = 0$, it suffices to show that $G^b(1) \in [-\infty, 0)$ and $H^b(0) \in [-\infty, 0)$.

If $1 \le \tau < (p+3)/2$, then it is easy to see from (4.1) that $G^b(1) = -\infty$. If $\tau < 1$, then it follows from (4.1) that

$$G^b(1) = -\frac{n-2}{2} \int_0^1 \frac{r^{n-1+\sigma}}{(1-r^{n-2})^\tau} dr \in (-\infty, 0).$$

Thus we obtain $G^b(1) \in [-\infty, 0)$. Similarly, if $(\sigma+2)/(n-2) \le p < (n+2+2\sigma)/(n-2)$, then it is easy to see from (4.2) that $H^b(0) = -\infty$. If $1 < p < (\sigma+2)/(n-2)$, then it follows from (4.2) that

$$H^b(0) = -\frac{n-2}{2} \int_0^1 \frac{r^{\sigma+1-(n-2)p}}{(1-r^{n-2})^{\tau-p-1}} dr \in (-\infty, 0).$$

Thus we obtain $H^b(0) \in [-\infty, 0)$. The proof of (ii) is now complete. Q.E.D.

Lemma 4.4 *Let* $p = (n+2+2\sigma)/(n-2)$.

(i) *If* $(p+3)/2 < \tau < p+1$, *then* $r_G^b = r_H^b = 1$.

(ii) *If* $\tau = (p+3)/2$, *then* $G^b(r) \equiv 0$ *on* $(0,1)$. *Moreover, the solutions of* (3.1) *and* (3.3) *are explicitly written as* (3.2) *and* (3.4), *respectively.*

(iii) *If* $\tau < (p+3)/2$, *then* $r_G^b = r_H^b = 0$.

Proof. For $p = (n+2+2\sigma)/(n-2)$, $S(r)$ is monotone in $(0,1)$ and $S(0) = 0$. Hence

$$S(r) \begin{cases} > 0 & \text{if } \tau > (p+3)/2, \\ < 0 & \text{if } \tau < (p+3)/2, \end{cases}$$

on $(0,1)$. Since $G^b(0) = 0 = H^b(1)$, we obtain (i) and (iii). If $\tau = (p+3)/2$, then $S(r) \equiv 0$ so that $G^b(r) \equiv 0$ on $(0,1)$. The exact solutions (3.2) and (3.4) are immediately obtained from (1.15) and (1.16). Q.E.D.

Lemma 4.5 *Let* $p > (n+2+2\sigma)/(n-2)$. *If* $\tau \le (p+3)/2$, *then* $r_G^b = r_H^b = 0$.

Proof. For $p > (n+2+2\sigma)/(n-2)$, we have $S(0) < 0$ and $S(1) \le 0$. Since $S(r) < \max\{S(0), S(1)\} \le 0$ on $(0,1)$, the inequalities $G_r^b(r) < 0$ and $H_r^b(r) < 0$ hold on $(0,1)$ in view of (4.3) and (4.4). Thus we obtain $r_G^b = r_H^b = 0$. Q.E.D.

Proofs of Corollaries 3.1 and 3.2. The proofs are immediately obtained from the above lemmas and Theorems 1.1 and 1.2. Q.E.D.

Proof of Corollary 3.3. By the Kelvin transformation

$$w(s;\gamma) := r^{n-2}u(r;\gamma), \quad s := r^{-1},$$

we have

$$r^2\frac{d}{dr} = -\frac{d}{ds}.$$

Hence

$$\begin{aligned} r^{n+2}\left(u_{rr} + \frac{n-1}{r}u_r\right) &= r \cdot r^2(r^{n-3} \cdot r^2 u_r)_r \\ &= s^{-1}\{s^{-n+3}(s^{n-2}w)_s\}_s \\ &= w_{ss} + \frac{n-1}{s}w_s. \end{aligned}$$

Moreover,

$$w(1;\gamma) = u(1;\gamma) = 0, \quad w_s(1;\gamma) = -u_r(1;\gamma) = -\gamma.$$

Thus the system (3.5) is transformed to the interior problem

$$
(4.6) \qquad \begin{cases} w_{ss} + \dfrac{n-1}{s} w_s + \dfrac{s^{\tilde{\sigma}}}{(1-s^{n-2})^{\tau}} w^p = 0, & s \in (0,1), \\ w(1;\gamma) = 0, \quad w_s(1;\gamma) = -\gamma < 0, \end{cases}
$$

where

$$
(4.7) \qquad \tilde{\sigma} := (n-2)p - \sigma + (n-2)\tau - n - 2.
$$

It is clear that the structure of Type R, C, S and M for (3.5) corresponds to the structure of Type R, C, S and M for (4.6). Furthermore, the exact solution (3.6) is transformed to (3.4).

By (4.7), we have

$$
\begin{aligned}
\tilde{\sigma} \le -2 \ &\Leftrightarrow\ p \le (n+\sigma)/(n-2) - \tau, \\
\tilde{\sigma} > -2 \ &\Leftrightarrow\ p > (n+\sigma)/(n-2) - \tau,
\end{aligned}
$$

and

$$
\begin{aligned}
p < (n+2+2\tilde{\sigma})/(n-2) \ &\Leftrightarrow\ p > (n+2+2\sigma)/(n-2) - 2\tau, \\
p = (n+2+2\tilde{\sigma})/(n-2) \ &\Leftrightarrow\ p = (n+2+2\sigma)/(n-2) - 2\tau, \\
p > (n+2+2\tilde{\sigma})/(n-2) \ &\Leftrightarrow\ p < (n+2+2\sigma)/(n-2) - 2\tau.
\end{aligned}
$$

Thus we obtain the conclusion by Corollary 3.2. Q.E.D.

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Figure caption

Figure 1: Structure of solutions for (3.1) with $\sigma > -2$.

Figure 2: Structure of solutions for (3.3) with $\sigma \leq -2$.

Figure 3: Structure of solutions for (3.3) with $\sigma > -2$.

Figure 4: Structure of solutions for (3.5) with $\sigma \leq 2(n-3)$.

Figure 5: Structure of solutions for (3.5) with $\sigma > 2(n-3)$.

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