Funkcialaj Ekvacioj
Volume 44, No. 3, pp 487--499

Energy Decay of Solutions for the Semilinear Dissipative Wave Equations in an Exterior Domain

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Abstract

Uniform energy decay of solutions for the semilinear wave equations with a linear dissipation will be given to the exterior mixed problems. In order to derive the total energy decay property of a solution, an useful inequality due to Ikehata-Matsuyama [3] will be used. In fact, we shall derive the decay rate such as $(1+t)^2E(t) \leq C$ for small initial datum with the compact support, where $E(t)$ represents the total energy.

1 Introduction

Let $\Omega \subset R^N (N \geq 2)$ be an exterior domain with smooth compact boundary $\partial\Omega$. Without loss of generality we may assume $0 \notin \bar{\Omega}$. In this paper we are concerned with the initial-boundary value problem for the semilinear dissipative wave equation:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = |u(t,x)|^{p-1}u(t,x), \quad (t,x) \in (0,\infty)\times\Omega, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{1.3}$$

Throughout this paper, $\|\cdot\|_q$ means the usual $L^q(\Omega)$-norm and in particular, we set $\|\cdot\| = \|\cdot\|_2$. Furthermore, we adopt

$$(f,g) = \int_\Omega f(x)g(x)dx$$

as the usual $L^2(\Omega)$-inner product. The total energy $E(t)$ to the equation (1.1) is defined by

$$E(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|\nabla u(t,\cdot)\|^2.$$

The research of the author was in part supported by Grant-in-Aid for Scientific Research (No.10740068), Ministry of Education, Science and Culture.

The main purpose of this paper is to derive a certain decay rate for the total energy $E(t)$ and L^2-norm of the solution $u(t,x)$ to the problem (1.1)-(1.3) with compact support initial data in an "exterior domain" through the multiplier method together with the semigroup theory. Our argument is based on the results due to Ikehata-Matsuyama [3] and Saeki-Ikehata [10] which derive the sharp decay estimates of the various norms of the solutions to the linear equation:

$$u_{tt} - \Delta u + u_t = 0.$$

For the related result, Nakao-Ono [9] studied the global solvability and energy decay to the Cauchy problem (1.1)-(1.2) with $\Omega = R^N$ through the modified potential-well method. Roughly speaking, they have derived the following results: let $1 + \frac{4}{N} \leq p < \frac{N+2}{[N-2]^+}$. Then, for small initial data $\|u_0\|_{H^1} + \|u_1\| \ll 1$ the Cauchy problem (1.1)-(1.2) with $\Omega = R^N$ has a global solution $u \in C([0,+\infty); H^1(R^N)) \cap C^1([0,+\infty); L^2(R^N))$ satisfying

$$\|u(t,\cdot)\|^2 \leq C, \quad E(t) \leq C(1+t)^{-1}.$$

For the present, it seems unknown whether the total energy $E(t)^{1/2}$ and more L^2-norm of a solution to the problem (1.1)-(1.3) in exterior domains decay faster than $(1+t)^{-1}$ or not. Our device is in the fact that we need not go through any so called the spectral analysis in order to obtain the decay rate as in Dan-Shibata [1].

Now before stating our main theorem we shall define a function $d(x)$ as follows:

$$d(x) = \begin{cases} |x| & N \geq 3, \\ |x| \log(B|x|) & N = 2, \end{cases} \tag{1.4}$$

where $B > 0$ is a constant such that $\inf_{x \in \Omega} |x| \geq \frac{2}{B} > 0$. We make some assumptions before introducing the main theorem.

(A.1) $1 + \frac{6}{N+2} < p \leq \frac{N}{N-2} \quad (N = 3)$,

(A.2) $1 + \frac{6}{N+2} < p < +\infty \quad (N = 2)$.

Let $\rho > 0$ be a real number such that $\partial\Omega \subset B_\rho$. Our final assumption is as follows: for each fixed $R > \rho$,

(A.3) $\mathit{supp}\, u_0 \cup \mathit{supp}\, u_1 \subset \Omega \cap B_R$.

Here $B_r = \{x \in R^N : |x| < r\}$. Further, set

$$I_0 = \|u_0\|_{H^1} + \|u_1\| + \|d(\cdot)(u_0 + u_1)\|.$$

Our result reads as follows:

Theorem 1. 1 *Let $N = 3$. Under the assumptions* (A.1) *and* (A.3), *there exists a real number $\delta > 0$ such that if the initial data further satisfies $I_0 < \delta$, then the problem* (1.1)-(1.3) *has a global solution $u \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ satisfying*

$$E(t) \leq C I_0^2 (1+t)^{-2}, \quad \|u(t,\cdot)\|^2 \leq C I_0^2 (1+t)^{-1}.$$

Theorem 1. 2 *Let $N = 2$. Under the assumptions* (A.2) *and* (A.3), *we have the same conclusion as in Theorem* 1.1.

Remark 1. 1 *In the assumptions* (A.1),(A.2), *we have* $1 + \dfrac{4}{N} > 1 + \dfrac{6}{N+2}$. *Therefore, the restriction to the range of p can be weaken in our case as comparing with that of* [8].

2 Proof of Theorem 1.1

In this section we will prove Theorem 1.1. First we shall prepare several facts concerning the linear problem:

$$u_{tt}(t,x) - \Delta u(t,x) + u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega, \tag{2.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \tag{2.2}$$

$$u|_{\partial\Omega} = 0, \quad t \in (0,\infty). \tag{2.3}$$

Define a semigroup $S(t) : H_0^1(\Omega) \times L^2(\Omega) \to H_0^1(\Omega) \times L^2(\Omega)$ by

$$S(t) : \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \mapsto \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix},$$

where $u(t,\cdot) \in C([0,+\infty); H_0^1(\Omega)) \cap C^1([0,+\infty); L^2(\Omega))$ is a unique solution to the "linear" problem (2.1)-(2.3). Then in [3] and [10] they have derived the sharp energy decay rates of the solution to the problem (2.1)-(2.3) using the following Hardy type inequality.

Lemma 2. 1 *Let $N \geq 2$. For each $u \in H_0^1(\Omega)$ it holds that*

$$\left(\int_\Omega \frac{|u(x)|^2}{d(x)^2} dx\right)^{\frac{1}{2}} \leq C^* \|\nabla u\|$$

with a function $d(x)$ defined in (1.4).

Theorem 2. 1 (Ikehata-Matsuyama [3]) *Let $N \geq 2$. If $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfies $\|d(\cdot)(u_0 + u_1)\| < +\infty$, then it holds that*

$$\|u(t,\cdot)\| \leq C_1 I_0 (1+t)^{-1/2}.$$

Set

$$\left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_E = \|v\| + \|\nabla u\|.$$

Theorem 2. 2 (Saeki-Ikehata [10]) *Let $N \geq 2$. If $[u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ further satisfies $\|d(\cdot)(u_0 + u_1)\| < +\infty$, then it holds that*

$$\left\|S(t)\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}\right\|_E \leq C_1 I_0 (1+t)^{-1}.$$

Theorem 2.2 plays an important role in this article, so that we shall give a brief proof of this one. Although in [10] we relied on the Nakao inequality, for making the dependence of the coefficient clear we will give an alternative simple proof.

Proof of Theorem 2.2. To begin with, we start with the well-known 2 identities given by the linear equation (2.1):

$$E(t) + \int_0^t \|u_t(s,\cdot)\|^2 ds = E(0), \tag{2.4}$$

$$\frac{d}{dt}(u_t(t,\cdot), u(t,\cdot)) + \|\nabla u(t,\cdot)\|^2 + \frac{1}{2}\frac{d}{dt}\|u(t,\cdot)\|^2 = \|u_t(t,\cdot)\|^2. \tag{2.5}$$

Because of (2.4), since the function $t \to E(t)$ is decreasing, we see

$$\frac{d}{dt}\{(1+t)^2 E(t)\} \le 2(1+t)E(t),$$

so that one has

$$(1+t)^2 E(t) \le E(0) + \int_0^t (1+s)(\|\nabla u(s,\cdot)\|^2 + \|u_t(s,\cdot)\|^2) ds. \tag{2.6}$$

On the other hand, multiplying the both sides of (2.5) by $(1+t)$ and integrating it over $[0,t]$ we have

$$\int_0^t (1+s)\|\nabla u(s,\cdot)\|^2 ds$$

$$= \int_0^t (1+s)\|u_t(s,\cdot)\|^2 ds - \frac{1}{2}\int_0^t (1+s)\frac{d}{ds}\|u(s,\cdot)\|^2 ds - \int_0^t (1+s)\frac{d}{ds}(u_t(s,\cdot), u(s,\cdot)) ds.$$

Integration by parts gives

$$\int_0^t (1+s)\|\nabla u(s,\cdot)\|^2 ds$$

$$= \int_0^t (1+s)\|u_t(s,\cdot)\|^2 ds - \frac{1}{2}(1+t)\|u(t,\cdot)\|^2 + \frac{1}{2}\|u_0\|^2 + \frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds$$

$$-(1+t)(u_t(t,\cdot), u(t,\cdot)) + (u_1, u_0) + \frac{1}{2}\|u(t,\cdot)\|^2 - \frac{1}{2}\|u_0\|^2.$$

Since $-E'(t) = \|u_t(t,\cdot)\|^2$ (see (2.4)), we get

$$\int_0^t (1+s)\|\nabla u(s,\cdot)\|^2 ds$$

$$\le -\int_0^t (1+s)E'(s) ds + (u_1, u_0) + (1+t)\|u_t(t,\cdot)\|\|u(t,\cdot)\|$$

$$+\frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds + \frac{1}{2}\|u(t,\cdot)\|^2$$

$$\le \int_0^t E(s) ds + E(0) + (u_1, u_0) + (1+t)\|u_t(t,\cdot)\|\|u(t,\cdot)\|$$

$$+\frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds + \frac{1}{2}\|u(t,\cdot)\|^2$$

$$\le \int_0^t E(s) ds + E(0) + (u_1, u_0) + \frac{(1+t)}{2}\|u_t(t,\cdot)\|^2 + \frac{(1+t)}{2}\|u(t,\cdot)\|^2$$

$$+\frac{1}{2}\int_0^t \|u(s,\cdot)\|^2 ds + \frac{1}{2}\|u(t,\cdot)\|^2. \tag{2.7}$$

In [3] (see also Theorem 2.1), we have already proven

$$(1+t)\|u(t,\cdot)\|^2 \le C_1^2 I_0^2, \quad \int_0^t \|u(s,\cdot)\|^2 ds \le C_1^2 I_0^2. \tag{2.8}$$

Furthermore, the following estimates are well-known and standard(for example,see [10]):

$$(1+t)\|u_t(t,\cdot)\|^2 \le 2(1+t)E(t) \le C_1(\|u_0\|_{H^1}^2 + \|u_1\|^2), \tag{2.9}$$

$$\int_0^t E(s)ds \le C_1(\|u_0\|_{H^1}^2 + \|u_1\|^2). \tag{2.10}$$

Finally, again we have

$$\int_0^t (1+s)\|u_t(s,\cdot)\|^2 ds = -\int_0^t (1+s)E'(s)ds \le E(0) + \int_0^t E(s)ds. \tag{2.11}$$

(2.6),(2.7) and (2.8)-(2.11) imply the desired estimates. ∎

Lemma 2. 2 (Gagliardo-Nirenberg) *Let $1 \le r \le q \le 2N/[N-2]^+$. Then, if $u \in H_0^1(\Omega)$, we have*

$$\|u\|_q \le K_0 \|u\|_r^{1-\theta} \|\nabla u\|^\theta,$$

where $K_0 > 0$ is a constant independent of u and

$$\theta = (1/r - 1/q)(1/N - 1/2 + 1/r)^{-1}.$$

Futhermore, we shall prepare the following well-known inequalities. For the sake of the reader's convenience, we will give the proof of (1) only, and the proof of (2) is left to the reader's exercise.

Lemma 2. 3 *If $\beta > 1$, then there exists a constant $C_\beta > 0$ depending only on β such that*

$$(1) \quad \int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\beta} ds \le C_\beta (1+t)^{-\frac{1}{2}},$$

$$(2) \quad \int_0^t (1+t-s)^{-1}(1+s)^{-\beta} ds \le C_\beta (1+t)^{-1}$$

for all $t \ge 0$.

Proof of (1). First we devide the left hand side of (1) into two parts:

$$\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\beta} ds \le I_1(t) + I_2(t),$$

where

$$I_1(t) = \int_0^{(1+t)/2} (1+t-s)^{-\frac{1}{2}}(1+s)^{-\beta} ds,$$

$$I_2(t) = \int_{(1+t)/2}^{1+t} (1+t-s)^{-\frac{1}{2}}(1+s)^{-\beta} ds.$$

Here

$$I_1(t) \le \int_0^{(1+t)/2} (1+t-(\frac{1+t}{2}))^{-\frac{1}{2}}(1+s)^{-\beta}ds$$

$$= (\frac{1+t}{2})^{-\frac{1}{2}} \int_0^{(1+t)/2} (1+s)^{-\beta}ds$$

$$= 2^{1/2}(\beta-1)^{-1}(1+t)^{-1/2}\{1-(1+\frac{1+t}{2})^{-(\beta-1)}\} \le 2^{1/2}(\beta-1)^{-1}(1+t)^{-1/2}$$

provided that $\beta - 1 > 0$.

Next

$$I_2(t) \le \int_{(1+t)/2}^{1+t} (1+t-s)^{-\frac{1}{2}} s^{-\beta} ds$$

$$\le (\frac{1+t}{2})^{-\beta} \int_{(1+t)/2}^{1+t} (1+t-s)^{-\frac{1}{2}} ds = 2^{\beta+1}2^{-1/2}(1+t)^{-(\beta-\frac{1}{2})}.$$

Since $\beta > 1$, we see that $\beta - \frac{1}{2} > \frac{1}{2}$. Thus we have

$$I_2(t) \le C_\beta (1+t)^{-1/2}$$

with some constant $C_\beta > 0$. These arguments imply the desired estimates. ∎

Now based on these decay estimates for the linear problem (2.1)-(2.3) we shall derive the decay property of a nonlinear problem (1.1)-(1.3). By a standard semigroup theory, the nonlinear problem (1.1)-(1.3) is rewritten as:

$$U(t) = S(t)U_0 + \int_0^t S(t-s)F(s)ds, \tag{2.12}$$

where $U(t) = \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix}$, $U_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$, $F(s) = \begin{bmatrix} 0 \\ f(u(s,\cdot)) \end{bmatrix}$ with $f(u)(x) = |u(x)|^{p-1}u(x)$.

Proposition 2. 1 *Let $N \ge 2$ and suppose $1 < p \le N/[N-2]^+$. For each $U_0 = [u_0, u_1] \in H_0^1(\Omega) \times L^2(\Omega)$ there exists a real number $T_1 = T_1(\|u_0\|_{H^1}, \|u_1\|) > 0$ such that the equation (2.12) has a unique solution $u \in C([0,T_1); H_0^1(\Omega)) \cap C^1([0,T_1); L^2(\Omega))$.*

We proceed our argument based on the way of Nakao [7]. In order to show the global existence, it suffices to obtain the a priori estimates for $E(t)$ and $\|u(t,\cdot)\|$ in the interval of existence. As a result of Theorems 2.1-2.2, first one has

Lemma 2. 4 *Under the assumptions as in Theorem 1.1, we have*

$$\|S(t)U_0\|_E \le C_1 I_0 (1+t)^{-1}$$

on $[0, T_1)$.

Furthermore, if

$$I(s) = \|f(u(s,\cdot))\| + \|d(\cdot)f(u(s,\cdot))\| < +\infty$$

for each $s \in [0,t]$ with $t \in [0,T_1)$, then from Theorem 2.2 we have

$$\|S(t-s)F(s)\|_E \le C_1 I(s)(1+t-s)^{-1}. \tag{2.13}$$

Thus from (2.12) one can estimate $U(t)$ as follows:

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_1 \int_0^t (1+t-s)^{-1} I(s) ds. \tag{2.14}$$

Take $K > 0$ so large and choose $T \in (0, T_1)$ so small such as

$$(1+t)\|U(t)\|_E \le K I_0 \quad on \quad [0, T), \tag{2.15}$$

$$(1+t)^{1/2}\|u(t)\| \le K I_0 \quad on \quad [0, T). \tag{2.16}$$

Since the initial data satisfies (A.3), we see that

$$supp\, u(t, \cdot) \subset \Omega \cap B_{R+t} \tag{2.17}$$

for each $t \in [0, T)$. So, in the case when $N = 3$ we can estimate as

$$\|d(\cdot) f(u(s, \cdot))\|^2 \le (R+s)^2 \int_\Omega |u(s, x)|^{2p} dx.$$

By applying Lemma 2.2 we see

$$\|d(\cdot) f(u(s, \cdot))\| \le (R+s)\|u(s, \cdot)\|_{2p}^p \le K_0 (R+s) \|u(s, \cdot)\|^{p(1-\theta)} \|\nabla u(s, \cdot)\|^{p\theta}$$

with $\theta = N(p-1)/2p \in (0, 1]$. Similarly one has

$$\|f(u(s, \cdot))\| \le K_0 \|u(s, \cdot)\|^{p(1-\theta)} \|\nabla u(s, \cdot)\|^{p\theta}.$$

Therefore, as long as (2.15)-(2.16) holds one gets

$$\|d(\cdot) f(u(s, \cdot))\| \le K_0 (R+s) \{K I_0 (1+s)^{-1/2}\}^{p(1-\theta)} \{K I_0 (1+s)^{-1}\}^{p\theta}$$
$$= K_0 (R+s) K^p I_0^p (1+s)^{-p(1+\theta)/2}.$$

Here note that $1 + \theta = (N(p-1) + 2p)/2p$. By these and similar estimates to $\|f(u(s, \cdot))\|$ one has

Lemma 2. 5 *As long as* (2.15)-(2.16) *hold on* $[0, T)$ *we have*

$$\|d(\cdot) f(u(t, \cdot))\| \le K_0 (R+t) K^p I_0^p (1+t)^{-N(p-1)/4 - p/2},$$

$$\|f(u(t, \cdot))\| \le K_0 K^p I_0^p (1+t)^{-N(p-1)/4 - p/2}.$$

By applying Lemma 2.5 to (2.14) we see that

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_1 \int_0^t (1+t-s)^{-1} K_0 K^p I_0^p \{(R+s)(1+s)^{-\gamma} + (1+s)^{-\gamma}\} ds$$

$$\le C_1 I_0 (1+t)^{-1} + C_1 K_0 K^p I_0^p \int_0^t (1+t-s)^{-1} (1+s)^{-\gamma} \{(1+R) + s\} ds$$

$$\le C_1 I_0 (1+t)^{-1} + C_1 (1+R) K_0 K^p I_0^p \int_0^t (1+t-s)^{-1} (1+s)^{1-\gamma} ds,$$

where

$$\gamma = \frac{N(p-1)}{4} + \frac{p}{2}.$$

Setting $\beta = \gamma - 1$, we see that $\beta > 1$ because of the assumption (A.1). Thus we have

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_1(1+R)K_0 K^p I_0^p \int_0^t (1+t-s)^{-1}(1+s)^{-\beta} ds,$$

so that from Lemma 2.3 it follows that

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_R K_0 K^p I_0^p (1+t)^{-1}$$

with some constant $C_R > 0$. Setting

$$Q_0(I_0, K) = C_1 + C_R K_0 K^p I_0^{p-1},$$

we get the following lemma.

Lemma 2. 6 *As long as* (2.15)-(2.16) *hold on* $[0,T)$ *we get*

$$\|U(t)\|_E \le I_0 Q_0(I_0, K)(1+t)^{-1}.$$

Next let us derive the L^2-estimates for the local solution $u(t,x)$ to the problem (1.1)-(1.3). Indeed, since (2.16) are valid on the interval $[0,T)$ we have from (2.12) that

$$\|u(t,\cdot)\| \le C_1 I_0 (1+t)^{-1/2} + C_1 \int_0^t (1+t-s)^{-1/2} I(s) ds.$$

Therefore, it follows from Lemma 2.5 that

$$\|u(t,\cdot)\| \le C_1 I_0 (1+t)^{-1/2}$$

$$+C_1 \int_0^t (1+t-s)^{-1/2} K_0 K^p I_0^p [(R+s)(1+s)^{-N(p-1)/4-p/2} + (1+s)^{-N(p-1)/4-p/2}] ds$$

$$\le C_1 I_0 (1+t)^{-1/2}$$

$$+C_1(1+R)K_0 K^p I_0^p \int_0^t (1+t-s)^{-1/2}(1+s)^{1-N(p-1)/4-p/2} ds$$

$$\le C_1 I_0 (1+t)^{-1/2} + C_R K_0 K^p I_0^p \int_0^t (1+t-s)^{-1/2}(1+s)^{-\beta} ds$$

with some generous constant $C_R > 0$, where $\beta = -1 + N(p-1)/4 + p/2 > 1$ because of (A.1). This together with Lemma 2.3 implies

$$\|u(t,\cdot)\| \le C_1 I_0 (1+t)^{-1/2} + C_R K_0 K^p I_0^p (1+t)^{-1/2}.$$

Thus we have

Lemma 2. 7 *As long as* (2.15)-(2.16) *hold on* $[0,T)$ *it follows that*

$$\|u(t,\cdot)\| \le I_0 Q_0(I_0, K)(1+t)^{-1/2}.$$

Take $K > C_1$ so large and take I_0 so small such as

$$C_R K_0 K^p I_0^{p-1} < K - C_1. \tag{2.18}$$

For such $K > 0$ and I_0 we have

$$Q_0(I_0, K) < K.$$

Therefore, by combining this with Lemmas 2.6-2.7 we see that

$$\|U(t)\|_E < K I_0 (1+t)^{-1}, \tag{2.19}$$

$$\|u(t,\cdot)\| < K I_0 (1+t)^{-1/2} \tag{2.20}$$

on $[0,T)$. Thus (2.15)-(2.16) and (2.19)-(2.20) show that under the assumption (2.18), the local solution $u(t,\cdot)$ exists globally in time and these estimates hold in fact for all $t \geq 0$. Taking $\delta = (\dfrac{K - C_1}{C_R K_0 K^p})^{1/(p-1)}$, the proof of Theorem 1.1 is now finished.

3 Proof of Theorem 1.2.

In this section we shall prove Theorem 1.2 which is the $N = 2$ dimensional case and this part demands a little technical skill in comparison with the previous section 2 in order to handle with the logarithmic function.

Let $N = 2$ and $1 + \dfrac{6}{N+2} < p < +\infty$. Since $d(x) = |x| \log(B|x|)$ in this case, because of (2.17), we have

$$\|d(\cdot) f(u(s,\cdot))\| \leq (R+s) \log(B(R+s)) \|u(s,\cdot)\|_{2p}^p.$$

Next take $\epsilon > 0$ so amall such as $\epsilon \in (0, 1/2) = (0, (4-N)/2N)$ and $\dfrac{4\epsilon}{N+2} < p - (1 + \dfrac{6}{N+2})$. Then we have

$$1 + \frac{6}{N+2} + \frac{4\epsilon}{N+2} < p < +\infty,$$

so that,

$$\gamma - (1+\epsilon) > 1, \quad (\gamma = N(p-1)/4 + p/2)$$

and

$$1 + \frac{6}{N+2} + \frac{4\epsilon}{N+2} < 1 + \frac{4}{N}.$$

For $\epsilon > 0$ above it holds that

$$\frac{\log(B(R+t))}{(R+t)^\epsilon} \leq C_*$$

with some constant $C_* > 0$ depending only on B, R and p. So it follows from the similar calculation to Lemma 2.5 we have

Lemma 3. 1 *Let $N = 2$. As long as (2.15)-(2.16) hold on $[0,T)$ one has*

$$\|d(\cdot) f(u(t,\cdot))\| \leq K_0 C_* (R+t)^{1+\epsilon} K^p I_0^p (1+t)^{-N(p-1)/4 - p/2},$$

$$\|f(u(t,\cdot))\| \leq K_0 K^p I_0^p (1+t)^{-N(p-1)/4 - p/2}.$$

Therefore, from the same derivation as in Lemma 2.6 we see that

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1}$$

$$+C_1 \int_0^t (1+t-s)^{-1} K_0 K^p I_0^p \{C_*(R+s)^{1+\epsilon}(1+s)^{-\gamma} + (1+s)^{-\gamma}\} ds$$

$$\le C_1 I_0 (1+t)^{-1} + C_R K_0 K^p I_0^p \int_0^t (1+t-s)^{-1}(1+s)^{-\gamma}(1+s)^{1+\epsilon} ds$$

with a generous constant $C_R = C_R(R, C_*) > 0$. This implies

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_R K_0 K^p I_0^p \int_0^t (1+t-s)^{-1}(1+s)^{-(\gamma-1-\epsilon)} ds,$$

with $\gamma = N(p-1)/4 + p/2$. Since

$$\gamma - 1 - \epsilon > 1,$$

it follows from Lemma 2.3 that

$$\|U(t)\|_E \le C_1 I_0 (1+t)^{-1} + C_R K_0 K^p I_0^p (1+t)^{-1}. \tag{3.1}$$

Thus we have arrived at the following result.

Lemma 3. 2 *Let* $N = 2$. *As long as* (2.15)-(2.16) *hold on* $[0, T)$ *we get*

$$\|U(t)\|_E \le I_0 Q_0(I_0, K)(1+t)^{-1}.$$

Let us derive the L^2-decay rate for the solution under the condition (2.15)-(2.16). Indeed, from Lemma 3.1 and the proof of Lemma 2.7 we also have

$$\|u(t)\| \le C_1 I_0 (1+t)^{-1/2}$$

$$+C_1 \int_0^t (1+t-s)^{-1/2} K_0 K^p I_0^p \{C_*(R+s)^{1+\epsilon}(1+s)^{-\gamma} + (1+s)^{-\gamma}\} ds$$

$$\le C_1 I_0 (1+t)^{-1/2} + C_R K_0 K^p I_0^p \int_0^t (1+t-s)^{-1/2}(1+s)^{-\gamma}(1+s)^{1+\epsilon} ds$$

with a generous constant $C_R > 0$. This means

$$\|u(t)\| \le C_1 I_0 (1+t)^{-1/2} + C_R K_0 K^p I_0^p \int_0^t (1+t-s)^{-1/2}(1+s)^{-(\gamma-1-\epsilon)} ds.$$

Therefore, it follows from Lemma 2.3 that

$$\|u(t)\| \le C_1 I_0 (1+t)^{-1/2} + C_R K_0 K^p I_0^p (1+t)^{-1/2}$$

with a generous constant $C_R > 0$. This yields the following decay estimate.

Lemma 3. 3 *Let* $N = 2$. *As long as* (2.15)-(2.16) *hold on* $[0, T)$ *we get*

$$\|u(t)\| \le I_0 Q_0(I_0, K)(1+t)^{-1/2}.$$

As in the proof of Theorem 1.1, from Lemmas 3.2 and 3.3 we have the desired statement.

Acknowledgement. The author wishes to thank Prof. Toshitaka Matsumoto, Hiroshima University, for his helpful comments. Furthermore, the author also would like to thank the referee for his useful advice.

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