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Global Solvability and Asymptotic Behaviour of a Hyperbolic Problem With Acoustic Boundary Conditions

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1. Introduction

This article presents a study of global solvability and asymptotic behaviour of solutions to the mixed problem for the nonlinear hyperbolic equation with nonlinear damping and acoustic boundary conditions:

$$u_{tt} - a(u)u_{xx} + g(u_t) = 0, \qquad \text{in } Q = (0,1)\times(0,T), \tag{1.1}$$

$$u(0,t) = 0, \qquad t \in (0,T), \tag{1.2}$$

$$-\rho u_t(1,t) = \delta_{tt}(t) + c_1\delta_t(t) + c_0\delta(t), \qquad t \in (0,T), \tag{1.3}$$

$$u_x(1,t) = \delta_t(t), \qquad t \in (0,T), \tag{1.4}$$

$$u(x,0) = u_0(x)\ ,\ u_t(x,0) = u_1(x),\ x \in (0,1), \quad \delta(0) = \delta_0; \tag{1.5}$$

where ρ, T, c_0, c_1 are positive constants, $a : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are differentiable functions.

In [3], J. T. Beale and S. I. Rosencrans considered acoustic boundary conditions for a gas, undergoing small irrotacional pertubations from the rest, in a domain $\Omega \subset \mathbb{R}^3$ with a smooth compact boundary $\partial\Omega = \Gamma$. They assumed that the boundary Γ is nonporous and locally reacting in the sense that wave motion along the boundary is negligible. Therefore, if $u(x,t)$ is the velocity potential and $\delta(x,t)$ denotes the normal displacement of a point $x \in \Gamma$ at time t, they must satisfy the equations:

$$u_{tt} - \Delta u = 0, \qquad \text{in} \quad \Omega\times(0,T); \tag{1.6}$$

$$\delta_{tt} - c_1\delta_t + c_0\delta = -\rho u_t, \qquad \text{on} \quad \Gamma\times(0,T); \tag{1.7}$$

$$\delta_t = \frac{\partial u}{\partial \nu}, \qquad \text{on} \quad \Gamma\times(0,T); \tag{1.8}$$

here ν is the unit outer normal, $c_1 = \frac{d}{m}$, $c_0 = \frac{k}{m}$, $\rho = \frac{\rho_0}{m}$; where m is the mass per unit area on the boundary, d is the resistivity of the boundary, k is the spring constant and ρ_0 is the unperturbed density of the gas.

The mixed problem for $(1.6)-(1.8)$ was considered by J. T. Beale which, using semigroup techniques, carried out a detailed analysis in bounded domain [1] and exterior domains [2]. Beale′s idea was to work with an equivalent initial value problem in the form $u_t = Au$ where A is an operator on a suitable Hilbert space H. For bounded domains, in [1], Beale gave a description of the spectrum of the operator A and proved that there is no uniform rate of decay for solutions to the mixed problem for $(1.6)-(1.8)$.

Recently, C. L. Frota and J. A. Goldstein [4] studied acoustic boundary conditions (1.7), (1.8) for the nonlinear Carrier equation with a nonlinear damping

$$u_{tt} - M(\int_\Omega u^2\,dx)\Delta u + \mid u_t \mid^\alpha u_t = 0. \tag{1.9}$$

The authors proved results on global solvability, uniqueness, regularity and continuous dependence of solutions on the parameters when the boundary Γ is made up of two disjoint pieces Γ_0, Γ_1, each having nonnempty interior. Acoustic boundary conditions $(1.7), (1.8)$ were imposed for $(x,t) \in \Gamma_1 \times (0,T)$ and the Dirichlet boundary condition $u(x,t) = 0$ was imposed for $(x,t) \in \Gamma_0 \times [0,T]$. Moreover, for dimensions $n = 2$ or $n = 3$, global solvability and uniqueness for the mixed problem for (1.9) with acoustic boundary conditions on the whole boundary Γ were stablished. This was obtained as a limit case when the positive measure of Γ_0 shrink to zero.

Global solvability and stability of the energy of the mixed problem for (1.9) with the Dirichlet boundary conditions was proved in [5].

In the present paper we consider the nonlinear wave equation (1.1) with local nonlinearities. The function $a(u)$ depends on a solution while the function $M(\int_\Omega u^2\,dx)$ in (1.9) depends on the L^2-norm of it. This difference makes the study of $(1.1)-(1.5)$ more complicated and we restrict it only to the one-dimensional case. Exponential and algebraic energy decay are proved under some appropriate assumptions on the damping $g(u_t)$ and when c_0 is sufficiently large.

2. Existence and uniqueness

In this section we prove the existence and uniqueness of global solutions to $(1.1)-(1.5)$. We denote the inner product and norm in $L^2(0,1)$ by

$$(u,v)=\int_0^1 u(x)v(x)\,dx, \qquad \|u\|=\left(\int_0^1 \mid u(x)\mid^2\ dx\right)^{\frac{1}{2}},$$

and we define $V=\{z\in H^1(0,1)\,;\ z(0)=0\}$ and the norm in V by

$$\|z\|_V=\|z_x\| \qquad z\in V.$$

The functional spaces are standard, see in Lions and Magenes [5].

Theorem 2.1. *Suppose that the functions a and g satisfy*

$$0<a_0\le a(\lambda), \tag{2.1}$$

$$\mid a'(\lambda)\mid\le\ k_0\,a(\lambda), \tag{2.2}$$

$$\mid g(\xi)\mid\le\ k_1\,(\mid\xi\mid+\mid\xi\mid^{\alpha+1}), \tag{2.3}$$

$$k_2\mid\xi\mid^{\alpha+2}\le\ g(\xi)\,\xi, \tag{2.4}$$

$$k_3\mid\xi\mid^{\alpha}\le\ g'(\xi), \tag{2.5}$$

for all $\lambda,\ \xi\ \in\mathbb{R}$, where k_i; $i=0,1,2,3$; are positive constants and $\alpha>1$. Let c_0, ρ be positive constants and $c_1\in\mathbb{R}$. Given $\delta_0\in\mathbb{R}$, $u_0\in V\cap H^2(0,1)$ and $u_1\in V\cap L^{2\alpha+2}(0,1)$, there exists a unique pair of functions $(u(x,t),\delta(t))$ which is a solution to $(1.1)-(1.5)$ from the class:

$$u,\ u'\in L^\infty(0,T;C([0,1])),\ \ u''\in L^\infty(0,T;L^2(0,1));$$
$$\delta\in C^1([0,T]),\ \ \delta''\in L^\infty(0,1).$$

Proof. Let $\{w_j\}$ be a basis of V, orthonormal in $L^2(0,1)$. For each $\epsilon\in(0,1]$ and $m\in\mathbb{N}$ we consider $\delta_{\epsilon m}:[0,t]\to\mathbb{R}$ and $u_{\epsilon m}(x,t)=\sum_{j=1}^m\varphi_{\epsilon mj}(t)\,w_j(x),\ x\in$

$(0,1)\,, t\in[0,T_{\epsilon m}]$, which are solutions to the following approximate problem

$$\left|\begin{array}{l}\left((\epsilon+\dfrac{1}{a(u_{\epsilon m}(t))})u''_{\epsilon m}(t),w_j\right)+\left(\dfrac{\partial u_{\epsilon m}}{\partial x}(t),\dfrac{\partial w_j}{\partial x}\right)-\delta'_{\epsilon m}(t)\,w_j(1)\\ +\left(\dfrac{g(u'_{\epsilon m}(t))}{a(u_{\epsilon m}(t))},w_j\right)=0,\qquad\qquad 1\le j\le m,\\ \rho u'_{\epsilon m}(1,t)+\delta''_{\epsilon m}(t)+c_1\delta'_{\epsilon m}(t)+c_0\delta_{\epsilon m}(t)=0,\\ u_{\epsilon m}(0)=u_{0m},\qquad u'_{\epsilon m}(0)=u_{1m},\qquad \delta_{\epsilon m}(0)=\delta_0,\qquad \delta'_{\epsilon m}(0)=\dfrac{\partial u_{0m}}{\partial x}(1);\end{array}\right.$$

where $\delta_0\in\mathbb{R}$, $u_{0m}=\sum_{j=1}^m(u_0,w_j)w_j$ and $u_{1m}=\sum_{j=1}^m(u_1,w_j)w_j$.

Since $\epsilon>0$, this approximate problem can be transformed into a normal system of ODE for the unknown functions $\delta_{\epsilon m}$ and $\varphi_{\epsilon mj}$, $1\le j\le m$. Therefore, it has local solutions, defined at some interval $[0,T_{\epsilon m}]$, which satisfy the following approximate equations:

$$\left((\epsilon+\frac{1}{a(u_{\epsilon m}(t))})u''_{\epsilon m}(t),v\right)+\left(\frac{\partial u_{\epsilon m}}{\partial x}(t),\frac{\partial v}{\partial x}\right)-\delta'_{\epsilon m}(t)\,v(1)+\left(\frac{g(u'_{\epsilon m}(t))}{a(u_{\epsilon m}(t))},v\right)=0,\qquad \text{for all}\, v\in V_m=[w_1,...,w_m], \tag{2.6}$$

$$-\rho u'_{\epsilon m}(1,t)\;=\;\delta''_{\epsilon m}(t)+c_1\delta'_{\epsilon m}(t)+c_0\delta_{\epsilon m}(t). \tag{2.7}$$

Estimate 1. Taking $v=2u'_{\epsilon m}(t)$ in (2.6) and multiplying (2.7) by $2\delta'_{\epsilon m}(t)$, we have

$$\frac{d}{dt}\left[\int_0^1(\epsilon+\frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx+\|u_{\epsilon m}(t)\|_V^2\right]-2\delta'_{\epsilon m}(t)\,u'_{\epsilon m}(1,t)$$
$$+2\int_0^1\frac{g(u'_{\epsilon m}(x,t))\,u'_{\epsilon m}(x,t)}{a(u_{\epsilon m}(x,t))}\,dx=-\int_0^1\frac{a'(u_{\epsilon m}(x,t))\,(u'_{\epsilon m}(x,t))^3}{(a(u_{\epsilon m}(x,t)))^2}\,dx\,;$$

$$-2\delta'_{\epsilon m}(t)\,u'_{\epsilon m}(1,t)=\frac{d}{dt}\left[\frac{1}{\rho}(\delta'_{\epsilon m}(t))^2+\frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2\right]+\frac{2c_1}{\rho}(\delta'_{\epsilon m}(t))^2.$$

From here we get

$$\frac{d}{dt}\left[\int_0^1(\epsilon+\frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx+\|u_{\epsilon m}(t)\|_V^2+\frac{1}{\rho}(\delta'_{\epsilon m}(t))^2\right.$$
$$\left.+\frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2\right]+2\int_0^1\frac{g(u'_{\epsilon m}(x,t))\,u'_{\epsilon m}(x,t)}{a(u_{\epsilon m}(x,t))}\,dx$$
$$=-\int_0^1\frac{a'(u_{\epsilon m}(x,t))\,(u'_{\epsilon m}(x,t))^3}{(a(u_{\epsilon m}(x,t)))^2}\,dx-\frac{2c_1}{\rho}(\delta'_{\epsilon m}(t))^2.$$

This equality, (2.2) and (2.4) give

$$\frac{d}{dt}\left[\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx + \|u_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho}(\delta'_{\epsilon m}(t))^2 + \frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2\right] + 2k_2\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))}\,dx \;\le\; k_0\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{3}}{a(u_{\epsilon m}(x,t))}\,dx + \frac{2\mid c_1\mid}{\rho}(\delta'_{\epsilon m}(t))^2.$$

Using Young's inequality for all $\eta > 0$, we obtain

$$\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{3}}{a(u_{\epsilon m}(x,t))}\,dx \le \eta\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))}\,dx + c(\eta)\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{2}}{a(u_{\epsilon m}(x,t))}\,dx.$$

Choosing η sufficiently small, we find

$$\frac{d}{dt}\left[\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx + \|u_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho}(\delta'_{\epsilon m}(t))^2 + \frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2\right] + A_1\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))}\,dx \;\le\; A_2\int_0^1 \frac{\mid u'_{\epsilon m}(x,t)\mid^{2}}{a(u_{\epsilon m}(x,t))}\,dx + A_3(\delta'_{\epsilon m}(t))^2;$$

here and in expressions to follow, A_i, $i \in \mathbb{N}$, denotes a positive constant which does not depend on ϵ, m and t. Integrating this inequality from 0 to $t \le T_{\epsilon m}$, we obtain

$$\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx + \|u_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho}(\delta'_{\epsilon m}(t))^2 + \frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2 + A_1\int_0^t\int_0^1 \frac{\mid u'_{\epsilon m}(x,\tau)\mid^{\alpha+2}}{a(u_{\epsilon m}(x,\tau))}\,dx\,d\tau \;\le\; A_4 + A_2\int_0^t\int_0^1 \frac{\mid u'_{\epsilon m}(x,\tau)\mid^{2}}{a(u_{\epsilon m}(x,\tau))}\,dx\,d\tau + A_3\int_0^t (\delta'_{\epsilon m}(\tau))^2 d\tau.$$

Gronwall's inequality yields

$$\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))})\,(u'_{\epsilon m}(x,t))^2\,dx + \|u_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho}(\delta'_{\epsilon m}(t))^2 + \frac{c_0}{\rho}(\delta_{\epsilon m}(t))^2 + A_1\int_0^t\int_0^1 \frac{\mid u'_{\epsilon m}(x,\tau)\mid^{\alpha+2}}{a(u_{\epsilon m}(x,\tau))}\,dx\,d\tau \;\le\; A_5. \tag{2.8}$$

Taking into account that $V \hookrightarrow H^1(0,1) \hookrightarrow C([0,1])$ and a is a continuos function, we can see that there exits a constant M, independent of ϵ, m and t, such that

$$0 < a_0 \leq a(u_{\epsilon m}(x,t)) \leq M; \qquad \text{for all } x \in (0,1)\ ,\ t \in [0, T_{\epsilon m}]. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\|u'_{\epsilon m}(t)\|^2 + \|u_{\epsilon m}(t)\|^2 + (\delta'_{\epsilon m}(t))^2 + (\delta_{\epsilon m}(t))^2 + \int_0^t \|u'_{\epsilon m}(\tau)\|_{\alpha+2}^{\alpha+2} d\tau \leq A_6. \tag{2.10}$$

This is the first estimate.

Estimate 2. First of all we put $t = 0$ in (2.6) and (2.7). It follows

$$\left((\epsilon + \frac{1}{a(u_{0m})})u''_{\epsilon m}(0), v\right) - (\frac{\partial^2 u_{0m}}{\partial x^2}, v) + (\frac{g(u_{1m})}{a(u_{0m})}, v) = 0,$$

$$\delta''_{\epsilon m}(0) = -\rho u_{1m}(1) - c_1 \frac{\partial u_{0m}}{\partial x}(1) - c_0 \delta_0.$$

Taking $v = u''_{\epsilon m}(0)$ in this equality, we find constants A_7 and A_8, such that

$$\|u''_{\epsilon m}(0)\| \leq A_7 \qquad \text{and} \qquad |\ \delta''_{\epsilon m}(0)\ | \leq A_8. \tag{2.11}$$

Differentiating (2.6) and (2.7) with respect to t, we have

$$\begin{aligned} &\left((\epsilon + \frac{1}{a(u_{\epsilon m}(t))})u'''_{\epsilon m}(t), v\right) + (\frac{\partial u'_{\epsilon m}}{\partial x}(t), \frac{\partial v}{\partial x}) - \delta''_{\epsilon m}(t) v(1) \\ &(\frac{g'(u'_{\epsilon m}(t))u''_{\epsilon m}(t)}{a(u_{\epsilon m}(t))}, v) - (\frac{g(u'_{\epsilon m}(t))u'_{\epsilon m}(t)a'(u_{\epsilon m}(t))}{[a(u_{\epsilon m}(t))]^2}, v) \\ &-(\frac{a'(u_{\epsilon m}(t))u'_{\epsilon m}(t)u''_{\epsilon m}(t)}{[a(u_{\epsilon m}(t)]^2}, v) = 0, \end{aligned} \tag{2.12}$$

$$\delta'''_{\epsilon m}(t) + c_1 \delta''_{\epsilon m}(t) + c_0 \delta'_{\epsilon m}(t) = -\rho u''_{\epsilon m}(1,t). \tag{2.13}$$

Taking $v = 2u''_{\epsilon m}(t)$ in (2.12) and multiplying (2.13) by $2\delta''_{\epsilon m}(t)$, we obtain

$$\begin{aligned} &\frac{d}{dt}\left[\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))})\ |\ u''_{\epsilon m}(x,t)\ |^2\ dx\ +\ \|u'_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho}(\delta''_{\epsilon m}(t))^2\right. \\ &\left.+\frac{c_0}{\rho}(\delta'_{\epsilon m}(t))^2\right] + 2\int_0^1 \frac{g'(u'_{\epsilon m}(x,t))\ |\ u''_{\epsilon m}(x,t)\ |^2}{a(u_{\epsilon m}(x,t))}\ dx = -\frac{2c_1}{\rho}(\delta''_{\epsilon m}(t))^2 \\ &+ \int_0^1 \frac{a'(u_{\epsilon m}(x,t))\, u'_{\epsilon m}(x,t)\ |\ u''_{\epsilon m}(x,t)\ |^2}{[a(u_{\epsilon m}(x,t))]^2} dx \\ &+2\int_0^1 \frac{a'(u_{\epsilon m}(x,t)) g(u'_{\epsilon m}(x,t)) u'_{\epsilon m}(x,t) u''_{\epsilon m}(x,t)}{[a(u_{\epsilon m}(x,t))]^2}\ dx. \end{aligned}$$

For all $\eta > 0$, Young's inequality gives

$$I_1 = \int_0^1 \frac{a'(u_{\epsilon m}(x,t))\, u'_{\epsilon m}(x,t) \ \mid u''_{\epsilon m}(x,t) \mid^2}{[a(u_{\epsilon m}(x,t))]^2} dx$$
$$\leq \eta \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + C(\eta) \int_0^1 \frac{\mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx;$$

$$I_2 = 2 \int_0^1 \frac{a'(u_{\epsilon m}(x,t)) g(u'_{\epsilon m}(x,t)) u'_{\epsilon m}(x,t) u''_{\epsilon m}(x,t)}{[a(u_{\epsilon m}(x,t))]^2} dx$$
$$\leq 2k_0 k_1 \Big(\int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^2 \mid u''_{\epsilon m}(x,t) \mid}{a(u_{\epsilon m}(x,t))} dx + \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha+2} \mid u''_{\epsilon m}(x,t) \mid}{a(u_{\epsilon m}(x,t))} dx \Big)$$
$$\leq \eta \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + C(\eta) \|u'_{\epsilon m}(t)\|_V^2 \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))} dx$$
$$+ 2k_0 k_1 \int_0^1 \frac{\mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + A_{11}\,;$$

$$I_3 = 2 \int_0^1 \frac{g'(u'_{\epsilon m}(x,t)) \ \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx \geq 2k_3 \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx\,;$$

here we have used (2.3) and (2.4).

Therefore,

$$\frac{d}{dt} \left[\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))}) \mid u''_{\epsilon m}(x,t) \mid^2 dx \ + \ \|u'_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho} (\delta''_{\epsilon m}(t))^2 \right.$$
$$\left. + \frac{c_0}{\rho} (\delta'_{\epsilon m}(t))^2 \right] + 2k_3 \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx \leq \frac{2 \mid c_1 \mid}{\rho} (\delta''_{\epsilon m}(t))^2$$
$$+ \eta \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + C(\eta) \|u'_{\epsilon m}(t)\|_V^2 \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))} dx$$
$$+ \int_0^1 \frac{\mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + A_{11}.$$

Choosing $\eta > 0$ sufficiently small we have

$$\frac{d}{dt} \left[\int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))}) \mid u''_{\epsilon m}(x,t) \mid^2 dx \ + \ \|u'_{\epsilon m}(t)\|_V^2 + \frac{1}{\rho} (\delta''_{\epsilon m}(t))^2 \right.$$
$$\left. + \frac{c_0}{\rho} (\delta'_{\epsilon m}(t))^2 \right] + A_{12} \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha} \mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx \leq \ A_{13} \Big((\delta''_{\epsilon m}(t))^2$$
$$+ \int_0^1 \frac{\mid u''_{\epsilon m}(x,t) \mid^2}{a(u_{\epsilon m}(x,t))} dx + \|u'_{\epsilon m}(t)\|_V^2 \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))} dx + 1 \Big).$$

Integrating this,we get

$$\Psi_{\epsilon m}(t) + \int_0^t \int_0^1 \frac{\mid u'_{\epsilon m}(x,\tau) \mid^\alpha \mid u''_{\epsilon m}(x,\tau) \mid^2}{a(u_{\epsilon m}(x,\tau))}\, dx\, d\tau$$
$$\leq\ A_{14}\,(1 + \int_0^t h_{\epsilon m}(\tau)\, \Psi_{\epsilon m}(\tau)\, d\tau),$$

where

$$\Psi_{\epsilon m}(t) = \int_0^1 (\epsilon + \frac{1}{a(u_{\epsilon m}(x,t))}) \mid u''_{\epsilon m}(x,t) \mid^2 dx\ +\ \|u'_{\epsilon m}(t)\|_V^2$$
$$+\,(\delta''_{\epsilon m}(t))^2\ +\ (\delta'_{\epsilon m}(t))^2,$$

and

$$h_{\epsilon m}(t) = 1 + \int_0^1 \frac{\mid u'_{\epsilon m}(x,t) \mid^{\alpha+2}}{a(u_{\epsilon m}(x,t))} dx.$$

From (2.10) we can see that there exists a constant A_{15} independent on ϵ, m, t such that

$$\int_0^t \mid h_{\epsilon m}(\tau) \mid\ d\tau \leq\ A_{15},$$

then $h_{\epsilon m} \in L^1(0,T)$. This and Gronwall's inequality give

$$\Psi_{\epsilon m}(t)\ \leq\ A_{16}\ e^{A_{14}\int_0^T h_{\epsilon m}(\tau)d\tau} = A_{17}, \qquad \text{for all } t \in [0,T]. \tag{2.14}$$

Hence

$$\|u''_{\epsilon m}(t)\|^2 + \|u'_{\epsilon m}(t)\|_V^2 + (\delta''_{\epsilon m}(t))^2 + (\delta'_{\epsilon m}(t))^2\ \leq\ A_{18},\ \text{for all } t \in [0,T], \tag{2.15}$$

and the second estimate is proved.

Using estimates (2.10) and (2.15) we can pass to the limit as $m \to \infty$ and $\epsilon \to 0$ in the approximate problem to get

$$(\frac{u''(t)}{a(u(t))}, v) + (\frac{\partial u}{\partial x}(t), \frac{\partial v}{\partial x}) - \delta'(t)v(1) + (\frac{g(u'(t))}{a(u(t))}, v) = 0, \tag{2.16}$$

for all $v \in V$, a.e. on $[0,T]$.

$$\rho u'(1,t) + \delta''(t) + c_1\delta'(t) + c_0\delta(t) = 0, \qquad \text{a.e. on } [0,T]. \tag{2.17}$$

$$u(x,0) = u_0(x), \qquad u'(x,0) = u_1(x), \qquad \text{a.e. on } [0,1]. \tag{2.18}$$

(2.19) $$\delta(0) = \delta_0, \qquad \delta'(0) = \frac{\partial u_0}{\partial x}(1).$$

Taking in (2.16) $v = \varphi \in \mathcal{D}(0,1)$, we have

$$\left(\frac{u''(t)}{a(u(t))} - \frac{\partial^2 u}{\partial x^2}(t) + \frac{g(u'(t))}{a(u(t))}\,,\,\varphi\right) = 0$$

which implies

$$u'' - a(u)u_{xx} + g(u') = 0 \qquad \text{a.e. in} \qquad (0,1)\times(0,T).$$

Uniqueness may be proved in the usual way. Theorem 2.1 is proved.

Remark 2.1. The result of Theorem 2.1 is valid even for $\alpha = 1$ if we assume that k_2 and k_3 are sufficiently large.

3. Asymptotic behaviour

In this section, we shall prove the exponential and algebraic decay of the energy of (1.1) – (1.5). Therefore, instead of (2.3) and (2.4) we make the following stronger assumptions

(3.1) $$\mid g(\xi) \mid \le \overline{k_1}\,(\mid \xi \mid^{\beta+1} + \mid \xi \mid^{\alpha+1}),$$

(3.2) $$\overline{k_2}\,(\mid \xi \mid^{\beta+2} + \mid \xi \mid^{\alpha+2}) \le g(\xi)\,\xi\,;$$

for all $\xi \in \mathbb{R}$, with $0 \le \beta \le 1$, where

(3.3) $$\overline{k_2} > \left[\frac{k_0(\alpha-1)}{2(\alpha-\beta)}\right]^{\frac{(\alpha-1)}{(\alpha-\beta)}} \left[\frac{k_0(1-\beta)}{(\alpha-\beta)}\right]^{\frac{(1-\beta)}{(\alpha-\beta)}}.$$

Theorem 3.1. *Let* $(u\,,\,\delta)$ *be a solution to the problem* (1.1) – (1.5). *Assume that* $c_1 > 0$ *and the coefficient* c_0 *in* (1.3) *satisfies the inequality*

(3.4) $$\frac{\rho}{c_0} < \frac{1}{\sqrt{8\overline{c}}},$$

where $\overline{c} > 0$ *is the constant of the embedding* $\|\,.\,\|_{C([0,1])} \le \overline{c}\,\|\,.\,\|_V$. *Then there are positive constants* K *and* C *such that*

(3.5) $$\begin{cases} h(t) \le Ke^{-Ct}, \ \textit{for all}\ t \ge 0, & \textit{if} \quad \beta = 0 \\ \textit{or} & \\ h(t) \le C(1+t)^{\frac{-2}{\beta}}, \ \textit{for all}\ t \ge 0, & \textit{if} \quad 0 < \beta \le 1 \end{cases}$$

here $h(t) = \|u'(t)\|^2 + \|u(t)\|_V^2 + \mid \delta'(t) \mid^2 + \mid \delta(t) \mid^2$.

Proof. Let us define a function $\Phi : [0, \infty] \to \mathbb{R}$ by

$$\Phi(t) = \int_0^1 \frac{(u'(x,t))^2}{a(u(x,t))}\,dx + \|u(t)\|_V^2 + \frac{1}{\rho}(\delta'(t))^2 + \frac{c_0}{\rho}(\delta(t))^2. \tag{3.6}$$

Multiplying (1.1) by $2u'(x,t)$ and integrating over $(0,1)$, we obtain

$$\frac{d}{dt}\left[\int_0^1 \frac{(u'(x,t))^2}{a(u(x,t))}\,dx + \|u(t)\|_V^2\right] - 2\,\delta'(t)\,u'(1,t)$$

$$+2\int_0^1 \frac{g(u'(x,t))\,u'(x,t)}{a(u(x,t))}dx = -\int_0^1 \frac{a'(u(x,t))\,(u'(x,t))^3}{[a(u(x,t))]^2}\,dx.$$

This and (1.3) give

$$\Phi'(t) + 2\int_0^1 \frac{g(u'(x,t))\,u'(x,t)}{a(u(x,t))}dx + \frac{2c_1}{\rho}(\delta'(t))^2 = -\int_0^1 \frac{a'(u(x,t))\,(u'(x,t))^3}{[a(u(x,t))]^2}dx. \tag{3.7}$$

From (2.2) and (3.2), we find

$$\Phi'(t) + 2\overline{k}_2\int_0^1 \frac{\mid u'(x,t)\mid^{\beta+2}}{a(u(x,t))}\,dx + 2\overline{k}_2\int_0^1 \frac{\mid u'(x,t)\mid^{\alpha+2}}{a(u(x,t))}dx + \frac{2c_1}{\rho}(\delta'(t))^2$$

$$\le k_0\int_0^1 \frac{\mid u'(x,t)\mid^3}{a(u(x,t))}dx.$$

By Young's inequality

$$\Phi'(t) + \frac{2c_1}{\rho}(\delta'(t))^2 + (2\overline{k}_2 - \epsilon)\int_0^1 \frac{\mid u'(x,t)\mid^{\alpha+2}}{a(u(x,t))}dx$$

$$+\left[2\overline{k}_2 - \frac{k_0^{\frac{(\alpha-\beta)}{(\alpha-1)}}(\alpha-1)}{(\alpha-\beta)}\left(\frac{(\alpha-\beta)}{(1-\beta)}\epsilon\right)^{-\frac{(1-\beta)}{(\alpha-1)}}\right]\int_0^1 \frac{\mid u'(x,t)\mid^{\beta+2}}{a(u(x,t))}\,dx \le 0,$$

for all $\epsilon > 0$.

Choosing $\epsilon = \overline{k}_2$, and taking into account (3.3), we obtain

$$\Phi'(t) + \frac{2c_1}{\rho}(\delta'(t))^2 + A_1\int_0^1 \frac{\mid u'(x,t)\mid^{\beta+2}}{a(u(x,t))}\,dx + \overline{k}_2\int_0^1 \frac{\mid u'(x,t)\mid^{\alpha+2}}{a(u(x,t))}dx \le 0, \tag{3.8}$$

where

$$A_1 = 2\overline{k}_2 - \frac{k_0^{\frac{(\alpha-\beta)}{(\alpha-1)}}(\alpha-1)}{(\alpha-\beta)}\left(\frac{(\alpha-\beta)}{(1-\beta)}\overline{k}_2\right)^{-\frac{(1-\beta)}{(\alpha-1)}} > 0.$$

Inequality (3.8) shows that Φ is a monotonically decreasing and bounded function for all $t \geq 0$. We define

$$\Theta(t) = [\Phi(t) - \Phi(t+1)]^{\frac{1}{\beta+2}}\ , \qquad t \geq 0.$$

Integrating (3.8) we get

$$\Phi(t) + \frac{2c_1}{\rho}\int_0^t \mid \delta'(\tau) \mid^2 d\tau \,+\, A_1 \int_0^t \int_0^1 \frac{\mid u'(x,\tau) \mid^{\beta+2}}{a(u(x,\tau))}\,dx\,d\tau$$
$$+\overline{k}_2 \int_0^t \int_0^1 \frac{\mid u'(x,\tau) \mid^{\alpha+2}}{a(u(x,\tau))} dx\,d\tau \;\leq\; \Phi(0), \tag{3.9}$$

and

$$\int_t^{t+1} \mid \delta'(\tau) \mid^2 d\tau + \int_t^{t+1} \|u'(\tau)\|^{\beta+2}_{L^{\beta+2}(0,1)} d\tau$$
$$+\int_t^{t+1} \|u'(\tau)\|^{\alpha+2}_{L^{\alpha+2}(0,1)} d\tau \leq\; A_2\,\Theta^{\beta+2}(t), \qquad \text{for all } t \geq 0. \tag{3.10}$$

By the mean value theorem for integrals, there exist $t_1 \in [t, t+\frac{1}{4}]$ and $t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|u'(t_i)\| \;\leq\; 2A_2^{\frac{1}{\beta+2}}\,\Theta(t)\ ,\ \mid \delta'(t_i) \mid \leq\; 2A_2^{\frac{1}{2}}\,\Theta^{\frac{\beta+2}{2}}(t), \text{ for } i \;=\; 1,\, 2. \tag{3.11}$$

Moreover, from (3.6), we obtain

$$\|u(t)\|_V \;\leq\; \Phi^{\frac{1}{2}}(t) \qquad \text{and} \qquad \mid \delta(t) \mid \leq \sqrt{\frac{\rho}{c_0}}\,\Phi^{\frac{1}{2}}(t). \tag{3.12}$$

Another simple consequence of the mean value theorem is that there exists a point $t^* \in [t_1, t_2]$ such that

$$\Phi(t^*) \leq 2\int_{t_1}^{t_2} \Phi(\tau)\,d\tau. \tag{3.13}$$

Integrating (3.7) from t^* to t, we find

$$\Phi(t) = \Phi(t^*) + 2\int_t^{t^*}\int_0^1 \frac{g(u'(x,\tau))\,u'(x,\tau)}{a(u(x,\tau))} dx\,d\tau$$
$$+\frac{2c_1}{\rho}\int_t^{t^*} \mid \delta'(\tau) \mid^2 d\tau + \int_t^{t^*}\int_0^1 \frac{a'(u(x,\tau)) \mid u'(x,\tau) \mid^3}{[a(u(x,\tau))]^2}\,dx\,d\tau. \tag{3.14}$$

To estimate the right hand side of (3.14), we observe that (2.1), (3.1) and (3.10) give

$$2\int_t^{t^*}\int_0^1 \frac{g(u'(x,\tau))\,u'(x,\tau)}{a(u(x,\tau))}dx\,d\tau \tag{3.15}$$

$$\leq \frac{2\overline{k}_1}{a_0}\Big(\int_t^{t^*} \|u'(\tau)\|^{\beta+2}_{L^{\beta+2}(0,1)}d\tau + \int_t^{t^*} \|u'(\tau)\|^{\alpha+2}_{L^{\alpha+2}(0,1)}d\tau\Big) \leq \frac{4\overline{k}_1 A_2}{a_0}\,\Theta^{\beta+2}(t);$$

$$\frac{2c_1}{\rho}\int_t^{t^*} \mid \delta'(\tau)\mid^2 d\tau \;\leq\; \frac{2c_1A_2}{a_0}\Theta^{\beta+2}(t); \tag{3.16}$$

$$\int_t^{t^*}\int_0^1 \frac{a'(u(x,\tau))\mid u'(x,\tau)\mid^3}{[a(u(x,\tau))]^2}\,dx\,d\tau \leq \frac{k_0}{a_0}\int_t^{t^*}\int_0^1 \mid u'(x,\tau)\mid^3 dx\,d\tau \tag{3.17}$$

$$\leq A_3\left(\int_t^{t^*} \|u'(\tau)\|^{\beta+2}_{L^{\beta+2}(0,1)}d\tau + \int_t^{t^*} \|u'(\tau)\|^{\alpha+2}_{L^{\alpha+2}(0,1)}d\tau\right) \leq A_4\Theta^{\beta+2}(t).$$

Taking into account (3.13) – (3.17), we find

$$\Phi(t) \;\leq\; 2\int_{t_1}^{t_2} \Phi(\tau)\,d\tau + A_5\Theta^{\beta+2}(t). \tag{3.18}$$

We need an estimate for the integral $\int_{t_1}^{t_2}\Phi(\tau)\,d\tau$. Using (3.6) and (3.10), we have

$$\int_{t_1}^{t_2}\Phi(\tau)\,d\tau \leq A_6(\Theta^2(t)+\Theta^{\beta+2}(t)\,)+\int_{t_1}^{t_2}\|u(\tau)\|^2_V d\tau+\frac{c_0}{\rho}\int_{t_1}^{t_2} \mid \delta(\tau)\mid^2 d\tau. \tag{3.19}$$

Multiplying (1.1) by $u(x,t)$ and integrating over $(0,1)$ we get

$$\|u(t)\|^2_V = -\frac{d}{dt}\int_0^1 \frac{u'(x,t)\,u(x,t)}{a(u(x,t))}dx + \int_0^1 \frac{\mid u'(x,t)\mid^2}{a(u(x,t))}dx$$
$$-\int_0^1 \frac{a'(u(x,t))\ \mid u'(x,t)\mid^2\ u(x,t)}{[a(u(x,t))]^2}dx + \delta'(t)\,u(1,t)$$
$$-\int_0^1 \frac{g(u'(x,t))\,u(x,t)}{a(u(x,t))}dx.$$

Integrating this from t_1 to t_2, we obtain

$$\int_{t_1}^{t_2}\|u(\tau)\|^2_V = \int_0^1 \frac{u'(x,t_1)\,u(x,t_1)}{a(u(x,t_1))}dx - \int_0^1 \frac{u'(x,t_2)\,u(x,t_2)}{a(u(x,t_2))}dx \tag{3.20}$$

$$+\int_{t_1}^{t_2}\int_0^1 \frac{\mid u'(x,\tau)\mid^2}{a(u(x,\tau))}dx\,d\tau - \int_{t_1}^{t_2}\int_0^1 \frac{a'(u(x,\tau))\ \mid u'(x,\tau)\mid^2\ u(x,\tau)}{[a(u(x,\tau))]^2}dx\,d\tau$$

$$+\int_{t_1}^{t_2} \delta'(\tau)\, u(1,\tau)\, d\tau - \int_{t_1}^{t_2}\int_0^1 \frac{g(u'(x,\tau))\, u(x,\tau)}{a(u(x,\tau))} dx\, d\tau.$$

Since $V \hookrightarrow C([0,1])$, making use of (3.10) – (3.12), we can estimate each term of the right hand side of (3.20) for all $\eta > 0$ as follows:

$$\int_0^1 \frac{u'(x,t_i)\, u(x,t_i)}{a(u(x,t_i))}\, dx \le \frac{2A_2^{\frac{1}{\beta+2}}}{a_0}\, \Theta(t)\, \Phi^{\frac{1}{2}}(t) \tag{3.21}$$

$$\le \eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right) + C(\eta)\, \Theta^2(t); \qquad \text{for } i = 1,2.$$

$$\int_{t_1}^{t_2}\int_0^1 \frac{|\, u'(x,\tau)\, |^2}{a(u(x,\tau))} dx\, d\tau \le \frac{A_2^{\frac{2}{\beta+2}}}{a_0} \Theta^2(t); \tag{3.22}$$

$$\int_{t_1}^{t_2}\int_0^1 \frac{a'(u(x,\tau))\ |\, u'(x,\tau)\, |^2\ u(x,\tau)}{[a(u(x,\tau))]^2} dx\, d\tau \tag{3.23}$$

$$\le \frac{k_0}{a_0} \int_{t_1}^{t_2} (\max_{0\le x\le 1} |\, u(x,\tau)\, |) \int_0^1 |\, u'(x,\tau)\, |^2\, dx\, d\tau$$

$$\le A_7 \int_{t_1}^{t_2} \|u(\tau)\|_V\, \|u'(\tau)\|^2 d\tau \le A_8 \Theta^2(t);$$

$$\int_{t_1}^{t_2} \delta'(\tau)\, u(1,\tau)\, d\tau \le \int_{t_1}^{t_2} (\max_{0\le x\le 1} |\, u(x,\tau)\, |)\ |\, \delta'(\tau)\, |\, d\tau \tag{3.24}$$

$$\le A_9 \int_{t_1}^{t_2} \|u(\tau)\|_V\ |\, \delta'(\tau)\, |\ d\tau \le \eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right) + C(\eta)\, \Theta^{\beta+2}(t);$$

$$\int_{t_1}^{t_2}\int_0^1 \frac{g(u'(x,\tau))\, u(x,\tau)}{a(u(x,\tau))} dx\, d\tau \tag{3.25}$$

$$\le \frac{\overline{k}_1}{a_0} \int_{t_1}^{t_2}\int_0^1 \left(|\, u'(x,\tau)\, |^{\beta+1} + |\, u'(x,\tau)\, |^{\alpha+1} \right)\ |\, u(x,\tau)\, |\, dx\, d\tau$$

$$\le \eta \int_{t_1}^{t_2} \left[\max_{0\le x\le 1} |\, u(x,\tau)\, |^{\beta} + \max_{0\le x\le 1} |\, u(x,\tau)\, |^{\alpha} \int_0^1 |\, u(x,\tau)\, |^2\, dx \right] d\tau$$

$$+C(\eta) \int_{t_1}^{t_2} \left(\|u'(\tau)\|^{\beta+2}_{L^{\beta+2}(0,1)} + \|u'(\tau)\|^{\alpha+2}_{L^{\alpha+2}(0,1)} \right) d\tau$$

$$\le C\, \eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right) + C(\eta) \Theta^{\beta+2}(t).$$

From (3.20) – (3.25), we have

$$\int_{t_1}^{t_2} \|u(\tau)\|_V^2 \, ds \;\le\; 4\eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right) + A(\eta) \left(\Theta^2(t) + \Theta^{\beta+2}(t)\right). \tag{3.26}$$

It remains to estimate the last term of (3.19). We multiply (1.3) by $\delta(t)$ and integrate from t_1 to t_2, to obtain

$$\frac{c_0}{\rho} \int_{t_1}^{t_2} \mid \delta(\tau) \mid^2 d\tau = u(1,t_1)\delta(t_1) - u(1,t_2)\delta(t_2) + \int_{t_1}^{t_2} u(1,\tau)\delta'(\tau)\, d\tau \tag{3.27}$$
$$+\frac{1}{\rho}\delta'(t_1)\delta(t_1) - \frac{1}{\rho}\delta'(t_2)\delta(t_2) + \frac{1}{\rho}\int_{t_1}^{t_2} \mid \delta'(\tau) \mid^2 d\tau - \frac{c_1}{\rho}\int_{t_1}^{t_2} \delta(\tau)\delta'(\tau)\, d\tau.$$

Let us estimate the right hand side of (3.27). First of all, using (3.12), we can see that

$$\mid u(1,t_i)\,\delta(t_i) \mid \le \overline{c}\sqrt{\frac{\rho}{c_0}}\,\Phi(t)\,, \qquad \text{for } i = 1,2. \tag{3.28}$$

Here $\overline{c}$ is given by (3.4), and we have used that $V \hookrightarrow C([0,1])$ and the fact that Φ is a monotonically decreasing function.

On the other hand,

$$\frac{1}{\rho}\delta'(t_i)\delta(t_i) \le \eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right) + C(\eta)\,\Theta^{\beta+2}(t), \qquad \text{for } \eta > 0,\;\; i = 1,2; \tag{3.29}$$

$$\frac{1}{\rho}\int_{t_1}^{t_2} \mid \delta'(\tau) \mid^2 \; d\tau \le \frac{A_2}{\rho}\,\Theta^{\beta+2}(t); \tag{3.30}$$

$$\frac{c_1}{\rho}\int_{t_1}^{t_2} \delta(\tau)\delta'(\tau)\, d\tau \le \frac{\epsilon c_0}{\rho}\int_{t_1}^{t_2} \mid \delta(\tau) \mid^2 \; d\tau + C(\epsilon)\Theta^{\beta+2}(t), \text{ for all } \epsilon \ge 0. \tag{3.31}$$

Taking into account (3.24), (3.27) and (3.28) – (3.31), we get

$$\frac{c_0}{\rho}(1-\epsilon)\int_{t_1}^{t_2} \mid \delta(\tau) \mid^2 d\tau \le 2\,\overline{c}\sqrt{\frac{\rho}{c_0}}\Phi(t) + 3\eta \left(\sup_{t\le\tau\le t+1} \Phi(\tau) \right)$$
$$+(3C(\eta) + \frac{A_2}{\rho} + C(\epsilon)\,)\Theta^{\beta+2}(t),$$

chosing $\epsilon = \frac{1}{2}$, we find

$$\frac{c_0}{\rho}\int_{t_1}^{t_2} \mid \delta(\tau) \mid^2 d\tau \le 4\,\overline{c}\sqrt{\frac{\rho}{c_0}}\Phi(t) \tag{3.32}$$

$$+6\eta \left(\sup_{t \le \tau \le t+1} \Phi(\tau) \right) + (6C(\eta) + A_{10})\, \Theta^{\beta+2}(t).$$

From (3.18), (3.19), (3.26) and (3.32), it follows

$$(1 - 8\,\overline{c}\sqrt{\frac{\rho}{c_0}})\, \Phi(t) \le A_{11}\eta \left(\sup_{t \le \tau \le t+1} \Phi(\tau) \right) + \overline{A}(\eta) \left(\Theta^2(t) + \Theta^{\beta+2}(t) \right).$$

This inequality and assumption (3.4), for $\eta > 0$ sufficiently small, imply

$$\sup_{t \le \tau \le t+1} \Phi(\tau) \le A_{12}(\Phi(0))\, \Theta^2(t), \qquad \text{for all } t \ge 0,$$

which can be rewritten as

$$\sup_{t \le \tau \le t+1} \Phi^{1+\frac{\beta}{2}}(\tau) \le C(\Phi(0))\, [\, \Phi(t) - \Phi(t+1)\,] \qquad \text{for all } t \ge 0. \tag{3.33}$$

By lemma of M. Nakao (see [6] and [7]), (3.5) holds, there with theorem 3.1 is proved.

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