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ANALYTICAL SIMPLIFICATION OF A NONLINEAR 2-SYSTEM WITH SIMPLIFIED EQUATIONS INVOLVING INFINITELY MANY TERMS

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To the memory of my brothers

§1. Introduction.

In a previous paper (M. Iwano [5]), the author studies a nonlinear 2-system of the form

$$\text{(A)} \qquad x^2\frac{dy}{dx} = (\mu + \alpha x)y + f(x,y,z), \quad x^2\frac{dz}{dx} = (-\nu + \beta x)z + g(x,y,z),$$

under the assumptions that

(i) *μ and ν are commensurable positive numbers.*

(ii) *α and β are real constants such that* $\alpha\mu + \beta\nu > 0$.

(iii) *$f(x,y,z)$ and $g(x,y,z)$ are holomorphic and bounded functions of (x,y,z) for $|x| < a, \quad |y| < b, \quad |z| < b$ and their Taylor series expansions in (x,y,z) involve neither the independent terms nor the linear terms with respect to y and z:*

$$\text{(1.1)} \qquad f(x,y,z) = \sum_{j+k\ge 2, i\ge 0} a_{i,j,k}x^iy^jz^k, \quad g(x,y,z) = \sum_{j+k\ge 2, i\ge 0} b_{i,j,k}x^iy^jz^k,$$

a and b being small positive constants.

To simplify the description, we assume that

$$\text{(1.2)} \qquad \mu = \nu = 1.$$

By virtue of the commensurability condition between $\mu(=1)$ and $\nu(=1)$, there remain infinitely many terms in the simplified equations, to which we can give any analytical meaning without applying a well known theorem named Borel-Ritt Theorem(for example, M. Hukuhara [1]). By applying a transformation of cubic polynomials in (x,u,v), we can reduce the values of $a_{0,j,k}$ for $j+k \le 3$ to zero except for $(j,k) = (2,1)$ and the values of $b_{0,j,k}$ for $j+k \le 3$ to zero except for $(j,k) = (1,2)$. It is already shown that, after such a transformation has been applied, the new values $a_{0,2,1} = \alpha'$ and $b_{0,1,2} = \beta'$ are given by

$$\text{(1.3)} \qquad \begin{cases} \alpha' = -a_{0,2,0}a_{0,1,1} + a_{0,1,1}b_{0,1,1} + \dfrac{2}{3}a_{0,0,2}b_{0,2,0} + a_{0,2,1}, \\ \beta' = b_{0,0,2}b_{0,1,1} - b_{0,1,1}a_{0,1,1} - \dfrac{2}{3}b_{0,2,0}a_{0,0,2} + b_{0,1,2}. \end{cases}$$

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By assuming that

(iv) *the sum δ of the quantities α' and β' is a nonzero quantity* :

$$\delta \equiv \alpha' + \beta' \neq 0, \tag{1.4}$$

the author succeeds in constructing a formal transformation which reduces the f to *the product of y and a quadratic form in yz* and g to *the product of z and a quadratic form in yz*. The result is clarified in the theorem below:

Theorem A. *Apply successively two formal transformations $\{y, z\} \to \{u, v\}$ and $\{u, v\} \to \{\eta, \zeta\}$ of the types* :

$$y = u + \sum_{i\geq 0, j+k\geq 2} p_{i,j,k} x^i u^j v^k, \quad z = v + \sum_{i\geq 0, j+k\geq 2} q_{i,j,k} x^i u^j v^k \tag{1.5}$$

and

$$u = \eta\big(1 + \sum_{i+j\geq 1, j\geq 1} p_{i,j} x^i (\eta\zeta)^j\big), \quad v = \zeta\big(1 + \sum_{i+j\geq 1, j\geq 1} q_{i,j} x^i (\eta\zeta)^j\big). \tag{1.6}$$

Equations (A) *are formally changed to equations of the form*

$$\text{(B)} \quad \begin{cases} x^2 \dfrac{d\eta}{dx} &= \eta(1 + \alpha x + \alpha' \eta\zeta + \frac{\gamma}{\delta} x \alpha(x) \eta\zeta + \alpha(x)(\eta\zeta)^2), \\ x^2 \dfrac{d\zeta}{dx} &= \zeta(-1 + \beta x + \beta' \eta\zeta + \frac{\gamma}{\delta} x \beta(x) \eta\zeta + \beta(x)(\eta\zeta)^2). \end{cases}$$

The $\alpha(x)$ and $\beta(x)$ are expressed by power series in x. The equations (B) *are called the simplified equations.*

But, unfortunately, we cannot given any analytical meaning to the power series $\alpha(x)$ and $\beta(x)$ by a natural manner (which means that they are solutions of algebraic equations or solutions of simple differential equations.) So, by means of Borel-Ritt Theorem, we define the $\alpha(x)$ and $\beta(x)$ as *holomorphic functions* such that *they are holomorphic and bounded in x for a domain of the form*

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_0 \tag{1.7}$$

and, moreover, admit asymptotic expansions in powers of x as x tends to the origin through (1.7). The ϵ is a preassinged sufficiently small positive quantity and the a_0 is a small constant.

The purpose of the present paper is to discuss analytical meaning of the formal transformation $\{y, z\} \to \{\eta, \zeta\}$ of the following form, which are obtained by the composite of (1.5) and (1.6):

$$T : y = \eta + \sum_{j+k\geq 2} R_{j,k}(x) \eta^j \zeta^k, \quad z = \zeta + \sum_{j+k\geq 2} S_{j,k}(x) \eta^j \zeta^k. \tag{1.8}$$

The essential problem on giving analytical meaning to the formal transformation is the construction of stable domains of solutions of the simplified equations (B). The character of the stable domains depends on the sign of the quantity $\gamma = \alpha + \beta - 1$. As the consequence, there is an essential distinction between the case of $\gamma \neq 0$ and the case of $\gamma = 0$ as will be shown in the theorems below. We shall write down the main results.

Theorem B_+. *Assume $\gamma > 0$. There is a transformation $\{y, z\} \to \{\eta, \zeta\}$*

$$y = \Phi_0(x, \eta, \zeta), \quad z = \Psi_0(x, \eta, \zeta), \tag{1.9}$$

which changes equations (A) *to equations* (B) *with the properties that*:

(i) *Φ_0 and Ψ_0 are expressed by*

$$\Phi_0(x, \eta, \zeta) = \Phi(x, \frac{\eta\zeta}{x}, \eta, \zeta), \quad \Psi_0(x, \eta, \zeta) = \Psi(x, \frac{\eta\zeta}{x}, \eta, \zeta), \tag{1.10}$$

where $\Phi(x, w, \eta, \zeta)$ and $\Psi(x, w, \eta, \zeta)$ are holomorphic and bounded in (x, w, η, ζ) in a domain of the form

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x_0| < a_0, \quad |w| < d_0, \quad |\eta| < c_0, \quad |\zeta| < c_0. \tag{1.11}$$

(ii) *The $\Phi(x, w, \eta, \zeta)$ and $\Psi(x, w, \eta, \zeta)$ admit uniformly convergent expansions in powers of η and ζ in domain* (1.11) *with coefficients holomorphic and bounded in (x, w) for*

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x_0| < a_0, \quad |w| < d_0. \tag{1.12}$$

The coefficients are expanded to convergent power series in w uniformly valid in domain (1.12)*, with coefficients which are functions holomorphic and bounded in x for*

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x_0| < a_0 \tag{1.13}$$

and admitting asymptotic expansions to powers in x as x tends to the origin through (1.13).

Theorem B_-. *Assume $\gamma < 0$. There is a transformation $\{y, z\} \to \{\eta, \zeta\}$ of the form* (1.9) *which changes equations* (A) *to equations* (B) *with the properties that*:

(i) *$\Phi_0(x, \eta, \zeta)$ and $\Psi_0(x, \eta, \zeta)$ are expressed by*

$$\Phi_0(x, \eta, \zeta) = \Phi(x, \frac{\eta\zeta}{x} + \frac{\gamma}{\delta}, \eta, \zeta), \quad \Psi_0(x, \eta, \zeta) = \Psi(x, \frac{\eta\zeta}{x} + \frac{\gamma}{\delta}, \eta, \zeta), \tag{1.14}$$

where $\Phi(x, w, \eta, \zeta)$ and $\Psi(x, w, \eta, \zeta)$ are holomorphic and bounded in (x, w, η, ζ) in a domain of the form (1.11).

(ii) *The $\Phi(x, w, \eta, \zeta)$ and $\Psi(x, w, \eta, \zeta)$ admit uniformly convergent expansions in powers of η and ζ in domain* (1.11) *with coefficients holomorphic and bounded in (x, w) for domain* (1.12)*. The coefficients are expanded to convergent power series in w uniformly valid in domain* (1.12)*, with coefficients which are functions holomorphic and bounded in x for domain* (1.13) *and admitting asymptotic expansions to powers in x as x tends to the origin through* (1.13).

Theorem $\mathbf{B}_0$. *Assume $\gamma = 0$. Let μ' and μ be any angles such that*

$$(1.15) \qquad -\frac{\pi}{2} + 2\epsilon < \mu' < \frac{\pi}{2} - \epsilon, \quad \frac{\pi}{2} + 2\epsilon < \mu < \frac{3\pi}{2} - \epsilon, \quad \mu - \mu' < \pi.$$

There is a transformation of the form (1.9)*, which changes* (A) *to* (B) *with the properties that*:

(i) *$\Phi_0(x,\eta,\zeta)$ and $\Psi_0(x,\eta,\zeta)$ are expressible by*(1.10)*, where the $\Phi(x,w,\eta,\zeta)$ and $\Psi(x,w,\eta,\zeta)$ are holomorphic and bounded functions in (x,w,η,ζ) in a domain of the form*

$$(1.16) \qquad \begin{cases} \mu' < \arg x < \mu, \quad 0 < |x| < a_0, \\ -\mu' - \dfrac{\pi}{2} < \arg w < -\mu + \dfrac{3\pi}{2}, \quad 0 < |w| < d_0, \quad |\eta| < c_0, \quad |\zeta| < c_0. \end{cases}$$

(ii) *The $\Phi(x,w,\eta,\zeta)$ and $\Psi(x,w,\eta,\zeta)$ admit uniformly convergent expansions in powers of η and ζ in doman* (1.16)*, with coefficients holomorphic and bounded in (x,w) for*

$$(1.17) \quad \mu' < \arg x < \mu, \ 0 < |x| < a_0, \quad -\mu' - \frac{\pi}{2} < \arg w < -\mu + \frac{3\pi}{2}, \ 0 < |w| < d_0.$$

The coefficients admit power series in w as asymptotic expansions uniformly valid in domain (1.17) *with coefficients, which are functions holomorphic and bounded in x for*

$$(1.18) \qquad \mu' < \arg x < \mu, \quad 0 < |x| < a_0$$

and admitting asymptotic expansions to powers in x as x tends to the origin through (1.18)*.*

The relations between the functions $\{\Phi_0, \Psi_0\}$ and formal power series (T) will be clarified in Theorem 5.B_+ and Theorem 5.B_0 in § 8, and in Theorem 5.B_- in § 9.

Remark. By the substitutions : $\mu \to -\mu', \mu' \to -\mu, \arg x \to -\arg x, \arg w \to -\arg w$ in Theorem B_0, a similar theorem holds.

§2. **Formal transformation of the first kind.**

We shall write the equations (A) in the form

$$(2.1) \quad x^2\frac{dy}{dx} = (1+\alpha x)y + \sum_{j+k\geq 2} f_{j,k}(x)y^jz^k, \ x^2\frac{dz}{dx} = (-1+\beta x)z + \sum_{j+k\geq 2} g_{j,k}(x)y^jz^k.$$

We want to reproduce formal transformation (1.6) in a slightly different manner from M. Iwano [5]. Consider a transformation of the form

$$(2.2.\mathrm{N}) \qquad T_N : y = u + \sum_{j+k=N} p_{j,k}(x)u^jv^k, \quad z = v + \sum_{j+k=N} q_{j,k}(x)u^jv^k.$$

The coefficients $p_{j,k}(x)$ and $q_{j,k}(x)$ are expressed by power series in x. Let

$$(2.3.\mathrm{N}) \qquad \begin{aligned} x^2\frac{du}{dx} &= (1+\alpha x)u + \sum_{j+k\geq 2} A_{j,k}(x)u^jv^k, \\ x^2\frac{dv}{dx} &= (-1+\beta x)v + \sum_{j+k\geq 2} B_{j,k}(x)u^jv^k. \end{aligned}$$

be the transformed equations.

The inverse transformation of (2.2.N) is wirtten in the form

$$u = y - \sum_{j+k=N} p_{j,k}(x) y^j z^k + [x;y,z]_{2N-1}, v = z - \sum_{j+k=N} q_{j,k}(x) y^j z^k + [x;y,z]_{2N-1}.$$

The symbol $[x;y,z]_M$ stands for an expression which is expressed by a convergent or a formal power series in y and z with coefficients holomorphic in x without the terms of degree less than M. By differentiation, we get

$$\begin{aligned} x^2 u' &= x^2 y' - \sum_{j+k=N} \left\{ x^2 \frac{p_{j,k}}{dx} + p_{j,k} \left[j \frac{x^2 y'}{y} + k \frac{x^2 z'}{z} \right] \right\} y^j z^k + [x;y,z]_{2N-1} \\ &= (1+\alpha x) y + \sum_{s+t \geq 2} f_{s,t}(x) y^s z^t \\ &\quad - \sum_{j+k=N} \left\{ x^2 \frac{p_{j,k}}{dx} + p_{j,k} \big[j - k + (j\alpha + k\beta) x + [x;y,z]_1 \big] \right\} y^j z^k + [x;y,z]_{2N-1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} x^2 u' &= (1+\alpha x)\big(u + \sum_{j+k=N} p_{j,k} u^j v^k\big) \\ &\quad + \sum_{s+t \geq 2} f_{s,t}(x) \big[u + \sum_{j+k=N} p_{j,k} u^j v^k\big]^s \big[v + \sum_{j+k=N} q_{j,k} u^j v^k\big]^t \\ &\quad - \sum_{j+k=N} \left\{ x^2 \frac{p_{j,k}}{dx} + (j - k + (j\alpha + k\beta)x) p_{j,k} \right\} u^j v^k + [x;u,v]_{N+1}. \end{aligned}$$

It immediately follows that, if the $p_{j,k}(x)$ and $q_{j,k}(x)$ are holomorphic in x in a certain domain, the expression $[x;u,v]_{N+1}$ can be expressed by a uniformly convergent expansion in u and v with coefficients holomorphic and bounded in x in the same domain.

Equating this and (2.3.N) gives

$$(2.4) \qquad A_{s,t}(x) = f_{s,t}(x) \quad \text{for} \quad s+t < N$$

and, for $j+k=N$,

$$(2.5) \qquad A_{j,k}(x) = -x^2 \frac{dp_{j,k}}{dx} + \big(k+1-j-[(j-1)\alpha + k\beta]x\big) p_{j,k} + f_{j,k}(x).$$

If $j \neq k+1$, we put $A_{j,k}(x) \equiv 0$ and the $p_{j,k}(x)$ are uniquely determined as solutions of (2.5) (with $A_{j,k}(x) \equiv 0$) in a domain of the form

$$(2.6) \qquad |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_1$$

in such a way that they are holomorphic and bounded in x and admit asymptotic expansions in powers in x as x tends to the origin through (2.6). To simplify the description, such a function will be called *a function with Property C* in domain

(2.6). And, when a solution of differential equation is a function with Property $\mathcal{C}$, such a solution is said to be *a solution with Property* $\mathcal{C}$.

When $j = k + 1$, we define $p_{j,k}(x) \equiv 0$ and, consequently, $A_{j,k}(x) = f_{j,k}(x)$.

In quite a similar way, from the second one in (2.2.N), we can derive

$$B_{s,t}(x) = g_{s,t}(x) \quad \text{for} \quad s + t < N \tag{2.7}$$

and, for $j + k = N$,

$$B_{j,k}(x) = -x^2 \frac{dq_{j,k}}{dx} + (k - 1 - j - [j\alpha + (k-1)\beta]x)q_{j,k} + g_{j,k}(x). \tag{2.8}$$

Thus, if $k \neq j + 1$, we put $B_{j,k}(x) \equiv 0$. The $q_{j,k}(x)$ are uniquely determined as solutions with Property $\mathcal{C}$ of equation (2.8) (with $B_{j,k}(x) \equiv 0$) in domain (2.6). If $k = j + 1$, we define $q_{j,k}(x) \equiv 0$ and, consequently, $B_{j,k}(x) = g_{j,k}(x)$.

We see that the $A_{j,k}(x)$ and $B_{j,k}(x)$ for $j + k > N$ are uniquely determined as functions with Property $\mathcal{C}$ in domain (2.6). Thus, we have the following

Assertion 1. *Make a composite transformation* $T_2 \circ T_3 \circ \cdots \circ T_N$ *for every* N *and let* N *tend to infinity. Then, we have a formal transformation of the form*

$$T : y = u + \sum_{j+k \geq 2} p_{j,k}(x)u^j v^k, \quad z = v + \sum_{j+k \geq 2} q_{j,k}(x)u^j v^k \tag{2.9}$$

with the properties that:

(i) *The coefficients* $p_{j,k}(x)$ *and* $q_{j,k}(x)$ *are functions of* x *with Property* $\mathcal{C}$ *in domain* (2.6).

(ii) *The formal transformation* (2.9) *changes formally equations* (2.1) *to equations of the form*

$$\begin{aligned} x^2 \frac{du}{dx} &= u(1 + \alpha x + \alpha' uv + \sum_{\ell=1}^{\infty} f_\ell(x)(uv)^\ell), \\ x^2 \frac{dv}{dx} &= v(-1 + \beta x + \beta' uv + \sum_{\ell=1}^{\infty} g_\ell(x)(uv)^\ell), \end{aligned} \tag{2.10}$$

where the $f_\ell(x)$ *and* $g_\ell(x)$ *are functions with Property* $\mathcal{C}$ *in domain* (2.6). *In particular,*

$$f_1(0) = \lim_{x \to 0} f_1(x) = 0, \quad g_1(0) = \lim_{x \to 0} g_1(x) = 0. \tag{2.11}$$

§3. Formal transformation of the second kind.

Assuming that the simplified equations have the form of equations (B) in Theorem A, we want to prove the following

Assertion 2. *There is a formal transformation of the form*

$$u = \eta(1 + \sum_{j=1}^{\infty} p_j(x)(\eta\zeta)^j), \quad v = \zeta(1 + \sum_{j=1}^{\infty} q_j(x)(\eta\zeta)^j) \tag{3.1}$$

which formally changes equations (2.10) *to the simplified equations* (B). *The coefficients* $p_j(x)$ *and* $q_j(x)$ *are functions with Property* $\mathcal{C}$ *in domain* (2.6).

Note. Borel-Ritt Theorem plays an essential role in giving analytical meaning to the equations (B).

Proof. Set

$$\rho(x) = \alpha(x) + \beta(x). \tag{3.2}$$

By differentiation of the first equation in (3.1), we have

$$\begin{aligned}
x^2\frac{du}{dx} &= x^2\frac{d\eta}{dx}\cdot\big(1+\sum_{j=1}^{\infty}p_j(x)(\eta\zeta)^j\big)\\
&+\eta\sum_{j=1}^{\infty}\Big\{x^2\frac{dp_j}{dx}+jp_j((\alpha+\beta)x+\delta\eta\zeta+\frac{\gamma}{\delta}x\rho(x)\eta\zeta+\rho(x)(\eta\zeta)^2)\Big\}(\eta\zeta)^j\\
&=\eta\Big[(1+\alpha x+\alpha'\eta\zeta+\frac{\gamma}{\delta}x\alpha(x)\eta\zeta+\alpha(x)(\eta\zeta)^2)(1+\sum_{j=1}^{\infty}p_j(x)(\eta\zeta)^j)\\
&+\sum_{j=1}^{\infty}\Big\{x^2\frac{dp_j}{dx}+jp_j\big((\alpha+\beta)x+\delta\eta\zeta+\frac{\gamma}{\delta}x\rho(x)\eta\zeta+\rho(x)(\eta\zeta)^2\big)\Big\}(\eta\zeta)^j\Big].
\end{aligned}\tag{3.3}$$

On the other hand,

$$\begin{aligned}
x^2\frac{du}{dx} &= u(1+\alpha x+\alpha' uv+\sum_{\ell=1}^{\infty}f_\ell(x)(uv)^\ell) = \eta\big(1+\sum_{j=1}^{\infty}p_j(x)(\eta\zeta)^j\big)\\
&\times\Big[\big(1+\alpha x+\alpha'\eta\zeta(1+\sum_{j=1}^{\infty}p_j(x)(\eta\zeta)^j\big)\big(1+\sum_{j=1}^{\infty}q_j(x)(\eta\zeta)^j\big)\\
&+\sum_{\ell=1}^{\infty}f_\ell(x)\Big\{1+\sum_{j=1}^{\infty}(p_j(x)+q_j(x))(\eta\zeta)^j\\
&\qquad+\big(\sum_{s=1}^{\infty}p_s(x)(\eta\zeta)^s\big)\big(\sum_{t=1}^{\infty}q_t(x)(\eta\zeta)^t\big)\Big\}^\ell(\eta\zeta)^\ell\Big].
\end{aligned}\tag{3.4}$$

Equate the expressions appearing in the right sides of (3.3) and (3.4) and divide them by the factor η. Rearrage them in powers of $\eta\zeta$ and cancel out the term $1+\alpha x$ from the both sides. Compare the coefficients of the like terms in powers of $\eta\zeta$. We get the equations

$$x^2\frac{dp_1}{dx} = -(\alpha+\beta)xp_1+f_1(x)-\frac{\gamma}{\delta}x\alpha(x), \tag{3.5.1}$$

$$\begin{aligned}
x^2\frac{dp_2}{dx} &= -2(\alpha+\beta)xp_2+[f_1(x)-\delta-\frac{\gamma}{\delta}x(\alpha(x)+\rho(x))]p_1(x)\\
&+(\alpha'+f_1(x))q_1(x)+f_2(x)-\alpha(x),
\end{aligned}\tag{3.5.2}$$

$$(3.5.j) \qquad x^2\frac{dp_j}{dx} = -j(\alpha+\beta)xp_j + \mathcal{A}_j(x, p_1(x), \cdots, p_{j-1}(x), q_1(x), \cdots, q_{j-1}(x)).$$

The function $\mathcal{A}_j(x,p,q) \equiv \mathcal{A}_j(x, p_1, \cdots, p_{j-1}, q_1, \cdots, q_{j-1})$ has the form

$$\begin{aligned}\mathcal{A}_j(x,p,q) &= -\big(\alpha' + (j-1)\delta + \frac{\gamma}{\delta}x((j-1)\rho(x) + \alpha(x)\big)p_{j-1} \\ (3.6.j) \qquad &- \big((j-2)\rho(x) + \alpha(x)\big)p_{j-2} + \alpha'\sum_{\ell=0}^{j-1} p_\ell \sum_{s+t=j-\ell-1} p_s q_t \\ &+ A_j(x, p_1, \cdots, p_{j-1}, q_1, \cdots, q_{j-1}), \quad (p_0 = q_0 \equiv 1)\end{aligned}$$

where the $A_j(x, p_1, \cdots, p_{j-1}, q_1, \cdots, q_{j-1})$ is a linear form of the $f_1(x), \cdots, f_j(x)$ whose coefficients are polynomials in $\{p_1, \cdots, p_{j-1}, q_1, \cdots, q_{j-1}\}$ and are symmetric forms in p and q.

Apply quite a same discussion to the second one in relations (3.1). Then we can derive the following equations:

$$(3.7.1) \qquad x^2\frac{dq_1}{dx} = -(\alpha+\beta)xq_1 + g_1(x) - \frac{\gamma}{\delta}x\beta(x),$$

$$\begin{aligned}(3.7.2) \qquad x^2\frac{dq_2}{dx} &= -2(\alpha+\beta)xq_2 + [g_1(x) - \delta - \frac{\gamma}{\delta}x(\beta(x) + \rho(x))]q_1(x) \\ &+ (\beta' + g_1(x))p_1(x) + g_2(x) - \beta(x),\end{aligned}$$

$$(3.7.j) \qquad x^2\frac{dq_j}{dx} = -j(\alpha+\beta)xq_j + \mathcal{B}_j(x, p_1(x), \cdots, p_{j-1}(x), q_1(x), \cdots, q_{j-1}(x)).$$

Every function $\mathcal{B}_j(x,p,q)$ can be obtained from the $\mathcal{A}_j(x,p,q)$ by the substitutions: $\alpha' \to \beta', \alpha(x) \to \beta(x), p \to q, q \to p, f(x) \to g(x)$. As was already shown, all these equations possess formal solutions which are expressed by power series in x. The origin $x = 0$ is aparently a regular singular point. In particular, $f_1(0) = g_1(0) = 0$. Hence, we can also verify directly the existence of a formal power series solutions for equations (3.5.1) and (3.7.1). Moreover, the coefficients appearing in the equations are functions with Property $\mathcal{C}$ in domain (2.6). Thus, $p_1(x)$ is uniquely determined as a solution of (3.5.1) with Property $\mathcal{C}$ in domain (2.6). This is also the case for $q_1(x)$. From the linear equations (3.5.2) and (3.7.2), we can uniquely determine functions $p_2(x)$ and $q_2(x)$ as unique solutions with Property $\mathcal{C}$ in domain (2.6). Inductively, using the linear equations (3.5.j) and (3.7.j), we can determine $p_j(x)$ and $q_j(x)$ as unique solutions with Property $\mathcal{C}$ in domain (2.6). **q.e.d.**

Thus we have proved the following

Theorem 1. *Assume that $\alpha + \beta > 0$ and $\alpha' + \beta' \neq 0$. By successive applications of formal transformations of two types:*

$$(3.8) \qquad y = u + \sum_{j+k\geq 2} p_{j,k}(x)u^j v^k, \quad z = v + \sum_{j+k\geq 2} q_{j,k}(x)u^j v^k$$

$$u=\eta\big(1+\sum_{\ell=1}^{\infty}p_\ell(x)(\eta\zeta)^\ell\big),\quad v=\zeta\big(1+\sum_{\ell=1}^{\infty}q_\ell(x)(\eta\zeta)^\ell\big), \tag{3.9}$$

equations (A) *are formally changed to equations* (B). *All the coefficients appearing in* (3.8) *and* (3.9) *are uniquely determined as solutions of certain linear differential equations with Property* $\mathcal{C}$ *in domain* (2.6).

The composite of the formal transformations (3.8) *and* (3.9) *yields a formal transformation:*$\{y,z\}\to\{\eta,\zeta\}$ *of the form*

$$y=\eta+\sum_{j+k\geq 2}R_{j,k}(x)\eta^j\zeta^k,\quad z=\zeta+\sum_{j+k\geq 2}S_{j,k}(x)\eta^j\zeta^k. \tag{3.10}$$

Here, the coefficients $R_{j,k}(x)$ *are given by*

$$R_{j,k}(x)=p_{j,k}(x)+\sum_{(s,t)\prec(j,k)}p_{s,t}(x)A_{s,t}(x),$$

where $A_{s,t}(x)$ *is a polynomial in* $p_m(x),q_n(x)$ *for* $m+n\leq\min\{j-s,k-t\}$. *The* $S_{j,k}(x)$ *have analogous meaning. The* $R_{j,k}(x)$ *and* $S_{j,k}(x)$ *are functions with Property* $\mathcal{C}$ *in domain* (2.6). *The symbol* $\preceq$ *stands for lexiographical order.*

Note. The assumption that $\alpha'+\beta'\neq 0$ is already used for the fact that equations (B) are reduced to a quadratic form in $\eta\zeta$ (see, M. Iwano [5]).

§4. New arrangement of the formal double power series (3.10) in a single power series.

Rearrange double power series (3.10) of η and ζ either in a single power series of η, for example,

$$y=\eta+\sum_{j=0}^{\infty}R_j(x,\zeta)\eta^j,\quad z=\zeta+\sum_{j=0}^{\infty}S_j(x,\zeta)\eta^j, \tag{4.1}$$

$$R_j(x,\zeta)=\sum_{k=0}^{\infty}R_{j,k}(x)\zeta^k,\quad S_j(x,\zeta)=\sum_{k=0}^{\infty}S_{j,k}(x)\zeta^k. \tag{4.2.j}$$

Obviously,

$$R_{j,k}(x)=S_{j,k}(x)=0\quad\text{for}\quad j+k\leq 1. \tag{4.3}$$

We want to prove the following

Theorem 2. *For every* j, *the power series* (4.2.j) *are uniformly convergent for* (x,ζ) *in a domain of the form*

$$\big|\arg x\mp\frac{\pi}{2}\big|<\pi-\epsilon,\quad 0<|x|<a_1,\quad |\zeta|<c_1, \tag{4.4}$$

so that their sums $R_j(x,\zeta)$ and $S_j(x,\zeta)$ become holomorphic and bounded functions in (x,ζ) for domain (4.4).

Proof. The proof is based on the construction of differential equations which are satisfied by $\{R_j(x,\zeta), S_j(x,\zeta)\}$. Since the formal power series (4.2.j) are formal solutions of those differential equations, we have only to apply our standard anlysis (for example, M. Iwano [1]) to obtain the conclusion of the theorem.

By the use of the first relation in (4.1), a direct computation yields

$$\begin{aligned}
x^2\frac{dy}{dx} &= x^2\frac{d\eta}{dx} + \sum_{j=0}^{\infty}\left\{x^2\frac{dR_j}{dx}\eta^j + jR_j\eta^{j-1}x^2\frac{d\eta}{dx}\right\}\\
&= x^2\frac{d\eta}{dx} + \sum_{j=0}^{\infty}\left\{\left(x^2\frac{\partial R_j}{\partial x} + \frac{\partial R_j}{\partial \zeta}x^2\frac{d\zeta}{dx}\right)\eta^j + jR_j\eta^{j-1}x^2\frac{d\eta}{dx}\right\}\\
&= \eta(1+\alpha x+\alpha'\eta\zeta+\frac{\gamma}{\delta}x\alpha(x)\eta\zeta+\alpha(x)(\eta\zeta)^2)\\
&\quad+\sum_{j=0}^{\infty}\left\{\left(x^2\frac{\partial R_j}{\partial x}+\frac{\partial R_j}{\partial\zeta}\zeta(-1+\beta x+\beta'\eta\zeta+\frac{\gamma}{\delta}x\beta(x)\eta\zeta+\beta(x)(\eta\zeta)^2)\right)\right.\\
&\quad\left.+jR_j(1+\alpha x+\alpha'\eta\zeta+\frac{\gamma}{\delta}x\alpha(x)\eta\zeta+\alpha(x)(\eta\zeta)^2))\right\}\eta^j.
\end{aligned} \tag{4.5}$$

Rearrange the last expression in powers of η. Then we have equation as

$$x^2\frac{dy}{dx} = \sum_{j=0}^{\infty}A_j(x,\zeta)\eta^j. \tag{4.6}$$

A simple inspection gives

$$A_0(x,\zeta) = x^2\frac{\partial R_0}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_0}{\partial\zeta},$$

$$\begin{aligned}
A_1(x,\zeta) &= x^2\frac{\partial R_1}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_1}{\partial\zeta} + (1+\alpha x)R_1\\
&\quad+\left[1+\alpha x+(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_0}{\partial\zeta}(x,\zeta)\right]_1,
\end{aligned}$$

$$\begin{aligned}
A_2(x,\zeta) &= x^2\frac{\partial R_2}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_2}{\partial\zeta} + 2(1+\alpha x)R_2\\
&\quad+\left[(1+R_1)(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta+(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_1}{\partial\zeta}+\beta(x)\zeta^3\frac{\partial R_0}{\partial\zeta}\right]_2,
\end{aligned}$$

$$\begin{aligned}
A_3(x,\zeta) &= x^2\frac{\partial R_3}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_3}{\partial\zeta} + 3(1+\alpha x)R_3 + \left[\alpha(x)\zeta^2\right.\\
&\quad\left.(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_2}{\partial\zeta}+\beta(x)\zeta^3\frac{\partial R_1}{\partial\zeta}+2(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta R_2+\alpha(x)\zeta^2R_1\right]_3,
\end{aligned}$$

$$\begin{aligned}
A_j(x,\zeta) &= x^2\frac{\partial R_j}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_j}{\partial\zeta} + j(1+\alpha x)R_j + \left[(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_{j-1}}{\partial\zeta}\right.\\
&\quad\left.+\beta(x)\zeta^3\frac{\partial R_{j-2}}{\partial\zeta}+(j-1)(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta R_{j-1}+(j-2)\alpha(x)\zeta^2R_{j-2}\right]_j.
\end{aligned}$$

Here the R_k and $\frac{\partial R_k}{\partial \zeta}$ $(k = j-2, j-1)$ appearing in the expressions in the brackets with the suffix j (for $j \geq 2$) are regarded as known functions of (x, ζ).

On the other direction, Taylor's formula implies

$$
\begin{aligned}
(1+\alpha x)y + f(x,y,z) &= (1+\alpha x)\eta + (1+\alpha x)\sum_{j=0}^{\infty} R_j(x,\zeta)\eta^j \\
&+ f(x, R_0, \zeta + S_0) + (A(x,\zeta)(R_1+1) + B(x,\zeta)S_1)\eta \\
&+ \sum_{j=0}^{\infty}\{A(x,\zeta)R_j + B(x,\zeta)S_j + \mathcal{P}_j(x,\zeta)\}\eta^j,
\end{aligned} \tag{4.7}
$$

where

$$
A(x,\zeta) = \frac{\partial f}{\partial y}(x, R_0(x,\zeta), \zeta + S_0(x,\zeta)), \quad B(x,\zeta) = \frac{\partial f}{\partial z}(x, R_0(x,\zeta), \zeta + S_0(x,\zeta)).
$$

And, $\mathcal{P}_j(x,\zeta)$ is given by

$$
\mathcal{P}_j(x,\zeta) = \sum_{s+t \leq j} \frac{\partial^{s+t} f}{\partial y^s \partial z^t}(x, R_0(x,\zeta), \zeta + S_0(x,\zeta)) A_{s,t}(x,\zeta),
$$

where $A_{s,t}(x,\zeta)$ is a polynomial in $R_1(x,\zeta), \cdots, R_{j-1}(x,\zeta), S_1(x,\zeta), \cdots, S_{j-1}(x,\zeta)$.

Similarily, from the second relation in (4.1), we can derive an equation as

$$
x^2 \frac{dz}{dx} = \sum_{j=0}^{\infty} B_j(x,\zeta)\eta^j, \tag{4.8}
$$

where the $B_j(x,\zeta)$ are expressed in terms of $\{S_j, S_{j-1}, S_{j-2}\}$ and their derivatives by formulas similar to $A_j(x,\zeta)$.

And,

$$
\begin{aligned}
(-1+\beta x)z + g(x,y,z) &= (-1+\beta x)(\zeta + S_0) + g(x, R_0, \zeta + S_0) \\
&+ (-1+\beta x)\sum_{j=1}^{\infty} S_j(x,\zeta)\eta^j + (C(x,\zeta)(R_1+1) + D(x,\zeta)S_1)\eta \\
&+ \sum_{j=0}^{\infty}\{C(x,\zeta)R_j + D(x,\zeta)S_j + \mathcal{Q}_j(x,\zeta)\}\eta^j,
\end{aligned} \tag{4.9}
$$

where

$$
C(x,\zeta) = \frac{\partial g}{\partial y}(x, R_0(x,\zeta), \zeta + S_0(x,\zeta)), \quad D(x,\zeta) = \frac{\partial g}{\partial z}(x, R_0(x,\zeta), \zeta + S_0(x,\zeta))
$$

and the $\mathcal{Q}_j(x,\zeta)$ have an analogous meaning to the $\mathcal{P}_j(x,\zeta)$.

Note that the expressions (4.6) and (4.7) have to be identity relations and this is also the same for the expressions (4.8) and (4.9). Compare the coefficients appearing in the like terms of η^j for $j = 0, 1, \cdots$. We can obtain the following equations:

$$
\left\{
\begin{aligned}
x^2 \frac{\partial R_0}{\partial x} + \zeta(-1+\beta x)\frac{\partial R_0}{\partial \zeta} &= (1+\alpha x)R_0 + f(x, R_0, \zeta + S_0), \\
x^2 \frac{\partial S_0}{\partial x} + \zeta(-1+\beta x)\frac{\partial S_0}{\partial \zeta} &= (-1+\beta x)S_0 + g(x, R_0, \zeta + S_0),
\end{aligned}
\right. \tag{4.10.0}
$$

$$
(4.10.j)\quad \begin{cases} x^2\dfrac{\partial R_j}{\partial x}+\zeta(-1+\beta x)\dfrac{\partial R_j}{\partial \zeta}=(1-j)(1+\alpha x)R_j+A(x,\zeta)R_j \\ \qquad\qquad +B(x,\zeta)S_j+F_j(x,\zeta), \\ x^2\dfrac{\partial S_j}{\partial x}+\zeta(-1+\beta x)\dfrac{\partial S_j}{\partial \zeta}=[(-1+\beta x)-j(1+\alpha x)]S_j+C(x,\zeta)R_j \\ \qquad\qquad +D(x,\zeta)S_j+G_j(x,\zeta), \end{cases}
$$

where

$$
F_1(x,\zeta)=A(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_0(x,\zeta)}{\partial\zeta},
$$

$$
G_1(x,\zeta)=C(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\left(\frac{\partial S_0(x,\zeta)}{\partial\zeta}+1\right),
$$

$$
\begin{aligned}
F_2(x,\zeta)&=\mathcal{P}_2(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_1(x,\zeta)}{\partial\zeta}\\
&\quad-\beta(x)\zeta^3\frac{\partial R_0(x,\zeta)}{\partial\zeta}-(\alpha'+\frac{\gamma}{\delta}x\beta(x))\zeta R_1(x,\zeta),\\
G_2(x,\zeta)&=C(x,\zeta)R_2+D(x,\zeta)S_2+\mathcal{Q}_2(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial S_1(x,\zeta)}{\partial\zeta}\\
&\quad-\beta(x)\zeta^3\frac{\partial S_0(x,\zeta)}{\partial\zeta}-(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta S_1(x,\zeta)-\beta(x)\zeta^3,
\end{aligned}
$$

$$
\begin{aligned}
F_3(x,\zeta)&=\mathcal{P}_3(x,\zeta)-\alpha(x)\zeta^2-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_2(x,\zeta)}{\partial\zeta}\\
&\quad-\beta(x)\zeta^3\frac{\partial R_1(x,\zeta)}{\partial\zeta}-2(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta R_2(x,\zeta)-\alpha(x)\zeta^2R_1(x,\zeta),\\
G_3(x,\zeta)&=\mathcal{Q}_3(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial S_2(x,\zeta)}{\partial\zeta}\\
&\quad-\beta(x)\zeta^3\frac{\partial S_1(x,\zeta)}{\partial\zeta}-2(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta S_2(x,\zeta)-\alpha(x)\zeta^2S_1(x,\zeta),
\end{aligned}
$$

$$
\begin{aligned}
F_j(x,\zeta)&=\mathcal{P}_j(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial R_{j-1}(x,\zeta)}{\partial\zeta}-\beta(x)\zeta^3\frac{\partial R_{j-2}(x,\zeta)}{\partial\zeta}\\
&\quad-(j-1)(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta R_{j-1}(x,\zeta)-(j-2)\alpha(x)\zeta^2R_{j-2}(x,\zeta),\\
G_j(x,\zeta)&=\mathcal{Q}_j(x,\zeta)-(\beta'+\frac{\gamma}{\delta}x\beta(x))\zeta^2\frac{\partial S_{j-1}(x,\zeta)}{\partial\zeta}-\beta(x)\zeta^3\frac{\partial S_{j-2}(x,\zeta)}{\partial\zeta}\\
&\quad-(j-1)(\alpha'+\frac{\gamma}{\delta}x\alpha(x))\zeta S_{j-1}(x,\zeta)-(j-2)\alpha(x)\zeta^2S_{j-2}(x,\zeta).
\end{aligned}
$$

Notice that these equations possess formal solutions which are expressed by the power series (4.2.j).

§5. Determinations of the coefficient functions $R_j(x,\zeta)$ and $S_j(x,\zeta)$.

Let x be restirctecd within the domain

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_1.$$

To determine the coefficients $R_j(x,\zeta)$ and $S_j(x,\zeta)$ as holomorphic functions, we regard the ζ as a holonorphic solution $\zeta = \tilde{\zeta}(x)$ of the differential equation

$$x^2 \frac{d\zeta}{dx} = \zeta(-1 + \beta x). \tag{5.1}$$

Corresponding to partial differential equations (4.10.0), we consider the nonlinear ordinary differential equation

$$\left\{ \begin{array}{l} x^2 \dfrac{dR_0}{dx} = (1 + \alpha x)R_0 + f(x, R_0, \zeta + S_0), \\ x^2 \dfrac{dS_0}{dx} = (-1 + \beta x)S_0 + g(x, R_0, \zeta + S_0), \end{array} \right. \tag{5.2.0}$$

The point $x = 0$ is an irregular type singular point. By our standard analysis (M. Iwano [1]), *the formal solution* (4.2.0) *is uniformly convergent for a domain of the form* (4.4): $|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_1, \quad |\zeta| < c_1$ *and the sum* $\{R_0(x,\zeta), S_0(x,\zeta)\}$ *represents a solution of equations* (5.2.0). The $R_0(x,\zeta)$ and $S_0(x,\zeta)$ are holomorphic and bounded functions in (x,ζ) for domain (4.4). A function like $R_0(x,\zeta)$ will be called *a function with Property* $\mathcal{A}$ *with respect to* ζ *in domain* (4.4). Substitute these functions for $\{R_0(x,\zeta), S_0(x,\zeta)\}$ appearing in the expressions $\{A(x,\zeta), B(x,\zeta)\}$ (see (4.7)), $\{C(x,\zeta), D(x,\zeta)\}$ (see (4.9)) and $\{F_k(x,\zeta), G_k(x,\zeta)\}$ for $k = 1, 2$. Then, the resulting expressions become functions with Property $\mathcal{A}$ with respect to ζ in domain (4.4).

In genereric, corresponding to partial differential equations (4.10.j), we consider the linear ordinary differential equations

$$\begin{array}{l} x^2 \dfrac{dR_j}{dx} = (1 - j)(1 + \alpha x)R_j + A(x,\zeta)R_j + B(x,\zeta)S_j + F_j(x,\zeta), \\ x^2 \dfrac{dS_j}{dx} = [-1 - j + (\beta - j\alpha)x]S_j + C(x,\zeta)R_j + D(x,\zeta)S_j + G_j(x,\zeta). \end{array} \tag{5.2.j}$$

Obviously, the power series (4.2.j) are a formal solution of (5.2.j). By induction on j, we can assume that the $F_j(x,\zeta)$ and $G_j(x,\zeta)$ are regarded as known functions with Property $\mathcal{A}$ with respect to ζ in domain (4.4). Hence, by our standard analysis, *regardless of the sign of the quantity* γ, *the formal solution* (4.2.j) *is uniformly convergent for domain* (4.4) *so that the sum* $\{R_j(x,\zeta), S_j(x,\zeta)\}$ *represents a solution of equations* (5.2.j). *The functions* $R_j(x,\zeta)$ *and* $S_j(x,\zeta)$ *have Property* $\mathcal{A}$ *with respect to* ζ *in* (x,ζ) *for domain*(4.4). This proves Theorem 2. **q.e.d.**

§6. Truncated differential equations.

In order to study an analytical meaning of the formal transformation (4.1), we put

$$R_{(N)}(x,\eta,\zeta) = \eta + \sum_{j=0}^{2N} R_j(x,\zeta)\eta^j, \quad S_{(N)}(x,\eta,\zeta) = \zeta + \sum_{j=0}^{2N} S_j(x,\zeta)\eta^j, \tag{6.1}$$

where N is any integer greater than a certain quantity N_0. Make a change of variables $(y, z) \to (u, v)$ by

$$y = R_{(N)}(x, u, v), \quad z = S_{(N)}(x, u, v). \tag{6.2}$$

When y and z are expressed by power series (4.1), it is easily verified that the formal solution $\{u, v\}$ of equations (6.2) are given by

$$u = \eta + \sum_{j=2N+1}^{\infty} \phi_j(x, \zeta)\eta^j, \quad v = \zeta + \sum_{j=2N+1}^{\infty} \psi_j(x, \zeta)\eta^j. \tag{6.3}$$

The $\phi_j(x, \zeta)$ and $\psi_j(x, \zeta)$ are functions with Property $\mathcal{A}$ with respect to ζ in domain (4.4).

To see this, we notice, by the assumption **(iii)** on f and g, that

$$R_0(x, 0) = 0, \quad S_0(x, 0) = 0, \quad \frac{\partial R_0}{\partial \zeta}(x, 0) = 0, \quad \frac{\partial S_0}{\partial \zeta}(x, 0) = 0. \tag{6.4}$$

We want to solve the following equation with respect to $\{u, v\}$:

$$u + \sum_{j=0}^{2N} R_j(x, v)u^j = \eta + \sum_{j=0}^{\infty} R_j(x, \zeta)\eta^j, v + \sum_{j=0}^{2N} S_j(x, v)u^j = \zeta + \sum_{j=0}^{\infty} S_j(x, \zeta)\eta^j. \tag{6.5}$$

Put

$$u = \eta + U, \quad v = \zeta + V.$$

Assume that U and V satisfy formally the order condition: $U = \mathrm{O}(\eta^2)$, $V = \mathrm{O}(\eta^2)$. Equations (6.5) are approximately written as in the form

$$\eta + (1 + R_1(x, \zeta) + \mathrm{O}(\eta))U + \big(\frac{\partial R_0(x, \zeta)}{\partial \zeta} + \mathrm{O}(\eta)\big)V + [\cdots] = \eta + \mathrm{O}(\eta^{2N+1}),$$

$$\zeta + (S_1(x, \zeta) + \mathrm{O}(\eta))U + \big(1 + \frac{\partial S_0(x, \zeta)}{\partial \zeta} + \mathrm{O}(\eta)\big)V + [\cdots] = \zeta + \mathrm{O}(\eta^{2N+1}).$$

The expressions in the brackets denote higher order terms in $\{U, V\}$. Observe that

$$\det \begin{pmatrix} 1 + R_1(x, 0) & \frac{\partial R_0}{\partial \zeta}(x, 0) \\ S_1(x, 0) & 1 + \frac{\partial S_0}{\partial \zeta}(x, 0) \end{pmatrix} = 1 \quad (\text{by}(6.4)). \tag{6.6}$$

Hence, we can find a formal solution $\{U, V\}$, which is expressed by power series in η with coefficients with Property $\mathcal{A}$ with respect to ζ in domain (4.4) and satisfies the order condition $U = \mathrm{O}(\eta^{2N+1})$, $V = \mathrm{O}(\eta^{2N+1})$.

It is not so hard to prove that the equations on $\{u, v\}$ are written as in the form (6.7.N)

$$x^2 \frac{du}{dx} = u(1 + \alpha x + (\alpha' + \frac{\gamma}{\delta} x\alpha(x))uv + \alpha(x)(uv)^2) + u^{2N+1}F_N(x, u, v),$$

$$x^2 \frac{dv}{dx} = v(-1 + \beta x + (\beta' + \frac{\gamma}{\delta} x\beta(x))uv + \beta(x)(uv)^2) + u^{2N+1}G_N(x, u, v).$$

The $F_N(x,u,v)$ and $G_N(x,u,v)$ are holomorphic and bounded functions of (x,u,v) for a domain of the form

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_N, \quad |u| < c_N, \quad |v| < c_N. \tag{6.8}$$

The a_N, d_N, c_N are sufficiently small constants depending on N. The equations (6.7.N) are called *truncated differential equations associated with the formal transformation* (4.1).

§7. Modified simplified equations and stable domains.

We introduce an auxiliary variable w by

$$w = \frac{\eta\zeta}{x} \quad \text{or} \quad w = \frac{\eta\zeta}{x} + \frac{\gamma}{\delta} \tag{7.1}$$

according as $\gamma \geq 0$ or $\gamma < 0$. By using simplified equations (B), we make formally the differential equation which is satified by the variable w. Eliminate $\eta\zeta$ appearing in (B) by the help of w, then we find a 3-system of equations on $\{\eta, \zeta, w\}$. Regard $\{\eta, \zeta, w\}$ as independent variables. The equations on $\{\eta, \zeta, w\}$ are called *the modified simplified differential equations* of equations (A). Before investigating the growth order of a solution $\{\eta(x), \zeta(x)\}$ of equations (B), we need to know the growth order of $\frac{\eta(x)\zeta(x)}{x}$ or $\frac{\eta(x)\zeta(x)}{x} + \frac{\gamma}{\delta}$ according as $\gamma \geq 0$ or $\gamma < 0$. So, it is a natural idea to introduce the new variable w.

As was already seen in M. Iwano [5], the modified simplified equations are given by

$$(\mathrm{B}_{+,0}) \qquad \begin{cases} x^2\dfrac{d\eta}{dx} = \eta\big(1 + \alpha x + \big(\alpha' + \frac{\gamma}{\delta}\alpha(x)x\big)xw + \alpha(x)x^2w^2\big), \\ x^2\dfrac{d\zeta}{dx} = \zeta\big(-1 + \beta x + \big(\beta' + \frac{\gamma}{\delta}\beta(x)x\big)xw + \beta(x)x^2w^2\big), \\ x\dfrac{dw}{dx} = \gamma w + \big(\delta + \frac{\gamma}{\delta}\rho(x)x\big)w^2 + \rho(x)xw^3. \end{cases}$$

or

$$(\mathrm{B}_{-}) \qquad \begin{cases} x^2\dfrac{d\eta}{dx} = \eta\big(1 + \alpha_1 x + \big(\alpha' - \frac{\gamma}{\delta}\alpha(x)x\big)xw + \alpha(x)x^2w^2\big), \\ x^2\dfrac{d\zeta}{dx} = \zeta\big(-1 + \beta_1 x + \big(\beta' - \frac{\gamma}{\delta}\beta(x)x\big)xw + \beta(x)x^2w^2\big), \\ x\dfrac{dw}{dx} = -\gamma w + \dfrac{\gamma^2}{\delta^2}\rho(x)xw + \big(\delta - \frac{2\gamma}{\delta}\rho(x)x\big)w^2 + \rho(x)xw^3, \end{cases}$$

$$\alpha_1 = \alpha - \frac{\alpha'\gamma}{\delta}, \qquad \beta_1 = \beta - \frac{\beta'\gamma}{\delta}. \tag{7.2}$$

Let $\{\hat{\eta}(x), \hat{\zeta}(x)\}$ be the solution of equations (B) subject to: $\{\hat{\eta}(x_0), \hat{\zeta}(x_0)\} = \{\eta_0, \zeta_0\}$. Denote by $\{\eta(x), \zeta(x), w(x)\}$ the solution of the modified simplified equations such that $\{\eta(x_0), \zeta(x_0), w(x_0)\} = \{\eta_0, \zeta_0, w_0\}$. Then, as was shown in M. Iwano [5], we have the identity relation $\{\hat{\eta}(x), \hat{\zeta}(x)\} = \{\eta(x), \zeta(x)\}$ if and only if the relation $w_0 = \frac{\eta_0\zeta_0}{x_0}$ or $w_0 = \frac{\eta_0\zeta_0}{x_0} + \frac{\gamma}{\delta}$ holds according as $\gamma \geq 0$ or $\gamma < 0$.

To construct stable domains of solutions of the modified simplified equations with respect to the curve $\Gamma(x_0)$, which joins the origin and the starting point x_0, as was done in M.Iwano [5], we need to simplify them to equations with a simpler form by using nonsingular transformations. This curve consists generally of two parts: Γ' and Γ''. The variable point x on Γ' is expressed by the formula $\frac{1}{x} = A + \sigma - iBe^{\kappa\sigma}, \quad 0 \le \sigma < \infty$. Here κ is a certain positive constant such that $1+\kappa\alpha > 0$ and $-1+\kappa\beta > 0$ and $A = \frac{A_0'}{(A_0')^2+(B_0')^2}, B = \frac{B_0'}{(A_0')^2+(B_0')^2}$, where $A_0' + iB_0'$ is the starting point of the curve Γ' and, at the same time, is the end point of the curve Γ''. The variable point x on the curve Γ'' is expressed by the formula $x = \left(|x_0|\frac{\cos\phi}{\cos\theta}\right)e^{i\phi}$, where $\theta = \arg x_0$. (See, M. Iwano [5]).

We shall write down the results only.

Case I. $\gamma > 0$.

Theorem 3.B$_+$. **(i)** *There is a nonsingular transformation* :$\{\eta, \zeta, w\} \to \{Y, Z, W\}$ *of the form*

$$(7.3)\quad \eta = Y(1+WP(x,W)), \quad \zeta = Z(1+WQ(x,W)), \quad w = W(1+W\Phi(x,W)),$$

which changes equations (B$_+$) *to the linear equations*

$$(\mathrm{B}_+')\qquad x^2\frac{dY}{dx} = Y(1+\alpha x), \quad x^2\frac{dZ}{dx} = Z(-1+\beta x), \quad x\frac{dW}{dx} = \gamma W.$$

The $P(x,W), Q(x,W), \Phi(x,W)$ *are functions with Property* $\mathcal{A}$ *with respect to* W *in a domain of the form*

$$(7.4)\qquad |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_0, \quad |W| < d_0.$$

(ii) *Stable domains of solutions of equations* (B$'_+$) *are given by the form*

$$(7.5)\qquad \begin{cases} |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_0\omega(\arg x), \\ |Y| < b_0\chi_\alpha(\arg x), \quad |Z| < b_0\chi_\beta(\arg x), \quad |W| < d_0\chi_\gamma(\arg x). \end{cases}$$

The $\omega(\phi)$ *and* $\chi_\xi(\phi)(\xi = \alpha, \beta, \gamma)$ *are strictly positive valued continuous functions defined in the* ϕ*-interval* $[-\pi+\epsilon, \pi-\epsilon]$. *They are expressed by the formulas*

$$(7.6)\qquad \omega(\phi) = \begin{cases} \dfrac{\cos\Omega}{\sin\epsilon}, & |\phi \mp \frac{\pi}{2}| \le \frac{\pi}{2} - \Omega, \\[2ex] \dfrac{|\cos\phi|}{\sin\epsilon}, & \frac{\pi}{2} - \Omega \le |\phi \mp \frac{\pi}{2}| \le \pi - \epsilon. \end{cases}$$

$$(7.7)\qquad \chi_\xi(\phi) = \begin{cases} (\cos\Omega)^\xi, & |\phi \mp \frac{\pi}{2}| \le \frac{\pi}{2} - \Omega, \\ |\cos\phi|^\xi, & \frac{\pi}{2} - \Omega \le |\phi \mp \frac{\pi}{2}| \le \pi - \epsilon. \end{cases}$$

The Ω *is a suitable angle between* $\frac{\pi}{4}$ *and* $\frac{\pi}{2}$.

(iii) *Stable domains of solutions of equations* (B$_+$) *in the* (x, η, ζ, w)*-space are the image of the* (x, Y, Z, W)*-stable domains* (7.5) *in the* (x, Y, Z, W)*-space by the topological mapping* (7.3).

Case II. $\gamma < 0$. Let Ω_1 be a suitable angle between $\frac{\pi}{4}$ and $\frac{\pi}{2}$.

Theorem 3.B$_-$. **(i)** *There is a nonsingular transformation* $\{\eta, \zeta, w\} \to \{Y, Z, W\}$*:*

(7.8) $\eta = Y(1 + WP_1(x, W)), \quad \zeta = Z(1 + WQ_1(x, W)), w = W(1 + \Phi_1(x, W)),$

by which the equations (B$_-$) *are changed to the linear equations*

$$(\mathrm{B}'_-) \qquad x^2\frac{dY}{dx} = Y(1 + \alpha_1 x), \quad x^2\frac{dZ}{dx} = Z(-1 + \beta_1 x), \quad x\frac{dW}{dx} = \gamma_1 W.$$

The $P_1(x, W), Q_1(x, W), \Phi_1(x, W)$ *are functions with Property* $\mathcal{A}$ *with respect to* W *in a domain of the form* (7.4). *In particular,* $\lim_{x\to 0} \Phi_1(x, 0) = 0$.

(ii) *Stable domains of solutions* $\{Y(x), Z(x), W(x)\}$ *of equations* (B$'_-$) *are of the form*

$$(7.9) \qquad \begin{cases} |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_0\omega_1(\arg x), \\ |Y| < b_0\chi_{\alpha_1}(\arg x), \quad |Z| < b_0\chi_{\beta_1}(\arg x), \quad |W| < d_0\chi_{\gamma_1}(\arg x). \end{cases}$$

The $\omega_1(\phi)$ *and* $\chi_\xi(\phi)(\xi = \alpha_1, \beta_1, \gamma_1)$ *are strictly positive valued continuous functions defined in the* ϕ*-interval* $[-\pi + \epsilon, \pi - \epsilon]$. *They are expressed by the formulas* (7.6) *and* (7.7) *if we replace the angle* Ω *by* Ω_1.

(iii) *Stable domains of solutions of equations* (B$_-$) *in the* (x, η, ζ, w)*-space are defined by the image of the* (x, Y, Z, W)*-stable domains* (7.9) *in the* (x, Y, Z, W)*-space by the topological mapping* (7.8).

Case III. $\gamma = 0$. As is shown in §13 in M. Iwano [5], the open angle of the x-domain in an $\{x, \eta, \zeta, w\}$-stable domain cannot be greater than π. Assume $\arg\delta = 0$. If otherwise, it is sufficient to rotate the W-plane by the angle $\arg\delta$ positively.

Theorem 3.B$_0$. **(i)** *There is a nonsingular transformation:*$\{\eta, \zeta, w\} \to \{Y, Z, W\}$ *of the form*

(7.10)

$\eta = Y(1 + xW^2P_0(x, W)), \zeta = Z(1 + xW^2Q_0(x, W)), w = W(1 + xW^2\Psi(x, W)),$

by which equations (B$_0$) *are changed to the equations*

$$(\mathrm{B}'_0) \quad x^2\frac{dY}{dx} = Y(1+\alpha x+\alpha' xW), \quad x^2\frac{dZ}{dx} = Z(-1+\beta x+\beta' xW), \quad x\frac{dW}{dx} = \delta W^2.$$

The $P_0(x, W), Q_0(x, W), \Psi(x, W)$ *are functions holomorphic and bounded in* (x, w) *in a domain which is sensibly the same as a domain of the form*

$$(7.11) \quad |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \; 0 < |x| < a_0, \; |\arg W + \arg\delta| < \pi - \epsilon, \; 0 < |W| < d_0.$$

Moreover, these functions admit asymptotic expansions in powers of W *(or* x *) uniformly valid in domain* (7.11) *with coefficients, which are functions with Property* $\mathcal{C}$ *with respect to* x *(or* W*) in the* x*-domain (or* W*-domain) in* (7.11).

(ii) *Let* μ *and* μ' *be any angles satisfying the inequalities either*

$$(7.12^-) \qquad \frac{\pi}{2} + 2\epsilon \le \mu < \frac{3\pi}{2} - \epsilon, -\frac{\pi}{2} + 2\epsilon < \mu' \le \frac{\pi}{2} - \epsilon, \quad \mu - \mu' < \pi$$

or

$$(7.12^+)\qquad -\frac{3\pi}{2}+\epsilon<\mu'\leq-\frac{\pi}{2}-2\epsilon, -\frac{\pi}{2}+\epsilon\leq\mu'<\frac{\pi}{2}-2\epsilon,\quad \mu-\mu'<\pi.$$

Stable domains of solutions of equations (B_0') *are given by the form*

$$(7.13)\qquad \begin{cases} \mu'<\arg x<\mu,\ 0<|x|<a_0\omega_0(\arg x), & W\in\mathcal{D}_W^{\mp}(\arg x;d_0)\\ |Y|<b_0\chi_{\alpha,\alpha'}(\arg x,W), & |Z|<b_0\chi_{\beta,\beta'}(\arg x,W).\end{cases}$$

The $\omega_0(\phi)$ *and* $\chi_{\xi,\tau}(\phi,W)(\{\xi,\tau\}=\{\alpha,\alpha'\},\{\beta,\beta'\})$ *are strictly positive valued continuous functions in* (ϕ,W) *defined in the product of* ϕ*-interval* $[\mu',\mu]$ *and* $W\in\mathcal{D}_W^{\mp}(\arg x;d_0)$. *They are expressed by the formulas*

$$(7.14)\qquad \omega(\phi)=\begin{cases}1, & \text{for}\quad |\phi\mp\frac{\pi}{2}|\leq\epsilon,\\ \frac{|\cos\phi|}{\sin\epsilon}, & \text{for}\quad \mu'\leq\phi\leq\pm\frac{\pi}{2}-\epsilon,\quad \pm\frac{\pi}{2}+\epsilon\leq\phi\leq\mu.\end{cases}$$

$$(7.15)\qquad \chi_{\xi,\tau}(\phi,W)=\begin{cases}|\sin\epsilon|^{\xi}|1+W\log(|\sin\epsilon|e^{i\phi})|^{\frac{\tau}{\delta}}, & \text{for}\ |\phi\mp\frac{\pi}{2}|\leq\epsilon,\\ |\cos\phi|^{\xi}|1+W\log(|\cos\phi|e^{i\phi})|^{\frac{\tau}{\delta}}, & \\ \qquad \text{for}\quad \mu'\leq\phi\leq\pm\frac{\pi}{2}-\epsilon,\quad \pm\frac{\pi}{2}+\epsilon\leq\phi\leq\mu. &\end{cases}$$

The domain $\mathcal{D}_W^{-}(\phi;d_0)$ *is described by the use of the parameters* $\phi(=\arg x),\Phi(=\arg W),\mu,\mu'$ *and it is contained in a sectorial domain of the form:* $-\mu'-\frac{\pi}{2}<\arg W<-\mu+\frac{3\pi}{2},\quad 0<|W|<d_0$, *while the domain* $\mathcal{D}_W^{+}(\phi;d_0)$ *is in a sectorial domain :* $-\mu'-\frac{3\pi}{2}<\Phi<-\mu+\frac{\pi}{2}, 0<W|<d_0$ *and it is obtained from* $\mathcal{D}_W^{-}(\phi;d_0)$ *by the substitutions of the parameters :*$\phi\to-\phi,\Phi\to-\Phi,\mu\to-\mu',\mu'\to-\mu$. *Here the double signs are to be chosen in the same order.*

(iii) *Stable domains of solutions of equations* (B_0) *in the* (x,η,ζ,w)*-space are given by the image of the* (x,Y,Z,W)*-stable domains* (7.13) *in the* (x,Y,Z,W)*-space by the topological mapping* (7.10).

A function like the $P_0(x,w)$ is called *a function with Property $\mathcal{B}$ with respect to* W(or x) in domain (7.11).

Note. An explicite shape of the domain $\mathcal{D}_W^{-}(\phi;d_0)$ is clarified in Theorem 5.2 in §13 in M. Iwano [5].

§8. Modified truncated equations and Existence theorem, Part I.(Case of $\gamma\geq 0$).

Let $\{\hat\eta(x),\hat\zeta(x)\}$ be a holomorphic solution of the simplified equations (B). Apply a changement of variables $(u,v)\to(\hat Y,\hat Z)$ by

$$(8.1)\qquad u=\frac{\hat\eta(x)}{1-\hat Y},\quad v=\frac{\hat\zeta(x)}{1-\hat Z}.$$

A direct calculation gives

$$x^2\frac{du}{dx}=\frac{x^2\frac{d\hat\eta(x)}{dx}}{1-\hat Y}+\frac{\hat\eta(x)}{(1-\hat Y)^2}x^2\frac{d\hat Y}{dx}.$$

Define $A_0(Y,Z), B_0(Y,Z), C_0(Y,Z)$ by

$$
(8.2)\qquad
\begin{aligned}
&A_0(Y,Z) = \frac{1}{1-Z} - (1-Y), \quad B_0(Y,Z) = \frac{1}{(1-Y)(1-Z)^2} - (1-Y),\\
&C_0(Y,Z) = \big[1 - \frac{1}{(1-Y)(1-Z)}\big]^2 (1-Y)
\end{aligned}
$$

and put

$$
(8.3)\qquad \hat{w}(x) = \frac{\hat{\eta}(x)\hat{\zeta}(x)}{x}.
$$

Then, by the use of equations (B) and (6.7.N), we have easily

$$
(8.4)\qquad
\begin{aligned}
x^2\frac{d\hat{Y}}{dx} &= \big(\alpha' x\hat{w}(x) + \frac{\gamma}{\delta}x^2\alpha(x)\hat{w}(x)\big)A_0(\hat{Y},\hat{Z}) + x^2\alpha(x)\hat{w}(x)^2 B_0(\hat{Y},\hat{Z})\\
&\quad + \frac{\hat{\eta}(x)^{2N}}{(1-\hat{Y})^{2N-1}}F_N\big(x, \frac{\hat{\eta}(x)}{1-\hat{Y}}, \frac{\hat{\zeta}(x)}{1-\hat{Z}}\big),\\
x^2\frac{d\hat{Z}}{dx} &= \big(\beta' x\hat{w}(x) + \frac{\gamma}{\delta}x^2\beta(x)\hat{w}(x)\big)A_0(\hat{Z},\hat{Y}) + x^2\beta(x)\hat{w}(x)^2 B_0(\hat{Z},\hat{Y})\\
&\quad + \frac{(1-\hat{Z})^2}{(1-\hat{Y})^{2N+1}}\frac{\hat{\eta}(x)^{2N+1}}{\hat{\zeta}(x)}G_N\big(x, \frac{\hat{\eta}(x)}{1-\hat{Y}}, \frac{\hat{\zeta}(x)}{1-\hat{Z}}\big).
\end{aligned}
$$

Let $\{\eta(x), \zeta(x), w(x)\}$ be a solution of the modified simplified differential equations (B_+) or (B_0) appearing in §7. Now, in equations (8.4), we make the substitutions:

$$
(8.5)\qquad \{\hat{\eta}(x), \hat{\zeta}(x), \hat{w}(x)\} \to \{\eta(x), \zeta(x), w(x)\}, \quad \{\hat{Y}, \hat{Z}\} \to \{Y, Z\}.
$$

Thus, we have the equations

$$
(8.6)\qquad
\begin{aligned}
x^2\frac{dY}{dx} &= \big(\alpha' xw(x) + \frac{\gamma}{\delta}x^2\alpha(x)w(x)\big)A_0(Y,Z) + x^2\alpha(x)w(x)^2 B_0(Y,Z)\\
&\quad + \frac{\eta(x)^{2N}}{(1-Y)^{2N-1}}F_N\big(x, \frac{\eta(x)}{1-Y}, \frac{\zeta(x)}{1-Z}\big),\\
x^2\frac{dZ}{dx} &= \big(\beta' xw(x) + \frac{\gamma}{\delta}x^2\beta(x)w(x)\big)A_0(Z,Y) + x^2\beta(x)w^2 B_0(Z,Y)\\
&\quad + \frac{(1-Z)^2}{(1-Y)^{2N+1}}\frac{\eta(x)^{2N+1}}{\zeta(x)}G_N\big(x, \frac{\eta(x)}{1-Y}, \frac{\zeta(x)}{1-Z}\big).
\end{aligned}
$$

The equations (8.6) are called *modified truncated differential equations associated with formal transformation* (4.1) or, simply, *modified truncated equations.* Write (8.6) simply as in the form

$$
(8.7)\quad x\frac{dY}{dx} = A_N(x, w(x), \eta(x), \zeta(x), Y, Z), \quad x\frac{dZ}{dx} = B_N(x, w(x), \eta(x), \zeta(x), Y, Z).
$$

The meaning of the $A_N(x, w, \eta, \zeta, Y, Z)$ and $B_N(x, w, \eta, \zeta, Y, Z)$ will be almost clear.

When a_N, d_N, c_N, b_0 are sufficiently small quantities, we have

$$
(8.8)\qquad
\begin{aligned}
&|A_0(Y,Z)| \le 2|Y| + 2|Z|, \quad |B_0(Y,Z)| \le 3|Y| + 3|Z|, \\
&\big|(1-Y)^{1-2N} F_N\big(x, \frac{\eta}{1-Y}, \frac{\zeta}{1-Z}\big)\big| \le L_N, \\
&\big|(1-Z)^2(1-Y)^{-2N-1} G_N\big(x, \frac{\eta}{1-Y}, \frac{\zeta}{1-Z}\big)\big| \le L_N
\end{aligned}
$$

for

$$
(8.9)\qquad |\arg x \mp \frac{\pi}{2}| < \pi - \epsilon,\ 0 < |x| < a_N, |\eta| < c_N,\ |\zeta| < c_N,\ |Y| < b_0,\ |Z| < b_0.
$$

L_N is a positive constant depending on N. Moreover, there is a constant M_0 such that

$$
(8.10)\qquad |\alpha'| \le M_0, \quad |\beta'| \le M_0, | \quad |\alpha(x)| \le M_0, \quad |\beta(x)| \le M_0.
$$

The functions A_N and B_N are holomorphic in $(x, w, \eta, \zeta, Y, Z)$ in a domain of the form

$$
\begin{cases}
|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_N, \quad |w| < d_N, \\
|\eta| < c_N, \quad 0 < |\zeta| < c_N, \quad |Y| < b_0, \quad |Z| < b_0,
\end{cases}
$$

but they are not bounded there. In order that the functions A_N and B_N may become bounded, we consider a subdomain of the form

$$
(8.11)\qquad
\begin{cases}
|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < a_N, \quad |w| < d_N, \\
\left|\dfrac{\eta^N}{x\zeta}\right| < \Delta, \quad 0 < |\zeta| < c_N, \quad |Y| < b_0, \quad |Z| < b_0,
\end{cases}
$$

where the Δ is a positive constant indpendent of N and

$$
(8.12)\qquad \Delta \ge 1.
$$

Then, we have

$$
(8.13)\qquad
\begin{aligned}
&|A_N(x, w, \eta, \zeta, Y, Z)| \le 14 M_0 d_N \max\{|Y|, |Z|\} + L_N \Delta |\eta|^N, \\
&|B_N(x, w, \eta, \zeta, Y, Z)| \le 14 M_0 d_N \max\{|Y|, |Z|\} + L_N \Delta \frac{|\eta|^{N+1}}{|\zeta|}.
\end{aligned}
$$

By utilizing a technique which was used in M. Iwano [2,3,4], we can prove the following

Theorem 4.B$_+$. *Assume $\gamma > 0$. Equations (8.7) have a unique solution $\{Y, Z\} = \{\varphi_N(x, w(x), \eta(x), \zeta(x)), \psi_N(x, w(x), \eta(x), \zeta(x))\}$ with the properties that*

(i) *$\varphi_N(x, w, \eta, \zeta)$ and $\psi_N(x, w, \eta, \zeta)$ are holomorphic and bounded functions of (x, w, η, ζ) for a domain, in the (x, w, η, ζ)-space, of the form*

$$
(8.14)\qquad
\begin{cases}
|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon, \quad 0 < |x| < {a_N}^0 (< a_N), \quad |w| < {d_N}^0 (< d_N), \\
\left|\dfrac{\eta^N}{x\zeta}\right| < \Delta, \quad 0 < |\zeta| < {c_N}^0 (< d_N)
\end{cases}
$$

(ii) *They satisfy there inequalities of the form*

$$\max\{|\varphi_N(x,w,\eta,\zeta)|, |\psi_N(x,w,\eta,\zeta)|\} \le K_N \max\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\}. \tag{8.15}$$

The ${a_N}^0, {d_N}^0, {c_N}^0$ *are sufficiently small constants and the* K_N *is a positive constants depending on* N.

Then, by using this theorem, we have the following

Theorem 5.B$_+$. *There is a transformation* $\{y,z\} \to \{\eta,\zeta\} : y = \Phi_0(x,\eta,\zeta), z = \Psi_0(x,\eta,\zeta)\}$, *which changes* (A) *to* (B) *in a domain of the form*

$$\left|\arg x \mp \frac{\pi}{2}\right| < \pi - \epsilon, \quad 0 < |x| < a^0, \quad |\eta| < c^0, \quad |\zeta| < c^0, \quad |\eta\zeta| < d^0|x|. \tag{8.16}$$

In other words, the $\Phi_0(x,\eta,\zeta)$ *and* $\Psi_0(x,\eta,\zeta)$ *admit convergent expansions to double power series in* η, ζ *of the form, for example,*

$$\Phi_0(x,\eta,\zeta) = \eta + \sum_{\ell=0}^{\infty} \mathcal{R}_{\ell,0}(x, \frac{\eta\zeta}{x})\eta^\ell + \sum_{\ell=1}^{\infty} \mathcal{R}_{0,\ell}(x, \frac{\eta\zeta}{x})\zeta^\ell, \tag{8.17}$$

which are obtained by algebraic rearrangement of (3.10), *whenever the values of* (x,η,ζ), *considered as points in the* (x,η,ζ)*-space, belong to domain* (8.16). *Here the coefficients are expressed by convergent power series in* w:

$$\mathcal{R}_{\ell,0}(x,w) = \sum_{k=1}^{\infty} \{R_{k+\ell,k}(x)x^k\}w^k, \quad \mathcal{R}_{0,\ell}(x,w) = \sum_{k=1}^{\infty} \{R_{k,k+\ell}(x)x^k\}w^k. \tag{8.18}$$

uniformly valid in (x,w) *for* $\left|\arg x \mp \frac{\pi}{2}\right| < \pi - \epsilon, \quad 0 < |x| < a^0, \quad |w| < d^0.$

Theorem 4.B$_0$. *Assume* $\gamma = 0$. *Let* μ *and* μ' *be any angles satisfying* (7.12$^-$). *Equations* (8.7) *have a unique solution* $\{Y,Z\} = \{\varphi_N(x,w(x),\eta(x),\zeta(x)), \psi_N(x, w(x),\eta(x),\zeta(x))\}$ *with the properties that:*

(i) $\varphi_N(x,w,\eta,\zeta)$ *and* $\psi_N(x,w,\eta,\zeta)$ *are holomorphic and bounded functions of* (x,w,η,ζ) *for a domain, in the* (x,w,η,ζ)*-space, of the form*

$$\begin{cases} \mu' < \arg x < \mu, \quad 0 < |x| < {a_N}^0 (< a_N), \\ -\mu' - \dfrac{\pi}{2} < \arg w < -\mu + \dfrac{3\pi}{2}, \quad 0 < |w| < {d_N}^0, \\ \left|\dfrac{\eta^N}{x\zeta}\right| < \Delta, \quad 0 < |\zeta| < {c_N}^0 (< d_N) \end{cases} \tag{8.19}$$

(ii) *They satisfy there inequalities of form* (8.15).

Then, by using this theorem, we have the following

Theorem 5.B$_0$. *There is a transformation* $\{y, z\} \to \{\eta, \zeta\} : y = \Phi_0(x, \eta, \zeta), z = \Psi_0(x, \eta, \zeta)$, *which changes* (A) *to* (B) *in a domain of the form*

$$(8.20) \qquad \begin{cases} \mu' < \arg x < \mu, \quad 0 < |x| < a^0, \\ -\mu' - \dfrac{\pi}{2} < \arg\big(\dfrac{\eta\zeta}{x}\big) < -\mu + \dfrac{3\pi}{2}, \qquad 0 < |\eta\zeta| < d^0|x|, \\ |\eta| < c^0, \quad |\zeta| < c^0. \end{cases}$$

In other words, the pair $\{\Phi_0(x, \eta, \zeta), \Psi_0(x, \eta, \zeta)\}$ *admits a convergent expansion to a double power series in* η, ζ *of the form* (8.17) *with coefficients given by* (8.18) *whenever the values of* (x, η, ζ), *considered as points in the* (x, η, ζ)*-space, belong to domain* (8.20). *But the coefficients* $\mathcal{R}_{\ell,0}(x, w), \mathcal{R}_{0,\ell}(x, w), \mathcal{S}_{\ell,0}(x, w), \mathcal{S}_{0,\ell}(x, w)$ *admit not convergent but asymptotic expansions to power series in* w *like* (8.18) *uniformly valid in* (x, w) *for*

$$\mu' < \arg x < \mu, \ 0 < |x| < a^0, \ -\mu' - \frac{\pi}{2} < \arg w < -\mu + \frac{3\pi}{2}, \ 0 < |w| < d^0.$$

Remark. When μ and μ' satisfy (7.12$^+$), similar theorems hold if we replace the domain of w by : $-\mu' - \frac{3\pi}{2} < \arg w < -\mu + \frac{\pi}{2}, \ 0 < |w| < d^0_N$.

§9. **Modified truncated equations and Existence theorem, Part II.(Case of** $\gamma < 0$).

In the case of $\gamma < 0$, it is convenient to write the simplified equations in the form

$$\begin{cases} x^2 \dfrac{d\eta}{dx} &= \eta\big(1 + \alpha_1 x + \big(\alpha' - \frac{\gamma}{\delta}\alpha(x)x\big)x\big(\dfrac{\eta\zeta}{x} + \dfrac{\gamma}{\delta}\big) + \alpha(x)x^2\big(\dfrac{\eta\zeta}{x} + \dfrac{\gamma}{\delta}\big)^2\big), \\ x^2 \dfrac{d\zeta}{dx} &= \zeta\big(-1 + \beta_1 x + \big(\beta' - \frac{\gamma}{\delta}\beta(x)x\big)x\big(\dfrac{\eta\zeta}{x} + \dfrac{\gamma}{\delta}\big) + \beta(x)x^2\big(\dfrac{\eta\zeta}{x} + \dfrac{\gamma}{\delta}\big)^2\big) \end{cases}$$

with $\quad \alpha_1 = \alpha - \frac{\alpha'\gamma}{\delta}, \qquad \beta_1 = \beta - \frac{\beta'\gamma}{\delta}$.

By using a holomorphic solution $\{\hat{\eta}(x), \hat{\zeta}(x)\}$ of simplified equations (B), we apply the changement of variables $\{u, v\} \to \{\hat{Y}, \hat{Z}\}$ to equations (6.7.N) by

$$(9.1) \qquad u = \frac{\hat{\eta}(x)}{1 - \big(\frac{\alpha'}{\beta'}x^{-\gamma}\hat{Y} + \hat{Z}\big)}, \quad v = \frac{\hat{\zeta}(x)}{1 - x^{-\gamma}\hat{Y} + \hat{Z}} \quad (\beta' \neq 0)$$

or

$$(9.1\text{-bis}) \qquad u = \frac{\hat{\eta}(x)}{1 - \big(x^{-\gamma}\hat{Y} + \hat{Z}\big)}, \quad v = \frac{\hat{\zeta}(x)}{1 + \hat{Z}}, \quad (\beta' = 0)$$

according as $\beta' \neq 0$ or $\beta' = 0$.

We assume that $\beta' \neq 0$. Since the case of $\beta' = 0$ is much simpler, so this case will not be discussed here.

Differentiating the first one in relations (9.1), we have

$$\begin{aligned} x^2\frac{du}{dx} = {} & \frac{\hat{\eta}}{1 - \dfrac{\alpha'}{\beta'}x^{-\gamma}\hat{Y} - \hat{Z}}\Bigg[1 + \alpha_1 x \\ & + \big(\alpha' - \frac{\gamma}{\delta}x\alpha(x)\big)\Big(\hat{\eta}\hat{\zeta} + \frac{\gamma}{\delta}x\Big) + \alpha(x)\Big(\hat{\eta}\hat{\zeta} + \frac{\gamma}{\delta}x\Big)^2\Bigg] \\ & + \frac{\hat{\eta}}{\big(1 - \dfrac{\alpha'}{\beta'}x^{-\gamma}\hat{Y} - \hat{Z}\big)^2}\Big\{-\gamma\frac{\alpha'}{\beta'}x^{-\gamma+1}\hat{Y} + \frac{\alpha'}{\beta'}x^{-\gamma}x^2\frac{d\hat{Y}}{dx} + x^2\frac{d\hat{Z}}{dx}\Big\}. \end{aligned}$$

In the other direction, substitute expressions (9.1) for $\{u, v\}$ appearing in the right hand side of (6.7.N). Since

$$\begin{aligned} &1+\alpha x+\alpha' uv+\frac{\gamma}{\delta}\alpha(x)xuv+\alpha(x)(uv)^2= \\ &=1+\alpha_1 x+(\alpha'-\frac{\gamma}{\delta}\alpha(x)x)(uv+\frac{\gamma}{\delta}x)+\alpha(x)(uv+\frac{\gamma}{\delta}x)^2, \end{aligned}$$

the right hand side expression, which has to coincide with $x^2\frac{du}{dx}$, becomes

$$\begin{aligned} x^2\frac{du}{dx} &= \frac{\hat\eta}{1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y-\hat Z}\Bigg[1+\alpha_1 x \\ &+\big(\alpha'-\frac{\gamma}{\delta}x\alpha(x)\big)\Bigg(\frac{\hat\eta\hat\zeta}{\big(1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y-\hat Z\big)\big(1-x^{-\gamma}\hat Y+\hat Z\big)}+\frac{\gamma}{\delta}x\Bigg) \\ &+\alpha(x)\Bigg(\frac{\hat\eta\hat\zeta}{\big(1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y-\hat Z\big)\big(1-x^{-\gamma}\hat Y+\hat Z\big)}+\frac{\gamma}{\delta}x\Bigg)^2\Bigg] \\ &+\frac{\hat\eta^{2N+1}}{\big(1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y-\hat Z\big)^{2N+1}}F_N\big(x,\frac{\hat\eta}{1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y-\hat Z},\frac{\hat\zeta}{1-x^{-\gamma}\hat Y+\hat Z}\big). \end{aligned}$$

To simplify the description, we write

$$\mathcal{Y}=\frac{\alpha'}{\beta'}x^{-\gamma}\hat Y+\hat Z,\quad \mathcal{Z}=x^{-\gamma}\hat Y-\hat Z,\quad \hat w(x)=\frac{\hat\eta(x)\hat\zeta(x)}{x}+\frac{\gamma}{\delta}. \tag{9.2}$$

¿From these two equations, by the use of (8.2) we have immediately

$$\begin{aligned} \frac{\alpha'}{\beta'}x^{-\gamma}x^2\frac{d\hat Y}{dx}+x^2\frac{d\hat Z}{dx} &= \frac{\gamma\alpha'}{\beta'}x^{-\gamma+1}\hat Y+\big(\alpha'-\frac{\gamma}{\delta}x\alpha(x)\big)\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)A_0(\mathcal{Y},\mathcal{Z}) \\ &-\big(\alpha'-\frac{\gamma}{\delta}x\alpha(x)\big)\frac{\gamma}{\delta}xA_0(\mathcal{Y},\mathcal{Z})+\alpha(x)\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)^2B_0(\mathcal{Y},\mathcal{Z}) \\ &+2\frac{\gamma}{\delta}\alpha(x)x\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)\big[A_0(\mathcal{Y},\mathcal{Z})-B_0(\mathcal{Y},\mathcal{Z})\big] \\ &+\frac{\gamma^2}{\delta^2}\alpha(x)x^2C_0(\mathcal{Y},\mathcal{Z})+\frac{\hat\eta^{2N}}{(1-\mathcal{Y})^{2N-1}}F_N\left(x,\frac{\hat\eta(x)}{1-\mathcal{Y}},\frac{\hat\zeta(x)}{1-\mathcal{Z}}\right). \end{aligned} \tag{9.3}$$

In quite a similar manner, by the use of the second one in equations (9.1), we obtain the relation

$$\begin{aligned} x^{-\gamma}x^2\frac{d\hat Y}{dx}-x^2\frac{d\hat Z}{dx} &= \gamma x^{-\gamma+1}\hat Y+\big(\beta'-\frac{\gamma}{\delta}x\beta(x)\big)\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)A_0(\mathcal{Z},\mathcal{Y}) \\ &-\big(\beta'-\frac{\gamma}{\delta}x\beta(x)\big)\frac{\gamma}{\delta}xA_0(\mathcal{Z},\mathcal{Y})+\beta(x)\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)^2B_0(\mathcal{Z},\mathcal{Y}) \\ &+2\frac{\gamma}{\delta}\beta(x)x\big(\hat\eta\hat\zeta+\frac{\gamma}{\delta}x\big)\big[A_0(\mathcal{Z},\mathcal{Y})-B_0(\mathcal{Z},\mathcal{Y})\big] \\ &+\frac{\gamma^2}{\delta^2}\beta(x)x^2C_0(\mathcal{Z},\mathcal{Y})+\frac{(1-\mathcal{Z})^2}{(1-\mathcal{Y})^{2N+1}}\frac{\hat\eta^{2N+1}}{\hat\zeta}G_N\left(x,\frac{\hat\eta(x)}{1-\mathcal{Y}},\frac{\hat\zeta(x)}{1-\mathcal{Z}}\right). \end{aligned} \tag{9.4}$$

Assuming that $|\mathcal{Y}|$ and $|\mathcal{Z}|$ are small, we have

$$A_0(\mathcal{Y},\mathcal{Z}) = \mathcal{Y}+\mathcal{Z}+[\mathcal{Y},\mathcal{Z}]_2 = \frac{\delta}{\beta'}x^{-\gamma}\hat{Y}+[x^{-\gamma}\hat{Y},\hat{Z}]_2,$$

$$B_0(\mathcal{Y},\mathcal{Z}) = 2(\mathcal{Y}+\mathcal{Z})+[\mathcal{Y},\mathcal{Z}]_2 = 2\frac{\delta}{\beta'}x^{-\gamma}\hat{Y}+[x^{-\gamma}\hat{Y},\hat{Z}]_2,$$

$$A_0(\mathcal{Y},\mathcal{Z})-B_0(\mathcal{Y},\mathcal{Z}) = -(\mathcal{Y}+\mathcal{Z})+[\mathcal{Y},\mathcal{Z}]_2 = -\frac{\delta}{\beta'}x^{-\gamma}\hat{Y}+[x^{-\gamma}\hat{Y},\hat{Z}]_2,$$

$$C_0(\mathcal{Y},\mathcal{Z}) = (\mathcal{Y}+\mathcal{Z})^2+[\mathcal{Y},\mathcal{Z}]_3 = [x^{-\gamma}\hat{Y},\hat{Z}]_2.$$

Here, for simplicity's sake, we use the symbol $[y,z]_2$ to express a convergent expansion in y and z neither the constant nor the linear terms. Cancel out the term $\frac{\gamma\alpha'}{\beta'}x^{-\gamma+1}\hat{Y}$ in (9.3) and the term $\gamma x^{-\gamma+1}\hat{Y}$ in (9.4). By adding expressions (9.3) and (9.4), we have the relation for $x^2\frac{d\hat{Y}}{dx}$. The relation for $x^2\frac{d\hat{Z}}{dx}$ is obtained by substructing the multiple of expression (9.4) by $\frac{\alpha'}{\beta'}$ from expression (9.3). It is very complicated to write down the whole expressions of the equations for $\{\hat{Y},\hat{Z}\}$. But, when we prove, according to our standard analysis, the existence of solutions of certain integral equations, the linear terms in $\hat{Y}$ and $\hat{Z}$ plays especially an important role in our analysis. So we want to pick up the linear terms in $x^{-\gamma}\hat{Y}$ and $\hat{Z}$ only.

Assuming that the linear terms $x^{-\gamma}\hat{Y}$ and $\hat{Z}$ are sufficiently small in modulus, we pick them up appearing in (9.3) and (9.4). After dividing the expressions by x, we can easily verify that the resulting equations are written in the form

$$\begin{aligned}(9.5)\qquad &\frac{\alpha'}{\beta'}x^{-\gamma}x\frac{d\hat{Y}}{dx}+x\frac{d\hat{Z}}{dx} = \frac{\delta}{\beta'}\big(\alpha'-\frac{\gamma}{\delta}x\alpha(x)\big)\hat{w}(x)x^{-\gamma}\hat{Y}\\ &+\frac{\gamma^2}{\delta\beta'}x\alpha(x)x^{-\gamma}\hat{Y}+\frac{2\delta}{\beta'}x\alpha(x)\hat{w}(x)^2x^{-\gamma}\hat{Y}-\frac{2\gamma}{\beta'}\alpha(x)x\hat{w}(x)x^{-\gamma}\hat{Y}\\ &+\big[x^{-\gamma}\hat{Y},\hat{Z}\big]_2+\frac{F_N}{\big(1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat{Y}-\hat{Z}\big)^{2N-1}}\frac{\hat{\eta}^{2N}}{x},\end{aligned}$$

$$\begin{aligned}(9.6)\qquad &x^{-\gamma}x\frac{d\hat{Y}}{dx}-x\frac{d\hat{Z}}{dx} = \frac{\delta}{\beta'}\big(\beta'-\frac{\gamma}{\delta}x\beta(x)\big)\hat{w}(x)x^{-\gamma}\hat{Y}\\ &+\frac{\gamma^2}{\delta\beta'}x\beta(x)x^{-\gamma}\hat{Y}+\frac{2\delta}{\beta'}\beta(x)x\hat{w}(x)^2x^{-\gamma}\hat{Y}-\frac{2\gamma}{\beta'}\beta(x)x\hat{w}(x)x^{-\gamma}\hat{Y}\\ &+\big[x^{-\gamma}\hat{Y},\hat{Z}\big]_2+\frac{(1-x^{-\gamma}\hat{Y}+\hat{Z})^2}{\big(1-\frac{\alpha'}{\beta'}x^{-\gamma}\hat{Y}-\hat{Z}\big)^{2N+1}}\frac{\hat{\eta}^{2N+1}}{x\hat{\zeta}}G_N.\end{aligned}$$

It should be emphasized that, in the equations excluding the terms of F_N and G_N, *the linear terms in $x^{-\gamma}\hat{Y}$ involve either x or $\hat{w}(x)$ as a factor and there are no linear terms in $\hat{Z}$* and *every expression $[x^{-\gamma}\hat{Y},\hat{Z}]_2$ contains either x or $\hat{w}(x)$ as a factor.*

Solve equations (9.5) and (9.6) with respect to $\{x^{-\gamma}x\frac{d\hat{Y}}{dx}, x\frac{d\hat{Z}}{dx}\}$. The resulting equations can be written as

$$(9.7)\qquad \begin{cases} x\dfrac{d\hat{Y}}{dx} &= x^{\gamma}A_N(x,\hat{w}(x),\hat{\eta}(x),\hat{\zeta}(x),x^{-\gamma}\hat{Y},\hat{Z}),\\ x\dfrac{d\hat{Z}}{dx} &= B_N(x,\hat{w}(x),\hat{\eta}(x),\hat{\zeta}(x),x^{-\gamma}\hat{Y},\hat{Z}).\end{cases}$$

Since $F_N = F_N\left(x, \frac{\eta}{1-\frac{\alpha'}{\beta'}Y-Z}, \frac{\zeta}{1-Y+Z}\right)$ and $G_N = G_N\left(x, \frac{\eta}{1-\frac{\alpha'}{\beta'}Y-Z}, \frac{\zeta}{1-Y+Z}\right)$, the F_N and G_N are regarded as holomoprhic and bounded functions of (x,η,ζ,Y,Z) in a domain of the form

$$(9.8)\qquad \begin{aligned} &|\arg x \mp \frac{\pi}{2}| < \pi-\epsilon, \quad 0<|x|<a_N, \quad |w|<d_N,\\ &|\eta|<c_N, \quad |\zeta|<c_N, \quad |Y|<b_0, \quad |Z|<b_0,\end{aligned}$$

where a_N, d_N, c_N, b_0 are sufficiently small constants. It turns out that the $A_N(x,w,\eta,\zeta,Y,Z)$ and $B_N(x,w,\eta,\zeta,Y,Z)$ are holomorphic functions of (x,w,η,ζ,Y,Z) in domain (9.8). But, they are not bounded in domain (9.8). So, in order that they may be bounded, we have to consider a subdomain of (9.8). There is a constant L_N depending on N such that

$$(9.9)\quad |1-\frac{\alpha'}{\beta'}Y-Z|^{1-2N}|F_N| \le L_N, \quad |1-Y+Z|^2|1-\frac{\alpha'}{\beta'}Y-Z|^{-1-2N}|G_N| \le L_N$$

for (x,η,ζ,Y,Z) in domain (9.8). It is not so hard to verify that there is a positive constant M, *independent of* N, such that

$$(9.10)\qquad \begin{aligned} &\max\{|A_N(x,w,\eta,\zeta,Y,Z)|, |B_N(x,w,\eta,\zeta,Y,Z)|\}\\ &\le M(|x|+|w|)\max\{|Y|,|Z|\} + L_N\Delta\max\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\}\end{aligned}$$

for (x,w,η,ζ,Y,Z) in a domain of the form

$$(9.11)\qquad \begin{aligned} &|\arg x \mp \frac{\pi}{2}| < \pi-\epsilon, \quad 0<|x|<a_N, \quad |w|<d_N,\\ &\frac{|\eta|^N}{|x\zeta|} < \Delta, \quad 0<|\zeta|<c_N, \quad |Y|<b_0, \quad |Z|<b_0.\end{aligned}$$

Here the a_N, c_N, d_N, b_0 are sufficiently small quantities which depend on N, but the Δ is any constant, independent of N, larger than the unity. The functions A_N and B_N are now bounded.

Let $\{\eta(x),\zeta(x),w(x)\}$ be a holomorphic and bounded solution of the modified simplified equations (B_-) appearing in §7. Now, in equations (9.7), we make the substitutions:

$$(9.12)\qquad \hat{\eta}(x)\to\eta(x), \quad \hat{\zeta}(x)\to\zeta(x), \quad \hat{w}(x)\to w(x), \quad \hat{Y}\to Y, \quad \hat{Z}\to Z.$$

Then we have the equations

$$(9.13)\qquad \begin{cases} x\dfrac{dY}{dx} &= x^{\gamma}A_N(x,w(x),\eta(x),\zeta(x),x^{-\gamma}Y,Z),\\ x\dfrac{dZ}{dx} &= B_N(x,w(x),\eta(x),\zeta(x),x^{-\gamma}Y,Z).\end{cases}$$

The equations (9.13) are called *the modified truncated equations associated with formal transformation* (4.1). Again apply the reasonings in M.Iwano [2, 3, 4]. Then we have the following theorems.

Theorem 4.B$_-$. *Equations* (9.13) *possess a unique solution* $\{Y, Z\} = \{\varphi_N(x, w(x), \eta(x), \zeta(x)), \psi_N(x, w(x), \eta(x), \zeta(x))\}$ *with the properties that:*

(i) $\varphi_N(x, w, \eta, \zeta)$ *and* $\psi_N(x, w, \eta, \zeta)$ *are holomorphic and bounded functions in* (x, w, η, ζ) *for a domain of the form*

$$\big|\arg x \mp \frac{\pi}{2}\big| < \pi - \epsilon, 0 < |x| < a_N{}^0, |w| < d_N{}^0, \left|\frac{\eta^N}{x\zeta}\right| < \Delta, 0 < |\zeta| < c_N{}^0. \tag{9.14}$$

(ii) *They satisfy there inequalities of the form*

$$\left\{\begin{aligned} &|\varphi_N(x, w, \eta, \zeta)| \le K_N |x|^\gamma \max\big\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\big\}, \\ &|\psi_N(x, w, \eta, \zeta)| \le K_N \max\big\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\big\}. \end{aligned}\right. \tag{9.15}$$

By using this theorem, we have the following

Theorem 5.B$_-$. *There is a transformation* $\{y, z\} \to \{\eta, \zeta\} : y = \Phi_0(x, \eta, \zeta), z = \Psi_0(x, \eta, \zeta)$*, which changes* (A) *to* (B) *in a domain of the form*

$$\big|\arg x \mp \frac{\pi}{2}\big| < \pi - \epsilon,\ 0 < |x| < a^0,\ |\eta| < c^0,\ |\zeta| < c^0,\ |\eta\zeta + \frac{\gamma}{\delta}x| < d^0|x|, \tag{9.16}$$

The a^0, c^0, d^0 *are sufficiently small positive constants.*

In other words, the $\Phi_0(x, \eta, \zeta)$ *and* $\Psi_0(x, \eta, \zeta)$ *admit convergent double power series expansions in* $\{\eta, \zeta\}$*, which are obtained by rearranging algebraically* (3.10) *as: for example,*

$$\Phi_0(x, \eta, \zeta) = \eta + \sum_{\ell=0}^{\infty} \mathcal{R}_{\ell,0}\big(x, \frac{\eta\zeta}{x} + \frac{\gamma}{\delta}\big)\eta^\ell + \sum_{\ell=1}^{\infty} \mathcal{R}_{0,\ell}\big(x, \frac{\eta\zeta}{x} + \frac{\gamma}{\delta}\big)\zeta^\ell, \tag{9.17}$$

whenever the values of (x, η, ζ)*, considered as points in* (x, η, ζ)*-space, belong to domain* (9.16)*. Here , for example, the coefficients* $\mathcal{R}_{\ell,0}(x, w)$ *are expressed by*

$$\begin{aligned} \mathcal{R}_{\ell,0}(x, w) &= \sum_{k=0}^{\infty} R_{k+\ell,k}(x) x^k (w - \frac{\gamma}{\delta})^k \\ &= \sum_{k=0}^{\infty} R_{k+\ell,k}(x) x^k \sum_{h=0}^{k} \frac{k!}{h!(k-h)!}\big(-\frac{\gamma}{\delta}\big)^{k-h} w^h = \sum_{h=0}^{\infty} \mathcal{R}_h(x) w^h, \\ \mathcal{R}_h(x) &= \sum_{k=h}^{\infty} \frac{k!}{h!(k-h)!}\big(-\frac{\gamma}{\delta}\big)^{k-h} R_{k+\ell,k}(x) x^k. \end{aligned} \tag{9.18}$$

The $\mathcal{R}_h(x)$ *are functions with Property* $\mathcal{C}$ *in* x *for the* x*-domain in* (9.16).

§10. Stable domains of the function $\frac{\eta(x)^N}{x\zeta(x)}$.

For the proofs of the existence theorems appearing in §8 and §9, we need to construct stable domains for the function $\frac{\eta(x)^N}{x\zeta(x)}$ with respect to the curve $\Gamma(x_0)$

(appearing in § 7), where $\{\eta(x), \zeta(x), w(x)\}$ is the solution of the modified simplified equations. Set

$$\lambda(x) = \frac{\eta(x)^N}{x\zeta(x)}. \tag{10.1}$$

We discuss the three cases according to the sign of γ separately.

Case I: $\gamma > 0$.

By the help of the modified simplified equations, a simple consideration shows that the $\lambda(x)$ satisfies the equation

$$\begin{aligned} x^2\frac{d\lambda}{dx} &= \lambda\{N+1+(N\alpha-\beta-1)x+(N\alpha'-\beta')xw \\ &\quad + \frac{\gamma}{\delta}(N\alpha(x)-\beta(x))x^2w+(N\alpha(x)-\beta(x))x^2w^2\}. \end{aligned} \tag{10.2}$$

Since $1+\kappa\alpha > 0, -1+\kappa\beta > 0$ are assumed to be satisfied, for the cases of $\gamma > 0$ and $\gamma = 0$, we take N large enough so as to have

$$N+1+\kappa(N\alpha-\beta-1) = N(1+\kappa\alpha)-(-1+\kappa\beta)-1 > 0. \tag{10.3}$$

By virtue of Lemma 1 in M. Iwano [5] and Theorem 3.B$_+$ in §7, we have at once the following

Proposition 1. *There is a nonsingular trasnformation of the form*

$$\lambda = \Lambda(1+W\mathcal{P}(x,W)) \quad (\text{with} \quad w = W(1+W\Phi(x,W))) \tag{10.4}$$

with the properties that:

(i) *The $\mathcal{P}(x,W)$ and $\Phi(x,W)$ are functions with Property $\mathcal{A}$ with respect to W in a domain of the form*

$$|\arg x \mp \frac{\pi}{2}| < \pi-\epsilon, \quad 0<|x|<a_0, \quad |W|<d_0. \tag{10.5}$$

(ii) *Transformation* (10.4) *changes equation* (10.2) *to the equation*

$$x^2\frac{d\Lambda}{dx} = \Lambda(N+1+(N\alpha-\beta-1)x). \tag{10.6}$$

(iii) *Stable domains of solutions of equations* (10.6) *are given by*

$$|\arg x \mp \frac{\pi}{2}| < \pi-\epsilon, \quad 0<|x|<a_0\omega(\arg x), \quad |\Lambda| < b_0\chi_{N\alpha-\beta-1}(\arg x), \tag{10.7}$$

where the $\omega(\phi)$ and $\chi_\xi(\phi)$ are given by formulas (7.6) *and* (7.7) *with the same angle* Ω.

(iv) *As x moves on the curve Γ', if N is large enough to have*

$$N \geq \frac{5(-1+\kappa\beta)}{2(1+\kappa\alpha)} + \frac{5\kappa}{2(1+\kappa\alpha)}, \tag{10.8}$$

we have

$$\frac{1}{|\Lambda|}\frac{d|\Lambda|}{ds_x} \geq \frac{N(1+\kappa\alpha)}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}}B^2e^{2\kappa\sigma}. \tag{10.9}$$

Note. Inequality (10.9) guarantees that the angle Ω can be chosen independently of N.

Proof. It is sufficient to prove (iv) only. As in M.Iwano [5], we have for $x \in \Gamma'$,

$$\frac{1}{|\Lambda|}\frac{d|\Lambda|}{ds_x} = \Re\left\{\frac{x^2}{\Lambda}\frac{d\Lambda}{dx}\frac{1}{x^2}\frac{dx}{ds_x}\right\} = \frac{\mathcal{E}_0(\sigma)}{\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}},$$

where

$$\begin{aligned}\mathcal{E}_0(\sigma) = \{&(N+1)+\kappa(N\alpha-\beta-1)\}B^2e^{2\kappa\sigma}\\ &+(N+1)(A+\sigma)^2+(N\alpha-\beta-1)(A+\sigma).\end{aligned}$$

Observe that

$$N+1+\kappa(N\alpha-\beta-1) = \frac{N(1+\kappa\alpha)}{4} + \frac{3N(1+\kappa\alpha)}{4} - (-1+\kappa\beta) - \kappa.$$

As was already shown in M. Iwano [5],

$$N|\alpha|(|A|+\sigma) \leq \frac{N(1+\kappa\alpha)}{4}B^2e^{2\kappa\sigma}, \quad \beta(|A|+\sigma) \leq \frac{-1+\kappa\beta}{4}B^2e^{2\kappa\sigma}.$$

The last inequality implies that $|A|+\sigma < \frac{\kappa}{4}B^2e^{2\kappa\sigma}$. Hence,

$$|(N\alpha-\beta-1)(A+\sigma)| \leq \frac{N(1+\kappa\alpha)}{4}B^2e^{2\kappa\sigma} + \frac{-1+\kappa\beta}{4}B^2e^{2\kappa\sigma} + \frac{\kappa}{4}B^2e^{2\kappa\sigma}.$$

It turns out that

$$\mathcal{E}_0(\sigma) \geq \frac{N(1+\kappa\alpha)}{4}B^2e^{2\kappa\sigma} + \left(\frac{2N(1+\kappa\alpha)}{4} - \frac{5(-1+\kappa\beta)}{4} - \frac{5\kappa}{4}\right)B^2e^{2\kappa\sigma}.$$

If, besides (10.3), N satisfies (10.8), we have (10.9).

It is easy to verify, as done in M. Iwano [5], that, when x travels on the curve Γ'', the solution $\Lambda(x)$ satisfies $|\Lambda(x)| < b_0\chi_{N\alpha-\beta-1}(\phi)$ provided that this inequality holds at the starting point $x_0 = |x_0|e^{i\theta}$. **q.e.d.**

Case II: $\gamma < 0$.

If we substitute $\{\alpha, \beta, \kappa, -\gamma, \Omega\}$ by $\{\alpha_1, \beta_1, \kappa_1, \gamma_1, \Omega_1\}$, the exactly same calculations imply a proposition analogous to Proposition 1. For example, the function $\lambda(x)$ is a solution of the differential equation

$$\begin{aligned}x^2\frac{d\lambda}{dx} = \lambda\{&N+1+(N\alpha_1-\beta_1-1)x+(N\alpha'-\beta')xw\\ &-\frac{\gamma}{\delta}(N\alpha(x)-\beta(x))x^2w+(N\alpha(x)-\beta(x))x^2w^2\}.\end{aligned}$$

which has the exactly same form as (10.2) by the above substitutions of the parameters $\{\alpha, \beta, \gamma\}$ which are involved in (10.2). For shortness' sake, this equation is refered to as equation (10.2_1) to which is assigned the number with a suffix 1 for the number of an original equation. By assumption, $1+\kappa_1\alpha_1 > 0$, $\quad -1+\kappa_1\beta_1 > 0$. Hence, we take N large enough to have inequality (10.3_1).

By applying Lemma $1'$ in M. Iwano [5], we have by the help of Theorem 3.B$_-$

Proposition 2. *There is a nonsingular transformation of the form* $\lambda = \Lambda(1 + W\mathcal{P}(x,W))$, (with $w = W(1 + \Phi_1(x,W))$ *with the properties that:*

(i) *The* $\mathcal{P}(x,W)$ *is a function with Property* $\mathcal{A}$ *with respect to* W *in a domain of the form* (10.5).

(ii) *Transformation* (10.4_1), *changes equation* (10.2_1), *to equation* (10.6_1).

(iii) *Stable domains of solutions of equations* (10.6_1) *are given by domains of the form* (10.7_1), *in which the* $\omega(\phi)$ *and* $\chi_\xi(\phi)$ *are expressed by formulas* (7.6) *and* (7.7), *where the* Ω *should be replaced by* Ω_1.

(iv) *As* x *moves on the curve* Γ', *if, besides* (10.3_1), N *satisfies an inequality of the form* (10.8_1), *the function* $\Lambda(x)$ *satifies an inequality of the form* (10.9_1).

The proof is immediate with no additional effort other than replacing the parameters by the above substitutions.

Case III:$\gamma = 0$.

The $\lambda(x)$ satisfies equation (10.2), but the simplified equation for the w-equation is quite different from those for the two cases of $\gamma \neq 0$. Let N be any integer satisfying (10.3). Owing to Lemma 2 in M. Iwano [5] and Theorem 3.B$_0$, we have

Proposition 3. *There is a nonsingular trasnformation of the form*

$$\lambda = \Lambda(1 + xW^2\mathcal{P}_0(x,W)), \quad (\text{with} \quad w = W(1 + xW^2\Psi(x,W))) \tag{10.10}$$

with the properties that

(i) *The* $\mathcal{P}_0(x,W)$ *is a function with Property* $\mathcal{B}$ *with respect to* W *(or* x*) in a domain which is sensibly the same as a domain of the form*

$$|\arg x \mp \frac{\pi}{2}| < \pi - \epsilon,\ 0 < |x| < a_0,\ |\arg W + \arg\delta| < \pi - \varepsilon,\ 0 < |W| < d_0. \tag{10.11}$$

(ii) *Transformation* (10.10) *changes equation* (10.2) *to the equation*

$$x^2\frac{d\Lambda}{dx} = \Lambda(N + 1 + (N\alpha - \beta - 1)x + (N\alpha' - \beta')xW). \tag{10.12}$$

(iii) *Let* μ' *and* μ *be any angles satisfying inequalities* (7.12^-) *or* (7.12^+). *Stable domains of solutions of equations* (10.12) *are given by*

$$\begin{aligned} \mu' < \arg x < \mu, \quad 0 < |x| < a_0\omega_0(\arg x), \quad W \in \mathcal{D}_W^{\mp}(\arg x; d_0),\\ |\Lambda| < b_0\chi_{N\alpha-\beta-1,N\alpha'-\beta'}(\arg x, W), \end{aligned} \tag{10.13}$$

where the $\omega_0(\phi)$ *and* $\chi_{\xi,\tau}(\phi,W)$ *are given by formulas* (7.14) *and* (7.15), *the domains* $\mathcal{D}_W^{\mp}(\phi;d_0)$ *are the same ones as that which appeared in Corollary* 3 *and Remark following Theorem* 5.2 (§13 *in M. Iwano* [5]).

(iv) *As* x *moves on the curve* Γ', *the* $\Lambda(x)$ *satisfies an inequality of the form* (10.9), *if* N *satisfies* (10.3) *and* (10.8).

The laborious part of the proof is to prove that the modulus of the function

$$\Phi(x) = \Lambda(x)(1 + W(x)\log(e^{i\phi}\sin\varepsilon))^{-\frac{N\alpha'-\beta'}{\delta}} \tag{10.14}$$

is an increasing function in s_x as x travels on the curve Γ'. We can easily verify this fact by repeating step by step the arguments which are used for the proof of Theorem 5.1 in § 11 in M. Iwano [5].

§11. **Evaluation of the integrals of the kernel functions.**

For functions of the form

$$\Psi_1(x) = \eta(x)^N, \quad \Psi_2(x) = \frac{\eta(x)^{N+1}}{\zeta(x)}, \tag{11.1}$$

which are the kind of kernels of integral equations, we need to evaluate the integrals

$$\mathcal{I}_i(x_0) = \left| \int_{\Gamma(x_0)} x^{-1}\Psi_i(x)dx \right| \quad (i = 1, 2), \tag{11.2}$$

where the $\Gamma(x_0)$ appeared in §7.

We have to discuss the three cases separately, since the modified simplified equations have slightly different forms according to the signs of γ.

1°. Case I :$\gamma > 0$. Assume that N satisfies

$$(N+1)(1+\kappa\alpha) - 6(-1+\kappa\beta) > 0. \tag{11.3}$$

(i). On the curve Γ'. As was shown in Theorem 2 in §9 (M.Iwano [5]), we have on the curve Γ',

$$\begin{aligned} &\frac{1}{|\eta|}\frac{d|\eta|}{ds_x} \geq \frac{(1+\kappa\alpha)B^2e^{2\kappa\sigma}}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}}, \\ &\frac{3(-1+\kappa\beta)B^2e^{2\kappa\sigma}}{2\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}} \geq \frac{1}{|\zeta|}\frac{d|\zeta|}{ds_x} \geq \frac{(-1+\kappa\beta)B^2e^{2\kappa\sigma}}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}}. \end{aligned} \tag{11.4}$$

A direct computation implies

$$\frac{d|\Psi_1(x)|}{ds_x} = N|\eta|^{N-1}\frac{d|\eta|}{ds_x} \geq \frac{N(1+\kappa\alpha)B^2e^{2\kappa\sigma}}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}}|\eta|^N.$$

Since, by the choice of the angle Ω, we have $1 \geq \kappa B$ and $|A| + \sigma < Be^{\kappa\sigma}$, it holds that $\frac{1}{|x|} = \sqrt{(A+\sigma)^2 + B^2e^{2\kappa\sigma}} \leq \sqrt{2}Be^{\kappa\sigma}$. Hence,

$$\frac{B^2e^{2\kappa\sigma}}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}} \geq \frac{Be^{\kappa\sigma}}{4\sqrt{2}\kappa} \geq \frac{1}{8\kappa|x|},$$

which gives

$$\frac{d|\Psi_1(x)|}{ds_x} \geq \frac{N(1+\kappa\alpha)}{8\kappa}|x|^{-1}|\Psi_1(x)|.$$

By integrating, we have at once

$$\int_0^{s_0'} |x|^{-1}|\Psi_1(x)|ds_x \leq \frac{8\kappa}{N(1+\kappa\alpha)}|\Psi_1(x_0')|, \tag{11.5}$$

where the x_0' is the starting point of the curve Γ' and s_0' is its arc length.

In a similar way, we have

$$\begin{aligned}\frac{d|\Psi_2(x)|}{ds_x} &= (N+1)\frac{|\eta|^N}{|\zeta|}\frac{d|\eta|}{ds_x} - \frac{|\eta|^{N+1}}{|\zeta|^2}\frac{d|\zeta|}{ds_x} \\ &\geq \frac{\{(N+1)(1+\kappa\alpha)-6(-1+\kappa\beta)\}B^2e^{2\kappa\sigma}}{4\sqrt{1+\kappa^2B^2e^{2\kappa\sigma}}}|\Psi_2(x)| \\ &\geq \frac{(N+1)(1+\kappa\alpha)-6(-1+\kappa\beta)}{8\kappa}|x|^{-1}|\Psi_2(x)|.\end{aligned}$$

Thus we have

$$\int_0^{s_0'} |x|^{-1}|\Psi_2(x)|ds_x \leq \frac{8\kappa}{(N+1)(1+\kappa\alpha)-6(-1+\kappa\beta)}|\Psi_2(x_0')|, \tag{11.6}$$

Note. If we apply the analysis in §9 in M. Iwano [5] to $\Psi_2(x)$ directly, we can find a slightly better estimation than (11.6). But, we don't need such an acurate estimation.

(**ii**). **On the curve** Γ''. The variable point x on the curve Γ'' is expressed by $x = \big(|x_0|\frac{\cos\phi}{\cos\theta}\big)e^{i\phi}$, where $\theta \leq \phi \leq \Omega$ or $\pi - \Omega \leq \phi \leq \theta$. The curve Γ'' should be connected to the curve Γ'. We want to evaluate the upper bound of the functions $|\Psi_i(x)|(i = 1, 2)$. Since $\{\eta, \zeta\}$ is a solution of the modified simplified equations, we have

$$\begin{aligned}x^2\frac{d\Psi_1}{dx} &= \Psi_1(N + N\alpha x + N\alpha' xw + \frac{N\gamma}{\delta}\alpha(x)x^2w + N\alpha(x)x^2w^2), \\ x^2\frac{d\Psi_2}{dx} &= \Psi_2\{(N+2+((N+1)\alpha-\beta)x + (\frac{\gamma}{\delta}+w)((N+1)\alpha(x)-\beta(x))x^2w\}.\end{aligned}$$

By virtue of Lemma 1 in § 5 (M.Iwano [5]), there are functions $P_i(x, W)(i = 1, 2)$ with Property $\mathcal{A}$ with respect to W in a domain of the form

$$|\arg x \mp \frac{\pi}{2}| < \pi - \varepsilon, \quad 0 < |x| < a_0, \quad |W| < d_0 \tag{11.7}$$

such that transformations of the form $\Psi_i = \Phi_i(1+WP_i(x, W))$ change the equations on Ψ_i to the equations

$$x^2\frac{d\Phi_1}{dx} = \Phi_1(N + N\alpha x), \quad x^2\frac{d\Phi_2}{dx} = \Phi_2\{(N+2+((N+1)\alpha-\beta)x\}. \tag{11.8}$$

Since a_0 and d_0 are sufficiently, we can assume that inequalities of the form

$$\frac{1}{2} < |1 + WP_i(x, W)| < 2 \tag{11.9}$$

hold in domain (11.7).

By integrating equations (11.8), we have

$$\Phi_1(x) = \Phi_1(x_0)e^{\frac{N}{x_0}-\frac{N}{x}}\big(\frac{x}{x_0}\big)^{N\alpha}, \Phi_2(x) = \Phi_2(x_0)e^{\frac{N+2}{x_0}-\frac{N+2}{x}}\big(\frac{x}{x_0}\big)^{(N+1)\alpha-\beta}. \tag{11.10}$$

On the curve Γ'', we have $\quad ds_x = -\frac{|x_0|}{\cos\theta} d\phi$. The ε is assumed to be sufficinetly small, so we can assume that $\dfrac{\pi}{4} < \Omega, \Omega_1 < \dfrac{\pi}{2} - \varepsilon$. Since, for example,

$$-\frac{\pi}{2} + \varepsilon < \theta < \phi \leq \Omega, \quad \text{or} \quad \pi - \Omega \leq \phi \leq \theta < \frac{3\pi}{2} - \varepsilon,$$

we have

$$\sin\varepsilon \leq \left|\frac{\cos\phi}{\cos\theta}\right| \leq \frac{1}{\sin\varepsilon}, \quad \frac{1}{|x|} \leq \frac{1}{|x_0|\sin\varepsilon}.$$

Remark. In the case of $\gamma = 0$, the Ω should be replaced by $\frac{\pi}{2} - \varepsilon$. In the case of $\gamma < 0$, we use Ω_1 instead of Ω.

Thus, in any case, we see that

$$\text{(11.11)} \qquad \text{arc length of the curve}\, \Gamma'' = \int_\theta^\Omega \frac{|x_0|}{|\cos\theta|} d\phi \leq \frac{\pi|x_0|}{\sin\varepsilon}.$$

¿From (11.10) we have at once

$$\text{(11.12)} \quad |\Phi_1(x)| \leq |\Phi_1(x_0)| \left(\frac{1}{\sin\varepsilon}\right)^{|N\alpha|}, \quad |\Phi_2(x)| \leq |\Phi_2(x_0)| \left(\frac{1}{\sin\varepsilon}\right)^{|(N+1)\alpha-\beta|}.$$

By the help of (11.9), we have

$$|\Psi_i(x)| < 2|\Phi_i(x)|, \quad |\Phi_i(x_0)| < 2|\Psi_i(x_0)|.$$

Hence, after a simple consideration, we have

$$\text{(11.13)} \qquad \begin{aligned} \left|\int_{\Gamma''} x^{-1}\Psi_1(x)dx\right| &\leq \frac{4\pi|\Psi_1(x_0)|}{(\sin\varepsilon)^{|N\alpha|+2}}, \\ \left|\int_{\Gamma''} x^{-1}\Psi_2(x)dx\right| &\leq \frac{4\pi|\Psi_2(x_0)|}{(\sin\varepsilon)^{|(N+1)\alpha-\beta|+2}}. \end{aligned}$$

(iii) On the curve $\Gamma(x_0)$. Since

$$|\Psi_1(x_0')| \leq \frac{4|\Psi_1(x_0)|}{(\sin\epsilon)^{N|\alpha|}}, \quad |\Psi_2(x_0')| \leq \frac{4|\Psi_2(x_0)|}{(\sin\epsilon)^{|(N+1)\alpha-\beta|}},$$

we obtain immediatly

$$\text{(11.14)} \qquad \begin{aligned} \mathcal{I}_1(x_0) &\leq \frac{4}{(\sin\epsilon)^{N|\alpha|+2}} \left(\frac{8\kappa(\sin\epsilon)^2}{N(1+\kappa\alpha)} + \pi\right)|\Psi_1(x_0)|, \quad (\text{by}(11.5)) \\ \mathcal{I}_2(x_0) &\leq \frac{4}{(\sin\epsilon)^{|(N+1)\alpha-\beta|+2}} \left(\frac{8\kappa(\sin\epsilon)^2}{M(N,\alpha,\beta)} + \pi\right)|\Psi_2(x_0)|, \quad (\text{by}(11.6)) \end{aligned}$$

with $\quad M(N,\alpha,\beta) = (N+1)(1+\kappa\alpha) - 6(-1+\kappa\beta)$.

2°. Case II: $\gamma < 0$.

In this case, we have to estimate, besides of (11.2), the integrals

$$\text{(11.15)} \qquad \mathcal{J}_i(x_0) = \left|\int_{\Gamma(x_0)} x^{\gamma-1}\Psi_i(x)dx\right| \quad (i = 1, 2)$$

We assume that N satisfies the inequality

$$(N+1)(1+\kappa_1\alpha_1) - 6(-1+\kappa_1\beta_1) - 16\gamma_1\kappa_1 > 0. \tag{11.16}$$

(i) On the curve Γ'. By virtue of Theorem 4 in §10 (M.Iwano [5]), it is proved that

$$\frac{1}{|\eta|}\frac{d|\eta|}{ds_x} \geq \frac{(1+\kappa_1\alpha_1)B^2e^{2\kappa_1\sigma}}{4\sqrt{1+\kappa_1^2B^2e^{2\kappa_1\sigma}}},$$
$$\frac{3(-1+\kappa_1\beta_1)B^2e^{2\kappa_1\sigma}}{2\sqrt{1+\kappa_1^2B^2e^{2\kappa_1\sigma}}} \geq \frac{1}{|\zeta|}\frac{d|\zeta|}{ds_x} \geq \frac{(-1+\kappa_1\beta_1)B^2e^{2\kappa_1\sigma}}{4\sqrt{1+\kappa_1^2B^2e^{2\kappa_1\sigma}}}.$$

Hence, by applying the same reasoning as in Case I, we have inequalities of the form

$$\begin{aligned}\int_0^{s_0'} |x|^{-1}|\Psi_1(x)|ds_x &\leq \frac{8\kappa_1\,|\Psi_1(x_0')|}{N(1+\kappa_1\alpha_1)},\\ \int_0^{s_0'} |x|^{-1}|\Psi_2(x)|ds_x &\leq \frac{8\kappa_1\,|\Psi_2(x_0')|}{(N+1)(1+\kappa_1\alpha_1)-6(-1+\kappa_1\beta_1)}.\end{aligned} \tag{11.17}$$

Observe that

$$\frac{1}{|x|}\frac{d|x|}{ds_x} = \frac{A+\sigma+\kappa_1B^2e^{2\kappa_1\sigma}}{\sqrt{1+\kappa_1^2B^2e^{2\kappa_1\sigma}}} \leq \frac{2\kappa_1B^2e^{2\kappa_1\sigma}}{\kappa_1Be^{\kappa_1\sigma}} = 2Be^{\kappa_1\sigma} \leq \frac{2}{|x|}.$$

Since

$$\frac{d}{ds_x}(|x|^\gamma|\Psi_1(x)|) = \gamma|x|^{\gamma-1}|\Psi_1(x)|\frac{d|x|}{ds_x} + |x|^\gamma\frac{d|\Psi_1(x)|}{ds_x},$$

we have

$$\frac{d}{ds_x}(|x|^\gamma|\Psi_1(x)|) \geq \big(2\gamma + \frac{N(1+\kappa_1\alpha_1)}{8\kappa_1}\big)|x|^{\gamma-1}|\Psi_1(x)|.$$

In quite a same manner, we get

$$\frac{d}{ds_x}(|x|^\gamma|\Psi_2(x)|) \geq \big(2\gamma + \frac{N(1+\kappa_1\alpha_1)-6(-1+\kappa_1\beta_1)}{8\kappa_1}\big)|x|^{\gamma-1}|\Psi_2(x)|.$$

By integration, we have at once

$$\begin{aligned}\int_0^{s_0'} |x|^{\gamma-1}|\Psi_1(x)|ds_x &\leq \frac{8\kappa_1\,|(x_0')^\gamma\Psi_1(x_0')|}{N(1+\kappa_1\alpha_1)+16\gamma\kappa_1},\\ \int_0^{s_0'} |x|^{\gamma-1}|\Psi_2(x)|ds_x &\leq \frac{8\kappa_1\,|(x_0')^\gamma\Psi_2(x_0')|}{(N+1)(1+\kappa_1\alpha_1)-6(-1+\kappa_1\beta_1)+16\gamma\kappa_1}.\end{aligned} \tag{11.18}$$

(ii) On the curve Γ''. If we replace $\{\alpha,\beta,\kappa\}$ by $\{\alpha_1,\beta_1,\kappa_1\}$ and if we notice that

$$|x|^\gamma = \left(|x_0|\frac{\cos\phi}{\cos\theta}\right)^\gamma \leq (|x_0|\sin\varepsilon)^\gamma = \frac{|x_0|^\gamma}{(\sin\varepsilon)^{\gamma_1}}, \quad (\gamma_1 = -\gamma)$$

the same reasoning will be applied. Thus, we have

$$
\begin{aligned}
(11.19)\qquad &\left|\int_{\Gamma''} x^{\gamma-1}\Psi_1(x)dx\right| \le \frac{4\pi|(x_0)^\gamma\Psi_1(x_0)|}{(\sin\varepsilon)^{|N\alpha_1|+\gamma_1+2}},\\
&\left|\int_{\Gamma''} x^{\gamma-1}\Psi_2(x)dx\right| \le \frac{4\pi|(x_0)^\gamma\Psi_2(x_0)|}{(\sin\varepsilon)^{|(N+1)\alpha_1-\beta_1|+\gamma_1+2}}.
\end{aligned}
$$

(iii) On the curve $\Gamma(x_0)$. If we notice that

$$
|\Psi_1(x_0')| \le \frac{4|\Psi_1(x_0)|}{(\sin\epsilon)^{N|\alpha_1|}},\quad |\Psi_2(x_0')| \le \frac{4|\Psi_2(x_0)|}{(\sin\epsilon)^{|(N+1)\alpha_1-\beta_1|}},
$$

we get immediately the evaluation

$$
\begin{aligned}
(11.20)\qquad &\mathcal{J}_1(x_0) \le \frac{4}{(\sin\epsilon)^{N|\alpha_1|+\gamma_1+2}}\left(\frac{8\kappa_1(\sin\epsilon)^2}{N(1+\kappa_1\alpha_1)-16\gamma_1\kappa_1}+\pi\right)|(x_0)^\gamma\,\Psi_1(x_0)|,\\
&\mathcal{J}_2(x_0) \le \frac{4}{(\sin\epsilon)^{|(N+1)\alpha_1-\beta_1|+\gamma_1+2}}\left(\frac{8\kappa_1(\sin\epsilon)^2}{M(N,\alpha_1,\beta_1,\gamma_1)}+\pi\right)|(x_0)^\gamma\,\Psi_2(x_0)|
\end{aligned}
$$

with $\quad M(N,\alpha_1,\beta_1,\gamma_1) = (N+1)(1+\kappa_1\alpha_1) - 6(-1+\kappa_1\beta_1) - 16\gamma_1\kappa_1.$

2°. Case III:$\gamma = 0$.

We assume that N satisfies inequality (11.3).

(i) On the curve Γ'. As is done in M. Iwano [5] (See Theorem 6 in § 11), we have inequalities of the form (11.4). By repeating an almost samme argument as in the case of $\gamma > 0$, we have inequalities of the form (11.5) and (11.6).

(ii) On the curve Γ''. One notices that the $\Psi_i(x)$ satisfy the same equations as those for the case of $\gamma > 0$. By virtue of Lemma 2 in §6 in M.Iwano [5], there exist functions $P_i(x,W)(i=1,2)$ with Property $\mathcal{B}$ with respect to W (or x) in a domain, which is sensibly the same as the domain of the form

$$
(11.21)\quad |\arg x \mp \frac{\pi}{2}| < \pi-\varepsilon, 0<|x|<a_0, \quad |\arg W + \arg\delta| < \pi-\varepsilon, 0<|W|<d_0
$$

such that the transformations $\Psi_i = \Phi_i(1+xW^2P_i(x,W))$, change equations on Ψ_i to the equations

$$
\begin{aligned}
(11.22)\qquad &x^2\frac{d\Phi_1}{dx} = \Phi_1(N+N\alpha x+N\alpha' xW),\\
&x^2\frac{d\Phi_2}{dx} = \Phi_2\{N+2+((N+1)\alpha-\beta)x+((N+1)\alpha'-\beta')xW\}.
\end{aligned}
$$

Since a_0 and d_0 are sufficiently, we can assume that inequalities of the form

$$
\frac{1}{2} < |1+xW^2P_i(x,W)| < 2
$$

hold in a domain, which is sensibly the same as domain (11.21).

By integration, we have

$$
\begin{aligned}
(11.23)\qquad &\Phi_1(x) = \Phi_1(x_0)e^{\frac{N}{x_0}-\frac{N}{x}}(\frac{x}{x_0})^{N\alpha}\left(\frac{W}{W_0}\right)^{N\alpha'},\\
&\Phi_2(x) = \Phi_2(x_0)e^{\frac{N+2}{x_0}-\frac{N+2}{x}}(\frac{x}{x_0})^{(N+1)\alpha-\beta}\left(\frac{W}{W_0}\right)^{(N+1)\alpha'-\beta'}.
\end{aligned}
$$

But, W is the solution of $x\frac{dW}{dx} = \delta W^2$. So, we get

$$W = \frac{W_0}{1 - \delta W_0 \log(\frac{x}{x_0})} = \frac{W_0}{1 - \delta W_0 \log\left(\frac{\cos\phi}{\cos\theta}\frac{e^{i\phi}}{e^{i\theta}}\right)}.$$

Since $\sin\varepsilon < \frac{\cos\phi}{\cos\theta} < \frac{1}{\sin\varepsilon}$, $|\log\left(\frac{\cos\phi}{\cos\theta}\frac{e^{i\phi}}{e^{i\theta}}\right)|$ is bounded when ϕ moves in the ϕ-interval. Hence, provided that $|W_0|$ is sufficiently small, we can assume that there holds an inequality of the form

$$\frac{1}{2} < \left|1 - \delta W_0 \log\left(\frac{\cos\phi}{\cos\theta}\frac{e^{i\phi}}{e^{i\theta}}\right)\right| < 2. \tag{11.24}$$

By repeating the same reasoning, which was used for obtaining (11.13), we have

$$\begin{aligned} \left|\int_{\Gamma''} x^{-1}\Psi_1(x)dx\right| &\leq \frac{2^{N|\alpha'|+2}\pi|\Psi_1(x_0)|}{(\sin\varepsilon)^{|N\alpha|+2}}, \\ \left|\int_{\Gamma''} x^{-1}\Psi_2(x)dx\right| &\leq \frac{2^{|(N+1)\alpha'-\beta'|+2}\pi|\Psi_2(x_0)|}{(\sin\varepsilon)^{|(N+1)\alpha-\beta|+2}}. \end{aligned} \tag{11.25}$$

(iii) On the curve $\Gamma(x_0)$. Hence we have immeadiately the estimations

$$\begin{aligned} \mathcal{I}_1(x_0) &\leq \frac{2^{N|\alpha'|+2}}{(\sin\epsilon)^{N|\alpha|+2}}\left(\frac{8\kappa(\sin\epsilon)^2}{N(1+\kappa\alpha)} + \pi\right)|\Psi_1(x_0)|, \\ \mathcal{I}_2(x_0) &\leq \frac{2^{|(N+1)\alpha'-\beta'|+2}}{(\sin\epsilon)^{|(N+1)\alpha-\beta|+2}}\left(\frac{8\kappa(\sin\epsilon)^2}{M(N,\alpha,\beta)} + \pi\right)|\Psi_2(x_0)|. \end{aligned} \tag{11.26}$$

§12. Sketch of the proof of the existence theorems.

We are only satisfied with the proofs of Theorem 4.B$_+$ and Theorem 5.B$_+$. The other existence theorems can be proved without any essential modification of the reasoning that follows below.

A domain of the form (8.14), in which a solution exists, should be repalced by a stable domain in the (x, w, η, ζ)-space. But, it is impossible to describe it in an explicit form. We use the following notation.

An x-stable domain $\mathcal{D}_x(a_N)$ is of the form $|\arg x \mp \frac{\pi}{2}| < \pi - \varepsilon$, $\quad 0 < |x| < a_N\omega(\arg x)$. (See Theorem 3.B$_+$). For each point $x \in \mathcal{D}_x(a_N)$ there is associated with a subdomain in the (w, η, ζ)-space as follows:

(i) To each $x \in \mathcal{D}_x(a_N)$, there corresponds a W-stable domain $\mathcal{D}_W(\arg x; d_N)$, which is a disc given by $|W| < d_N\chi_\gamma(\arg x)$. Then, a w-stable domain $\mathcal{K}_w(\arg x; d_N)$ is the image of the W-stable domain $\mathcal{D}_W(\arg x; d_N)$ by the mapping $w = W(1 + W\Phi(x, W))$. Then the correspondence between $\mathcal{D}_x(a_N) \times \mathcal{K}_w(\arg x; d_N)$ and $\mathcal{D}_x(a_N) \times \mathcal{D}_W(\arg x; d_N)$ is in a one-to-one manner.

(ii) For each point $\{x, w\} \in \mathcal{D}_x(a_N) \times \mathcal{K}_w(\arg x; d_N)$, a ζ-stable domain $\mathcal{K}_\zeta(\arg x; c_N)$ is defined by the image of the Z-stable domain $\mathcal{D}_Z(\arg x; c_N) : |Z| < c_N\chi_\beta(\arg x)$ by the mapping $\zeta = Z(1 + WQ(x, W))$.

(iii) Let $\lambda = \frac{\eta^N}{x\zeta}$. For each pair $\{x, w\} \in \mathcal{D}_x(a_N) \times \mathcal{K}_w(\arg x; d_N)$, a λ-stable domain $\mathcal{K}_\lambda(\arg x; \Delta)$ is given by the image of the Λ-stable domain $\mathcal{D}_\Lambda(\arg x; \Delta)$: $|\Lambda| < \Delta\chi_{N\alpha-\beta-1}(\arg x)$ by the mapping $\lambda = \Lambda(1 + W\mathcal{P}(x, W))$.

The $\arg x$ in the symbol $\mathcal{K}_w(\arg x; d_N)$, for example, means that to the point x there corresponds *an open disk with the radius which varies continuously in* $\arg x$ and d_N is considered as the size of a domain so that it shrinks to a single point as $d_N \to 0$.

To prove the existence of solutions for equations (8.7), we utilize the fixed point technique, diviced by Prof. Masuo Hukuhara [2, 3]. Denote by $\mathcal{F}$ the set of pairs $\{\varphi, \psi\}$ of functions $\varphi(x, w, \eta, \zeta)$ and $\psi(x, w, \eta, \zeta)$ which are holomorphic and bounded in (x, w, η, ζ) in a domain of the form

$$(12.1)\qquad \begin{cases} |\arg x \mp \dfrac{\pi}{2}| < \pi - \varepsilon, \quad 0 < |x| < a_N \omega(\arg x), \\ w \in \mathcal{K}_w(\arg x; d_N), \quad \zeta \in \mathcal{K}_\zeta(\arg x; c_N) \setminus \{0\}, \quad \dfrac{\eta^N}{x\zeta} \in \mathcal{K}_\lambda(\arg x; \Delta) \end{cases}$$

and satisfy there inequalities of the form

$$(12.2)\qquad \max\{|\varphi(x, w, \eta, \zeta)|, |\psi(x, w, \eta, \zeta)|\} \le K_N \max\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\}.$$

Let $(x_0, w_0, \eta_0, \zeta_0)$ be an arbitrary point in domain (12.1) and define the holomorphic solution $\{\eta(x), \zeta(x), w(x)\}$ of the modified simplified equations (B_+) satisfying the initial condition $\{\eta, \zeta, w\} = \{\eta_0, \zeta_0, w_0\}$ at $x = x_0$. Put

$$(12.3)\qquad \begin{aligned} \mathcal{A}_N(x) &= A_N(x, w(x), \eta(x), \zeta(x), \Phi(x), \Psi(x)), \\ \mathcal{B}_N(x) &= B_N(x, w(x), \eta(x), \zeta(x), \Phi(x), \Psi(x)), \end{aligned}$$

with $\Phi(x) = \varphi(x, w(x), \eta(x), \zeta(x))$, $\Psi(x) = \psi(x, w(x), \eta(x), \zeta(x))$. Since domain (12.1) is a stable domain for the solution $\{\eta(x), \zeta(x), w(x)\}$ with respect to the curve $\Gamma(x_0)$, which appears in §7, the $\mathcal{A}_N(x)$ and $\mathcal{B}_N(x)$ are holomorphic and bounded functions in x on the curve $\Gamma(x_0) \setminus \{0\}$. Define the pair $\{\Phi(x_0, w_0, \eta_0, \zeta_0), \Psi(x_0, w_0, \eta_0, \zeta_0)\}$ by the integrals

$$(12.4)\qquad \begin{aligned} \Phi(x_0, w_0, \eta_0, \zeta_0) &= \int_{\Gamma(x_0)} x^{-1} \mathcal{A}_N(x)\,dx, \\ \Psi(x_0, w_0, \eta_0, \zeta_0) &= \int_{\Gamma(x_0)} x^{-1} \mathcal{B}_N(x)\,dx. \end{aligned}$$

Define the mapping $\mathcal{T}$ by the correspondence

$$(12.5)\qquad \mathcal{T} : \{\varphi, \psi\} \to \{\Phi, \Psi\}.$$

We want to prove that *this mapping has a fixed point* which corresponds to a solution of (8.7) and that *a solution of equations* (8.7) *satisfying the order condition* $\max\{|Y|, |Z|\} = \mathrm{O}\big(\max\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\}\big)$ *is unique.* But we need a lengthy and tiresome reasoning. So we are satisfied with the proof of inequalities of the form

$$(12.6)\qquad \max\{|\Phi(x_0, w_0, \eta_0, \zeta_0)|, |\Psi(x_0, w_0, \eta_0, \zeta_0)|\} \le K_N \max\{|\eta_0|^N, \frac{|\eta_0|^{N+1}}{|\zeta_0|}\}.$$

We evaluate the integrals of $x^{-1}\mathcal{A}_N(x)$ and $x^{-1}\mathcal{B}_N(x)$ along the path of integration $\Gamma(x_0)$. They are dominated by the integrals

$$\int_0^{s_0} |x|^{-1}|\mathcal{A}_N(x)|ds_x \quad \text{and} \quad \int_0^{s_0} |x|^{-1}|\mathcal{B}_N(x)|ds_x, \tag{12.7}$$

where s_0 denotes the arc length of the curve $\Gamma(x_0)$. Thanks to (8.13), we have

$$|\mathcal{A}_N(x)|, |\mathcal{B}_N(x)| \leq (14M_0 d_N K_N + L_N \varDelta) \max\{|\eta(x)|^N, \frac{|\eta(x)|^{N+1}}{|\zeta(x)|}\}. \tag{12.8}$$

By virtue of (11.14), we obtain at once

$$\begin{aligned} &\int_0^{s_0} |x|^{-1} \max\{|\eta(x)|^N, \frac{|\eta(x)|^{N+1}}{|\zeta(x)|}\} ds_x \\ &\leq \frac{4}{(\sin\epsilon)^{H(N,\alpha,\beta)+2}}\left(\frac{8\kappa(\sin\epsilon)^2}{K(N,\alpha,\beta)} + \pi\right) \max\{|\eta_0|^N, \frac{|\eta_0|^{N+1}}{|\zeta_0|}\} \end{aligned} \tag{12.9}$$

with

$$H(N,\alpha,\beta) = \max\{N|\alpha|, |(N+1)\alpha - \beta|\},$$
$$K(N,\alpha,\beta) = \min\{N(1+\kappa\alpha), (N+1)(1+\kappa\alpha) - 6(-1+\kappa\beta)\}.$$

Now we take the d_N sufficiently small to have

$$\frac{14M_0}{(\sin\varepsilon)^{H(N,\alpha,\beta)+2}}\left(\frac{8\kappa(\sin\epsilon)^2}{K(N,\alpha,\beta)} + \pi\right) d_N < \frac{1}{32}. \tag{12.10}$$

Take K_N large enough to have

$$K_N = \frac{32 L_N \varDelta}{(\sin\varepsilon)^{H(N,\alpha,\beta)+2}}\left(\frac{8\kappa(\sin\epsilon)^2}{K(N,\alpha,\beta)} + \pi\right). \tag{12.11}$$

Since

$$K_N \max\{|\eta|^N, \frac{|\eta|^{N+1}}{|\zeta|}\} < K_N \varDelta |x| \max\{|\zeta|, |\eta|\},$$

we need to choose the value of a_N sufficiently small so that

$$K_N \varDelta |x| \max\{|\zeta|, |\eta|\} < b_0, \tag{12.12}$$

where the b_0 is the same quantity which appeared in (8.9). In this manner, we have

$$\int_0^{s_0} |x|^{-1} \max\{|\eta(x)|^N, \frac{|\eta(x)|^{N+1}}{|\zeta(x)|}\} ds_x \leq \frac{K_N}{4} \max\{|\eta_0|^N, \frac{|\eta_0|^{N+1}}{|\zeta_0|}\}, \tag{12.13}$$

which leads (12.6). The factor $\frac{1}{4}$ will be useful for the proof of the uniqueness of our solution.

In the case of $\gamma < 0$, a stable domain, corresponding to domain (12.1), has an almost same form as (12.1) if we substitute $\{\Omega, \alpha, \beta\}$ by $\{\Omega_1, \alpha_1, \beta_1\}$.

In the case when $\gamma = 0$, only if we have only to define a stable domain corresponding to (12.1), the analysis for the proof of the existence theorem can be carried out in quite a same way. But, w-stable domain has quite a different form from the other two cases. The stable domain (12.1) should be replaced by a domain of the form

$$\mu' < \arg x < \mu, \quad 0 < |x| < a_N \omega_0(\arg x),$$
$$w \in \mathcal{K}_w^0(\arg x, d_N) \quad \zeta \in \mathcal{K}_\zeta^0(\arg x; c_N) \setminus \{0\}, \quad \frac{\eta^N}{x\zeta} \in \mathcal{K}_\lambda^0(\arg x; \Delta).$$

Here the $\mathcal{K}_w^0(\arg x; d_N)$ is not a disc but is a sectorial domain contained in the sector either $-\mu' - \frac{\pi}{2} < \arg w < -\mu + \frac{3\pi}{2}$ or $-\mu' - \frac{3\pi}{2} < \arg w < -\mu + \frac{\pi}{2}$ according as the pair $\{\mu, \mu'\}$ satisfies (7.12$^-$) or (7.12$^+$). This domain is the image of the W-stable domain $\mathcal{D}_W^{\mp}(\arg x; d_N)$, which is defined in Theorem 5.2 (§13 in M. Iwano [5]) and Remark following it, by the mapping : $w = W(1+xW^2\Psi(x, W))$. The $\mathcal{K}_\zeta^0(\arg x; c_N)$ is the image of the Z-stable domain $\mathcal{D}_Z^0(\arg x; c_N)$: $|Z| < c_N\chi_{\beta,\beta'}(\arg x, W)$ by the mapping $\zeta = Z(1 + xW^2Q_0(x, W))$. The function $\chi_{\beta,\beta'}(\phi, W)$ is given in Theorem 3.B$_0$ in §7. The $\mathcal{K}_\lambda^0(\arg x; \Delta)$ has an analogous meaning to $\mathcal{K}_\lambda(\arg x; \Delta)$ in the case of $\gamma > 0$, where the $\chi_{N\alpha-\beta-1}(\phi)$ should be replaced by the function $\chi_{N\alpha-\beta-1, N\alpha'-\beta'}(\phi, W)$ given by (7.15).

In this way, Theorem 4.B$_+$, Theorem 4.B$_0$ and Theorem 4.B$_-$ are proved.

§13. Domain of holomorphy of the solution.

Let $\{\hat{\eta}(x), \hat{\zeta}(x)\}$ be a solution of the simplified equations (R). Set

$$\hat{w}(x) = \frac{\hat{\eta}(x)\hat{\zeta}(x)}{x} \quad (\gamma \geq 0), \quad \hat{w}(x) = \frac{\hat{\eta}(x)\hat{\zeta}(x)}{x} + \frac{\gamma}{\delta} \quad (\gamma < 0).$$

Let $\{\varphi_N, \psi_N\}$ be a fixed point of the mapping $\mathcal{T}$. Then, according to our standard analysis (for example, M. Iwano [1]) we can prove that $\{\hat{Y}(x), \hat{Z}(x)\} = \{\varphi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)), \psi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x))\}$ represents a solution of truncated differential equations (8.5). Further, put

$$u = \frac{\hat{\eta}(x)}{1 - \hat{Y}(x)}, \quad v = \frac{\hat{\zeta}(x)}{1 - \hat{Z}(x)} \quad (\gamma \geq 0),$$
$$u = \frac{\hat{\eta}(x)}{1 - \frac{\alpha'}{\beta'}x^{-\gamma}\hat{Y}(x) - \hat{Z}(x)}, \quad v = \frac{\hat{\zeta}(x)}{1 - x^{-\gamma}\hat{Y}(x) + \hat{Z}(x)}. \quad (\gamma < 0)$$

Set

$$(13.1) \quad \begin{cases} \Phi_N(x, w, \eta, \zeta) &= R_{(N)}\left(x, \dfrac{\eta}{1 - \varphi_N(x, w, \eta, \zeta)}, \dfrac{\zeta}{1 - \psi_N(x, w, \eta, \zeta)}\right), \\ \Psi_N(x, w, \eta, \zeta) &= S_{(N)}\left(x, \dfrac{\eta}{1 - \varphi_N(x, w, \eta, \zeta)}, \dfrac{\zeta}{1 - \psi_N(x, w, \eta, \zeta)}\right), \end{cases}$$

or

$$(13.2) \quad \begin{cases} \Phi_N(x, w, \eta, \zeta) &= R_{(N)}\left(x, \dfrac{\eta}{1 - \frac{\alpha'}{\beta'}x^{-\gamma}\varphi_N - \psi_N}, \dfrac{\zeta}{1 - x^{-\gamma}\varphi_N + \psi_N}\right), \\ \Psi_N(x, w, \eta, \zeta) &= S_{(N)}\left(x, \dfrac{\eta}{1 - \frac{\alpha'}{\beta'}x^{-\gamma}\varphi_N - \psi_N}, \dfrac{\zeta}{1 - x^{-\gamma}\varphi_N + \psi_N}\right). \end{cases}$$

according as $\gamma \geq 0$ or $\gamma < 0$. Then, thanks to $\{(6.2), (8.1)\}$ or $\{(6.2), (9.1)\}$, the pair $\{y, z\} = \{\Phi_N(x), \Psi_N(x)\}$ of functions $\Phi_N(x) = \Phi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x))$ and $\Psi_N(x) = \Psi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x))$ represents a solution of equations (A). We have the following

Proposition 4. *For $N' > N$, the identity relations*

$$\Phi_N(x) = \Phi_{N'}(x), \quad \Psi_N(x) = \Psi_{N'}(x) \tag{13.3}$$

hold provided that N is larger than a certain integer.

.

We are satisfied with the proof for the cases of $\gamma > 0$ and $\gamma = 0$. First,

$$\begin{aligned} y = \Phi_{N'}(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)) &= R_{(N')}\big(x, \frac{\hat{\eta}}{1-\varphi_{N'}}, \frac{\hat{\zeta}}{1-\psi_{N'}}\big) \\ &= \frac{\hat{\eta}}{1-\varphi_{N'}} + \sum_{j=0}^{2N'} R_j\big(x, \frac{\hat{\eta}}{1-\varphi_{N'}}, \frac{\hat{\zeta}}{1-\psi_{N'}}\big)\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^j = \frac{\hat{\eta}}{1-\varphi_{N'}} \\ &+ \sum_{j=0}^{2N} R_j\big(x, \frac{\hat{\eta}}{1-\varphi_{N'}}, \frac{\hat{\zeta}}{1-\psi_{N'}}\big)\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^j + \mathrm{O}\{\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^{2N+1}\}. \end{aligned} \tag{13.4}$$

We obtain a similar formula for $z = \Psi_{N'}(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x))$. Solve the equations

$$\begin{aligned} \Phi_{N'}(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)) &= R_{(N)}(x, u, v), \\ \Psi_{N'}(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)) &= S_{(N)}(x, u, v) \end{aligned} \tag{13.5}$$

with respect to $\{u, v\}$. It is easy to see that the solution $\{u, v\}$ satisfies

$$u(x) = \frac{\eta(x)}{1-\varphi_{N'}} + \mathrm{O}\{\big(\frac{\eta(x)}{1-\varphi_{N'}}\big)^{2N+1}\}, v(x) = \frac{\zeta(x)}{1-\psi_{N'}} + \mathrm{O}\{\big(\frac{\eta(x)}{1-\varphi_{N'}}\big)^{2N+1}\}. \tag{13.6}$$

We will show that if we put

$$u(x) = \frac{\hat{\eta}(x)}{1-\hat{Y}}, \quad v(x) = \frac{\hat{\zeta}(x)}{1-\hat{Z}}, \tag{13.7}$$

the pair $\{\hat{Y}, \hat{Z}\}$ coincides with the solution $\{\varphi_N, \psi_N\}$ of equations (8.4):

$$\hat{Y}(x) = \varphi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)), \quad \hat{Z}(x) = \psi_N(x, \hat{w}(x), \hat{\eta}(x), \hat{\zeta}(x)), \tag{13.8}$$

which leads to relations (13.3).

To prove (13.8), we solve equations (13.6) and (13.7) with respect to $\{\hat{Y}, \hat{Z}\}$. Assume that $N' > N$. First, we have

$$1 - \hat{Y} = (1-\varphi_{N'})\big(1 + \mathrm{O}\{\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^{2N}\}\big) = 1 - \varphi_{N'} + \mathrm{O}\{\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^{2N}\}.$$

Hence,

$$\begin{aligned} \hat{Y} = \varphi_{N'} + \mathrm{O}\{\big(\frac{\hat{\eta}}{1-\varphi_{N'}}\big)^{2N}\} &= \mathrm{O}\big(\max\{|\hat{\eta}|^{N'}, \frac{|\hat{\eta}|^{N'+1}}{|\hat{\zeta}|}\}\big) + \mathrm{O}(|\hat{\eta}|^{2N}) \\ &= \mathrm{O}\big(\max\{|\hat{\eta}(x)|^N, \frac{|\hat{\eta}(x)|^{N+1}}{|\hat{\zeta}(x)|}\}\big). \end{aligned}$$

Similarly, $1-\hat{Z}=(1-\psi_{N'})\big(1+(\dfrac{\hat{\zeta}}{1-\psi_{N'}})^{-1}\mathrm{O}\{(\dfrac{\hat{\eta}}{1-\varphi_{N'}})^{2N+1}\}\big)$.
Hence,

$$\begin{aligned}\hat{Z}&=\psi_{N'}+\mathrm{O}\big(\frac{|\hat{\eta}|^{2N+1}}{|\zeta|}\big)=\mathrm{O}\big(\max\{|\hat{\eta}|^{N'},\frac{|\hat{\eta}|^{N'+1}}{|\hat{\zeta}|}\}\big)+\mathrm{O}\big(\frac{|\hat{\eta}|^{2N+1}}{|\hat{\zeta}|}\big)\\&=\mathrm{O}\big(\max\{|\hat{\eta}(x)|^{N},\frac{|\hat{\eta}(x)|^{N+1}}{|\hat{\zeta}(x)|}\}\big).\end{aligned}$$

This shows that the pair $\{\hat{Y},\hat{Z}\}$ is a solution of truncated equations (8.4) satisfying the order condition $\max\{|\hat{Y}|,|\hat{Z}|\}=\mathrm{O}\big(\max\{|\hat{\eta}(x)|^{N},\frac{|\hat{\eta}(x)|^{N+1}}{|\hat{\zeta}(x)|}\}\big)$. But, by virtue of the uniqueness of our solution, such a solution must coincide with the solution $\{\varphi_N(x,\hat{w}(x),\hat{\eta}(x),\hat{\zeta}(x)),\psi_N(x,\hat{w}(x),\hat{\eta}(x),\,\hat{\zeta}(x))\}$, which proves (13.8).

Write as $\Phi(x,w,\eta,\zeta)=\Phi_N(x,w,\eta,\zeta)$, $\Psi(x,w,\eta,\zeta)=\Psi_N(x,w,\eta,\zeta)$. Define $\Phi_0(x,\eta,\zeta)$ and $\Psi_0(x,\eta,\zeta)$ by

$$\text{(13.9)}\quad\begin{aligned}&\Phi_0(x,\eta,\zeta)=\Phi(x,\frac{\eta\zeta}{x},\eta,\zeta),\ \Psi_0(x,\eta,\zeta)=\Psi(x,\frac{\eta\zeta}{x},\eta,\zeta)\ (\gamma\geq 0),\\&\Phi_0(x,\eta,\zeta)=\Phi(x,\frac{\eta\zeta}{x}+\frac{\gamma}{\delta},\eta,\zeta),\ \Psi_0(x,\eta,\zeta)=\Psi(x,\frac{\eta\zeta}{x}+\frac{\gamma}{\delta},\eta,\zeta)\ (\gamma<0).\end{aligned}$$

Then we assert that *the $\Phi_0(x,\eta,\zeta)$ and $\Psi_0(x,\eta,\zeta)$ are holomorphic and bounded functions in (x,η,ζ) in a domain of the form*

$$\begin{cases}|\arg x\mp\frac{\pi}{2}|<\pi-\varepsilon,\quad 0<|x|<a_0,\\ |\eta|<c_0,\quad|\zeta|<c_0,\quad|\eta\zeta|<d_0|x|,\quad(\text{for}\quad\gamma>0),\end{cases}$$

$$\begin{cases}|\arg x\mp\frac{\pi}{2}|<\pi-\varepsilon,\quad 0<|x|<a_0,\\ |\eta|<c_0,\quad|\zeta|<c_0,\quad\big|\frac{\eta\zeta}{x}+\frac{\gamma}{\delta}\big|<d_0,\quad(\text{for}\quad\gamma<0),\end{cases}$$

$$\begin{cases}\mu'<\arg x<\mu,\quad 0<|x|<a_0,\quad|\eta|<c_0,\quad|\zeta|<c_0,\\ -\mu'-\frac{\pi}{2}<\arg\big(\frac{\eta\zeta}{x}\big)<-\mu+\frac{3\pi}{2},\quad 0<|\eta\zeta|<d_0|x|,\quad(\text{for}\quad\gamma=0).\end{cases}$$

To prove this fact, choose a point (x_0,η_0,ζ_0) in an arbitrary way from one of these three domains. When $\gamma>0$, choose a positive integer N so large that $\frac{|\eta_0|^N}{|x_0\zeta_0|}<\Delta$. The function $\Phi_0(x,\eta,\zeta)$ is expressed by $\Phi_0(x,\eta,\zeta)=\Phi_N(x,\frac{\eta\zeta}{x},\eta,\zeta)$, where $\Phi_N(x,w,\eta,\zeta)$ is defined by formula (13.1) or (13.2) according as $\gamma\geq 0$ or $\gamma<0$. And, the $R_{(N)}(x,u,v)$ is holomorphic and bounded in (x,u,v) in a domain of the form $|\arg x\mp\frac{\pi}{2}|<\pi-\varepsilon$, $0<|x|<a_0$, $|u|<b_0$, $|v|<b_0$. However, in the case of $\gamma>0$, for example, the $\varphi_N(x,w,\eta,\zeta)$ and $\psi_N(x,w,\eta,\zeta)$, considered as functions of (x,w,η,ζ), are holomorphic and bounded in a neighborhood of the point (x_0,w_0,η_0,ζ_0) with $w_0=\frac{\eta_0\zeta_0}{x_0}$. Hence, the $\varphi_N(x,\frac{\eta\zeta}{x},\eta,\zeta)$ and $\psi_N(x,\frac{\eta\zeta}{x},\eta,\zeta)$, considered as functions of (x,η,ζ), are holomorphic in a neighborhood of the point (x_0,η_0,ζ_0).

Thus, it turns out that the $\Phi_0(x,\eta,\zeta)$ and $\Psi_0(x,\eta,\zeta)$ become holomorphic at the point (x_0,η_0,ζ_0).

The case of $\gamma = 0$ or $\gamma < 0$ can be treated in quite a same way. This completes the proofs of our main theorems in §**1**, namely Theorem 5.B_+, Theorem 5.B_- and Theorem 5.B_0.

References

M. Hukuhara

[1] Sur les propriétés asymptotiques des solutions d'un système d'équations différentielles linéaires contenant un paramètre. Mem. Fac. Engrg. Kyushu Imp. Univ.,**8** (1937), pp.249-280.

[2] Sur l'existence des points invariants d'une transformation dans l'espece fonctionnel. Japanese J. Math. **20**(1950),pp.1-4.

[3] (avec T.Kimura et T. Matuda) Équations différentielles ordinaires du premier ordre dans le champ complex. Publ. Math. Soc. Japan **7** (1961).

M.Iwano

[1] Analytic integration of a system of nonlinear ordinary differential equations with an irregular type singular point, Ann. Pura Appl.,(IV), **82**(1969), pp.189-256.

[2] New study on the convergence of a formal transformations, I, Funkcial. Ekvac., **38** (1995),pp.443-472.

[3] New study on the convergence of a formal transformations, II, Tôhoku Math. Jour. **47** (1995),pp.475-4497.

[4] General solution of a nonliear (m+n)-system with an irregular type singularity without Poincaré condition, Ibid. **172** (1997), pp.257-285.

[5] Simplified equations of a nonlinear 2-system with commensurable coefficients in the leading terms at an irregular type singularity without Poincaré condition.